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Computation of tight closure

Lecture 2

Geometric interpretation in dimension two

We will restrict now to the two-dimensional normal graded case in order to work on the corresponding smooth projective curve.

Let R be a two-dimensional standard-graded normal domain over an algebraically closed field K. Let $C = \text{Proj } R$ be the corresponding smooth projective curve and let

$$I = (f_1, \ldots, f_n)$$

be an R_+-primary homogeneous ideal with generators of degrees $d_1, \ldots, d_n$. Then we get on C the short exact sequence

$$0 \longrightarrow \text{Syz}\,(f_1, \ldots, f_n)(m) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_C(m - d_i) \stackrel{f_1,\ldots,f_n}{\longrightarrow} \mathcal{O}_C(m) \longrightarrow 0\,.$$

Here $\text{Syz}\,(f_1, \ldots, f_n)(m)$ is a vector bundle, called the *syzygy bundle*, of rank $n-1$ and of degree

$$((n-1)m - \sum_{i=1}^{n} d_i) \deg (C)\,.$$

Thus a homogeneous element f of degree m defines a cohomology class $\delta(f) \in H^1(C, \text{Syz}\,(f_1, \ldots, f_n)(m))$. Again we have that $f \in I^*$ if and only if $\delta(f)$ is tightly zero. The advantages to work on the projective curve are:

(1) We may work in dimension 1.
(2) The projective curve is smooth, we do not need to worry about singularities.
(3) We can use the well-developed theory of vector bundles on curves, in particular the notion of degree, of semistable bundles and the existence of moduli spaces.
(4) We can use ampleness results. Tight closure is then related to positivity and negativity of bundles.
(5) We can work within projective bundles, so that everything can be embedded into a smooth projective situation.

The following example shows already that one can not expect a sharp degree bound for primary non-parameter ideals. It also shows that one can compute tight closure whenever we have a nice decomposition of the syzygy bundle. This is always the case when the ideal has finite projective dimension. The notion of the strong Harder-Narasimhan filtration which we introduce below is a replacement of such a decomposition.

EXAMPLE 2.1. We consider the ideal $I = (x^4, y^4, xy^3)$ in $S = K[x, y]$ and in finite graded extensions $S \subseteq R$ (e.g. $R = K[x, y, z]/(F)$, where F is a homogeneous integral equation for z over $K[x, y]$) and describe an algorithm to compute the tight closure I^*. The graded resolution of the ideal is

$$0 \longrightarrow S(-5) \oplus S(-7) \longrightarrow S(-4) \oplus S(-4) \oplus S(-4) \stackrel{x^4,y^4,xy^3}{\longrightarrow} S \longrightarrow S/I \longrightarrow 0\,,$$

where the map on the left is given by sending the generators to

$$(0, x, -y) \text{ and } (y^3, 0, -x^3)\,.$$

On the projective line this corresponds to

$$0 \longrightarrow \mathcal{O}(-5) \oplus \mathcal{O}(-7) = \operatorname{Syz}(x^4, y^4, xy^3) \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-4) \oplus \mathcal{O}(-4) \stackrel{x^4,y^4,xy^3}{\longrightarrow} \mathcal{O} \longrightarrow 0.$$

This sequence is also exact over the curve $C = \operatorname{Proj}(R)$ and can be used to compute the tight closure of the ideal in R. A homogeneous element $h \in R$ of degree m yields a cohomology class in

$$H^1(C, \operatorname{Syz}(x^4, y^4, xy^3)(m)) \cong H^1(C, \mathcal{O}_C(m-5)) \oplus H^1(C, \mathcal{O}_C(m-7))\,,$$

which can be easily computed using Čech cohomology. On $D_+(x)$, h comes from $\left(\frac{h}{x^4}, 0, 0\right)$ and on $D_+(y)$ it comes from $\left(0, \frac{h}{y^4}, 0\right)$. Their difference, the syzygy

$$\left(\frac{h}{x^4}, -\frac{h}{y^4}, 0\right)$$

equals

$$-\frac{h}{xy^4}\,(0, x, -y) + \frac{h}{x^4y^3}\,(y^3, 0, -x^3)\,.$$

Hence the components of this cohomology class are

$$-\frac{h}{xy^4} \in H^1(C, \mathcal{O}_C(m-5)) \text{ and } \frac{h}{x^4y^3} \in H^1(C, \mathcal{O}_C(m-7))\,.$$

Therefore whether h belongs to the tight closure of I depends on these two components, which both correspond to a parameter situation.

