

I. *A Discourse concerning the Methods of Approximation in the Extraction of Surd Roots.*
By John Wallis, S. T. D. and Savilian Professor of Geometry at Oxford.

THE several Methods of Approximation, which have been mentioned of late Years, for Extracting the Roots of Simple or Affected Equations, gives me occasion of saying somewhat of that Subject.

It is agreed by all, (and, I think, demonstrated by the *Greeks* long ago) that if a Number proposed be not a true Square, it is in vain for to hope for a just Quadratick Root thereof, explicable by Rational Numbers, Integers, or Fracted: And therefore, in such cases, we must content our selves with Approximations (somewhat near the truth) without pretending to Accuracy.

And so, for the Cubick Root, of what is not a perfect Cube. And the like for Superiour Powers.

Now the Ancients (being aware of this) had their Methods of Approximation in such cases; whereof some have been derived down even to this day. Of which we shall speak more anon.

But since the Methods of Decimal Fractions (as they are now wont to be called) have come into Practice, it hath been usual to prosecute such Extractions (beyond the place of Unites) in the places of Decimal Parts to what Accuracy we please; whereby the former Methods of Approaches have been (not so much forgotten, as) neglected.

Not that if such Approaches by Decimals were always the most speedy, or the most exact; (for no Man doubts but that $\frac{1}{4}$ is a more Simple, and more Intelligible Notation of that Quantity, than 0.125, or $\frac{125}{1000}$: And $\frac{1}{2}$, not only a more Brief, but a more Accurate designation of the Square Root of $\frac{1}{2}$, than 0.333333, &c.) But, because Fractions reduced to the Decimal form, are more convenient for subsequent Operations, when there is occasion for a further Progress.

Mr. *Newton's* Method of Approximation for the Extracting Roots, even of Affected Equations, I have given some Account

count of in my *English Algebra*; and somewhat more fully in the *Latin Edition*: where I give an Account also of Mr. *Raphson's* Method; which I need not here repeat, because it is to be seen there.

Since which time, Monsieur *De L'agny* hath published his Method of Approximation, principally for single Equations, or Extracting the Root of a single Power.

And Mr. *Halley* hath since improved this Method, with a further Advantage; especially as to Affected Equations.

But I need not repeat either of them, because they are both published. That of *De L'agny* in a Treatise by it self; and that of Mr. *Halley*, in the *Philos. Transact.* Numb. 210, for the Month of *May*, 1694.

These may all, or any of them, be of good use (each in their own way) for making more speedy Approaches, and by greater Leaps, in many cases, than *Vieta's* Method (prosecuting the Extraction in Decimal Parts) pursued and improved by Mr. *Oughtred* and Mr. *Harriot* of our own, and by others abroad; especially as to Simple Equations, if we suppose such Extractions to be pursued to the full extent.

As for instance, if we would Extract the Root of an imperfect Surfold, (or a Power of five Dimensions) to have its Root true as far as the sixth place of Decimal Parts. In order to this, we are to add (or suppose to be added) six Punctations of Cyphers, (or, six times five Cyphers, that is Thirty Cyphers) beyond the place of Unites in the Number proposed. If now we pursue the whole Operation to the utmost of those Thirty places, the Work would be long and tedious.

But if we make use of Mr. *Oughtred's* Expedient, (for Multiplication, Division, Extraction of Roots, and other like Operations,) by neglecting so much of this long Process, as is afterward to be cut off and thrown away as useless, (which, I think, is generally practised) the Work will be much abridged, and the Advantage of the other Methods much less considerable.

That is, If (for ascertaining the sixth place of Decimal Parts) we add six Cyphers (instead of thirty) and one or two more (the better to secure what from the consequent places may in the Operation be transmitted hither;) and pursue the Operation thus far, neglecting the following place:
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(which are not likely to influence the Figure due to the sixth place of Decimal Parts of the Root sought:) this long Process will be much shortened.

And if we further consider, what Preparative Operations are to be made in some of those other Methods, before we come to the prescribed Division for giving the Root desired; the Advantage (though considerable) will not be so great as may at first be apprehended; especially as to Affected Equations, in which the Parodical Powers have great Coefficients. As will soon appear in Practice, if we come to apply them to particular cases.

