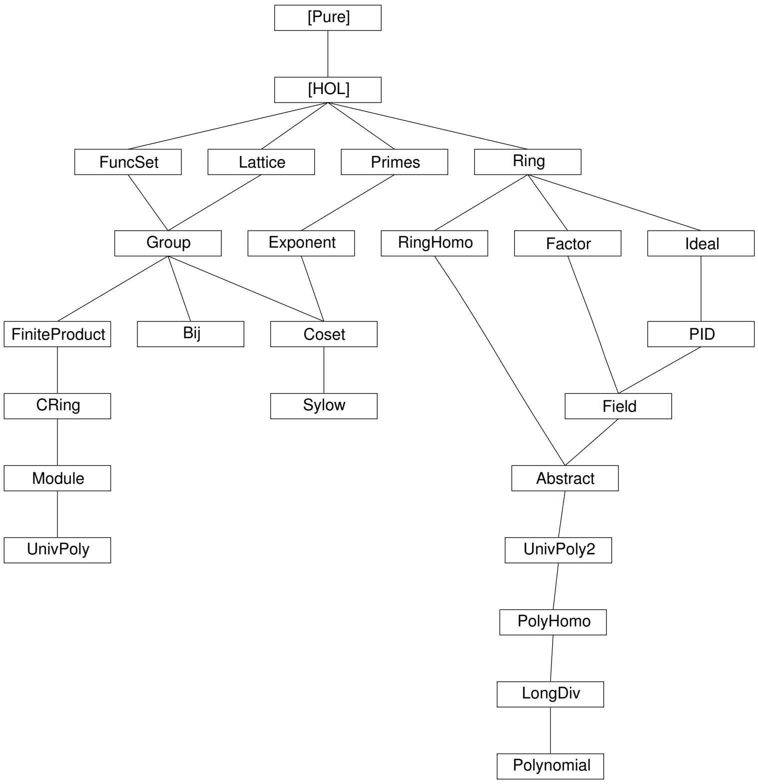
The Isabelle/HOL Algebra Library

Clemens Ballarin
Florian Kammuüller
Lawrence C Paulson

October 1, 2005

Contents


[Pure]
[HOL]
FuncSet
Lattice
Primes
Ring
Group
Exponent
RingHomo
Factor
Ideal
FiniteProduct
Bij
Coset
PID
CRing
Sylow
Field
Module
Abstract
UnivPoly
UnivPoly2
PolyHomo
LongDiv
Polynomial

1 Lattice: Orders and Lattices

theory *Lattice* **imports** *Main* **begin**

Object with a carrier set.

record *'a partial-object* =
  carrier :: *'a set*

1.1 Partial Orders

record *'a order* = *'a partial-object* +
  le :: [*'a*, *'a*] => *bool* (**infixl** $\sqsubseteq_1$ *50*)

locale *partial-order* = *struct L* +
  assumes *refl* [*intro*, *simp*]:
    $x \in carrier\ L ==> x \sqsubseteq x$
    and *anti-sym* [*intro*]:
    $[|\ x \sqsubseteq y;\ y \sqsubseteq x;\ x \in carrier\ L;\ y \in carrier\ L\ |] ==> x = y$
    and *trans* [*trans*]:
    $[|\ x \sqsubseteq y;\ y \sqsubseteq z;$
    $x \in carrier\ L;\ y \in carrier\ L;\ z \in carrier\ L\ |] ==> x \sqsubseteq z$

constdefs (**structure** *L*)
  less :: [-, *'a*, *'a*] => *bool* (**infixl** $\sqsubset_1$ *50*)
  $x \sqsubset y == x \sqsubseteq y\ \&\ x\ {}^\sim= y$

  — Upper and lower bounds of a set.
  Upper :: [-, *'a set*] => *'a set*
  $Upper\ L\ A == \{u.\ (ALL\ x.\ x \in A \cap carrier\ L\ --> x \sqsubseteq u)\} \cap$
    $carrier\ L$

  Lower :: [-, *'a set*] => *'a set*
  $Lower\ L\ A == \{l.\ (ALL\ x.\ x \in A \cap carrier\ L\ --> l \sqsubseteq x)\} \cap$
    $carrier\ L$

  — Least and greatest, as predicate.
  least :: [-, *'a*, *'a set*] => *bool*
  $least\ L\ l\ A == A \subseteq carrier\ L\ \&\ l \in A\ \&\ (ALL\ x : A.\ l \sqsubseteq x)$

  greatest :: [-, *'a*, *'a set*] => *bool*
  $greatest\ L\ g\ A == A \subseteq carrier\ L\ \&\ g \in A\ \&\ (ALL\ x : A.\ x \sqsubseteq g)$

  — Supremum and infimum
  sup :: [-, *'a set*] => *'a* ($\bigsqcup_1$- [*90*] *90*)
  $\bigsqcup A == THE\ x.\ least\ L\ x\ (Upper\ L\ A)$

  inf :: [-, *'a set*] => *'a* ($\bigsqcap_1$- [*90*] *90*)
  $\bigsqcap A == THE\ x.\ greatest\ L\ x\ (Lower\ L\ A)$

  join :: [-, *'a*, *'a*] => *'a* (**infixl** $\sqcup_1$ *65*)

$x \sqcup y == sup\ L\ \{x,\ y\}$

$meet :: [\text{-},\ 'a,\ 'a] => 'a$ (**infixl** $\sqcap_1$ *70*)
$x \sqcap y == inf\ L\ \{x,\ y\}$

1.1.1 Upper

lemma *Upper-closed* [*intro*, *simp*]:
Upper L A $\subseteq$ *carrier L*
by (*unfold Upper-def*) *clarify*

lemma *UpperD* [*dest*]:
includes *struct L*
shows [| $u \in$ *Upper L A*; $x \in A$; $A \subseteq$ *carrier L* |] ==> $x \sqsubseteq u$
by (*unfold Upper-def*) *blast*

lemma *Upper-memI*:
includes *struct L*
shows [| !! y. $y \in A$ ==> $y \sqsubseteq x$; $x \in$ *carrier L* |] ==> $x \in$ *Upper L A*
by (*unfold Upper-def*) *blast*

lemma *Upper-antimono*:
$A \subseteq B$ ==> *Upper L B* $\subseteq$ *Upper L A*
by (*unfold Upper-def*) *blast*

1.1.2 Lower

lemma *Lower-closed* [*intro*, *simp*]:
Lower L A $\subseteq$ *carrier L*
by (*unfold Lower-def*) *clarify*

lemma *LowerD* [*dest*]:
includes *struct L*
shows [| $l \in$ *Lower L A*; $x \in A$; $A \subseteq$ *carrier L* |] ==> $l \sqsubseteq x$
by (*unfold Lower-def*) *blast*

lemma *Lower-memI*:
includes *struct L*
shows [| !! y. $y \in A$ ==> $x \sqsubseteq y$; $x \in$ *carrier L* |] ==> $x \in$ *Lower L A*
by (*unfold Lower-def*) *blast*

lemma *Lower-antimono*:
$A \subseteq B$ ==> *Lower L B* $\subseteq$ *Lower L A*
by (*unfold Lower-def*) *blast*

1.1.3 least

lemma *least-carrier* [*intro*, *simp*]:
shows *least L l A* ==> $l \in$ *carrier L*
by (*unfold least-def*) *fast*

lemma *least-mem*:
  least L l A ==> $l \in A$
  by (*unfold least-def*) *fast*

lemma (**in** *partial-order*) *least-unique*:
  [| *least L x A*; *least L y A* |] ==> $x = y$
  by (*unfold least-def*) *blast*

lemma *least-le*:
  includes *struct L*
  shows [| *least L x A*; $a \in A$ |] ==> $x \sqsubseteq a$
  by (*unfold least-def*) *fast*

lemma *least-UpperI*:
  includes *struct L*
  assumes *above*: !! *x*. $x \in A$ ==> $x \sqsubseteq s$
    and *below*: !! *y*. $y \in$ *Upper L A* ==> $s \sqsubseteq y$
    and *L*: $A \subseteq$ *carrier L* $s \in$ *carrier L*
  shows *least L s* (*Upper L A*)
proof −
  have *Upper L A* $\subseteq$ *carrier L* **by** *simp*
  moreover from *above L* **have** $s \in$ *Upper L A* **by** (*simp add*: *Upper-def*)
  moreover from *below* **have** *ALL x* : *Upper L A*. $s \sqsubseteq x$ **by** *fast*
  ultimately show *?thesis* **by** (*simp add*: *least-def*)
qed

1.1.4 greatest

lemma *greatest-carrier* [*intro*, *simp*]:
  shows *greatest L l A* ==> $l \in$ *carrier L*
  by (*unfold greatest-def*) *fast*

lemma *greatest-mem*:
  greatest L l A ==> $l \in A$
  by (*unfold greatest-def*) *fast*

lemma (**in** *partial-order*) *greatest-unique*:
  [| *greatest L x A*; *greatest L y A* |] ==> $x = y$
  by (*unfold greatest-def*) *blast*

lemma *greatest-le*:
  includes *struct L*
  shows [| *greatest L x A*; $a \in A$ |] ==> $a \sqsubseteq x$
  by (*unfold greatest-def*) *fast*

lemma *greatest-LowerI*:
  includes *struct L*
  assumes *below*: !! *x*. $x \in A$ ==> $i \sqsubseteq x$

    and *above*: !! y. $y \in$ *Lower L A* ==> $y \sqsubseteq i$
    and *L*: $A \subseteq$ *carrier L* $i \in$ *carrier L*
  shows *greatest L i* (*Lower L A*)
proof −
  have *Lower L A* $\subseteq$ *carrier L* **by** *simp*
  moreover from *below L* **have** $i \in$ *Lower L A* **by** (*simp add*: *Lower-def*)
  moreover from *above* **have** *ALL x* : *Lower L A*. $x \sqsubseteq i$ **by** *fast*
  ultimately show *?thesis* **by** (*simp add*: *greatest-def*)
qed

1.2 Lattices

locale *lattice* = *partial-order* +
  assumes *sup-of-two-exists*:
    [| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> *EX s. least L s* (*Upper L* {x, y})
    and *inf-of-two-exists*:
    [| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> *EX s. greatest L s* (*Lower L* {x, y})

lemma *least-Upper-above*:
  includes *struct L*
  shows [| *least L s* (*Upper L A*); $x \in A$; $A \subseteq$ *carrier L* |] ==> $x \sqsubseteq s$
  by (*unfold least-def*) *blast*

lemma *greatest-Lower-above*:
  includes *struct L*
  shows [| *greatest L i* (*Lower L A*); $x \in A$; $A \subseteq$ *carrier L* |] ==> $i \sqsubseteq x$
  by (*unfold greatest-def*) *blast*

1.2.1 Supremum

lemma (**in** *lattice*) *joinI*:
  [| !!*l*. *least L l* (*Upper L* {x, y}) ==> *P l*; $x \in$ *carrier L*; $y \in$ *carrier L* |]
  ==> *P* ($x \sqcup y$)
proof (*unfold join-def sup-def*)
  assume *L*: $x \in$ *carrier L* $y \in$ *carrier L*
    and *P*: !!*l*. *least L l* (*Upper L* {x, y}) ==> *P l*
  with *sup-of-two-exists* **obtain** *s* **where** *least L s* (*Upper L* {x, y}) **by** *fast*
  with *L* **show** *P* (*THE l. least L l* (*Upper L* {x, y}))
    by (*fast intro*: *theI2 least-unique P*)
qed

lemma (**in** *lattice*) *join-closed* [*simp*]:
  [| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqcup y \in$ *carrier L*
  by (*rule joinI*) (*rule least-carrier*)

lemma (**in** *partial-order*) *sup-of-singletonI*:
  $x \in$ *carrier L* ==> *least L x* (*Upper L* {x})
  by (*rule least-UpperI*) *fast+*

lemma (**in** *partial-order*) *sup-of-singleton* [*simp*]:

```
  includes struct L
  shows x ∈ carrier L ==> ⨆{x} = x
  by (unfold sup-def) (blast intro: least-unique least-UpperI sup-of-singletonI)
```

Condition on A: supremum exists.

```
lemma (in lattice) sup-insertI:
  [| !!s. least L s (Upper L (insert x A)) ==> P s;
  least L a (Upper L A); x ∈ carrier L; A ⊆ carrier L |]
  ==> P (⨆(insert x A))
proof (unfold sup-def)
  assume L: x ∈ carrier L  A ⊆ carrier L
    and P: !!l. least L l (Upper L (insert x A)) ==> P l
    and least-a: least L a (Upper L A)
  from L least-a have La: a ∈ carrier L by simp
  from L sup-of-two-exists least-a
  obtain s where least-s: least L s (Upper L {a, x}) by blast
  show P (THE l. least L l (Upper L (insert x A)))
  proof (rule theI2)
    show least L s (Upper L (insert x A))
    proof (rule least-UpperI)
      fix z
      assume z ∈ insert x A
      then show z ⊑ s
      proof
        assume z = x then show ?thesis
          by (simp add: least-Upper-above [OF least-s] L La)
      next
        assume z ∈ A
        with L least-s least-a show ?thesis
          by (rule-tac trans [where y = a]) (auto dest: least-Upper-above)
      qed
    next
      fix y
      assume y: y ∈ Upper L (insert x A)
      show s ⊑ y
      proof (rule least-le [OF least-s], rule Upper-memI)
        fix z
        assume z: z ∈ {a, x}
        then show z ⊑ y
        proof
          have y': y ∈ Upper L A
            apply (rule subsetD [where A = Upper L (insert x A)])
            apply (rule Upper-antimono) apply clarify apply assumption
            done
          assume z = a
          with y' least-a show ?thesis by (fast dest: least-le)
        next
          assume z ∈ {x}
          with y L show ?thesis by blast
```

```
      qed
    qed (rule Upper-closed [THEN subsetD])
  next
    from L show insert x A ⊆ carrier L by simp
    from least-s show s ∈ carrier L by simp
  qed
next
  fix l
  assume least-l: least L l (Upper L (insert x A))
  show l = s
  proof (rule least-unique)
    show least L s (Upper L (insert x A))
    proof (rule least-UpperI)
      fix z
      assume z ∈ insert x A
      then show z ⊑ s
      proof
        assume z = x then show ?thesis
          by (simp add: least-Upper-above [OF least-s] L La)
      next
        assume z ∈ A
        with L least-s least-a show ?thesis
          by (rule-tac trans [where y = a]) (auto dest: least-Upper-above)
      qed
    next
      fix y
      assume y: y ∈ Upper L (insert x A)
      show s ⊑ y
      proof (rule least-le [OF least-s], rule Upper-memI)
        fix z
        assume z: z ∈ {a, x}
        then show z ⊑ y
        proof
          have y': y ∈ Upper L A
            apply (rule subsetD [where A = Upper L (insert x A)])
            apply (rule Upper-antimono) apply clarify apply assumption
            done
          assume z = a
          with y' least-a show ?thesis by (fast dest: least-le)
        next
          assume z ∈ {x}
          with y L show ?thesis by blast
        qed
      qed (rule Upper-closed [THEN subsetD])
    next
      from L show insert x A ⊆ carrier L by simp
      from least-s show s ∈ carrier L by simp
    qed
  qed
```

