A Discourse concerning the Measure of the Airs *resistance to Bodies moved in it. By the Learned* John Wallis *S. T. D. & R. S. Soc.*

1. THat the Air (and the like of any other *Medium*) doth considerably give resistance to Bodies moved in it, (and doth thereby abate their Celerity and Force:) is generally admitted. And Experience doth attest it: For otherwise, a Cannon Bullet projected Horizontally, should (supposing the Celerity and Force undiminished) strike as hard against a perpendicular Wall, erected at a great distance, as near at hand: which we find it doth not.

2. But at what Rate, or in what Proportion, such resistance is; and (consequently, at what Rate the Celerity and Force is continually diminished) seems not to have been so well examined. Whence it is, that the Motion of a Project (secluding this Consideration) is commonly reputed to describe a Parabolick Line; as arising from an Uniform or equal Celerity in the Line of Projection, and a Celerity uniformly accelerated in the Line of Descent: which two so compounded, do create a Parabola.

3. In order to the computation hereof; I first premise this *Lemma*, (as the most rational that doth occur for my first footing,) That (supposing other things equal) the resistance is proportional to the Celerity. For in a double Celerity, there is to be removed (in the same time) twice as much Air, (which is a double Impediment) in a treble, thrice as much; and so in other Proportions.

4. Suppose we then the Force impressed (and consequently the Celerity, if there were no resistance) as 1; the resistance as r. (which must be less than the Force, or else the Force would not prevail over the Impediment, to create a Motion.) And therefore the effective Force at a first Moment, is to be reputed as $1-r$: That is, so much as the

the Force impreſſed, is more than the Impediment or Reſiſtance.

5. Be it as $1-r$ to 1; ſo 1 to m. (which m is therefore greater than 1.)

6. And therefore the effective Force (and conſequently the Celerity) as to a firſt Moment, is to be $\frac{1}{m}$ of what it would be, had there been no reſiſtance.

7. This $\frac{1}{m}$ is alſo the remaining Force after ſuch firſt Moment; and this remaining Force is (for the ſame Reaſon) to be proportionally abated as to a ſecond Moment: that is we are to take $\frac{1}{m}$ thereof, that is $\frac{1}{mm}$ of the impreſſed Force. And for a third Moment (at equal diſtance of time) $\frac{1}{mmm}$; for a fourth $\frac{1}{m^4}$; and ſo onward infinitely.

8. Becauſe the length diſpatched (in equal times) is proportional to the Celerities; the Lines of Motion (anſwering to thoſe equal Times)are to be as $\frac{1}{m}$, $\frac{1}{m^2}$, $\frac{1}{m^3}$, $\frac{1}{m^4}$, *&c.* of what they would have been, in the ſame Times, had there been no reſiſtance.

9. This therefore is a Geometrical Progreſſion; and (becauſe of m greater than 1) continually decreaſing.

10. This decreaſing Progreſſion infinitely continued (determining in the ſame point of Reſt, where the Motion is ſuppoſed to expire) is yet of a Finite Magnitude; and equal to $\frac{1}{m-1}$ of what it would have been in ſo much Time, if there had been no reſiſtance. As is demonſtrated in my Algebra, *Chap.* 95. *Prop.* 8. For (as I have elſewhere demonſtrated) the Sum or Aggregate of a Geometrical Progreſſion is $\frac{VR-A}{R-1}$ (ſuppoſing V the greateſt term, A the leaſt, and R the common multiplyer.) That is $\frac{VR}{R-1} - \frac{A}{R-1}$. Now in the preſent Caſe, (ſuppoſing the Progreſſion infinitely continued) the leaſt term A,

be-

becomes infinitely small, or = o. And consequently $\frac{A}{R-1}$ doth also vanish, and thereby the Aggregate becomes $=\frac{VR}{R-1}$. That is (as will appear by dividing VR by $R-1$;) $V+\frac{V}{R}+\frac{V}{RR}+\frac{V}{R^3}+$ &c. $=\frac{VR}{R-1}$; (supposing the Progression to begin at $V=1$.) That is (dividing all by R, that so the Progression may begin at $\frac{V}{R}=\frac{1}{m}$:) $\frac{V}{R-1}=\frac{V}{R}+\frac{V}{RR}+\frac{V}{R^3}+$ &c. That is, in our present Case (because of $V=1$, & $R=m$:) $\frac{1}{m}+\frac{1}{mm}+\frac{1}{m^3}$ &c. $=\frac{1}{m-1}$. That is, (putting $n=m-1$) $\frac{1}{n}$ of what it would have been if there had been no resistance.