First of all, if $m \geq 7$, then both degrees are non-negative and therefore these classes are tightly 0 by (the proof of) Fakt *****. If $m = 6$, we only have to look at the second component inside $H^1(C, \mathcal{O}_C(-1))$. For the monomial y^3z^3 the second component is 0 (independent of F), hence it belongs to the tight closure, though the first component need not be 0. The monomial $x^2y^2z^2$ yields $\frac{z^2}{x^2y}$, which is not 0 unless the equation has low degree. This class is (with some exceptions in small characteristics) not tightly 0. Hence $x^2y^2z^2$ does not belong to the tight closure. For $m = 5$, still only the second component is interesting, therefore y^3z^2 belongs to the tight closure, but

xy^2z^2 not (under the same restrictions). For $m \leq 4$ both components lie in negative degree, so an element will belong to the tight closure only if it belongs to the ideal itself. For the element y^3z the second component is 0, but not the first component.

Torsors

A cohomology class $c \in H^1(C, \mathcal{S})$ in a locally free sheaf $\mathcal{S}$ has a geometric realization (or a geometric model), namely a so-called $\mathcal{S}$-torsor (or a principal fiber bundle). This is an affine-linear bundle over C on which $\mathcal{S}$ acts by translations. The relation between cohomology classes and $\mathcal{S}$-torsors work over any noetherian separated scheme by a general construction. We mention an alternative description of the torsor corresponding to a first cohomology class in a locally free sheaf which is better suited for the projective situation.

Remark 2.2. Let $\mathcal{S}$ denote a locally free sheaf on a scheme X. For a cohomology class $c \in H^1(X, \mathcal{S})$ one can construct a geometric object: Because of $H^1(X, \mathcal{S}) \cong \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{S})$, the class defines an extension

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}' \longrightarrow \mathcal{O}_X \longrightarrow 0\,.$$

This extension is such that under the connecting homomorphism of cohomology, $1 \in \Gamma(X, \mathcal{O}_X)$ is sent to $c \in H^1(X, \mathcal{S})$. The extension yields projective subbundles[1]

$$\mathbb{P}(\mathcal{S}^\vee) \subset \mathbb{P}(\mathcal{S}'^\vee)\,.$$

If V is the corresponding geometric vector bundle, one may think of $\mathbb{P}(\mathcal{S}^\vee)$ as $\mathbb{P}(V)$ which consists for every base point $x \in X$ of all the lines in the fiber V_x running through the zero point. The projective subbundle $\mathbb{P}(V)$ has codimension one inside $\mathbb{P}(V')$, for every point it is a projective space lying (linearly) inside a projective space of one dimension higher. The complement is then over every point an affine space. One can show that the global complement

$$T = \mathbb{P}(\mathcal{S}'^\vee) \setminus \mathbb{P}(\mathcal{S}^\vee)$$

is another model for the torsor given by the cohomology class. The advantage of this viewpoint is that we may work, in particular when X is projective, in an entirely projective setting.

Properties of a cohomology class are equivalent to geometric properties of the corresponding torsor T. The property of being tightly zero (itself equivalent to $f \in I^*$, if the cohomology class is $\delta(f)$) is equivalent to the property that T is not an affine variety (i.e. not isomorphic to the spectrum of a ring). This rests on interpretation of tight closure as solid closure. For this

[1] $\mathcal{S}^\vee$ denotes the dual bundle. According to our convention, the geometric vector bundle corresponding to a locally free sheaf $\mathcal{T}$ is given by Spec $(\oplus_{k \geq 0} S^k(\mathcal{T}))$ and the projective bundle is Proj $(\oplus_{k\geq 0} S^k(\mathcal{T}))$, where S^k denotes the kth symmetric power.

(non) affineness property, positivity (ampleness) properties of the bundle are crucial.

Semistability of vector bundles

For inclusion and exclusion results we need the concept of (Mumford) semistability.

DEFINITION 2.3. Let $\mathcal{S}$ be a vector bundle on a smooth projective curve C. It is called *semistable*, if $\frac{\deg(\mathcal{T})}{\operatorname{rk}(\mathcal{T})} \leq \frac{\deg(\mathcal{S})}{\operatorname{rk}(\mathcal{S})}$ for all subbundles $\mathcal{T}$.

Suppose that the base field has positive characteristic $p > 0$. Then $\mathcal{S}$ is called *strongly semistable*, if all (absolute) Frobenius pull-backs $F^{e*}(\mathcal{S})$ are semistable.

An important property of a semistable bundle of negative degree is that it can not have any global section $\neq 0$. Note that a semistable vector bundle need not be strongly semistable, the following is probably the simplest example.