  qed
qed

lemma (**in** *lattice*) *finite-sup-least*:
  $[|$ *finite A*; $A \subseteq$ *carrier L*; A ~= $\{\}$ $|]$ ==> *least L* ($\bigsqcup A$) (*Upper L A*)
proof (*induct set*: *Finites*)
  case *empty*
  then show *?case* **by** *simp*
next
  case (*insert x A*)
  show *?case*
  proof (*cases* $A = \{\}$)
    case *True*
    with *insert* **show** *?thesis* **by** (*simp add*: *sup-of-singletonI*)
  next
    case *False*
    with *insert* **have** *least L* ($\bigsqcup A$) (*Upper L A*) **by** *simp*
    with - **show** *?thesis*
      by (*rule sup-insertI*) (*simp-all add*: *insert* [*simplified*])
  qed
qed

lemma (**in** *lattice*) *finite-sup-insertI*:
  assumes *P*: !!*l*. *least L l* (*Upper L* (*insert x A*)) ==> *P l*
    and *xA*: *finite A* $x \in$ *carrier L* $A \subseteq$ *carrier L*
  shows *P* ($\bigsqcup$ (*insert x A*))
proof (*cases* $A = \{\}$)
  case *True* **with** *P* **and** *xA* **show** *?thesis*
    by (*simp add*: *sup-of-singletonI*)
next
  case *False* **with** *P* **and** *xA* **show** *?thesis*
    by (*simp add*: *sup-insertI finite-sup-least*)
qed

lemma (**in** *lattice*) *finite-sup-closed*:
  $[|$ *finite A*; $A \subseteq$ *carrier L*; A ~= $\{\}$ $|]$ ==> $\bigsqcup A \in$ *carrier L*
proof (*induct set*: *Finites*)
  case *empty* **then show** *?case* **by** *simp*
next
  case *insert* **then show** *?case*
    by − (*rule finite-sup-insertI*, *simp-all*)
qed

lemma (**in** *lattice*) *join-left*:
  $[|$ $x \in$ *carrier L*; $y \in$ *carrier L* $|]$ ==> $x \sqsubseteq x \sqcup y$
  by (*rule joinI* [*folded join-def*]) (*blast dest*: *least-mem*)

lemma (**in** *lattice*) *join-right*:
  $[|$ $x \in$ *carrier L*; $y \in$ *carrier L* $|]$ ==> $y \sqsubseteq x \sqcup y$

```
  by (rule joinI [folded join-def]) (blast dest: least-mem)

lemma (in lattice) sup-of-two-least:
  [| x ∈ carrier L; y ∈ carrier L |] ==> least L (⨆{x, y}) (Upper L {x, y})
proof (unfold sup-def)
  assume L: x ∈ carrier L  y ∈ carrier L
  with sup-of-two-exists obtain s where least L s (Upper L {x, y}) by fast
  with L show least L (THE xa. least L xa (Upper L {x, y})) (Upper L {x, y})
  by (fast intro: theI2 least-unique)
qed

lemma (in lattice) join-le:
  assumes sub: x ⊑ z  y ⊑ z
    and L: x ∈ carrier L  y ∈ carrier L  z ∈ carrier L
  shows x ⊔ y ⊑ z
proof (rule joinI)
  fix s
  assume least L s (Upper L {x, y})
  with sub L show s ⊑ z by (fast elim: least-le intro: Upper-memI)
qed

lemma (in lattice) join-assoc-lemma:
  assumes L: x ∈ carrier L  y ∈ carrier L  z ∈ carrier L
  shows x ⊔ (y ⊔ z) = ⨆{x, y, z}
proof (rule finite-sup-insertI)
  — The textbook argument in Jacobson I, p 457
  fix s
  assume sup: least L s (Upper L {x, y, z})
  show x ⊔ (y ⊔ z) = s
  proof (rule anti-sym)
    from sup L show x ⊔ (y ⊔ z) ⊑ s
      by (fastsimp intro!: join-le elim: least-Upper-above)
  next
    from sup L show s ⊑ x ⊔ (y ⊔ z)
    by (erule-tac least-le)
      (blast intro!: Upper-memI intro: trans join-left join-right join-closed)
  qed (simp-all add: L least-carrier [OF sup])
qed (simp-all add: L)

lemma join-comm:
  includes struct L
  shows x ⊔ y = y ⊔ x
  by (unfold join-def) (simp add: insert-commute)

lemma (in lattice) join-assoc:
  assumes L: x ∈ carrier L  y ∈ carrier L  z ∈ carrier L
  shows (x ⊔ y) ⊔ z = x ⊔ (y ⊔ z)
proof −
  have (x ⊔ y) ⊔ z = z ⊔ (x ⊔ y) by (simp only: join-comm)
```

also from L **have** ... $= \bigsqcup\{z,\ x,\ y\}$ **by** (*simp add*: *join-assoc-lemma*)
also from L **have** ... $= \bigsqcup\{x,\ y,\ z\}$ **by** (*simp add*: *insert-commute*)
also from L **have** ... $= x \sqcup (y \sqcup z)$ **by** (*simp add*: *join-assoc-lemma*)
finally show *?thesis* **.**
qed

1.2.2 Infimum

lemma (**in** *lattice*) *meetI*:
[| !!*i*. *greatest L i* (*Lower L* $\{x,\ y\}$) ==> *P i*;
$x \in$ *carrier L*; $y \in$ *carrier L* |]
==> $P\ (x \sqcap y)$
proof (*unfold meet-def inf-def*)
assume *L*: $x \in$ *carrier L* $y \in$ *carrier L*
and *P*: !!*g*. *greatest L g* (*Lower L* $\{x,\ y\}$) ==> *P g*
with *inf-of-two-exists* **obtain** *i* **where** *greatest L i* (*Lower L* $\{x,\ y\}$) **by** *fast*
with *L* **show** *P* (*THE g*. *greatest L g* (*Lower L* $\{x,\ y\}$))
by (*fast intro*: *theI2 greatest-unique P*)
qed

lemma (**in** *lattice*) *meet-closed* [*simp*]:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqcap y \in$ *carrier L*
by (*rule meetI*) (*rule greatest-carrier*)

lemma (**in** *partial-order*) *inf-of-singletonI*:
$x \in$ *carrier L* ==> *greatest L x* (*Lower L* $\{x\}$)
by (*rule greatest-LowerI*) *fast+*

lemma (**in** *partial-order*) *inf-of-singleton* [*simp*]:
includes *struct L*
shows $x \in$ *carrier L* ==> $\bigsqcap\ \{x\} = x$
by (*unfold inf-def*) (*blast intro*: *greatest-unique greatest-LowerI inf-of-singletonI*)

Condition on A: infimum exists.

lemma (**in** *lattice*) *inf-insertI*:
[| !!*i*. *greatest L i* (*Lower L* (*insert x A*)) ==> *P i*;
greatest L a (*Lower L A*); $x \in$ *carrier L*; $A \subseteq$ *carrier L* |]
==> *P* ($\bigsqcap$(*insert x A*))
proof (*unfold inf-def*)
assume *L*: $x \in$ *carrier L* $A \subseteq$ *carrier L*
and *P*: !!*g*. *greatest L g* (*Lower L* (*insert x A*)) ==> *P g*
and *greatest-a*: *greatest L a* (*Lower L A*)
from *L greatest-a* **have** *La*: $a \in$ *carrier L* **by** *simp*
from *L inf-of-two-exists greatest-a*
obtain *i* **where** *greatest-i*: *greatest L i* (*Lower L* $\{a,\ x\}$) **by** *blast*
show *P* (*THE g*. *greatest L g* (*Lower L* (*insert x A*)))
proof (*rule theI2*)
show *greatest L i* (*Lower L* (*insert x A*))
proof (*rule greatest-LowerI*)

```
      fix z
      assume z ∈ insert x A
      then show i ⊑ z
      proof
        assume z = x then show ?thesis
          by (simp add: greatest-Lower-above [OF greatest-i] L La)
      next
        assume z ∈ A
        with L greatest-i greatest-a show ?thesis
          by (rule-tac trans [where y = a]) (auto dest: greatest-Lower-above)
      qed
    next
      fix y
      assume y: y ∈ Lower L (insert x A)
      show y ⊑ i
      proof (rule greatest-le [OF greatest-i], rule Lower-memI)
        fix z
        assume z: z ∈ {a, x}
        then show y ⊑ z
        proof
          have y': y ∈ Lower L A
            apply (rule subsetD [where A = Lower L (insert x A)])
            apply (rule Lower-antimono) apply clarify apply assumption
            done
          assume z = a
          with y' greatest-a show ?thesis by (fast dest: greatest-le)
        next
          assume z ∈ {x}
          with y L show ?thesis by blast
        qed
      qed (rule Lower-closed [THEN subsetD])
    next
      from L show insert x A ⊆ carrier L by simp
      from greatest-i show i ∈ carrier L by simp
    qed
  next
    fix g
    assume greatest-g: greatest L g (Lower L (insert x A))
    show g = i
    proof (rule greatest-unique)
      show greatest L i (Lower L (insert x A))
      proof (rule greatest-LowerI)
        fix z
        assume z ∈ insert x A
        then show i ⊑ z
        proof
          assume z = x then show ?thesis
            by (simp add: greatest-Lower-above [OF greatest-i] L La)
        next
```

```
          assume z ∈ A
          with L greatest-i greatest-a show ?thesis
            by (rule-tac trans [where y = a]) (auto dest: greatest-Lower-above)
        qed
      next
        fix y
        assume y: y ∈ Lower L (insert x A)
        show y ⊑ i
        proof (rule greatest-le [OF greatest-i], rule Lower-memI)
          fix z
          assume z: z ∈ {a, x}
          then show y ⊑ z
          proof
            have y': y ∈ Lower L A
              apply (rule subsetD [where A = Lower L (insert x A)])
              apply (rule Lower-antimono) apply clarify apply assumption
              done
            assume z = a
            with y' greatest-a show ?thesis by (fast dest: greatest-le)
          next
            assume z ∈ {x}
            with y L show ?thesis by blast
          qed
        qed (rule Lower-closed [THEN subsetD])
      next
        from L show insert x A ⊆ carrier L by simp
        from greatest-i show i ∈ carrier L by simp
      qed
    qed
  qed
qed

lemma (in lattice) finite-inf-greatest:
  [| finite A; A ⊆ carrier L; A ~= {} |] ==> greatest L (⨅ A) (Lower L A)
proof (induct set: Finites)
  case empty then show ?case by simp
next
  case (insert x A)
  show ?case
  proof (cases A = {})
    case True
    with insert show ?thesis by (simp add: inf-of-singletonI)
  next
    case False
    from insert show ?thesis
    proof (rule-tac inf-insertI)
      from False insert show greatest L (⨅ A) (Lower L A) by simp
    qed simp-all
  qed
```

qed

lemma (**in** *lattice*) *finite-inf-insertI*:
  assumes *P*: !!*i*. *greatest L i* (*Lower L* (*insert x A*)) ==> *P i*
    and *xA*: *finite A* $x \in$ *carrier L* $A \subseteq$ *carrier L*
  shows *P* ($\bigsqcap$ (*insert x A*))
proof (*cases* $A = \{\}$)
  case *True* **with** *P* **and** *xA* **show** *?thesis*
    by (*simp add*: *inf-of-singletonI*)
next
  case *False* **with** *P* **and** *xA* **show** *?thesis*
    by (*simp add*: *inf-insertI finite-inf-greatest*)
qed

lemma (**in** *lattice*) *finite-inf-closed*:
  [| *finite A*; $A \subseteq$ *carrier L*; *A* ~= {} |] ==> $\bigsqcap A \in$ *carrier L*
proof (*induct set*: *Finites*)
  case *empty* **then show** *?case* **by** *simp*
next
  case *insert* **then show** *?case*
    by (*rule-tac finite-inf-insertI*) (*simp-all*)
qed

lemma (**in** *lattice*) *meet-left*:
  [| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqcap y \sqsubseteq x$
  by (*rule meetI* [*folded meet-def*]) (*blast dest*: *greatest-mem*)

lemma (**in** *lattice*) *meet-right*:
  [| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqcap y \sqsubseteq y$
  by (*rule meetI* [*folded meet-def*]) (*blast dest*: *greatest-mem*)

lemma (**in** *lattice*) *inf-of-two-greatest*:
  [| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==>
  greatest L ($\bigsqcap$ {*x*, *y*}) (*Lower L* {*x*, *y*})
proof (*unfold inf-def*)
  assume *L*: $x \in$ *carrier L* $y \in$ *carrier L*
  with *inf-of-two-exists* **obtain** *s* **where** *greatest L s* (*Lower L* {*x*, *y*}) **by** *fast*
  with *L*
  show *greatest L* (*THE xa*. *greatest L xa* (*Lower L* {*x*, *y*})) (*Lower L* {*x*, *y*})
  by (*fast intro*: *theI2 greatest-unique*)
qed

lemma (**in** *lattice*) *meet-le*:
  assumes *sub*: $z \sqsubseteq x$ $z \sqsubseteq y$
    and *L*: $x \in$ *carrier L* $y \in$ *carrier L* $z \in$ *carrier L*
  shows $z \sqsubseteq x \sqcap y$
proof (*rule meetI*)
  fix *i*
  assume *greatest L i* (*Lower L* {*x*, *y*})

```
  with sub L show z ⊑ i by (fast elim: greatest-le intro: Lower-memI)
qed

lemma (in lattice) meet-assoc-lemma:
  assumes L: x ∈ carrier L  y ∈ carrier L  z ∈ carrier L
  shows x ⊓ (y ⊓ z) = ⨅{x, y, z}
proof (rule finite-inf-insertI)
```

