Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 27

Warm-up-exercises

EXERCISE 27.1. Determine for which $a \in \mathbb{R}$ the function

$$a \longmapsto \int_{-1}^{2} at^2 - a^2 t \, dt$$

has a maximum or a minimum.

EXERCISE 27.2. According to recent studies the student's attention skills during the day are described by the following function

$$[8, 18] \longrightarrow \mathbb{R}, \, x \longmapsto f(x) = -x^2 + 25x - 100.$$

Here x is the time in hours and $y = f(x)$ is the attention measured in micro-credit points per second. When should one start a one and a half hour lecture, such that the total attention skills are optimal? How many micro-credit points will be added during this lecture?

EXERCISE 27.3. Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

be an increasing function and $b \in \mathbb{R}$. Show that the sequence $f(n)$, $n \in \mathbb{N}$, converges to b if and only if

$$\lim_{x \to +\infty} f(x) = b$$

holds, i.e. if the limit of the function for $x \to +\infty$ is b.

EXERCISE 27.4. Let I be an interval, r a boundary point of I and

$$f : I \longrightarrow \mathbb{R}$$

a continuous function. Prove that the existence of the improper integral

$$\int_{a}^{r} f(t) \, dt$$

does not depend on the choice of the starting point $a \in I$.

EXERCISE 27.5. Let $I =]r, s[$ be a bounded open interval and

$$f : I \longrightarrow \mathbb{R}$$

a continuous function, which can be extended continuously to $[r, s]$. Prove that the improper integral

$$\int_r^s f(t)\, dt$$

exists.

EXERCISE 27.6. Formulate and prove computation rules for improper integrals (analogous to Lemma 23.5).

EXERCISE 27.7. Decide whether the improper integral

$$\int_1^\infty \frac{x^2 - 3x + 5}{x^4 + 2x^3 + 5x + 8}\, dx$$

exists.

EXERCISE 27.8. Determine the improper integral

$$\int_0^\infty e^{-t}\, dt\,.$$

EXERCISE 27.9. Let $I = [a, b]$ be a bounded interval and let

$$f :]a, b[\longrightarrow \mathbb{R}$$

be a continuous function. Let $(x_n)_{n\in\mathbb{N}}$ be a decreasing sequence in I with limit a and let $(y_n)_{n\in\mathbb{N}}$ be an increasing sequence in I with limit b. Assume that the improper integral $\int_a^b f(t)\, dt$ exists. Prove that the sequence

$$w_n = \int_{x_n}^{y_n} f(t)\, dt$$

converges to this improper integral.

Hand-in-exercises

EXERCISE 27.10. (2 points)

Compute the energy that would be necessary to move the Earth, starting from the current position relative to the Sun, infinitely far away from the Sun.

Exercise 27.11. (5 points)

Decide whether the improper integral

$$\int_0^\infty \frac{1}{(x+1)\sqrt{x}}\, dx$$

exists and compute it in case of existence.

Exercise 27.12. (5 points)

Give an example of a not bounded, continuous function

$$f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0},$$

such that the improper integral $\int_0^\infty f(t)\, dt$ exists.

Exercise 27.13. (2 points)

Decide whether the improper integral

$$\int_{-1}^1 \frac{1}{\sqrt{1-t^2}}\, dt$$

exists and compute it in case of existence.

Exercise 27.14. (4 points)

Decide whether the improper integral

$$\int_1^\infty \frac{x^3 - 3x + 5}{x^4 + 2x^3 + 5x + 8}\, dx$$

exists.

Exercise 27.15. (6 points)

Decide whether the improper integral

$$\int_0^\infty \frac{\sin x}{x}\, dx$$

exists.

(Do not try to find an antiderivative for the integrand.)