EXAMPLE 2.4. Let C be the smooth Fermat quartic given by $x^4+y^4+z^4$ and consider on it the syzygy bundle $\operatorname{Syz}(x,y,z)$ (which is also the restricted cotangent bundle from the projective plane). This bundle is semistable. Suppose that the characteristic is 3. Then its Frobenius pull-back is $\operatorname{Syz}(x^3,y^3,z^3)$. The curve equation gives a global nontrivial section of this bundle of total degree 4. But the degree of $\operatorname{Syz}(x^3,y^3,z^3)(4)$ is negative, hence it can not be semistable anymore.

EXAMPLE 2.5. Let $R = K[x,y,z]/(x^3+y^3+z^3)$, where K is a field of positive characteristic $p \neq 3$, $I = (x^2,y^2,z^2)$, and $C = \operatorname{Proj}(R)$. The equation $x^3 + y^3 + z^3 = 0$ yields the short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \operatorname{Syz}(x^2,y^2,z^2)(3) \longrightarrow \mathcal{O}_C \longrightarrow 0\,.$$

This shows that $\operatorname{Syz}(x^2,y^2,z^2)$ is strongly semistable.

For a strongly semistable vector bundle $\mathcal{S}$ on C and a cohomology class $c \in H^1(C,\mathcal{S})$ with corresponding torsor we obtain the following affineness criterion.

THEOREM 2.6. *Let C denote a smooth projective curve over an algebraically closed field K and let $\mathcal{S}$ be a strongly semistable vector bundle over C together with a cohomology class $c \in H^1(C,\mathcal{S})$. Then the torsor $T(c)$ is an affine scheme if and only if* $\deg(\mathcal{S}) < 0$ *and $c \neq 0$ ($F^e(c) \neq 0$ for all e in positive characteristic*[2]*).*

[2] Here one has to check only finitely many es and there exist good estimates how far one has to go. Also, in a relative situation, this is only an extra condition for finitely many prime numbers.

This result rests on the ampleness of $\mathcal{S}'^{\vee}$ occuring in the dual exact sequence $0 \to \mathcal{O}_C \to \mathcal{S}'^{\vee} \to \mathcal{S}^{\vee} \to 0$ given by c (work of Hartshorne and Gieseker). It implies for a strongly semistable syzygy bundle the following *degree formula* for tight closure.

THEOREM 2.7. *Suppose that* $\operatorname{Syz}(f_1, \ldots, f_n)$ *is strongly semistable. Then*

$$R_m \subseteq I^* \text{ for } m \geq \frac{\sum d_i}{n-1} \text{ and (for almost all prime numbers) } R_m \cap I^* \subseteq I \text{ for } m < \frac{\sum d_i}{n-1}.$$

We indicate the proof of the inclusion result. The degree condition implies that $c = \delta(f) \in H^1(C, \mathcal{S})$ is such that $\mathcal{S} = \operatorname{Syz}(f_1, \ldots, f_n)(m)$ has nonnegative degree. Then also all Frobenius pull-backs $F^*(\mathcal{S})$ have nonnegative degree. Let $\mathcal{L} = \mathcal{O}(k)$ be a twist of the tautological line bundle on C such that its degree is larger than the degree of ω_C^{-1}, the dual of the canonical sheaf. Let $z \in H^0(C, \mathcal{L})$ be a non-zero element. Then $zF^{e*}(c) \in H^1(C, F^{e*}(\mathcal{S}) \otimes \mathcal{L})$, and by Serre duality we have

$$H^1(C, F^{e*}(\mathcal{S}) \otimes \mathcal{L}) \cong H^0(C, F^{e*}(\mathcal{S}^*) \otimes \mathcal{L}^{-1} \otimes \omega_C)^{\vee}.$$

On the right hand side we have a semistable sheaf of negative degree, which can not have a nontrivial section. Hence $zF^{e*} = 0$ and therefore f belongs to the tight closure.

Harder-Narasimhan filtration

In general, there exists an exact criterion depending on c and the *strong Harder-Narasimhan filtration* of $\mathcal{S}$. For this we give the definition of the Harder-Narasimhan filtration.

DEFINITION 2.8. Let $\mathcal{S}$ be a vector bundle on a smooth projective curve C over an algebraically closed field K. Then the (uniquely determined) filtration

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_{t-1} \subset \mathcal{S}_t = \mathcal{S}$$

of subbundles such that all quotient bundles $\mathcal{S}_k/\mathcal{S}_{k-1}$ are semistable with decreasing slopes $\mu_k = \mu(\mathcal{S}_k/\mathcal{S}_{k-1})$, is called the *Harder-Narasimhan filtration* of $\mathcal{S}$.