The textbook argument in Jacobson I, p 457

```
  fix i
  assume inf: greatest L i (Lower L {x, y, z})
  show x ⊓ (y ⊓ z) = i
  proof (rule anti-sym)
    from inf L show i ⊑ x ⊓ (y ⊓ z)
      by (fastsimp intro!: meet-le elim: greatest-Lower-above)
  next
    from inf L show x ⊓ (y ⊓ z) ⊑ i
    by (erule-tac greatest-le)
      (blast intro!: Lower-memI intro: trans meet-left meet-right meet-closed)
  qed (simp-all add: L greatest-carrier [OF inf])
qed (simp-all add: L)

lemma meet-comm:
  includes struct L
  shows x ⊓ y = y ⊓ x
  by (unfold meet-def) (simp add: insert-commute)

lemma (in lattice) meet-assoc:
  assumes L: x ∈ carrier L  y ∈ carrier L  z ∈ carrier L
  shows (x ⊓ y) ⊓ z = x ⊓ (y ⊓ z)
proof -
  have (x ⊓ y) ⊓ z = z ⊓ (x ⊓ y) by (simp only: meet-comm)
  also from L have ... = ⨅ {z, x, y} by (simp add: meet-assoc-lemma)
  also from L have ... = ⨅ {x, y, z} by (simp add: insert-commute)
  also from L have ... = x ⊓ (y ⊓ z) by (simp add: meet-assoc-lemma)
  finally show ?thesis .
qed
```

1.3 Total Orders

```
locale total-order = lattice +
  assumes total: [| x ∈ carrier L; y ∈ carrier L |] ==> x ⊑ y | y ⊑ x
```

Introduction rule: the usual definition of total order

```
lemma (in partial-order) total-orderI:
  assumes total: !!x y. [| x ∈ carrier L; y ∈ carrier L |] ==> x ⊑ y | y ⊑ x
  shows total-order L
proof (rule total-order.intro)
  show lattice-axioms L
```

```
  proof (rule lattice-axioms.intro)
    fix x y
    assume L: x ∈ carrier L  y ∈ carrier L
    show EX s. least L s (Upper L {x, y})
    proof -
      note total L
      moreover
      {
        assume x ⊑ y
        with L have least L y (Upper L {x, y})
          by (rule-tac least-UpperI) auto
      }
      moreover
      {
        assume y ⊑ x
        with L have least L x (Upper L {x, y})
          by (rule-tac least-UpperI) auto
      }
      ultimately show ?thesis by blast
    qed
  next
    fix x y
    assume L: x ∈ carrier L  y ∈ carrier L
    show EX i. greatest L i (Lower L {x, y})
    proof -
      note total L
      moreover
      {
        assume y ⊑ x
        with L have greatest L y (Lower L {x, y})
          by (rule-tac greatest-LowerI) auto
      }
      moreover
      {
        assume x ⊑ y
        with L have greatest L x (Lower L {x, y})
          by (rule-tac greatest-LowerI) auto
      }
      ultimately show ?thesis by blast
    qed
  qed
qed (assumption | rule total-order-axioms.intro)+
```

1.4 Complete lattices

```
locale complete-lattice = lattice +
  assumes sup-exists:
    [| A ⊆ carrier L |] ==> EX s. least L s (Upper L A)
    and inf-exists:
```

```
  [| A ⊆ carrier L |] ==> EX i. greatest L i (Lower L A)
```

Introduction rule: the usual definition of complete lattice

```
lemma (in partial-order) complete-latticeI:
  assumes sup-exists:
    !!A. [| A ⊆ carrier L |] ==> EX s. least L s (Upper L A)
    and inf-exists:
    !!A. [| A ⊆ carrier L |] ==> EX i. greatest L i (Lower L A)
  shows complete-lattice L
proof (rule complete-lattice.intro)
  show lattice-axioms L
    by (rule lattice-axioms.intro) (blast intro: sup-exists inf-exists)+
qed (assumption | rule complete-lattice-axioms.intro)+

constdefs (structure L)
  top :: - => 'a (⊤1)
  ⊤ == sup L (carrier L)

  bottom :: - => 'a (⊥1)
  ⊥ == inf L (carrier L)


lemma (in complete-lattice) supI:
  [| !!l. least L l (Upper L A) ==> P l; A ⊆ carrier L |]
  ==> P (⨆A)
proof (unfold sup-def)
  assume L: A ⊆ carrier L
    and P: !!l. least L l (Upper L A) ==> P l
  with sup-exists obtain s where least L s (Upper L A) by blast
  with L show P (THE l. least L l (Upper L A))
  by (fast intro: theI2 least-unique P)
qed

lemma (in complete-lattice) sup-closed [simp]:
  A ⊆ carrier L ==> ⨆A ∈ carrier L
  by (rule supI) simp-all

lemma (in complete-lattice) top-closed [simp, intro]:
  ⊤ ∈ carrier L
  by (unfold top-def) simp

lemma (in complete-lattice) infI:
  [| !!i. greatest L i (Lower L A) ==> P i; A ⊆ carrier L |]
  ==> P (⨅A)
proof (unfold inf-def)
  assume L: A ⊆ carrier L
    and P: !!l. greatest L l (Lower L A) ==> P l
  with inf-exists obtain s where greatest L s (Lower L A) by blast
  with L show P (THE l. greatest L l (Lower L A))
```

```
  by (fast intro: theI2 greatest-unique P)
qed

lemma (in complete-lattice) inf-closed [simp]:
  A ⊆ carrier L ==> ⨅A ∈ carrier L
  by (rule infI) simp-all

lemma (in complete-lattice) bottom-closed [simp, intro]:
  ⊥ ∈ carrier L
  by (unfold bottom-def) simp
```

Jacobson: Theorem 8.1

```
lemma Lower-empty [simp]:
  Lower L {} = carrier L
  by (unfold Lower-def) simp

lemma Upper-empty [simp]:
  Upper L {} = carrier L
  by (unfold Upper-def) simp

theorem (in partial-order) complete-lattice-criterion1:
  assumes top-exists: EX g. greatest L g (carrier L)
    and inf-exists:
      !!A. [| A ⊆ carrier L; A ~= {} |] ==> EX i. greatest L i (Lower L A)
  shows complete-lattice L
proof (rule complete-latticeI)
  from top-exists obtain top where top: greatest L top (carrier L) ..
  fix A
  assume L: A ⊆ carrier L
  let ?B = Upper L A
  from L top have top ∈ ?B by (fast intro!: Upper-memI intro: greatest-le)
  then have B-non-empty: ?B ~= {} by fast
  have B-L: ?B ⊆ carrier L by simp
  from inf-exists [OF B-L B-non-empty]
  obtain b where b-inf-B: greatest L b (Lower L ?B) ..
  have least L b (Upper L A)
apply (rule least-UpperI)
   apply (rule greatest-le [where A = Lower L ?B])
    apply (rule b-inf-B)
   apply (rule Lower-memI)
    apply (erule UpperD)
     apply assumption
    apply (rule L)
   apply (fast intro: L [THEN subsetD])
  apply (erule greatest-Lower-above [OF b-inf-B])
  apply simp
 apply (rule L)
apply (rule greatest-carrier [OF b-inf-B])
done
```

```
  then show EX s. least L s (Upper L A) ..
next
  fix A
  assume L: A ⊆ carrier L
  show EX i. greatest L i (Lower L A)
  proof (cases A = {})
    case True then show ?thesis
      by (simp add: top-exists)
  next
    case False with L show ?thesis
      by (rule inf-exists)
  qed
qed
```

1.5 Examples

1.5.1 Powerset of a set is a complete lattice

```
theorem powerset-is-complete-lattice:
  complete-lattice (| carrier = Pow A, le = op ⊆ |)
  (is complete-lattice ?L)
proof (rule partial-order.complete-latticeI)
  show partial-order ?L
    by (rule partial-order.intro) auto
next
  fix B
  assume B ⊆ carrier ?L
  then have least ?L (⋃ B) (Upper ?L B)
    by (fastsimp intro!: least-UpperI simp: Upper-def)
  then show EX s. least ?L s (Upper ?L B) ..
next
  fix B
  assume B ⊆ carrier ?L
  then have greatest ?L (⋂ B ∩ A) (Lower ?L B)
```

$\bigcap B$ is not the infimum of B: $\bigcap\{\} = UNIV$ which is in general bigger than A!

```
    by (fastsimp intro!: greatest-LowerI simp: Lower-def)
  then show EX i. greatest ?L i (Lower ?L B) ..
qed
```

An other example, that of the lattice of subgroups of a group, can be found in Group theory (Section 3.9).

end

2 Group: Groups

theory *Group* **imports** *FuncSet Lattice* **begin**

3 Monoids and Groups

Definitions follow [2].

3.1 Definitions

record *'a monoid* = *'a partial-object* +
  mult :: $['a, 'a] \Rightarrow 'a$ (**infixl** $\otimes\imath$ *70*)
  one :: *'a* ($\mathbf{1}\imath$)

constdefs (**structure** *G*)
  m-inv :: (*'a*, *'b*) *monoid-scheme* => *'a* => *'a* (*inv*ı - [*81*] *80*)
  inv x == (*THE y*. $y \in$ *carrier G* & $x \otimes y = \mathbf{1}$ & $y \otimes x = \mathbf{1}$)

  Units :: - => *'a set*
  — The set of invertible elements
  Units G == {*y*. $y \in$ *carrier G* & ($\exists x \in$ *carrier G*. $x \otimes y = \mathbf{1}$ & $y \otimes x = \mathbf{1}$)}

consts
  pow :: [(*'a*, *'m*) *monoid-scheme*, *'a*, *'b*::*number*] => *'a* (**infixr** '(ˆ')ı *75*)

defs (**overloaded**)
  nat-pow-def: *pow G a n* == *nat-rec* $\mathbf{1}_G$ (%*u b*. $b \otimes_G a$) *n*
  int-pow-def: *pow G a z* ==
    let p = *nat-rec* $\mathbf{1}_G$ (%*u b*. $b \otimes_G a$)
    in if neg z then inv_G (*p* (*nat* ($-z$))) *else p* (*nat z*)

locale *monoid* = *struct G* +
  assumes *m-closed* [*intro*, *simp*]:
      $[\![x \in carrier\ G;\ y \in carrier\ G]\!] \Longrightarrow x \otimes y \in carrier\ G$
    and *m-assoc*:
      $[\![x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G]\!]$
      $\Longrightarrow (x \otimes y) \otimes z = x \otimes (y \otimes z)$
    and *one-closed* [*intro*, *simp*]: $\mathbf{1} \in carrier\ G$
    and *l-one* [*simp*]: $x \in carrier\ G \Longrightarrow \mathbf{1} \otimes x = x$
    and *r-one* [*simp*]: $x \in carrier\ G \Longrightarrow x \otimes \mathbf{1} = x$

lemma *monoidI*:
  includes *struct G*
  assumes *m-closed*:
    !!*x y*. [| $x \in$ *carrier G*; $y \in$ *carrier G* |] ==> $x \otimes y \in$ *carrier G*
   and *one-closed*: $\mathbf{1} \in$ *carrier G*
   and *m-assoc*:
    !!*x y z*. [| $x \in$ *carrier G*; $y \in$ *carrier G*; $z \in$ *carrier G* |] ==>
    $(x \otimes y) \otimes z = x \otimes (y \otimes z)$
   and *l-one*: !!*x*. $x \in$ *carrier G* ==> $\mathbf{1} \otimes x = x$
   and *r-one*: !!*x*. $x \in$ *carrier G* ==> $x \otimes \mathbf{1} = x$
  shows *monoid G*
  by (*fast intro*!: *monoid.intro intro*: *prems*)

lemma (**in** *monoid*) *Units-closed* [*dest*]:
  $x \in \mathit{Units}\ G \Longrightarrow x \in \mathit{carrier}\ G$
  by (*unfold Units-def*) *fast*

lemma (**in** *monoid*) *inv-unique*:
  assumes *eq*: $y \otimes x = \mathbf{1}$ $x \otimes y' = \mathbf{1}$
    and *G*: $x \in \mathit{carrier}\ G$ $y \in \mathit{carrier}\ G$ $y' \in \mathit{carrier}\ G$
  shows $y = y'$
proof –
  from *G eq* **have** $y = y \otimes (x \otimes y')$ **by** *simp*
  also from *G* **have** ... $= (y \otimes x) \otimes y'$ **by** (*simp add*: *m-assoc*)
  also from *G eq* **have** ... $= y'$ **by** *simp*
  finally show *?thesis* **.**
qed

lemma (**in** *monoid*) *Units-one-closed* [*intro*, *simp*]:
  $\mathbf{1} \in \mathit{Units}\ G$
  by (*unfold Units-def*) *auto*

lemma (**in** *monoid*) *Units-inv-closed* [*intro*, *simp*]:
  $x \in \mathit{Units}\ G \Longrightarrow \mathit{inv}\ x \in \mathit{carrier}\ G$
  apply (*unfold Units-def m-inv-def*, *auto*)
  apply (*rule theI2*, *fast*)
   apply (*fast intro*: *inv-unique*, *fast*)
  done

lemma (**in** *monoid*) *Units-l-inv*:
  $x \in \mathit{Units}\ G \Longrightarrow \mathit{inv}\ x \otimes x = \mathbf{1}$
  apply (*unfold Units-def m-inv-def*, *auto*)
  apply (*rule theI2*, *fast*)
   apply (*fast intro*: *inv-unique*, *fast*)
  done

lemma (**in** *monoid*) *Units-r-inv*:
  $x \in \mathit{Units}\ G \Longrightarrow x \otimes \mathit{inv}\ x = \mathbf{1}$
  apply (*unfold Units-def m-inv-def*, *auto*)
  apply (*rule theI2*, *fast*)
   apply (*fast intro*: *inv-unique*, *fast*)
  done

lemma (**in** *monoid*) *Units-inv-Units* [*intro*, *simp*]:
  $x \in \mathit{Units}\ G \Longrightarrow \mathit{inv}\ x \in \mathit{Units}\ G$
proof –
  assume *x*: $x \in \mathit{Units}\ G$
  show $\mathit{inv}\ x \in \mathit{Units}\ G$
    by (*auto simp add*: *Units-def*
      intro: *Units-l-inv Units-r-inv x Units-closed* [*OF x*])
qed

lemma (**in** *monoid*) *Units-l-cancel* [*simp*]:
[| $x \in$ *Units G*; $y \in$ *carrier G*; $z \in$ *carrier G* |] ==>
($x \otimes y = x \otimes z$) = ($y = z$)
proof
assume *eq*: $x \otimes y = x \otimes z$
and *G*: $x \in$ *Units G* $y \in$ *carrier G* $z \in$ *carrier G*
then have (*inv* $x \otimes x$) $\otimes y$ = (*inv* $x \otimes x$) $\otimes z$
by (*simp add*: *m-assoc Units-closed*)
with *G* **show** $y = z$ **by** (*simp add*: *Units-l-inv*)
next
assume *eq*: $y = z$
and *G*: $x \in$ *Units G* $y \in$ *carrier G* $z \in$ *carrier G*
then show $x \otimes y = x \otimes z$ **by** *simp*
qed

lemma (**in** *monoid*) *Units-inv-inv* [*simp*]:
$x \in$ *Units G* ==> *inv* (*inv* x) = x
proof –
assume *x*: $x \in$ *Units G*
then have *inv* $x \otimes$ *inv* (*inv* x) = *inv* $x \otimes x$
by (*simp add*: *Units-l-inv Units-r-inv*)
with *x* **show** *?thesis* **by** (*simp add*: *Units-closed*)
qed

lemma (**in** *monoid*) *inv-inj-on-Units*:
inj-on (*m-inv G*) (*Units G*)
proof (*rule inj-onI*)
fix *x y*
assume *G*: $x \in$ *Units G* $y \in$ *Units G* **and** *eq*: *inv* x = *inv* y
then have *inv* (*inv* x) = *inv* (*inv* y) **by** *simp*
with *G* **show** $x = y$ **by** *simp*
qed

lemma (**in** *monoid*) *Units-inv-comm*:
assumes *inv*: $x \otimes y = \mathbf{1}$
and *G*: $x \in$ *Units G* $y \in$ *Units G*
shows $y \otimes x = \mathbf{1}$
proof –
from *G* **have** $x \otimes y \otimes x = x \otimes \mathbf{1}$ **by** (*auto simp add*: *inv Units-closed*)
with *G* **show** *?thesis* **by** (*simp del*: *r-one add*: *m-assoc Units-closed*)
qed