$$
\begin{array}{l}
R-1)\,VR\,(\,V+\frac{V}{R},\ \frac{V}{RR},\ \&c.\\
\quad VR-V\\
\hline
\quad +V\\
\quad +V-\frac{V}{R}\\
\hline
\quad +\frac{V}{R}\\
\quad +\frac{V}{R}-\frac{V}{RR}\\
\hline
\quad +\frac{V}{RR}\\
\quad \&c.
\end{array}
$$

11. This infinite Progression is fitly expressed by an ordinate in the exterior Hyperbola, parallel to one of the Asymptotes; and the several Member of that, by the several Members of this, cut in continual Proportion. As is there demonstrated at *Prop.* 15. For let *S H*, (*vid.* Fig. III.) be an Hyperbola between the Asymptotes *A B*, *AF*: And let the ordinate *D H* (in the exteriour Hyperbola, parallel to *A F*,) represent the impressed force undiminished; or the Line to be described in such time, by a Celerity answerable to such undiminished force. And let *B S* (a like ordinate) be $\frac{1}{m}$ thereof; which therefore, being less than

D H

DH (as being equal to a Part of it) will be further than it from AF. In AB (which I put $=1$) let Bd be such a Part thereof, as is BS of DH. Now because (as is well known) all the inscribed Parellelograms, in the exteriour Hyperbola, AS, AH, &c. are equal; and therefore their sides reciprocal: Therefore as $Ad = 1 - \frac{1}{m}$ (supposing Bd to be taken, from B toward A,) to $AB = 1$, (or as $m - 1$ to m:) so is $BS = \frac{1}{m} DH$, to dh, which is therefore equal to $\frac{1}{m-1}$ of DH; that is (as will appear by dividing 1, by $m-1$,) to $\frac{1}{m} + \frac{1}{mm} + \frac{1}{m^3}$ &c. of DH.

$$m - 1)\ 1\ (\frac{1}{m} + \frac{1}{mm} + \frac{1}{m^3} + \text{\&c.}$$

$$\begin{array}{l} 1 - \frac{1}{m} \\ \hline +\frac{1}{m} \\ +\frac{1}{m} - \frac{1}{mm} \\ \hline +\frac{1}{mm} \\ +\frac{1}{mm} - \frac{1}{mmm} \\ \hline \text{\&c.} \quad +\frac{1}{mmm} \end{array}$$

Or if Bd be taken beyond B; then as $Ad = 1 + \frac{1}{m}$, to $AB = 1$, or as $m + 1$ to m, so is $\frac{1}{m} DH$ to dh, which is therefore equal to $\frac{1}{m+1}$ DH; that is (as will appear by like dividing of 1 by $m+1$;) $=$ to $\frac{1}{m} - \frac{1}{mm} + \frac{1}{m^3} -$ &c. of DH.

12. Let such ordinate dh, or (equal to it in the Asymptote) AF, be so divided in L, M, N &c. (by perpendiculars cutting the Hyperbola in l, m, n, &c.) as that FL, LM, MN be as $\frac{1}{m}$, $\frac{1}{mm}$, $\frac{1}{m^3}$ &c. That is, so continually decreasing, as that each antecedent be to its consequent, as 1 to $\frac{1}{m}$, or as m to 1. See *Fig.* IV

13. This is done by taking AF, AL, AN, &c. in such proportion. For, of continual proportionals the differences are also continually proportional, and in the same pro-

proportion. For let A, B, C, D, &c. be ſuch proportionals; and their differences a, b, c, &c. That is $A - B = a$, $B - C = b$, $C - D = c$, &c.