The number μ_1 is called the *maximal slope* and the number μ_t is called the *minimal slope* of $\mathcal{S}$. In Example 2.1 the Harder-Narasimhan filtration of $\operatorname{Syz}(x^4, y^4, xy^3)$ is

$$\mathcal{O}_C(-5) \subset \operatorname{Syz}(x^4, y^4, xy^3),$$

the quotient is $\mathcal{O}_C(-7)$. The (strong) Harder-Narasimhan filtration is a replacement for the easy decomposition we had in this example.

The Harder-Narasimhan filtration exists uniquely (by a Theorem of Harder and Narasimhan). A Harder-Narasimhan filtration is called *strong* if all the quotients $\mathcal{S}_k/\mathcal{S}_{k-1}$ are strongly semistable. A Harder-Narasimhan filtration is not strong in general, however, by a Theorem of A. Langer, there exists some Frobenius pull-back $F^{e*}(\mathcal{S})$ such that its Harder-Narasimhan filtration is strong.

THEOREM 2.9. *Let C denote a smooth projective curve over an algebraically closed field K and let $\mathcal{S}$ be a vector bundle over C together with a cohomology class $c \in H^1(C, \mathcal{S})$. Let*

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots \subset \mathcal{S}_{t-1} \subset \mathcal{S}_t = \mathcal{S}$$

be a strong Harder-Narasimhan filteration. Then the torsor $T(c)$ is an affine scheme if and only if the following (inductively defined property starting with t) holds: there is an i such that $\deg(\mathcal{S}_i/\mathcal{S}_{i-1}) < 0$ and the image of c in this sheaf is $\neq 0$ (and also the Frobenius pull-backs of this class are $\neq 0$).

REMARK 2.10. A notion of strong semistabilty exists also on higher dimensional normal projective varieties (depending on a polarization). If Syz_d is the top-dimensional syzygy bundle for a homogeneous ideal $I \subseteq R$ and m is such that the minimal slope of $\mathrm{Syz}_d(m)$ is nonnegative, then $R_{\geq m} \subseteq I^*$. It is however more difficult to determine the degree of this syzygy bundle and to obtain exclusion results. Even if this bundle is strongly semistable there is no known inclusion/exclusion degree bound.

We describe two important consequences from this characterization of tight closure in terms of vector bundles.

Plus closure

Recall that the plus closure I^+ of an ideal I is given by $f \in I^+$ if and only if there exists a finite extension (of domains) $R \subseteq S$ such that $f \in IS$. In terms of the torsor this is equivalent to the property that there exists a projective curve inside the torsor, or that the corresponding cohomology class can be annihilated by a finite morphism of projective curves. Over a finite field, the same criterion along the strong Harder-Narasimhan filtration which holds for tight closure also holds for (graded) plus closure. Therefore we get.

THEOREM 2.11. *Let R be a standard-graded, two-dimensional normal domain over (the algebraic closure of) a finite field. Let I be an R_+-primary graded ideal. Then*

$$I^* = I^+ .$$

This is also true for non-primary graded ideals and also for submodules in finitely generated graded submodules. Moreover, G. Dietz has shown that one can get rid also of the graded assumption (of the ideal or module, but not of the ring).

Test exponents

The problem with an algorithmic computation of tight closure is that we have to check infinitely many conditions. For a test element z (a well established theory) a *test exponent* is a number e_0 such that $zf^q \in I^{[q]}$ for all $q = p^e$ and $e \leq e_0$ implies $f \in I^*$. This makes also sense for a restricted class of ideals. But even for parameter ideals nothing substantial is known.

The following variant is more promising and has the same computational effect: Let τ denote the test ideal of R. We call e_0 a *test ideal exponent* (for a class of ideals) if

$$zf^q \in I^{[q]} \text{ for all } z \in \tau$$

and for all $q = p^e$ and $e \leq e_0$ implies $f \in I^*$. For this one has to know the test ideal, but this is known in many cases. For the class of parameter ideals in the Gorenstein case this works, because then $I^* = (I : \tau)$ and so we can take even 0 as test ideal exponent.

The methods from above allow us to extend this to homogeneous primary ideals in a standard-graded two-dimensional domain over a finite field. The test ideal exponent is however huge and not suitable for computations. It depends on the genus, the number of ideal generators and most importantly on the number of elements in the field (via the finite number of semistable bundles in the moduli space).

THEOREM 2.12. *Let R be a standard-graded, two-dimensional (geometrically) normal Gorenstein domain over a finite field. Fix n and suppose that $p \geq 4(g-1)(n-1)^3$, where g is the genus of the curve. Then there exists a test ideal component for the class of primary homogeneous ideals generated by n elements.*

The finite field assumption in the last two statements is necessary. Both proofs rely on the fact that for fixed rank and degree there exists only finitely many semistable sheaves defined over the field with these numerical data.