Power

lemma (**in** *monoid*) *nat-pow-closed* [*intro*, *simp*]:
$x \in$ *carrier G* ==> x (^) (*n*::*nat*) $\in$ *carrier G*
by (*induct n*) (*simp-all add*: *nat-pow-def*)

lemma (**in** *monoid*) *nat-pow-0* [*simp*]:

```
  x (^) (0::nat) = 1
  by (simp add: nat-pow-def)

lemma (in monoid) nat-pow-Suc [simp]:
  x (^) (Suc n) = x (^) n ⊗ x
  by (simp add: nat-pow-def)

lemma (in monoid) nat-pow-one [simp]:
  1 (^) (n::nat) = 1
  by (induct n) simp-all

lemma (in monoid) nat-pow-mult:
  x ∈ carrier G ==> x (^) (n::nat) ⊗ x (^) m = x (^) (n + m)
  by (induct m) (simp-all add: m-assoc [THEN sym])

lemma (in monoid) nat-pow-pow:
  x ∈ carrier G ==> (x (^) n) (^) m = x (^) (n * m::nat)
  by (induct m) (simp, simp add: nat-pow-mult add-commute)
```

A group is a monoid all of whose elements are invertible.

```
locale group = monoid +
  assumes Units: carrier G <= Units G


lemma (in group) is-group: group G
  by (rule group.intro [OF prems])

theorem groupI:
  includes struct G
  assumes m-closed [simp]:
      !!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y ∈ carrier G
    and one-closed [simp]: 1 ∈ carrier G
    and m-assoc:
      !!x y z. [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
      (x ⊗ y) ⊗ z = x ⊗ (y ⊗ z)
    and l-one [simp]: !!x. x ∈ carrier G ==> 1 ⊗ x = x
    and l-inv-ex: !!x. x ∈ carrier G ==> ∃ y ∈ carrier G. y ⊗ x = 1
  shows group G
proof −
  have l-cancel [simp]:
    !!x y z. [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
    (x ⊗ y = x ⊗ z) = (y = z)
  proof
    fix x y z
    assume eq: x ⊗ y = x ⊗ z
      and G: x ∈ carrier G  y ∈ carrier G  z ∈ carrier G
    with l-inv-ex obtain x-inv where xG: x-inv ∈ carrier G
      and l-inv: x-inv ⊗ x = 1 by fast
    from G eq xG have (x-inv ⊗ x) ⊗ y = (x-inv ⊗ x) ⊗ z
```

```
      by (simp add: m-assoc)
    with G show y = z by (simp add: l-inv)
  next
    fix x y z
    assume eq: y = z
      and G: x ∈ carrier G  y ∈ carrier G  z ∈ carrier G
    then show x ⊗ y = x ⊗ z by simp
  qed
  have r-one:
    !!x. x ∈ carrier G ==> x ⊗ 1 = x
  proof −
    fix x
    assume x: x ∈ carrier G
    with l-inv-ex obtain x-inv where xG: x-inv ∈ carrier G
      and l-inv: x-inv ⊗ x = 1 by fast
    from x xG have x-inv ⊗ (x ⊗ 1) = x-inv ⊗ x
      by (simp add: m-assoc [symmetric] l-inv)
    with x xG show x ⊗ 1 = x by simp
  qed
  have inv-ex:
    !!x. x ∈ carrier G ==> ∃ y ∈ carrier G. y ⊗ x = 1 & x ⊗ y = 1
  proof −
    fix x
    assume x: x ∈ carrier G
    with l-inv-ex obtain y where y: y ∈ carrier G
      and l-inv: y ⊗ x = 1 by fast
    from x y have y ⊗ (x ⊗ y) = y ⊗ 1
      by (simp add: m-assoc [symmetric] l-inv r-one)
    with x y have r-inv: x ⊗ y = 1
      by simp
    from x y show ∃ y ∈ carrier G. y ⊗ x = 1 & x ⊗ y = 1
      by (fast intro: l-inv r-inv)
  qed
  then have carrier-subset-Units: carrier G <= Units G
    by (unfold Units-def) fast
  show ?thesis
    by (fast intro!: group.intro monoid.intro group-axioms.intro
      carrier-subset-Units intro: prems r-one)
qed

lemma (in monoid) monoid-groupI:
  assumes l-inv-ex:
    !!x. x ∈ carrier G ==> ∃ y ∈ carrier G. y ⊗ x = 1
  shows group G
  by (rule groupI) (auto intro: m-assoc l-inv-ex)

lemma (in group) Units-eq [simp]:
  Units G = carrier G
proof
```

```
  show Units G <= carrier G by fast
next
  show carrier G <= Units G by (rule Units)
qed

lemma (in group) inv-closed [intro, simp]:
  x ∈ carrier G ==> inv x ∈ carrier G
  using Units-inv-closed by simp

lemma (in group) l-inv [simp]:
  x ∈ carrier G ==> inv x ⊗ x = 1
  using Units-l-inv by simp
```

3.2 Cancellation Laws and Basic Properties

```
lemma (in group) l-cancel [simp]:
  [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
   (x ⊗ y = x ⊗ z) = (y = z)
  using Units-l-inv by simp

lemma (in group) r-inv [simp]:
  x ∈ carrier G ==> x ⊗ inv x = 1
proof −
  assume x: x ∈ carrier G
  then have inv x ⊗ (x ⊗ inv x) = inv x ⊗ 1
    by (simp add: m-assoc [symmetric] l-inv)
  with x show ?thesis by (simp del: r-one)
qed

lemma (in group) r-cancel [simp]:
  [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
   (y ⊗ x = z ⊗ x) = (y = z)
proof
  assume eq: y ⊗ x = z ⊗ x
    and G: x ∈ carrier G  y ∈ carrier G  z ∈ carrier G
  then have y ⊗ (x ⊗ inv x) = z ⊗ (x ⊗ inv x)
    by (simp add: m-assoc [symmetric] del: r-inv)
  with G show y = z by simp
next
  assume eq: y = z
    and G: x ∈ carrier G  y ∈ carrier G  z ∈ carrier G
  then show y ⊗ x = z ⊗ x by simp
qed

lemma (in group) inv-one [simp]:
  inv 1 = 1
proof −
  have inv 1 = 1 ⊗ (inv 1) by (simp del: r-inv)
  moreover have ... = 1 by simp
```

finally show *?thesis* **.**
qed

lemma (**in** *group*) *inv-inv* [*simp*]:
x ∈ *carrier G* ==> *inv* (*inv x*) = *x*
using *Units-inv-inv* **by** *simp*

lemma (**in** *group*) *inv-inj*:
inj-on (*m-inv G*) (*carrier G*)
using *inv-inj-on-Units* **by** *simp*

lemma (**in** *group*) *inv-mult-group*:
[| *x* ∈ *carrier G*; *y* ∈ *carrier G* |] ==> *inv* (*x* ⊗ *y*) = *inv y* ⊗ *inv x*
proof −
assume *G*: *x* ∈ *carrier G* *y* ∈ *carrier G*
then have *inv* (*x* ⊗ *y*) ⊗ (*x* ⊗ *y*) = (*inv y* ⊗ *inv x*) ⊗ (*x* ⊗ *y*)
by (*simp add*: *m-assoc l-inv*) (*simp add*: *m-assoc* [*symmetric*])
with *G* **show** *?thesis* **by** (*simp del*: *l-inv*)
qed

lemma (**in** *group*) *inv-comm*:
[| *x* ⊗ *y* = **1**; *x* ∈ *carrier G*; *y* ∈ *carrier G* |] ==> *y* ⊗ *x* = **1**
by (*rule Units-inv-comm*) *auto*

lemma (**in** *group*) *inv-equality*:
[|*y* ⊗ *x* = **1**; *x* ∈ *carrier G*; *y* ∈ *carrier G*|] ==> *inv x* = *y*
apply (*simp add*: *m-inv-def*)
apply (*rule the-equality*)
apply (*simp add*: *inv-comm* [*of y x*])
apply (*rule r-cancel* [*THEN iffD1*], *auto*)
done

Power

lemma (**in** *group*) *int-pow-def2*:
a (ˆ) (*z*::*int*) = (*if neg z then inv* (*a* (ˆ) (*nat* (−*z*))) *else a* (ˆ) (*nat z*))
by (*simp add*: *int-pow-def nat-pow-def Let-def*)

lemma (**in** *group*) *int-pow-0* [*simp*]:
x (ˆ) (*0*::*int*) = **1**
by (*simp add*: *int-pow-def2*)

lemma (**in** *group*) *int-pow-one* [*simp*]:
1 (ˆ) (*z*::*int*) = **1**
by (*simp add*: *int-pow-def2*)

3.3 Subgroups

locale *subgroup* = *var H* + *struct G* +
assumes *subset*: *H* ⊆ *carrier G*

and *m-closed* [*intro*, *simp*]: $[\![x \in H; y \in H]\!] \Longrightarrow x \otimes y \in H$
and *one-closed* [*simp*]: $\mathbf{1} \in H$
and *m-inv-closed* [*intro*,*simp*]: $x \in H \Longrightarrow inv\ x \in H$

declare (**in** *subgroup*) *group.intro* [*intro*]

lemma (**in** *subgroup*) *mem-carrier* [*simp*]:
$x \in H \Longrightarrow x \in carrier\ G$
using *subset* **by** *blast*

lemma *subgroup-imp-subset*:
$subgroup\ H\ G \Longrightarrow H \subseteq carrier\ G$
by (*rule subgroup.subset*)

lemma (**in** *subgroup*) *subgroup-is-group* [*intro*]:
includes *group G*
shows *group* $(G(\!|carrier := H|\!))$
by (*rule groupI*) (*auto intro*: *m-assoc l-inv mem-carrier*)

Since H is nonempty, it contains some element x. Since it is closed under inverse, it contains *inv x*. Since it is closed under product, it contains $x \otimes inv\ x = \mathbf{1}$.

lemma (**in** *group*) *one-in-subset*:
$[|\ H \subseteq carrier\ G;\ H \neq \{\};\ \forall\, a \in H.\ inv\ a \in H;\ \forall\, a{\in}H.\ \forall\, b{\in}H.\ a \otimes b \in H\ |]$
$==> \mathbf{1} \in H$
by (*force simp add*: *l-inv*)

A characterization of subgroups: closed, non-empty subset.

lemma (**in** *group*) *subgroupI*:
assumes *subset*: $H \subseteq carrier\ G$ **and** *non-empty*: $H \neq \{\}$
and *inv*: $!!a.\ a \in H \Longrightarrow inv\ a \in H$
and *mult*: $!!a\ b.\ [\![a \in H; b \in H]\!] \Longrightarrow a \otimes b \in H$
shows *subgroup H G*
proof (*simp add*: *subgroup-def prems*)
show $\mathbf{1} \in H$ **by** (*rule one-in-subset*) (*auto simp only*: *prems*)
qed

declare *monoid.one-closed* [*iff*] *group.inv-closed* [*simp*]
monoid.l-one [*simp*] *monoid.r-one* [*simp*] *group.inv-inv* [*simp*]

lemma *subgroup-nonempty*:
$\sim subgroup\ \{\}\ G$
by (*blast dest*: *subgroup.one-closed*)

lemma (**in** *subgroup*) *finite-imp-card-positive*:
$finite\ (carrier\ G) ==> 0 < card\ H$
proof (*rule classical*)
assume $finite\ (carrier\ G)\ \sim\ 0 < card\ H$
then have *finite H* **by** (*blast intro*: *finite-subset* [*OF subset*])

with *prems* **have** *subgroup* {} *G* **by** *simp*
with *subgroup-nonempty* **show** *?thesis* **by** *contradiction*
qed

3.4 Direct Products

constdefs
DirProd :: - ⇒ - ⇒ ('a × 'b) *monoid* (**infixr** ×× *80*)
$G \times\times H \equiv (\!|carrier = carrier\ G \times carrier\ H,$
$mult = (\lambda(g, h)\ (g', h').\ (g \otimes_G g', h \otimes_H h')),$
$one = (\mathbf{1}_G, \mathbf{1}_H)|\!)$

lemma *DirProd-monoid*:
includes *monoid G* + *monoid H*
shows *monoid* (*G* ×× *H*)
proof −
from *prems*
show *?thesis* **by** (*unfold monoid-def DirProd-def*, *auto*)
qed

Does not use the previous result because it's easier just to use auto.

lemma *DirProd-group*:
includes *group G* + *group H*
shows *group* (*G* ×× *H*)
by (*rule groupI*)
(*auto intro*: *G.m-assoc H.m-assoc G.l-inv H.l-inv*
simp add: *DirProd-def*)

lemma *carrier-DirProd* [*simp*]:
$carrier\ (G \times\times H) = carrier\ G \times carrier\ H$
by (*simp add*: *DirProd-def*)

lemma *one-DirProd* [*simp*]:
$\mathbf{1}_{G \times\times H} = (\mathbf{1}_G, \mathbf{1}_H)$
by (*simp add*: *DirProd-def*)

lemma *mult-DirProd* [*simp*]:
$(g, h) \otimes_{(G \times\times H)} (g', h') = (g \otimes_G g', h \otimes_H h')$
by (*simp add*: *DirProd-def*)

lemma *inv-DirProd* [*simp*]:
includes *group G* + *group H*
assumes *g*: $g \in carrier\ G$
and *h*: $h \in carrier\ H$
shows $m\text{-}inv\ (G \times\times H)\ (g, h) = (inv_G\ g, inv_H\ h)$
apply (*rule group.inv-equality* [*OF DirProd-group*])
apply (*simp-all add*: *prems group-def group.l-inv*)
done