Then, becauſe A, B, C, D, &c. are in continual proport:
That is A. B :: B. C :: C. D :: &c.
And dividing A − B. B :: B − C. C :: C − D. D :: &c.
That is *a*. B :: *b*. C :: *d*. D :: &c.
And alternly *a*. *b*. *c*. &c. :: B. C. D. &c. :: A. B. C. &c.
That is, in continual proportion as A to B, or as *m* to 1.

14. This being done; the Hyperbolick ſpaces *Fl*, *Lm*, *Mn*, &c. are equal. As is demonſtrated by *Gregory San-Vincent*; and as ſuch is commonly admitted.

15. So that *Fl*, *Lm*, *Mn*, &c. may fitly repreſent equal times, in which are diſpatched unequal lengths, repreſented by *FL*, *LM*, *MN*, &c.

16. And becauſe they are in number infinite (though equal to a finite Magnitude) the duration is infinite: And conſequenlty the impreſſed force, and motion thence ariſing, never to be wholly extinguiſhed (without ſome further impediment) but perpetually approaching to *A*, in the nature of Aſymptotes.

17. The ſpaces *Fl*, *Fm*, *Fn*, &c. are therefore as Logarithms (in Arithmetical progreſſion increaſing) anſwering to the lines *AF*, *AL*, *AM*, &c.; or to *FL*, *LM*, *MN*, &c. in Geometrical progreſſion decreaſing,

18. Becauſe *FL*, *LM*, *MN*, &c. are as $\frac{1}{m}$, $\frac{1}{mm}$, $\frac{1}{m^3}$, &c (infinitely) terminated at *A*; therefore (by ¶ 10) their Aggregate *FA* or *dh*, is to *DH*, (ſo much length as would have been diſpatched, in the ſame time, by ſuch impreſſed force undiminiſhed) as 1 to $m - 1 = n$.

19. If therefore we take, as 1 to *n*, ſo *AF* to *DH*; this will repreſent the length to be diſpatched, in the ſame time, by ſuch undiminiſhed force.

20. And if ſuch *DH* be ſuppoſed to be divided into equal parts innumerable (and therefore infinitely ſmall;) theſe anſwer to thoſe (as many) parts unequal in *FA*, or *hd*.

21. But, what is the proportion of r to 1, or (which depends on it) of $1-r$ to 1, or 1 to m; remains to be inquired by experiment.

22. If the progreſſion be not infinitely continued; but end (ſuppoſe) at N, and its leaſt term be $A=MN$: then, out of $\frac{V}{R-1}=\frac{1}{m}+\frac{1}{mm}+\frac{1}{m^3}$, &c. is to be ſubducted $\frac{A}{R-1}$ (as at ¶ 10) that is (as by diviſion will appear) $\frac{A}{R}+\frac{A}{R^2}+\frac{A}{R^3}$ &c. That is (in our preſent caſe) $\frac{a}{m}+\frac{a}{mm}+\frac{a}{m^3}$ &c. And ſo the Aggregate will be $\frac{1-a}{m}+\frac{1-a}{mm}+\frac{1-a}{m^3}$ &c. $=\frac{1-a}{n}$.

And thus as to the line of Projection, in which (ſecluding the reſiſtance) the motion is reputed uniform; diſpatching equal lengths in equal times. Conſider we next the line of Deſcent.

23. In the Deſcent of Heavy Bodies, it is ſuppoſed, that to each moment of time, there is ſuperadded a new Impulſe of Gravity to what was before: And each of theſe, ſecluding the conſideration of the Airs reſiſtance, to proceed equally (from their ſeveral beginnings) through the ſucceeding moments. As (in the erect lines) 1 1 1 1 &c. 1 1 1 &c, 1 1 &c. 1 &c. and ſo con- 1
tinually as in the line of of Projection. 1 1

24. Hence ariſeth (in the tranſverſe lines) 1 1 1
for the firſt moment 1, for the ſecond $1+1$, 1 1 1 1
for the third $1+1+1$, and ſo forth, in A- &c.
rithmetical progreſſion: As are the Ordinates in a Triangle, at equal diſtance.

