Prof. Dr. H. Brenner **Medellin 2012**

Vector bundles, forcing algebras and local cohomology

Lecture 4

Forcing algebras and induced torsors

As T_U is a V_U-torsor, and as every V-torsor is represented by a unique cohomology class, there should be a natural cohomology class coming from the forcing data. To see this, let R be a noetherian ring and $I = (f_1, \ldots, f_n)$ be an ideal. Then on $U = D(I)$ we have the short exact sequence

$$0 \longrightarrow \operatorname{Syz}(f_1, \ldots, f_n) \longrightarrow \mathcal{O}_U^n \longrightarrow \mathcal{O}_U \longrightarrow 0\,.$$

An element $f \in R$ defines an element $f \in \Gamma(U, \mathcal{O}_U)$ and hence a cohomology class $\delta(f) \in H^1(U, \operatorname{Syz}(f_1, \ldots, f_n))$. Hence f defines in fact a $\operatorname{Syz}(f_1, \ldots, f_n)$-torsor over U. We will see that this torsor is induced by the forcing algebra given by $f_1, \ldots, f_n$ and f.

Theorem 4.1. *Let R denote a noetherian ring, let $I = (f_1, \ldots, f_n)$ denote an ideal and let $f \in R$ be another element. Let $c \in H^1(D(I), \operatorname{Syz}(f_1, \ldots, f_n))$ be the corresponding cohomology class and let $B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f)$ denote the forcing algebra for these data. Then the scheme $\operatorname{Spec}(B)|_{D(I)}$ together with the natural action of the syzygy bundle on it is isomorphic to the torsor given by c.*

Proof. We compute the cohomology class $\delta(f) \in H^1(U, \operatorname{Syz}(f_1, \ldots, f_n))$ and the cohomology class given by the forcing algebra. For the first computation we look at the short exact sequence

$$0 \longrightarrow \operatorname{Syz}(f_1, \ldots, f_n) \longrightarrow \mathcal{O}_U^n \longrightarrow \mathcal{O}_U \longrightarrow 0\,.$$

On $D(f_i)$, the element f is the image of $(0, \ldots, 0, \frac{f}{f_i}, 0, \ldots, 0)$ (the non-zero entry is at the ith place). The cohomology class is therefore represented by the family of differences

$$(0, \ldots, 0, \frac{f}{f_i}, 0, \ldots, 0, -\frac{f}{f_j}, 0, \ldots, 0) \in \Gamma(D(f_i) \cap D(f_j), \operatorname{Syz}(f_1, \ldots, f_n))\,.$$

On the other hand, there are isomorphisms

$$V|_{D(f_i)} \longrightarrow T|_{D(f_i)},\ (s_1, \ldots, s_n) \longmapsto (s_1, \ldots, s_{i-1}, s_i + \frac{f}{f_i}, s_{i+1}, \ldots, s_n).$$

The difference of two such isomorphisms on $D(f_if_j)$ is the same as before. $\square$

EXAMPLE 4.2. Let $(R, \mathfrak{m})$ denote a two-dimensional normal local noetherian domain and let f and g be two parameters in R. On $D(\mathfrak{m})$ we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_U \cong \mathrm{Syz}\,(f,g) \longrightarrow \mathcal{O}_U^2 \xrightarrow{f,g} \mathcal{O}_U \longrightarrow 0$$

and its corresponding long exact sequence of cohomology,

$$0 \longrightarrow R \longrightarrow R^2 \xrightarrow{f,g} R \xrightarrow{\delta} H^1(U, \mathcal{O}) \longrightarrow \ldots\,.$$

The connecting homomorphisms δ sends an element $h \in R$ to $\frac{h}{fg}$. The torsor given by such a cohomology class $c = \frac{h}{fg} \in H^1(U, \mathcal{O}_X)$ can be realized by the forcing algebra

$$R[T_1, T_2]/(fT_1 + gT_2 - h)\,.$$

Note that different forcing algebras may give the same torsor, because the torsor depends only on the spectrum of the forcing algebra restricted to the punctured spectrum of R. For example, the cohomology class $\frac{1}{fg} = \frac{fg}{f^2g^2}$ defines one torsor, but the two quotients yield the two forcing algebras $R[T_1, T_2]/(fT_1 + gT_2 - 1)$ and $R[T_1, T_2]/(f^2T_1 + g^2T_2 - fg)$, which are quite different. The fiber over the maximal ideal of the first one is empty, whereas the fiber over the maximal ideal of the second one is a plane.

If R is regular, say $R = K[X, Y]$ (or the localization of this at (X, Y) or the corresponding power series ring) then the first cohomology classes are K-linear combinations of $\frac{1}{x^iy^j}$, $i, j \geq 1$. They are realized by the forcing algebras $K[X, Y]/(X^iT_1 + Y^jT_2 - 1)$. Since the fiber over the maximal ideal is empty, the spectrum of the forcing algebra equals the torsor. Or, the other way round, the torsor is itself an affine scheme.

The closure operations we have considered in the second lecture can be characterized by some property of the forcing algebra. However, they can not be characterized by a property of the corresponding torsor alone. For example, for $R = K[X, Y]$, we may write

$$\frac{1}{XY} = \frac{X}{X^2Y} = \frac{XY}{X^2Y^2} = \frac{X^2Y^2}{X^3Y^3},$$

so the torsors given by the forcing algebras

$$R[T_1, T_2]/(XT_1 + YT_2 + 1),$$
$$R[T_1, T_2]/(X^2T_1 + YT_2 + X),$$
$$R[T_1, T_2]/(X^2T_1 + Y^2T_2 + XY) \text{ and}$$
$$R[T_1, T_2]/(X^3T_1 + Y^3T_2 + X^2Y^2)$$

are all the same (the restriction over $D(X, Y)$), but there global properties are quite different. We have a non-surjection, a surjective non submersion, a submersion which does not admit (for $K = \mathbb{C}$) a continuous section and a map which admits a continuous section.