But, without disparaging these Methods (which are really considerable, and well worthy encouragement) that which I here intend, is, to shew the true Foundation of the Methods used by the Ancients, (however since neglected) and the just Improvement of them. Which though Anciently scarce applyed beyond the Quadratick, or perhaps the Cubick Root, (for with the Higher Powers they did not much trouble themselves) yet are equally applicable (by due adjustments) to the Superiour Powers also.

I shall begin with the Square Root: For which the Ancient Method is to this purpose.

From the proposed Non-quadrate (suppose N) subtract (in the usual manner) the greatest Square in Integers therein contained (suppose $Aq.$) The remainder (suppose $B = 2AE + Eq$) is to be the Numerator of a Fraction, for designing the near value of E (the remaining part of the Root sought $A + E = \sqrt{N}$), whose Denominator or Divisor is to be $2A$ (the double Root of the subtracted Square) or $2A + 1$ (that double Root increased by 1) the true value falling between these two; sometime the one, sometime the other, being nearest to the true value. But (for avoiding of Negative Numbers) the latter is commonly directed.

This Method Monsieur *De L'agny* affirms to be more than 200 Years old: And it is so; for I find it in *Lucas Pacciolus* (otherwise called *Lucas de Burgo*, or *de Burgo Sancti Sepulchri*) Printed at *Venice* in the Year 1494 (if not even sooner than so, for I find there have been several Editions of it.) And how much older than so, I cannot tell: For he doth not deliver it as a new Invention of his own, but as a received Practice, and derived from the *Moors* or *Arabs*, from whom they

they had their *Algorithm*, or Practice of Arithmetick by the Ten Numeral Figures now in use.

And it is continued down hitherto in Books of Practical Arithmetick in all Languages, which teach the Extraction of the Square Root, and (therein) this Method of Approximation, in case of a Non-quadrate.

The true ground of the Rule is this: Aq being (by Construction) the greatest Integer Square contained in N , 'tis evident that E must be less than 1 ; (otherwise not Aq , but the Square of $A + 1$, or some greater than so, would be the greatest Integer Square contained in N .) Now if the Remainder $B = 2AE + Eq$ be divided by $2A$, the Result will be too great for E , (the Divisor being too little; for it should be $2A + E$, to make the Quotient E .) But if (to rectifie this) we diminish the Quotient, by increasing the Divisor, adding 1 to it, it then becomes too little; because the Divisor is now too big. For (E being less than 1) $2A + 1$ is more than $2A + E$; and therefore too big.

As for instance; If the Non-quadrate proposed be $N = 5$, the greatest Integer Square therein contained is $Aq = 4$ (the Square of $A = 2$;) which being subtracted, leaves $N - Aq = 5 - 4 = 1 = B = 2AE + Eq$. Which divided by $2A = 4$, gives $\frac{1}{4}$: But divided by $2A + 1 = 4 + 1 = 5$, gives $\frac{1}{5}$. That too great, and this too little for E . And therefore the true Root ($A + E = \sqrt{N}$) is less than $2\frac{1}{4} = 2.25$, but greater than $2\frac{1}{5} = 2.2$: And this was Anciently thought an Approach near enough.

If this Approach be not now thought near enough, the same Process may be again repeated; and that as oft as is thought necessary.

Take now for A , $2\frac{1}{5} = 2.2$, whose Square is $4.84 = Aq$, (now considered as an Integer in the second place of Decimal Parts.) This subtracted from 5.00 , (or, which is the same, 0.84 the excess of this Square above the former, from 1 which was then the remainder,) leaves a new remainder $B = 0.16$: which divided by $2A = 4.4$, gives $\frac{.16}{4.4} = \frac{2}{55} = 0.03636 +$, too much. But divided by $2A + 1 = 4.5$, it gives $\frac{.16}{4.5} = \frac{8}{225} = 0.03555 +$, too little. The true Value (between these two) being 2.236 *proxime*, whose Square is 4.999696 .
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If this be not thought near enough, Subtract this Square from 5.000000: The Remainder $B = 0.000304$, divided by $2A = 4.472$, or by $2A + 1 = 4.473$, gives (either way) 0.000068 —; which added to $A = 2.236$, makes 2.236068 —, somewhat too big; but $2.236067 +$ would be much more too little.