This alternative proof of the previous result demonstrates interpret. It uses

Prod.inv-equality (available after *interpret*) instead of *group.inv-equality* [*OF DirProd-group*].

lemma
  includes *group G* + *group H*
  assumes *g*: $g \in$ *carrier G*
    and *h*: $h \in$ *carrier H*
  shows *m-inv* $(G \times\times H)$ $(g, h) = (inv_G\ g,\ inv_H\ h)$
proof −
  interpret *Prod*: *group* [$G \times\times H$]
    by (*auto intro*: *DirProd-group group.intro group.axioms prems*)
  show *?thesis* **by** (*simp add*: *Prod.inv-equality g h*)
qed

3.5 Homomorphisms and Isomorphisms

constdefs (**structure** *G* **and** *H*)
  hom :: - => - => ('*a* => '*b*) *set*
  hom G H ==
    {*h*. $h \in$ *carrier G* −> *carrier H* &
      $(\forall x \in$ *carrier G*. $\forall y \in$ *carrier G*. $h\ (x \otimes_G y) = h\ x \otimes_H h\ y)\}$

lemma *hom-mult*:
  [| $h \in$ *hom G H*; $x \in$ *carrier G*; $y \in$ *carrier G* |]
  ==> $h\ (x \otimes_G y) = h\ x \otimes_H h\ y$
  by (*simp add*: *hom-def*)

lemma *hom-closed*:
  [| $h \in$ *hom G H*; $x \in$ *carrier G* |] ==> $h\ x \in$ *carrier H*
  by (*auto simp add*: *hom-def funcset-mem*)

lemma (**in** *group*) *hom-compose*:
    [|$h \in$ *hom G H*; $i \in$ *hom H I*|] ==> *compose* (*carrier G*) *i h* $\in$ *hom G I*
apply (*auto simp add*: *hom-def funcset-compose*)
apply (*simp add*: *compose-def funcset-mem*)
done

3.6 Isomorphisms

constdefs
  iso :: - => - => ('*a* => '*b*) *set* (**infixr** $\cong$ *60*)
  $G \cong H$ == {*h*. $h \in$ *hom G H* & *bij-betw h* (*carrier G*) (*carrier H*)}

lemma *iso-refl*: (%*x*. *x*) $\in G \cong G$
by (*simp add*: *iso-def hom-def inj-on-def bij-betw-def Pi-def*)

lemma (**in** *group*) *iso-sym*:
    $h \in G \cong H \Longrightarrow$ *Inv* (*carrier G*) $h \in H \cong G$
apply (*simp add*: *iso-def bij-betw-Inv*)
apply (*subgoal-tac Inv* (*carrier G*) $h \in$ *carrier H* $\rightarrow$ *carrier G*)

```
 prefer 2 apply (simp add: bij-betw-imp-funcset [OF bij-betw-Inv])
apply (simp add: hom-def bij-betw-def Inv-f-eq funcset-mem f-Inv-f)
done

lemma (in group) iso-trans:
     [|h ∈ G ≅ H; i ∈ H ≅ I|] ==> (compose (carrier G) i h) ∈ G ≅ I
by (auto simp add: iso-def hom-compose bij-betw-compose)

lemma DirProd-commute-iso:
  shows (λ(x,y). (y,x)) ∈ (G ×× H) ≅ (H ×× G)
by (auto simp add: iso-def hom-def inj-on-def bij-betw-def Pi-def)

lemma DirProd-assoc-iso:
  shows (λ(x,y,z). (x,(y,z))) ∈ (G ×× H ×× I) ≅ (G ×× (H ×× I))
by (auto simp add: iso-def hom-def inj-on-def bij-betw-def Pi-def)
```

Basis for homomorphism proofs: we assume two groups G and H, with a homomorphism h between them

```
locale group-hom = group G + group H + var h +
  assumes homh: h ∈ hom G H
  notes hom-mult [simp] = hom-mult [OF homh]
    and hom-closed [simp] = hom-closed [OF homh]

lemma (in group-hom) one-closed [simp]:
  h 1 ∈ carrier H
  by simp

lemma (in group-hom) hom-one [simp]:
  h 1 = 1_H
proof −
  have h 1 ⊗_H 1_H = h 1 ⊗_H h 1
    by (simp add: hom-mult [symmetric] del: hom-mult)
  then show ?thesis by (simp del: r-one)
qed

lemma (in group-hom) inv-closed [simp]:
  x ∈ carrier G ==> h (inv x) ∈ carrier H
  by simp

lemma (in group-hom) hom-inv [simp]:
  x ∈ carrier G ==> h (inv x) = inv_H (h x)
proof −
  assume x: x ∈ carrier G
  then have h x ⊗_H h (inv x) = 1_H
    by (simp add: hom-mult [symmetric] del: hom-mult)
  also from x have ... = h x ⊗_H inv_H (h x)
    by (simp add: hom-mult [symmetric] del: hom-mult)
  finally have h x ⊗_H h (inv x) = h x ⊗_H inv_H (h x) .
  with x show ?thesis by (simp del: H.r-inv)
```

qed

3.7 Commutative Structures

Naming convention: multiplicative structures that are commutative are called *commutative*, additive structures are called *Abelian*.

3.8 Definition

locale *comm-monoid* = *monoid* +
  assumes *m-comm*: $[\![x \in \mathit{carrier}\ G;\ y \in \mathit{carrier}\ G]\!] \Longrightarrow x \otimes y = y \otimes x$

lemma (**in** *comm-monoid*) *m-lcomm*:
  $[\![x \in \mathit{carrier}\ G;\ y \in \mathit{carrier}\ G;\ z \in \mathit{carrier}\ G]\!] \Longrightarrow$
   $x \otimes (y \otimes z) = y \otimes (x \otimes z)$
proof −
  assume *xyz*: $x \in \mathit{carrier}\ G\ \ y \in \mathit{carrier}\ G\ \ z \in \mathit{carrier}\ G$
  from *xyz* **have** $x \otimes (y \otimes z) = (x \otimes y) \otimes z$ **by** (*simp add*: *m-assoc*)
  also from *xyz* **have** ... $= (y \otimes x) \otimes z$ **by** (*simp add*: *m-comm*)
  also from *xyz* **have** ... $= y \otimes (x \otimes z)$ **by** (*simp add*: *m-assoc*)
  finally show *?thesis* **.**
qed

lemmas (**in** *comm-monoid*) *m-ac* = *m-assoc m-comm m-lcomm*

lemma *comm-monoidI*:
  includes *struct G*
  assumes *m-closed*:
    !!$x\ y$. [| $x \in \mathit{carrier}\ G$; $y \in \mathit{carrier}\ G$ |] ==> $x \otimes y \in \mathit{carrier}\ G$
   and *one-closed*: $\mathbf{1} \in \mathit{carrier}\ G$
   and *m-assoc*:
    !!$x\ y\ z$. [| $x \in \mathit{carrier}\ G$; $y \in \mathit{carrier}\ G$; $z \in \mathit{carrier}\ G$ |] ==>
    $(x \otimes y) \otimes z = x \otimes (y \otimes z)$
   and *l-one*: !!x. $x \in \mathit{carrier}\ G$ ==> $\mathbf{1} \otimes x = x$
   and *m-comm*:
    !!$x\ y$. [| $x \in \mathit{carrier}\ G$; $y \in \mathit{carrier}\ G$ |] ==> $x \otimes y = y \otimes x$
  shows *comm-monoid G*
  using *l-one*
   by (*auto intro*!: *comm-monoid.intro comm-monoid-axioms.intro monoid.intro*
     intro: *prems simp*: *m-closed one-closed m-comm*)

lemma (**in** *monoid*) *monoid-comm-monoidI*:
  assumes *m-comm*:
    !!$x\ y$. [| $x \in \mathit{carrier}\ G$; $y \in \mathit{carrier}\ G$ |] ==> $x \otimes y = y \otimes x$
  shows *comm-monoid G*
  by (*rule comm-monoidI*) (*auto intro*: *m-assoc m-comm*)

lemma (**in** *comm-monoid*) *nat-pow-distr*:
[| $x \in$ *carrier* G; $y \in$ *carrier* G |] ==>
$(x \otimes y)$ (^) (n::*nat*) = x (^) $n \otimes y$ (^) n
by (*induct* n) (*simp*, *simp add*: *m-ac*)

locale *comm-group* = *comm-monoid* + *group*

lemma (**in** *group*) *group-comm-groupI*:
assumes *m-comm*: !!x y. [| $x \in$ *carrier* G; $y \in$ *carrier* G |] ==>
$x \otimes y = y \otimes x$
shows *comm-group* G
by (*fast intro*: *comm-group.intro comm-monoid-axioms.intro*
is-group prems)

lemma *comm-groupI*:
includes *struct* G
assumes *m-closed*:
!!x y. [| $x \in$ *carrier* G; $y \in$ *carrier* G |] ==> $x \otimes y \in$ *carrier* G
and *one-closed*: $\mathbf{1} \in$ *carrier* G
and *m-assoc*:
!!x y z. [| $x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* G |] ==>
$(x \otimes y) \otimes z = x \otimes (y \otimes z)$
and *m-comm*:
!!x y. [| $x \in$ *carrier* G; $y \in$ *carrier* G |] ==> $x \otimes y = y \otimes x$
and *l-one*: !!x. $x \in$ *carrier* G ==> $\mathbf{1} \otimes x = x$
and *l-inv-ex*: !!x. $x \in$ *carrier* G ==> $\exists\, y \in$ *carrier* G. $y \otimes x = \mathbf{1}$
shows *comm-group* G
by (*fast intro*: *group.group-comm-groupI groupI prems*)

lemma (**in** *comm-group*) *inv-mult*:
[| $x \in$ *carrier* G; $y \in$ *carrier* G |] ==> *inv* $(x \otimes y)$ = *inv* $x \otimes$ *inv* y
by (*simp add*: *m-ac inv-mult-group*)

3.9 Lattice of subgroups of a group

theorem (**in** *group*) *subgroups-partial-order*:
partial-order (| *carrier* = $\{H$. *subgroup* H $G\}$, *le* = *op* $\subseteq$ |)
by (*rule partial-order.intro*) *simp-all*

lemma (**in** *group*) *subgroup-self*:
subgroup (*carrier* G) G
by (*rule subgroupI*) *auto*

lemma (**in** *group*) *subgroup-imp-group*:
subgroup H G ==> *group* (G(| *carrier* := H |))
using *subgroup.subgroup-is-group* [*OF* - *group.intro*] **.**

lemma (**in** *group*) *is-monoid* [*intro*, *simp*]:
monoid G

```
  by (auto intro: monoid.intro m-assoc)

lemma (in group) subgroup-inv-equality:
  [| subgroup H G; x ∈ H |] ==> m-inv (G (| carrier := H |)) x = inv x
apply (rule-tac inv-equality [THEN sym])
  apply (rule group.l-inv [OF subgroup-imp-group, simplified], assumption+)
 apply (rule subsetD [OF subgroup.subset], assumption+)
apply (rule subsetD [OF subgroup.subset], assumption)
apply (rule-tac group.inv-closed [OF subgroup-imp-group, simplified], assumption+)
done

theorem (in group) subgroups-Inter:
  assumes subgr: (!!H. H ∈ A ==> subgroup H G)
    and not-empty: A ~= {}
  shows subgroup (⋂A) G
proof (rule subgroupI)
  from subgr [THEN subgroup.subset] and not-empty
  show ⋂A ⊆ carrier G by blast
next
  from subgr [THEN subgroup.one-closed]
  show ⋂A ~= {} by blast
next
  fix x assume x ∈ ⋂A
  with subgr [THEN subgroup.m-inv-closed]
  show inv x ∈ ⋂A by blast
next
  fix x y assume x ∈ ⋂A y ∈ ⋂A
  with subgr [THEN subgroup.m-closed]
  show x ⊗ y ∈ ⋂A by blast
qed

theorem (in group) subgroups-complete-lattice:
  complete-lattice (| carrier = {H. subgroup H G}, le = op ⊆ |)
    (is complete-lattice ?L)
proof (rule partial-order.complete-lattice-criterion1)
  show partial-order ?L by (rule subgroups-partial-order)
next
  have greatest ?L (carrier G) (carrier ?L)
    by (unfold greatest-def) (simp add: subgroup.subset subgroup-self)
  then show ∃G. greatest ?L G (carrier ?L) ..
next
  fix A
  assume L: A ⊆ carrier ?L and non-empty: A ~= {}
  then have Int-subgroup: subgroup (⋂A) G
    by (fastsimp intro: subgroups-Inter)
  have greatest ?L (⋂A) (Lower ?L A)
    (is greatest ?L ?Int -)
  proof (rule greatest-LowerI)
    fix H
```

```
    assume H: H ∈ A
    with L have subgroupH: subgroup H G by auto
    from subgroupH have groupH: group (G (| carrier := H |)) (is group ?H)
      by (rule subgroup-imp-group)
    from groupH have monoidH: monoid ?H
      by (rule group.is-monoid)
    from H have Int-subset: ?Int ⊆ H by fastsimp
    then show le ?L ?Int H by simp
  next
    fix H
    assume H: H ∈ Lower ?L A
    with L Int-subgroup show le ?L H ?Int by (fastsimp intro: Inter-greatest)
  next
    show A ⊆ carrier ?L by (rule L)
  next
    show ?Int ∈ carrier ?L by simp (rule Int-subgroup)
  qed
  then show ∃ I. greatest ?L I (Lower ?L A) ..
qed

end
```

4 FiniteProduct: Product Operator for Commutative Monoids

theory *FiniteProduct* **imports** *Group* **begin**

Instantiation of locale *LC* of theory *Finite-Set* is not possible, because here we have explicit typing rules like $x \in$ *carrier* G. We introduce an explicit argument for the domain D.

```
consts
  foldSetD :: ['a set, 'b => 'a => 'a, 'a] => ('b set * 'a) set

inductive foldSetD D f e
  intros
    emptyI [intro]: e ∈ D ==> ({}, e) ∈ foldSetD D f e
    insertI [intro]: [| x ~: A; f x y ∈ D; (A, y) ∈ foldSetD D f e |] ==>
                     (insert x A, f x y) ∈ foldSetD D f e

inductive-cases empty-foldSetDE [elim!]: ({}, x) ∈ foldSetD D f e

constdefs
  foldD :: ['a set, 'b => 'a => 'a, 'a, 'b set] => 'a
  foldD D f e A == THE x. (A, x) ∈ foldSetD D f e

lemma foldSetD-closed:
  [| (A, z) ∈ foldSetD D f e ; e ∈ D; !!x y. [| x ∈ A; y ∈ D |] ==> f x y ∈ D
```

```
    |] ==> z ∈ D
  by (erule foldSetD.elims) auto

lemma Diff1-foldSetD:
  [| (A − {x}, y) ∈ foldSetD D f e; x ∈ A; f x y ∈ D |] ==>
   (A, f x y) ∈ foldSetD D f e
  apply (erule insert-Diff [THEN subst], rule foldSetD.intros)
    apply auto
  done

lemma foldSetD-imp-finite [simp]: (A, x) ∈ foldSetD D f e ==> finite A
  by (induct set: foldSetD) auto

lemma finite-imp-foldSetD:
  [| finite A; e ∈ D; !!x y. [| x ∈ A; y ∈ D |] ==> f x y ∈ D |] ==>
   EX x. (A, x) ∈ foldSetD D f e
proof (induct set: Finites)
  case empty then show ?case by auto
next
  case (insert x F)
  then obtain y where y: (F, y) ∈ foldSetD D f e by auto
  with insert have y ∈ D by (auto dest: foldSetD-closed)
  with y and insert have (insert x F, f x y) ∈ foldSetD D f e
    by (intro foldSetD.intros) auto
  then show ?case ..
qed
```