25. And ſuch are the continual increments of the Diameter, or of the ordinates in the exterior Parabola, anſwering to the interior Ordinates, or Segments of the Tangent

gent, equally increaſing. As is known, and commonly admitted.

26. If we take-in the conſideration of the Airs reſiſtence; we are then for each of theſe equal progreſſions, to ſubſtitute a decreaſing progreſſion Geometrical; in like manner (and for the ſame reaſons) as in the line of Projection.

27. Hence ariſeth, for the firſt moment $\frac{1}{m}$; for the ſecond $\frac{1}{m} + \frac{1}{m^2}$; for the third $\frac{1}{m} + \frac{1}{m^2} + \frac{1}{m^3}$ &c. And ſuch is therefore the Deſcent of a heavy Body falling by its own weight. The ſeveral impulſes of Gravity being ſuppoſed equal.

$$\begin{array}{llll} \frac{1}{m} & & & \\ \frac{1}{m^2} & \frac{1}{m} & & \\ \frac{1}{m^3} & \frac{1}{m^2} & \frac{1}{m} & \\ \frac{1}{m^4} & \frac{1}{m^3} & \frac{1}{m^2} & \frac{1}{m} \end{array}$$

28. That is (in the figure of ¶ 12) as *FL*, *FM*, *FN*, &c, in the line of Deſcent, anſwering to *FL*, *LM*, *MN*, &c. in the line of Projection.

29. But though the Progreſſions for the line of Projection, are like to each of thoſe many in the line of Deſcent: it is not to be thence inferred, that therefore $\frac{1}{m}$ in the one, is equal to $\frac{1}{m}$ in the other: But in the line of Projection (ſuppoſe) $\frac{1}{m}f$ (ſuch a part of the force impreſſed, and a celerity anſwerable:) in the line of Deſcent, $\frac{1}{m}g$ (ſuch a part of the Impulſe of Gravity.)

30. Thoſe for the line of Deſcent (of the ſame Body) are all equal, each to other: Becauſe *g* (the new Impulſe of Gravity) in each moment is ſuppoſed to be the ſame.

31. But what is the proportion of *f* to *g* (that of the force impreſſed, to the Impulſe of Gravity in each Body) remains to be enquired by Experiment.

32. This proportion being found as to one known force; the ſame is thence known as to any other force

(who's proportion to this is given) in the same uniform *Medium*.

33. And this being known as to one *Medium*; the same is thence known as to any other *Medium*, the proportion of who's resistance to that of this is known.

34. If a heavy body be projected downward in a pendicular line; it descends therefore at the rate $\frac{1}{m}$, $\frac{1}{mm}$, $\frac{1}{m^3}$, &c. of f (the impressed force) increased by $\frac{1}{m}$, $\frac{1}{m} + \frac{1}{m^2}$, $\frac{1}{m} + \frac{1}{m^2} + \frac{1}{m^3}$ &c. of g the impulse of Gravity : (by ¶ 7. & ¶ 27.) Because both forces are here united.

35. If in a perpendicular projection upwards; it ascends in the rate of the former, abated by that of the latter. Because here the impulse of Gravity is contrary to the force impressed.

36. When therefore this latter (continually increasing) becomes equal to that former (continually decreasing) it then ceaseth to ascend; and doth thenceforth descend at the rate wherein the latter continually exceeds the former.

37. In an Horizontal or Oblique projection: If to a Tangent who's increments are as *FL*, *LM*, *MN*, &c; that is as $\frac{1}{m}f$, &c. be fitted Ordinates (at a given angle) who's increments are as *FL*, *FM*, *FN*, &c. that is as $\frac{1}{m}g$, &c: The Curve answering to the compound of these Motions, is that wherein the Project is is to move.