Which gives us the Square Root of 5, adjusted to the sixth place of Decimal Parts, at three steps. And by the same Method, if it be thought needful, we may proceed further.

It were easie to compound the Process of two or more steps into one, and give (for the Rule) the Result of such Composition. Which would make it seem more Intricate and Mysterious, to amuse the Reader: But I choose to make it as plain as I can (and therefore keep the several steps separate,) that the Reader (for his satisfaction) may clearly see the true ground of the Process. But of this enough.

Proceed we now to the Cubick Root. For which (consonant to the Qadratick) the Rule is this:

From the Non-Cubick proposed, (suppose N ,) subtract the greatest Cube in Integers therein contained, (suppose Ac ;) the remainder (suppose $B = 3AqE + 3AEq + Ec$,) is to be the Numerator of a Fraction for designing the value of E , (the remaining part of the Root sought $A + E = \sqrt[3]{cN}$.) To this Numerator, if (for the Denominator or Divisor) we subjoyn $3Aq$, the Result will certainly be too great for E , because the Divisor is too little: (For it should be $3Aq + 3AE + Eq$, to give the true value of E .) If, for the Divisor, we take $3Aq + 3A + 1$, it will certainly be too little, because the Divisor is too great. (For E , by construction, is less than 1.) It must therefore (between these limits) be more than this latter. And therefore this latter Result being added to A , will give a Root whose Cube may be subtracted from the Non-Cubick proposed, in order to another step.

This Approach I find in *Wingate's Arithmetick*, Published in the Year 1630, and must therefore be at least so old; how much older I cannot tell.

But if, for the Divisor, we take $3Aq + 3A$, (or even less than so) the Result may be too great; or (in case B be small) it may be too little, (and oft is so.)

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Which comes to pass from hence ; because E (by Construction) is less than 1 ; and therefore $3AE$ less than $3A$; and perhaps so much as that the addition of Eq will not redress it. And when it so happens, $3Aq + 3A$, is a better Divisor than $3Aq + 3A + 1$, (or even somewhat less than either.) But because it doth not always so happen (though for the most part it doth) the Rule doth rather direct the other ; as which doth certainly give a Root less than the true value, whose Cube may always be subtracted from the Non-Cubick proposed. The design being to have such a Cube as (being subtracted) may leave another B, to be ordered in like manner for a new Approach.

But, for the most part, $3Aq$ may be safely taken for the Divisor. For, though the Result will then be somewhat too big, yet the excess may be so small, as to be neglected ; or, at least, we may thence easily judge what Number (somewhat less than it) may be safely taken. And if we chance to take it somewhat too big, the Inconvenience will be but this, that B for the next step will be a Negative. Of which case we shall speak anon.

Thus, for instance ; if the Non-Cube proposed be $9 = N$. The greatest Integer Cube therein contained is $8 = Ac$, (whose Cubick Root is $A = 2$.) Which Cube subtracted, leaves $9 - 8 = 1 = B = 3AqE + 3AEq + Ec$. This divided by $3Aq = 12$, gives $\frac{1}{12} = 0.08333 +$, too big for E. But the same divided by $3Aq + 3A + 1 = 12 + 6 + 1 = 19$, gives $\frac{1}{19} = 0.05263 +$, too little. Or if but by $3Aq + 3A = 12 + 6 = 18$, it gives $\frac{1}{18} = \frac{5}{36} = 0.05555 +$, yet too little. For the Cube of $A + 0.06 = 2.06$, is but $8.742 -$, which is short of 9. And so much short of it, that we may safely take 2.07 as not too big : Or perhaps 2.08, (which if it chance to be too big, it will not be much too big ; of which case we are to speak anon :) And, upon tryal, it will be found not too big ; for the Cube of 2.08, is but 8.998912 .