4.1 Left-commutative operations

```
locale LCD =
  fixes B :: 'b set
  and D :: 'a set
  and f :: 'b => 'a => 'a    (infixl · 70)
  assumes left-commute:
    [| x ∈ B; y ∈ B; z ∈ D |] ==> x · (y · z) = y · (x · z)
  and f-closed [simp, intro!]: !!x y. [| x ∈ B; y ∈ D |] ==> f x y ∈ D

lemma (in LCD) foldSetD-closed [dest]:
  (A, z) ∈ foldSetD D f e ==> z ∈ D
  by (erule foldSetD.elims) auto

lemma (in LCD) Diff1-foldSetD:
  [| (A − {x}, y) ∈ foldSetD D f e; x ∈ A; A ⊆ B |] ==>
  (A, f x y) ∈ foldSetD D f e
  apply (subgoal-tac x ∈ B)
   prefer 2 apply fast
  apply (erule insert-Diff [THEN subst], rule foldSetD.intros)
    apply auto
  done
```

```
lemma (in LCD) foldSetD-imp-finite [simp]:
  (A, x) ∈ foldSetD D f e ==> finite A
  by (induct set: foldSetD) auto

lemma (in LCD) finite-imp-foldSetD:
  [| finite A; A ⊆ B; e ∈ D |] ==> EX x. (A, x) ∈ foldSetD D f e
proof (induct set: Finites)
  case empty then show ?case by auto
next
  case (insert x F)
  then obtain y where y: (F, y) ∈ foldSetD D f e by auto
  with insert have y ∈ D by auto
  with y and insert have (insert x F, f x y) ∈ foldSetD D f e
    by (intro foldSetD.intros) auto
  then show ?case ..
qed

lemma (in LCD) foldSetD-determ-aux:
  e ∈ D ==> ∀ A x. A ⊆ B & card A < n --> (A, x) ∈ foldSetD D f e -->
    (∀ y. (A, y) ∈ foldSetD D f e --> y = x)
  apply (induct n)
   apply (auto simp add: less-Suc-eq)
  apply (erule foldSetD.cases)
   apply blast
  apply (erule foldSetD.cases)
   apply blast
  apply clarify
```

force simplification of *card A < card (insert ...)*.

```
  apply (erule rev-mp)
  apply (simp add: less-Suc-eq-le)
  apply (rule impI)
  apply (rename-tac Aa xa ya Ab xb yb, case-tac xa = xb)
   apply (subgoal-tac Aa = Ab)
    prefer 2 apply (blast elim!: equalityE)
   apply blast
```

case *xa ∉ xb*.

```
  apply (subgoal-tac Aa − {xb} = Ab − {xa} & xb ∈ Aa & xa ∈ Ab)
   prefer 2 apply (blast elim!: equalityE)
  apply clarify
  apply (subgoal-tac Aa = insert xb Ab − {xa})
   prefer 2 apply blast
  apply (subgoal-tac card Aa ≤ card Ab)
   prefer 2
   apply (rule Suc-le-mono [THEN subst])
   apply (simp add: card-Suc-Diff1)
  apply (rule-tac A1 = Aa − {xb} in finite-imp-foldSetD [THEN exE])
```

```
    apply (blast intro: foldSetD-imp-finite finite-Diff)
   apply best
  apply assumption
 apply (frule (1) Diff1-foldSetD)
  apply best
 apply (subgoal-tac ya = f xb x)
  prefer 2
  apply (subgoal-tac Aa ⊆ B)
   prefer 2 apply best
  apply (blast del: equalityCE)
 apply (subgoal-tac (Ab - {xa}, x) ∈ foldSetD D f e)
  prefer 2 apply simp
 apply (subgoal-tac yb = f xa x)
  prefer 2
  apply (blast del: equalityCE dest: Diff1-foldSetD)
 apply (simp (no-asm-simp))
 apply (rule left-commute)
   apply assumption
  apply best
 apply best
 done

lemma (in LCD) foldSetD-determ:
  [| (A, x) ∈ foldSetD D f e; (A, y) ∈ foldSetD D f e; e ∈ D; A ⊆ B |]
  ==> y = x
  by (blast intro: foldSetD-determ-aux [rule-format])

lemma (in LCD) foldD-equality:
  [| (A, y) ∈ foldSetD D f e; e ∈ D; A ⊆ B |] ==> foldD D f e A = y
  by (unfold foldD-def) (blast intro: foldSetD-determ)

lemma foldD-empty [simp]:
  e ∈ D ==> foldD D f e {} = e
  by (unfold foldD-def) blast

lemma (in LCD) foldD-insert-aux:
  [| x ~: A; x ∈ B; e ∈ D; A ⊆ B |] ==>
    ((insert x A, v) ∈ foldSetD D f e) =
    (EX y. (A, y) ∈ foldSetD D f e & v = f x y)
  apply auto
  apply (rule-tac A1 = A in finite-imp-foldSetD [THEN exE])
     apply (fastsimp dest: foldSetD-imp-finite)
    apply assumption
   apply assumption
  apply (blast intro: foldSetD-determ)
  done

lemma (in LCD) foldD-insert:
    [| finite A; x ~: A; x ∈ B; e ∈ D; A ⊆ B |] ==>
```

```
    foldD D f e (insert x A) = f x (foldD D f e A)
  apply (unfold foldD-def)
  apply (simp add: foldD-insert-aux)
  apply (rule the-equality)
   apply (auto intro: finite-imp-foldSetD
    cong add: conj-cong simp add: foldD-def [symmetric] foldD-equality)
  done

lemma (in LCD) foldD-closed [simp]:
  [| finite A; e ∈ D; A ⊆ B |] ==> foldD D f e A ∈ D
proof (induct set: Finites)
  case empty then show ?case by (simp add: foldD-empty)
next
  case insert then show ?case by (simp add: foldD-insert)
qed

lemma (in LCD) foldD-commute:
  [| finite A; x ∈ B; e ∈ D; A ⊆ B |] ==>
   f x (foldD D f e A) = foldD D f (f x e) A
  apply (induct set: Finites)
   apply simp
  apply (auto simp add: left-commute foldD-insert)
  done

lemma Int-mono2:
  [| A ⊆ C; B ⊆ C |] ==> A Int B ⊆ C
  by blast

lemma (in LCD) foldD-nest-Un-Int:
  [| finite A; finite C; e ∈ D; A ⊆ B; C ⊆ B |] ==>
   foldD D f (foldD D f e C) A = foldD D f (foldD D f e (A Int C)) (A Un C)
  apply (induct set: Finites)
   apply simp
  apply (simp add: foldD-insert foldD-commute Int-insert-left insert-absorb
   Int-mono2 Un-subset-iff)
  done

lemma (in LCD) foldD-nest-Un-disjoint:
  [| finite A; finite B; A Int B = {}; e ∈ D; A ⊆ B; C ⊆ B |]
    ==> foldD D f e (A Un B) = foldD D f (foldD D f e B) A
  by (simp add: foldD-nest-Un-Int)
```

— Delete rules to do with *foldSetD* relation.

```
declare foldSetD-imp-finite [simp del]
  empty-foldSetDE [rule del]
  foldSetD.intros [rule del]
declare (in LCD)
  foldSetD-closed [rule del]
```

4.2 Commutative monoids

We enter a more restrictive context, with $f :: 'a => 'a => 'a$ instead of $'b => 'a => 'a$.

locale $ACeD$ =
  fixes $D :: 'a\ set$
    and $f :: 'a => 'a => 'a$ (**infixl** $\cdot$ *70*)
    and $e :: 'a$
  assumes *ident* [*simp*]: $x \in D ==> x \cdot e = x$
    and *commute*: $[|\ x \in D;\ y \in D\ |] ==> x \cdot y = y \cdot x$
    and *assoc*: $[|\ x \in D;\ y \in D;\ z \in D\ |] ==> (x \cdot y) \cdot z = x \cdot (y \cdot z)$
    and *e-closed* [*simp*]: $e \in D$
    and *f-closed* [*simp*]: $[|\ x \in D;\ y \in D\ |] ==> x \cdot y \in D$

lemma (**in** $ACeD$) *left-commute*:
  $[|\ x \in D;\ y \in D;\ z \in D\ |] ==> x \cdot (y \cdot z) = y \cdot (x \cdot z)$
proof –
  assume D: $x \in D\ y \in D\ z \in D$
  then have $x \cdot (y \cdot z) = (y \cdot z) \cdot x$ **by** (*simp add*: *commute*)
  also from D **have** ... $= y \cdot (z \cdot x)$ **by** (*simp add*: *assoc*)
  also from D **have** $z \cdot x = x \cdot z$ **by** (*simp add*: *commute*)
  finally show *?thesis* **.**
qed

lemmas (**in** $ACeD$) AC = *assoc commute left-commute*

lemma (**in** $ACeD$) *left-ident* [*simp*]: $x \in D ==> e \cdot x = x$
proof –
  assume D: $x \in D$
  have $x \cdot e = x$ **by** (*rule ident*)
  with D **show** *?thesis* **by** (*simp add*: *commute*)
qed

lemma (**in** $ACeD$) *foldD-Un-Int*:
  [| *finite* A; *finite* B; $A \subseteq D$; $B \subseteq D$ |] ==>
    foldD D f e A $\cdot$ *foldD D f e B* =
    foldD D f e (A *Un* B) $\cdot$ *foldD D f e* (A *Int* B)
  apply (*induct set*: *Finites*)
   apply (*simp add*: *left-commute LCD.foldD-closed* [*OF LCD.intro* [*of D*]])
  apply (*simp add*: *AC insert-absorb Int-insert-left*
    LCD.foldD-insert [*OF LCD.intro* [*of D*]]
    LCD.foldD-closed [*OF LCD.intro* [*of D*]]
    Int-mono2 Un-subset-iff)
  done

lemma (**in** $ACeD$) *foldD-Un-disjoint*:
  [| *finite* A; *finite* B; A *Int* B = {}; $A \subseteq D$; $B \subseteq D$ |] ==>
    foldD D f e (A *Un* B) = *foldD D f e A* $\cdot$ *foldD D f e B*
  by (*simp add*: *foldD-Un-Int*

left-commute LCD.foldD-closed [*OF LCD.intro* [*of D*]] *Un-subset-iff*)

4.3 Products over Finite Sets

constdefs (**structure** *G*)
finprod :: [('*b*, '*m*) *monoid-scheme*, '*a* => '*b*, '*a set*] => '*b*
finprod G f A == *if finite A*
then foldD (*carrier G*) (*mult G o f*) **1** *A*
else arbitrary

syntax
-finprod :: *index* => *idt* => '*a set* => '*b* => '*b*
((*3*⨂ --:-. -) [*1000*, *0*, *51*, *10*] *10*)
syntax (*xsymbols*)
-finprod :: *index* => *idt* => '*a set* => '*b* => '*b*
((*3*⨂ --∈-. -) [*1000*, *0*, *51*, *10*] *10*)
syntax (*HTML* **output**)
-finprod :: *index* => *idt* => '*a set* => '*b* => '*b*
((*3*⨂ --∈-. -) [*1000*, *0*, *51*, *10*] *10*)
translations
⨂ı*i*:*A*. *b* == *finprod* ◇ı (%*i*. *b*) *A*
— Beware of argument permutation!

lemma (**in** *comm-monoid*) *finprod-empty* [*simp*]:
finprod G f {} = **1**
by (*simp add*: *finprod-def*)

declare *funcsetI* [*intro*]
funcset-mem [*dest*]

lemma (**in** *comm-monoid*) *finprod-insert* [*simp*]:
[| *finite F*; *a* ∉ *F*; *f* ∈ *F* -> *carrier G*; *f a* ∈ *carrier G* |] ==>
finprod G f (*insert a F*) = *f a* ⊗ *finprod G f F*
apply (*rule trans*)
apply (*simp add*: *finprod-def*)
apply (*rule trans*)
apply (*rule LCD.foldD-insert* [*OF LCD.intro* [*of insert a F*]])
apply *simp*
apply (*rule m-lcomm*)
apply *fast*
apply *fast*
apply *assumption*
apply (*fastsimp intro*: *m-closed*)
apply *simp*+
apply *fast*
apply (*auto simp add*: *finprod-def*)
done