38. This Curve (being hitherto without a name) may be called *Linea Projectorum*; the line of Projects, or things projected; which resembles a Parabola deformed.

39. The Celerity and Tendency, as to each point of this line, is determined by a Tangent at that Point.

40. And that against which it makes the greatest stroke or

or percuſſion, is that which (at that point) is at right angles to that Tangent.

41. If the Projection (at ¶ 27.) be not infinitely continued, but terminate (ſuppoſe) at *N*, ſo that the laſt term in the firſt Column or Series erect be a; and conſequently in the ſecond, $m\,a$; in the third, $m\,m\,a$, &c. (each Series having one term fewer than that before it :) then (for the ſame reaſons as at ¶ 22.) the Aggregates of the ſeveral Columns (or erect Series) will be $\frac{1-a}{n}$, $\frac{1-ma}{n}$, $\frac{1-mma}{n}$, and ſo forth, till (the multiple of a becoming $=1$) the progreſſion expire.

42. Now all the abatements here, a, $m\,a$, $m\,m\,a$, &c. are the ſame with the terms of the firſt Column taken backward. For a is the laſt, $m\,a$ the next before it; and ſo of the reſt.

43. And the Aggregate of all the Numerators is ſo many times 1 as is the number of terms (ſuppoſe t,) wanting the firſt Column; that is $t-\frac{1-a}{n}$, or $\frac{nt-1+a}{n}$; & this again divided by the common denominator n, becomes $\frac{nt-1+a}{nn}$. And therefore $\frac{nt-1+a}{nn}g$, is the line of deſcent by its own Gravity.

44. If therefore this be added to a projecting force downward in a perpendicular; or ſubducted from ſuch projecting force upward; that is, to or from $\frac{1-a}{n}f$: The Deſcent in the firſt caſe will be $\frac{1-a}{n}f+\frac{nt-1+a}{nn}g$; and the Aſcent in the other caſe $\frac{1-a}{n}f-\frac{nt-1+a}{nn}g$. And in this latter caſe, when the ablative part becomes equal to the poſitive part, the Aſcent is at the higheſt: and

 thence-

thenceforth (the ablative part exceeding the poſitive) it will deſcend.

45. In an Horizontal or Oblique projection; having taken $\frac{1-a}{n}f$ in the line of Projection, and thence (at the Angle given) $\frac{nt-1+a}{nn}g$ in the line of Deſcent; the point in the Curve anſwering to theſe, is the place of the Project anſwering to that moment.

46. I am aware of ſome Objections to be made, whether to ſome points of the Proceſs, or to ſome of the Suppoſitions. But I ſaw not well how to wave it, without making the Computation much more perplexed. And in a matter ſo nice, and which muſt depend upon Phyſical Obſervations, t'will be hard to attain ſuch accuracy as not to ſtand in need of ſome allowances.

47. Somewhat might have been further added to direct the Experiments ſuggeſted at ¶ 21. and 31. But that may be done at leiſure, after deliberation had, which way to attempt the Experiment.

48. The like is to be ſaid of the different reſiſtence which different Bodies may meet with in the ſame *Medium*, according to their different Gravities (extenſively or intenſively conſidered) and their different figures and Poſitions in Motion. Whereof we have hitherto taken no account; but ſuppoſed them, as to all theſe, to be alike and equal.

Poſt-ſcript.

49. The computation in ¶ 41, 42, 43, may (if that be alſo deſired) be thus repreſented by Lines and Spaces. The Ablatives *a*, *m a*, *m m a*, &c. (being the ſame with the firſt Column taken backward) are fitly repreſented by the ſegments of *N F* (beginning at *N*) in Figure IV. and V. and therefore by Parallelograms on theſe Baſes, aſſuming the common hight of *F h*, or *N Q*: the Aggregate of which

is

is *N h*, or *F Q*. And, ſo many times 1, by ſo many equal ſpaces, on the ſame Baſes, between the ſame Parallels terminated at the Hyperbola: The Aggregate of which is *h F N Q n*. From whence if we ſubduct the Aggregate of Ablatives *F Q*; the remaining trilinear *h Q n*, repreſents the Deſcent.