If this first step be not near enough : This Cube subtracted from 9.000000 , leaves a new $B = 0.001088$, which divided by $3Aq = 12.9796$, gives $0.000084 -$; which will be somewhat too big, but not much. (For E is now so small, as that $3AE$ may be safely neglected, and Eq much more.) So that if to 2.08, we add $0.000084 -$, the Result 2.080084 will be too big, but 2.080083 will be more too little. (As will ap-

pear if we take the Cube of each.) So that either of them, at the second step, gives the true Root within an Unite in the sixth place of Decimal Parts.

But when I say, *Taking the Cube of each*, (which I do, that the thing may be more clearly apprehended) it is not necessary that we trouble our selves with the whole Cube. For, A^3 being already subtracted, for finding $B = 3A^2E + 3AE^2 + E^3$, we have no more to try, but whether $3A^2E + 3AE^2 + E^3$ be greater or less than B , according as we take 0.000084 , or 0.000083 , for E .

Which may conveniently be done in this manner: Take $3A + E$, and Multiply this by E , (or E by it) so have we $3AE + E^2$. To this add $3A^2$, and Multiply the whole by E , (so have we $3A^2E + 3AE^2 + E^3$), to see whether this be greater or less than B .

That is, in the present case, if we take $E = 0.000084$, and add to this $3A = 6.24$, then is $6.240084 = 3A + E$. This multiplied by $E = 0.000084$, is $3AE + E^2 = 0.000524$. To which if we add $3A^2 = 12.9792$, it is $3A^2 + 3AE + E^2 = 12.979724$. Which multiplied again by $E = 0.000084$, is $0.0010902 + = 3A^2E + 3AE^2 + E^3$, which is more than $B = 0.001088$.

But if we take $E = 0.000083$, and proceed as before, we shall have $3A^2E + 3AE^2 + E^3 = 0.001077 +$, which is less than $B = 0.001088$. And therefore (if we subtract that from this) the Remainder, 0.000011 , will be another B for the next step, if we please to proceed further.

Hitherto I have pursued the Method most affected by the Ancients, in seeking a Square or Cube (and the like of other Powers) always less than the just value, that it might be subtracted from the Number proposed, leaving B a Positive Remainder; thereby avoiding Negative Numbers.

But since the Arithmetick of Negatives is now so well understood, it may in this (and other Operations of like Nature) be advisable, to take the next greater (in case that be nearer to the true value) rather than the next lesser. Of which I took notice in my *Commercium Epistolicum*, *Epist.* 19. *Jan.* 2. 1657. in a case more intricate than this is. And which I elsewhere advise, in seeking the *Greatest Common Divisor* of two Numbers, in order to the abridging a Fraction, or otherwise.

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According to this Notion, for the Square Root of 5, I would say, it is (2 +) somewhat more than 2; and enquire, How much more? But for the Square Root of 8, I would say, it is (3 -) somewhat less than 3; and inquire; How much less? Taking (in both cases) that which is nearest to the just Value.

Thus, in the Cubick Root before us; I would take for B (in the last Enquiry) 0.000084 - (where, for the next step, we have $B = -0.000002$,) rather than 0.000083 + (where, for the next step, we should have $B = +0.000011$.) In the latter case, we are to Divide $B = +0.000011$, by $3Aq = 12.980236 -$, to find (by the Quotient) how much is to be added to 0.000083. In the other case, we are to Divide $B = +0.000002$, by $3Aq = 12.980248$, to find (by the Quotient) what is to be abated of 0.000084. So have we

$$\frac{0.000011}{12.980236} = 0.00000085 + \text{to be added to } 6.240083; \text{ Or}$$

$$\frac{-0.000002}{12.980248} = 0.00000015 + \text{to be abated of } 6.240084. \text{ (Or}$$

it may suffice, in either, to Divide by 12.98 +, or even by 13. -, without being incumbred with a long Divisor) either of which gives us, for the Root sought, 2.08008385 *proxime*. True (at the third step) to the Eighth place of Decimal Parts. And if this be not near enough, the Cube of this, compared with the Number proposed, will give us another B for the next step. And so onwards as far as we please.