lemma (**in** *comm-monoid*) *finprod-one* [*simp*]:

```
  finite A ==> (⨂ i:A. 1) = 1
proof (induct set: Finites)
  case empty show ?case by simp
next
  case (insert a A)
  have (%i. 1) ∈ A -> carrier G by auto
  with insert show ?case by simp
qed

lemma (in comm-monoid) finprod-closed [simp]:
  fixes A
  assumes fin: finite A and f: f ∈ A -> carrier G
  shows finprod G f A ∈ carrier G
using fin f
proof induct
  case empty show ?case by simp
next
  case (insert a A)
  then have a: f a ∈ carrier G by fast
  from insert have A: f ∈ A -> carrier G by fast
  from insert A a show ?case by simp
qed

lemma funcset-Int-left [simp, intro]:
  [| f ∈ A -> C; f ∈ B -> C |] ==> f ∈ A Int B -> C
  by fast

lemma funcset-Un-left [iff]:
  (f ∈ A Un B -> C) = (f ∈ A -> C & f ∈ B -> C)
  by fast

lemma (in comm-monoid) finprod-Un-Int:
  [| finite A; finite B; g ∈ A -> carrier G; g ∈ B -> carrier G |] ==>
     finprod G g (A Un B) ⊗ finprod G g (A Int B) =
     finprod G g A ⊗ finprod G g B
— The reversed orientation looks more natural, but LOOPS as a simprule!
proof (induct set: Finites)
  case empty then show ?case by (simp add: finprod-closed)
next
  case (insert a A)
  then have a: g a ∈ carrier G by fast
  from insert have A: g ∈ A -> carrier G by fast
  from insert A a show ?case
    by (simp add: m-ac Int-insert-left insert-absorb finprod-closed
          Int-mono2 Un-subset-iff)
qed

lemma (in comm-monoid) finprod-Un-disjoint:
  [| finite A; finite B; A Int B = {};
```

```
   g ∈ A −> carrier G; g ∈ B −> carrier G |]
  ==> finprod G g (A Un B) = finprod G g A ⊗ finprod G g B
 apply (subst finprod-Un-Int [symmetric])
   apply (auto simp add: finprod-closed)
 done

lemma (in comm-monoid) finprod-multf:
 [| finite A; f ∈ A −> carrier G; g ∈ A −> carrier G |] ==>
  finprod G (%x. f x ⊗ g x) A = (finprod G f A ⊗ finprod G g A)
proof (induct set: Finites)
 case empty show ?case by simp
next
 case (insert a A) then
 have fA: f ∈ A −> carrier G by fast
 from insert have fa: f a ∈ carrier G by fast
 from insert have gA: g ∈ A −> carrier G by fast
 from insert have ga: g a ∈ carrier G by fast
 from insert have fgA: (%x. f x ⊗ g x) ∈ A −> carrier G
  by (simp add: Pi-def)
 show ?case
  by (simp add: insert fA fa gA ga fgA m-ac)
qed

lemma (in comm-monoid) finprod-cong':
 [| A = B; g ∈ B −> carrier G;
   !!i. i ∈ B ==> f i = g i |] ==> finprod G f A = finprod G g B
proof −
 assume prems: A = B g ∈ B −> carrier G
  !!i. i ∈ B ==> f i = g i
 show ?thesis
 proof (cases finite B)
  case True
  then have !!A. [| A = B; g ∈ B −> carrier G;
   !!i. i ∈ B ==> f i = g i |] ==> finprod G f A = finprod G g B
  proof induct
   case empty thus ?case by simp
  next
   case (insert x B)
   then have finprod G f A = finprod G f (insert x B) by simp
   also from insert have ... = f x ⊗ finprod G f B
   proof (intro finprod-insert)
    show finite B .
   next
    show x ~: B .
   next
    assume x ~: B !!i. i ∈ insert x B ⟹ f i = g i
     g ∈ insert x B → carrier G
    thus f ∈ B −> carrier G by fastsimp
   next
```

```
        assume x ~: B !!i. i ∈ insert x B ⟹ f i = g i
          g ∈ insert x B → carrier G
        thus f x ∈ carrier G by fastsimp
      qed
      also from insert have ... = g x ⊗ finprod G g B by fastsimp
      also from insert have ... = finprod G g (insert x B)
      by (intro finprod-insert [THEN sym]) auto
      finally show ?case .
    qed
    with prems show ?thesis by simp
  next
    case False with prems show ?thesis by (simp add: finprod-def)
  qed
qed

lemma (in comm-monoid) finprod-cong:
  [| A = B; f ∈ B -> carrier G = True;
     !!i. i ∈ B ==> f i = g i |] ==> finprod G f A = finprod G g B

  by (rule finprod-cong') force+
```

Usually, if this rule causes a failed congruence proof error, the reason is that the premise *g ∈ B −> carrier G* cannot be shown. Adding *Pi-def* to the simpset is often useful. For this reason, *comm-monoid.finprod-cong* is not added to the simpset by default.

```
declare funcsetI [rule del]
  funcset-mem [rule del]

lemma (in comm-monoid) finprod-0 [simp]:
  f ∈ {0::nat} -> carrier G ==> finprod G f {..0} = f 0
by (simp add: Pi-def)

lemma (in comm-monoid) finprod-Suc [simp]:
  f ∈ {..Suc n} -> carrier G ==>
   finprod G f {..Suc n} = (f (Suc n) ⊗ finprod G f {..n})
by (simp add: Pi-def atMost-Suc)

lemma (in comm-monoid) finprod-Suc2:
  f ∈ {..Suc n} -> carrier G ==>
   finprod G f {..Suc n} = (finprod G (%i. f (Suc i)) {..n} ⊗ f 0)
proof (induct n)
  case 0 thus ?case by (simp add: Pi-def)
next
  case Suc thus ?case by (simp add: m-assoc Pi-def)
qed

lemma (in comm-monoid) finprod-mult [simp]:
  [| f ∈ {..n} -> carrier G; g ∈ {..n} -> carrier G |] ==>
     finprod G (%i. f i ⊗ g i) {..n::nat} =
```

```
    finprod G f {..n} ⊗ finprod G g {..n}
  by (induct n) (simp-all add: m-ac Pi-def)

end
```

5 Exponent: The Combinatorial Argument Underlying the First Sylow Theorem

```
theory Exponent imports Main Primes begin

constdefs
  exponent      :: [nat, nat] => nat
  exponent p s == if prime p then (GREATEST r. p^r dvd s) else 0
```

5.1 Prime Theorems

```
lemma prime-imp-one-less: prime p ==> Suc 0 < p
by (unfold prime-def, force)

lemma prime-iff:
    (prime p) = (Suc 0 < p & (∀ a b. p dvd a*b --> (p dvd a) | (p dvd b)))
apply (auto simp add: prime-imp-one-less)
apply (blast dest!: prime-dvd-mult)
apply (auto simp add: prime-def)
apply (erule dvdE)
apply (case-tac k=0, simp)
apply (drule-tac x = m in spec)
apply (drule-tac x = k in spec)
apply (simp add: dvd-mult-cancel1 dvd-mult-cancel2)
done

lemma zero-less-prime-power: prime p ==> 0 < p^a
by (force simp add: prime-iff)


lemma zero-less-card-empty: [| finite S; S ≠ {} |] ==> 0 < card(S)
by (rule ccontr, simp)


lemma prime-dvd-cases:
    [| p*k dvd m*n;  prime p |]
     ==> (∃ x. k dvd x*n & m = p*x) | (∃ y. k dvd m*y & n = p*y)
apply (simp add: prime-iff)
apply (frule dvd-mult-left)
apply (subgoal-tac p dvd m | p dvd n)
 prefer 2 apply blast
```

```
apply (erule disjE)
apply (rule disjI1)
apply (rule-tac [2] disjI2)
apply (erule-tac n = m in dvdE)
apply (erule-tac [2] n = n in dvdE, auto)
apply (rule-tac [2] k = p in dvd-mult-cancel)
apply (rule-tac k = p in dvd-mult-cancel)
apply (simp-all add: mult-ac)
done


lemma prime-power-dvd-cases [rule-format (no-asm)]: prime p
    ==> ∀ m n. p^c dvd m*n -->
        (∀ a b. a+b = Suc c --> p^a dvd m | p^b dvd n)
apply (induct-tac c)
 apply clarify
 apply (case-tac a)
  apply simp
 apply simp

apply simp
apply clarify
apply (erule prime-dvd-cases [THEN disjE], assumption, auto)

 apply (case-tac a)
  apply simp
 apply clarify
 apply (drule spec, drule spec, erule (1) notE impE)
 apply (drule-tac x = nat in spec)
 apply (drule-tac x = b in spec)
 apply simp
 apply (blast intro: dvd-refl mult-dvd-mono)

apply (case-tac b)
 apply simp
apply clarify
apply (drule spec, drule spec, erule (1) notE impE)
apply (drule-tac x = a in spec)
apply (drule-tac x = nat in spec, simp)
apply (blast intro: dvd-refl mult-dvd-mono)
done


lemma div-combine:
    [| prime p; ~ (p ^ (Suc r) dvd n);  p^(a+r) dvd n*k |]
     ==> p ^ a dvd k
by (drule-tac a = Suc r and b = a in prime-power-dvd-cases, assumption, auto)
```

```
lemma Suc-le-power: Suc 0 < p ==> Suc n <= p^n
apply (induct-tac n)
apply (simp (no-asm-simp))
apply simp
apply (subgoal-tac 2 * n + 2 <= p * p^n, simp)
apply (subgoal-tac 2 * p^n <= p * p^n)

apply (rule order-trans)
prefer 2 apply assumption
apply (drule-tac k = 2 in mult-le-mono2, simp)
apply (rule mult-le-mono1, simp)
done


lemma power-dvd-bound: [|p^n dvd a;  Suc 0 < p;  0 < a|] ==> n < a
apply (drule dvd-imp-le)
apply (drule-tac [2] n = n in Suc-le-power, auto)
done
```

5.2 Exponent Theorems

```
lemma exponent-ge [rule-format]:
    [|p^k dvd n;  prime p;  0<n|] ==> k <= exponent p n
apply (simp add: exponent-def)
apply (erule Greatest-le)
apply (blast dest: prime-imp-one-less power-dvd-bound)
done

lemma power-exponent-dvd: 0<s ==> (p ^ exponent p s) dvd s
apply (simp add: exponent-def)
apply clarify
apply (rule-tac k = 0 in GreatestI)
prefer 2 apply (blast dest: prime-imp-one-less power-dvd-bound, simp)
done

lemma power-Suc-exponent-Not-dvd:
    [|(p * p ^ exponent p s) dvd s;  prime p |] ==> s=0
apply (subgoal-tac p ^ Suc (exponent p s) dvd s)
 prefer 2 apply simp
apply (rule ccontr)
apply (drule exponent-ge, auto)
done

lemma exponent-power-eq [simp]: prime p ==> exponent p (p^a) = a
apply (simp (no-asm-simp) add: exponent-def)
apply (rule Greatest-equality, simp)
apply (simp (no-asm-simp) add: prime-imp-one-less power-dvd-imp-le)
done
```

lemma exponent-equalityI:
!r::nat. (p^r dvd a) = (p^r dvd b) ==> exponent p a = exponent p b
by (simp (no-asm-simp) add: exponent-def)

lemma exponent-eq-0 [simp]: ¬ prime p ==> exponent p s = 0
by (simp (no-asm-simp) add: exponent-def)

lemma exponent-mult-add1:
[| 0 < a; 0 < b |]
==> (exponent p a) + (exponent p b) <= exponent p (a * b)
apply (case-tac prime p)
apply (rule exponent-ge)
apply (auto simp add: power-add)
apply (blast intro: prime-imp-one-less power-exponent-dvd mult-dvd-mono)
done

lemma exponent-mult-add2: [| 0 < a; 0 < b |]
==> exponent p (a * b) <= (exponent p a) + (exponent p b)
apply (case-tac prime p)
apply (rule leI, clarify)
apply (cut-tac p = p **and** s = a*b **in** power-exponent-dvd, auto)
apply (subgoal-tac p ^ (Suc (exponent p a + exponent p b)) dvd a * b)
apply (rule-tac [2] le-imp-power-dvd [THEN dvd-trans])
prefer 3 **apply** assumption
prefer 2 **apply** simp
apply (frule-tac a = Suc (exponent p a) **and** b = Suc (exponent p b) **in** prime-power-dvd-cases)
apply (assumption, force, simp)
apply (blast dest: power-Suc-exponent-Not-dvd)
done

lemma exponent-mult-add:
[| 0 < a; 0 < b |]
==> exponent p (a * b) = (exponent p a) + (exponent p b)
by (blast intro: exponent-mult-add1 exponent-mult-add2 order-antisym)

lemma not-divides-exponent-0: ~ (p dvd n) ==> exponent p n = 0
apply (case-tac exponent p n, simp)
apply (case-tac n, simp)
apply (cut-tac s = n **and** p = p **in** power-exponent-dvd)
apply (auto dest: dvd-mult-left)
done

lemma exponent-1-eq-0 [simp]: exponent p (Suc 0) = 0
apply (case-tac prime p)

apply (*auto simp add*: *prime-iff not-divides-exponent-0*)
done

5.3 Lemmas for the Main Combinatorial Argument

lemma *le-extend-mult*: [| *0* < *c*; *a* <= *b* |] ==> *a* <= *b* * (*c*::*nat*)
apply (*rule-tac P* = %*x*. *x* <= *b* * *c* **in** *subst*)
apply (*rule mult-1-right*)
apply (*rule mult-le-mono*, *auto*)
done

lemma *p-fac-forw-lemma*:
[| *0* < (*m*::*nat*); *0*<*k*; *k* < *p*^*a*; (*p*^*r*) *dvd* (*p*^*a*)* *m* − *k* |] ==> *r* <= *a*
apply (*rule notnotD*)
apply (*rule notI*)
apply (*drule contrapos-nn* [*OF* - *leI*, *THEN notnotD*], *assumption*)
apply (*drule-tac m* = *a* **in** *less-imp-le*)
apply (*drule le-imp-power-dvd*)
apply (*drule-tac n* = *p* ^ *r* **in** *dvd-trans*, *assumption*)
apply (*frule-tac m* = *k* **in** *less-imp-le*)
apply (*drule-tac c* = *m* **in** *le-extend-mult*, *assumption*)
apply (*drule-tac k* = *p* ^ *a* **and** *m* = (*p* ^ *a*) * *m* **in** *dvd-diffD1*)
prefer *2* **apply** *assumption*
apply (*rule dvd-refl* [*THEN dvd-mult2*])
apply (*drule-tac n* = *k* **in** *dvd-imp-le*, *auto*)
done

lemma *p-fac-forw*: [| *0* < (*m*::*nat*); *0*<*k*; *k* < *p*^*a*; (*p*^*r*) *dvd* (*p*^*a*)* *m* − *k* |]
==> (*p*^*r*) *dvd* (*p*^*a*) − *k*
apply (*frule-tac k1* = *k* **and** *i* = *p* **in** *p-fac-forw-lemma* [*THEN le-imp-power-dvd*], *auto*)
apply (*subgoal-tac p*^*r dvd p*^*a**m)
prefer *2* **apply** (*blast intro*: *dvd-mult2*)
apply (*drule dvd-diffD1*)
apply *assumption*
prefer *2* **apply** (*blast intro*: *dvd-diff*)
apply (*drule less-imp-Suc-add*, *auto*)
done

lemma *r-le-a-forw*: [| *0* < (*k*::*nat*); *k* < *p*^*a*; *0* < *p*; (*p*^*r*) *dvd* (*p*^*a*) − *k* |] ==> *r* <= *a*
by (*rule-tac m* = *Suc 0* **in** *p-fac-forw-lemma*, *auto*)

lemma *p-fac-backw*: [| *0*<*m*; *0*<*k*; *0* < (*p*::*nat*); *k* < *p*^*a*; (*p*^*r*) *dvd p*^*a* − *k* |]