50. If to this of Gravity, be joyned a projecting Force; which is to the impulſe of Gravity as *h K* to *h F* (be it greater, leſs, or equal) taken in the ſame line: the ſame parallels determine proportional Parallelograms, whoſe Aggregate is *K Q*.

51. And therefore if this be a Perpendicular Projection downwards; then *h K k n* (the ſumme of this with the former) repreſents the Deſcent.

52. If it be a Perpendicular upwards; then the difference of theſe two repreſents the Motion: which ſo long as *K Q* is the greater, is Aſcendent: but Deſcendent when *h Q n* becomes greater: and it is then at the higheſt when they be equal.

53. If the Projection be not in the ſame Perpendicular, (but Horizontal, or Oblique) then *K Q* repreſents the Tangent of the Curve; and *h Q n* the Ordinates to that Tangent, at the given Angle.

54. But the Computation before given I take to be of better uſe than this repreſentation in Figure. Becauſe in ſuch Mathematical enquiries, I chooſe to ſeparate (as much as may be) what purely concerns Proportions; and conſider it abſtractly from lines or other matter wherewith it is incumbered.

As to the queſtion propoſed; whether the reſiſtance of the *Medium* do not always take off ſuch a proportional part of the force moving through it, as is the Specifick Gravity of the *Medium* to that of the Body moved in it: (for, if ſo, it will ſave us the trouble of Obſervation.)

I think this can by no means be admitted. For there be many other things of conſideration herein, beſide the Intenſive Gravity (or, as ſome call it, the Specifick Gravity) of the *Medium*.

A viſcous *Medium* ſhall more reſiſt, than one more fluid, though of like Intenſive Gravity.

And a ſharp Arrow ſhall bore his way more eaſily through the *Medium*, than a blunt headed Bolt, though of equal weight, and like intenſive Gravity.

And the ſame Pyramide with the Point, than with the Baſe forward.

And many other like varieties, intended in my ¶ 48.

But this I think may be admitted, namely, That different *Mediums*, equally liquid, (and other circumſtances alike,) do in ſuch proportion reſiſt, as is their Intenſive Gravity. Becauſe there is, in ſuch Proportion, a heavier object to be removed, by the ſame Force. Which is one of the things to which ¶ 33. refers.

And again: The heavyer Project once in motion, (being equally ſwift, and all other circumſtances alike) moves through the ſame *Medium* in ſuch proportion more ſtrongly, as is its Intenſive Gravity. For now the Force is in ſuch proportion greater, for the removal of the ſame reſiſtance. And this part of what my ¶ 32. inſinuates.

But where there is a complication of theſe conſiderations one with another, and with many other circumſtances whereof each is ſeverally to be conſidered: there muſt be reſpect had to all of them.