Now, what is said of the Cube, is easily applicable to the Higher Powers.

I shall omit that of the Biquadrate; because here perhaps it may be thought most advisable, to Extract the Square Root of the Number proposed; and then the Square Root of that Root.

But if we would do it at once, we are from N (the Number proposed, being not a Biquadrate) to Subtract Aqq (the greatest Biquadrate contained in it) to find the Remainder $B = 4AcE + 6AqEq + 4AEc + Eqq$. Which Remainder, if we Divide by $4Ac$, the Quotient will certainly be too big for E, (though perhaps not much:) If by $4Ac + 6Aq + 4A + 1$, it will certainly be too little (for reasons before mentioned.) And we are to use our discretion in taking some intermediate

intermediate Number. And if we chance not to hit on the nearest, the Inconvenience will be but this, that our Leap will not be so great as otherwise it might be. Which will be rectified by another B at the next step.

For the Surfolid (of five Dimensions) we are, from N (the Number propos'd, being not a perfect Surfolid) to Subtract Aqc (the greatest Surfolid therein contained) to find the Remainder $B = 5 AqqE + 10 AcEq + 10 AqEc + 5 AEqq + Eqc$. Which (as before) if we Divide by $5 Aqq$, the Result will be somewhat too big, (because the Divisor is too little :) If by $5 Aqq + 10 Ac + 10 Aq + 5 A + 1$, the Result will certainly be less than the true E. The just value of E being somewhat between these two, where we are to use our discretion, what Intermediate Number to take. Which according as it proves too great or too little, is to be rectified at the next step.

If, to direct us in the choice of such intermediate Number, we should Multiply Rules or Precepts for such choice, the Trouble of observing them, would be more than the Advantage to be gained by it. And, for the most part, it will be safe enough (and least trouble) to Divide by $5 Aqq$, which gives a Quotient somewhat too big: Which we may either Rectify at Discretion (by taking a Number somewhat less) or proceed to another B, (Affirmative or Negative, as the case shall require) and so onward to what exactness we please. (Which is, for substance, in a manner coincident with Mr. Raphson's Method, even for Affected Equations.)

Thus, in the present case; If the Number propos'd be $N = 33$, then is $Aqc = 32$, and $B = 33 - 32 = 1 = 5 AqqE + 10 AcEq + 10 AqEc + 5 AEqq + Eqc$. Which if we Divide by $5 Aqq = 5 \times 16 = 80$, the Result $\frac{1}{80} = 0.0125$, is somewhat too big for E, but not much. And if we examine it, by taking the Surfolid of 2.0125 , or of $2\frac{1}{80}$, we shall find a Negative B (for the next step) but not very considerable. Or if we think it considerable, we may proceed further to another step, or more than so.

The like Method may be applied (and with more Advantage) in the Higher Powers, according as the Composition of each Power requires.

And the same Method may be of use (with good Advantage) in long Numbers (if duly applied) even before we
come

come to the place of Unites, for the same will equally hold there also. Which the Reader may easily apprehend without a long Discourse upon it.

How far this Method may be coincident with some of those before mentioned, I do not trouble my self to enquire; nor whether, or for what causes, all or any of those may be more eligible. My design being only to shew the true Natural ground, from whence such Rules of Approach are (or might have been) derived; and by which (if there be occasion) they may be examined. And if I have done this, it is what I did propose.

In Affected Equations (especially where the Coefficients are great, and some Affirmatives, others Negatives,) the Cases will be more perplexed. And to Multiply Rules for each Case, would (I conceive) increase the Trouble, with no great Advantage. Which therefore I leave to the Prudence of each (as occasion shall require) to take some Intermediate, between a greater and a lesser. Or if they please to accommodate that above mentioned (out of *Commerc. Epistol.*) to the present case, which is there applied to a Case not less intricate. Or to make use of some of the Methods delivered by others to this purpose. Where this (withal) is to be considered, That such Affected Equations are capable of more Roots than one, according to the Number of Dimensions to which they arise.