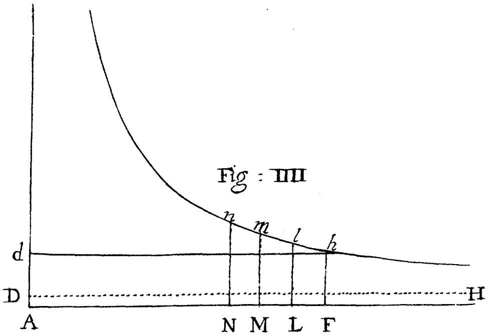

Fig: IIII

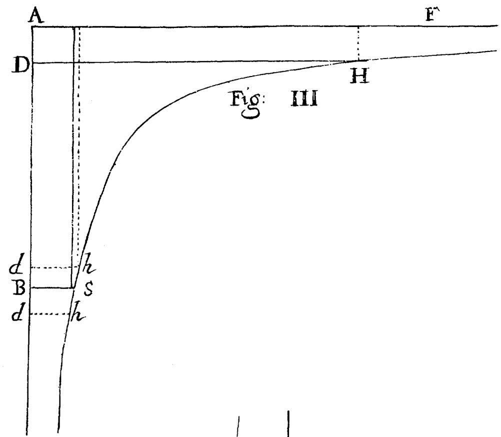

Fig: III

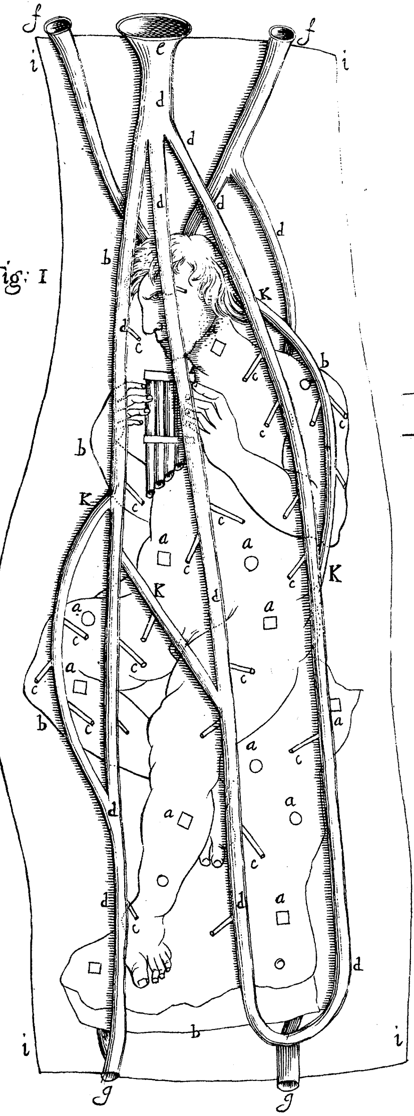

Fig: I

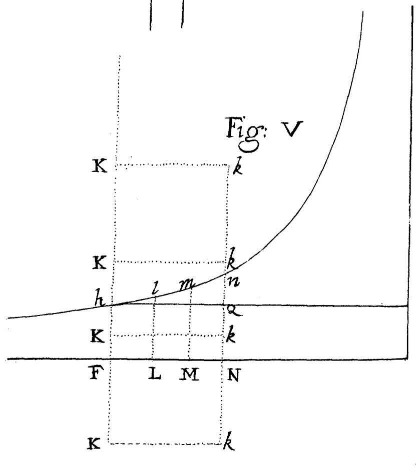

Fig: V

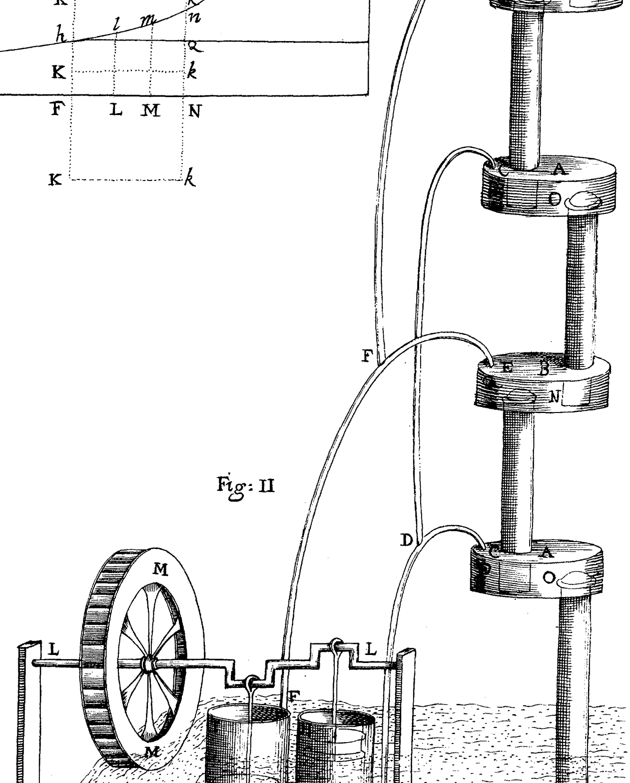

Fig: II