==> (*p*^*r*) *dvd* (*p*^*a*)**m* − *k*
apply (*frule-tac k1* = *k* **and** *i* = *p* **in** *r-le-a-forw* [*THEN le-imp-power-dvd*], *auto*)
apply (*subgoal-tac p*^*r dvd p*^*a**m)

```
 prefer 2 apply (blast intro: dvd-mult2)
apply (drule dvd-diffD1)
  apply assumption
 prefer 2 apply (blast intro: dvd-diff)
apply (drule less-imp-Suc-add, auto)
done
```

```
lemma exponent-p-a-m-k-equation: [| 0<m; 0<k; 0 < (p::nat);  k < p^a |]
      ==> exponent p (p^a * m - k) = exponent p (p^a - k)
apply (blast intro: exponent-equalityI p-fac-forw p-fac-backw)
done
```

Suc rules that we have to delete from the simpset

```
lemmas bad-Sucs = binomial-Suc-Suc mult-Suc mult-Suc-right
```

```
lemma p-not-div-choose-lemma [rule-format]:
     [| ∀ i. Suc i < K --> exponent p (Suc i) = exponent p (Suc(j+i))|]
      ==> k<K --> exponent p ((j+k) choose k) = 0
apply (case-tac prime p)
 prefer 2 apply simp
apply (induct-tac k)
apply (simp (no-asm))

apply (subgoal-tac 0 < (Suc (j+n) choose Suc n) )
 prefer 2 apply (simp add: zero-less-binomial-iff, clarify)
apply (subgoal-tac exponent p ((Suc (j+n) choose Suc n) * Suc n) =
                  exponent p (Suc n))
```

First, use the assumed equation. We simplify the LHS to *exponent p* (*Suc* (*j* + *n*) *choose Suc n*) + *exponent p* (*Suc n*) the common terms cancel, proving the conclusion.

```
 apply (simp del: bad-Sucs add: exponent-mult-add)
```

Establishing the equation requires first applying *Suc-times-binomial-eq* ...

```
apply (simp del: bad-Sucs add: Suc-times-binomial-eq [symmetric])
```

...then *exponent-mult-add* and the quantified premise.

```
apply (simp del: bad-Sucs add: zero-less-binomial-iff exponent-mult-add)
done
```

```
lemma p-not-div-choose:
     [| k<K;  k<=n;
       ∀ j. 0<j & j<K --> exponent p (n - k + (K - j)) = exponent p (K -
j)|]
      ==> exponent p (n choose k) = 0
apply (cut-tac j = n-k and k = k and p = p in p-not-div-choose-lemma)
  prefer 3 apply simp
```

```
 prefer 2 apply assumption
apply (drule-tac x = K - Suc i in spec)
apply (simp add: Suc-diff-le)
done


lemma const-p-fac-right:
     0 < m ==> exponent p ((p^a * m - Suc 0) choose (p^a - Suc 0)) = 0
apply (case-tac prime p)
 prefer 2 apply simp
apply (frule-tac a = a in zero-less-prime-power)
apply (rule-tac K = p^a in p-not-div-choose)
   apply simp
  apply simp
 apply (case-tac m)
  apply (case-tac [2] p^a)
   apply auto

apply (subgoal-tac 0<p)
 prefer 2 apply (force dest!: prime-imp-one-less)
apply (subst exponent-p-a-m-k-equation, auto)
done

lemma const-p-fac:
     0 < m ==> exponent p (((p^a) * m) choose p^a) = exponent p m
apply (case-tac prime p)
 prefer 2 apply simp
apply (subgoal-tac 0 < p^a * m & p^a <= p^a * m)
 prefer 2 apply (force simp add: prime-iff)
```

A similar trick to the one used in *p-not-div-choose-lemma*: insert an equation; use *exponent-mult-add* on the LHS; on the RHS, first transform the binomial coefficient, then use *exponent-mult-add*.

```
apply (subgoal-tac exponent p ((( (p^a) * m) choose p^a) * p^a) =
                    a + exponent p m)
 apply (simp del: bad-Sucs add: zero-less-binomial-iff exponent-mult-add prime-iff)
```

one subgoal left!

```
apply (subst times-binomial-minus1-eq, simp, simp)
apply (subst exponent-mult-add, simp)
apply (simp (no-asm-simp) add: zero-less-binomial-iff)
apply arith
apply (simp del: bad-Sucs add: exponent-mult-add const-p-fac-right)
done


end
```

6 Coset: Cosets and Quotient Groups

theory *Coset* **imports** *Group Exponent* **begin**

constdefs (**structure** *G*)
  r-coset :: [-, 'a set, 'a] ⇒ 'a set (**infixl** #>ı *60*)
  H #> *a* ≡ ⋃*h*∈*H*. {*h* ⊗ *a*}

  l-coset :: [-, 'a, 'a set] ⇒ 'a set (**infixl** <#ı *60*)
  a <# *H* ≡ ⋃*h*∈*H*. {*a* ⊗ *h*}

  RCOSETS :: [-, 'a set] ⇒ ('a set)set (*rcosets*ı - [*81*] *80*)
  rcosets H ≡ ⋃*a*∈*carrier G*. {*H* #> *a*}

  set-mult :: [-, 'a set ,'a set] ⇒ 'a set (**infixl** <#>ı *60*)
  H <#> *K* ≡ ⋃*h*∈*H*. ⋃*k*∈*K*. {*h* ⊗ *k*}

  SET-INV :: [-,'a set] ⇒ 'a set (*set'-inv*ı - [*81*] *80*)
  set-inv H ≡ ⋃*h*∈*H*. {*inv h*}

locale *normal* = *subgroup* + *group* +
  assumes *coset-eq*: (∀ *x* ∈ *carrier G*. *H* #> *x* = *x* <# *H*)

syntax
  @*normal* :: ['a set, ('a, 'b) *monoid-scheme*] ⇒ *bool* (**infixl** ⊲ *60*)

translations
  H ⊲ *G* == *normal H G*

6.1 Basic Properties of Cosets

lemma (**in** *group*) *coset-mult-assoc*:
    [| *M* ⊆ *carrier G*; *g* ∈ *carrier G*; *h* ∈ *carrier G* |]
    ==> (*M* #> *g*) #> *h* = *M* #> (*g* ⊗ *h*)
by (*force simp add*: *r-coset-def m-assoc*)

lemma (**in** *group*) *coset-mult-one* [*simp*]: *M* ⊆ *carrier G* ==> *M* #> **1** = *M*
by (*force simp add*: *r-coset-def*)

lemma (**in** *group*) *coset-mult-inv1*:
    [| *M* #> (*x* ⊗ (*inv y*)) = *M*; *x* ∈ *carrier G* ; *y* ∈ *carrier G*;
      M ⊆ *carrier G* |] ==> *M* #> *x* = *M* #> *y*
apply (*erule subst* [*of* **concl**: %*z*. *M* #> *x* = *z* #> *y*])
apply (*simp add*: *coset-mult-assoc m-assoc*)
done

lemma (**in** *group*) *coset-mult-inv2*:
    [| *M* #> *x* = *M* #> *y*; *x* ∈ *carrier G*; *y* ∈ *carrier G*; *M* ⊆ *carrier G* |]

$==> M \#> (x \otimes (inv\ y)) = M$
apply (*simp add*: *coset-mult-assoc* [*symmetric*])
apply (*simp add*: *coset-mult-assoc*)
done

lemma (**in** *group*) *coset-join1*:
$[|\ H \#> x = H;\ \ x \in carrier\ G;\ \ subgroup\ H\ G\ |] ==> x \in H$
apply (*erule subst*)
apply (*simp add*: *r-coset-def*)
apply (*blast intro*: *l-one subgroup.one-closed sym*)
done

lemma (**in** *group*) *solve-equation*:
$[\![subgroup\ H\ G;\ x \in H;\ y \in H]\!] \Longrightarrow \exists h{\in}H.\ y = h \otimes x$
apply (*rule bexI* [*of - y* $\otimes$ (*inv x*)])
apply (*auto simp add*: *subgroup.m-closed subgroup.m-inv-closed m-assoc*
subgroup.subset [*THEN subsetD*])
done

lemma (**in** *group*) *repr-independence*:
$[\![y \in H \#> x;\ \ x \in carrier\ G;\ subgroup\ H\ G]\!] \Longrightarrow H \#> x = H \#> y$
by (*auto simp add*: *r-coset-def m-assoc* [*symmetric*]
subgroup.subset [*THEN subsetD*]
subgroup.m-closed solve-equation)

lemma (**in** *group*) *coset-join2*:
$[\![x \in carrier\ G;\ \ subgroup\ H\ G;\ \ x{\in}H]\!] \Longrightarrow H \#> x = H$
— Alternative proof is to put $x = \mathbf{1}$ in *repr-independence*.
by (*force simp add*: *subgroup.m-closed r-coset-def solve-equation*)

lemma (**in** *group*) *r-coset-subset-G*:
$[|\ H \subseteq carrier\ G;\ x \in carrier\ G\ |] ==> H \#> x \subseteq carrier\ G$
by (*auto simp add*: *r-coset-def*)

lemma (**in** *group*) *rcosI*:
$[|\ h \in H;\ H \subseteq carrier\ G;\ x \in carrier\ G|] ==> h \otimes x \in H \#> x$
by (*auto simp add*: *r-coset-def*)

lemma (**in** *group*) *rcosetsI*:
$[\![H \subseteq carrier\ G;\ x \in carrier\ G]\!] \Longrightarrow H \#> x \in rcosets\ H$
by (*auto simp add*: *RCOSETS-def*)

Really needed?

lemma (**in** *group*) *transpose-inv*:
$[|\ x \otimes y = z;\ \ x \in carrier\ G;\ \ y \in carrier\ G;\ \ z \in carrier\ G\ |]$
$==> (inv\ x) \otimes z = y$
by (*force simp add*: *m-assoc* [*symmetric*])

lemma (**in** *group*) *rcos-self*: $[|\ x \in carrier\ G;\ subgroup\ H\ G\ |] ==> x \in H \#> x$

apply (*simp add*: *r-coset-def*)
apply (*blast intro*: *sym l-one subgroup.subset* [*THEN subsetD*]
subgroup.one-closed)
done

6.2 Normal subgroups

lemma *normal-imp-subgroup*: $H \lhd G \Longrightarrow$ *subgroup H G*
by (*simp add*: *normal-def subgroup-def*)

lemma (**in** *group*) *normalI*:
subgroup H G $\Longrightarrow$ ($\forall x \in$ *carrier G*. H #> $x = x$ <# H) $\Longrightarrow H \lhd G$
by (*simp add*: *normal-def normal-axioms-def prems*)

lemma (**in** *normal*) *inv-op-closed1*:
$[\![x \in$ *carrier G*; $h \in H]\!] \Longrightarrow$ (*inv x*) $\otimes h \otimes x \in H$
apply (*insert coset-eq*)
apply (*auto simp add*: *l-coset-def r-coset-def*)
apply (*drule bspec*, *assumption*)
apply (*drule equalityD1* [*THEN subsetD*], *blast*, *clarify*)
apply (*simp add*: *m-assoc*)
apply (*simp add*: *m-assoc* [*symmetric*])
done

lemma (**in** *normal*) *inv-op-closed2*:
$[\![x \in$ *carrier G*; $h \in H]\!] \Longrightarrow x \otimes h \otimes$ (*inv x*) $\in H$
apply (*subgoal-tac inv* (*inv x*) $\otimes h \otimes$ (*inv x*) $\in H$)
apply (*simp add*:)
apply (*blast intro*: *inv-op-closed1*)
done

Alternative characterization of normal subgroups

lemma (**in** *group*) *normal-inv-iff*:
($N \lhd G$) =
(*subgroup N G* & ($\forall x \in$ *carrier G*. $\forall h \in N$. $x \otimes h \otimes$ (*inv x*) $\in N$))
(**is** - = *?rhs*)
proof
assume *N*: $N \lhd G$
show *?rhs*
by (*blast intro*: *N normal.inv-op-closed2 normal-imp-subgroup*)
next
assume *?rhs*
hence *sg*: *subgroup N G*
and *closed*: $\bigwedge x.\ x \in$*carrier G* $\Longrightarrow \forall h \in N.\ x \otimes h \otimes$ *inv x* $\in N$ **by** *auto*
hence *sb*: $N \subseteq$ *carrier G* **by** (*simp add*: *subgroup.subset*)
show $N \lhd G$
proof (*intro normalI* [*OF sg*], *simp add*: *l-coset-def r-coset-def*, *clarify*)
fix *x*
assume *x*: $x \in$ *carrier G*

show $(\bigcup h{\in}N.\ \{h \otimes x\}) = (\bigcup h{\in}N.\ \{x \otimes h\})$
proof
show $(\bigcup h{\in}N.\ \{h \otimes x\}) \subseteq (\bigcup h{\in}N.\ \{x \otimes h\})$
proof *clarify*
fix n
assume n: $n \in N$
show $n \otimes x \in (\bigcup h{\in}N.\ \{x \otimes h\})$
proof
from *closed* [*of inv x*]
show $inv\ x \otimes n \otimes x \in N$ **by** (*simp add*: *x n*)
show $n \otimes x \in \{x \otimes (inv\ x \otimes n \otimes x)\}$
by (*simp add*: *x n m-assoc* [*symmetric*] *sb* [*THEN subsetD*])
qed
qed
next
show $(\bigcup h{\in}N.\ \{x \otimes h\}) \subseteq (\bigcup h{\in}N.\ \{h \otimes x\})$
proof *clarify*
fix n
assume n: $n \in N$
show $x \otimes n \in (\bigcup h{\in}N.\ \{h \otimes x\})$
proof
show $x \otimes n \otimes inv\ x \in N$ **by** (*simp add*: *x n closed*)
show $x \otimes n \in \{x \otimes n \otimes inv\ x \otimes x\}$
by (*simp add*: *x n m-assoc sb* [*THEN subsetD*])
qed
qed
qed
qed
qed

6.3 More Properties of Cosets

lemma (**in** *group*) *lcos-m-assoc*:
$[|\ M \subseteq carrier\ G;\ g \in carrier\ G;\ h \in carrier\ G\ |]$
$==>\ g <\#\ (h <\#\ M) = (g \otimes h) <\#\ M$
by (*force simp add*: *l-coset-def m-assoc*)

lemma (**in** *group*) *lcos-mult-one*: $M \subseteq carrier\ G ==> \mathbf{1} <\#\ M = M$
by (*force simp add*: *l-coset-def*)

lemma (**in** *group*) *l-coset-subset-G*:
$[|\ H \subseteq carrier\ G;\ x \in carrier\ G\ |] ==> x <\#\ H \subseteq carrier\ G$
by (*auto simp add*: *l-coset-def subsetD*)

lemma (**in** *group*) *l-coset-swap*:
$[\![y \in x <\#\ H;\ \ x \in carrier\ G;\ \ subgroup\ H\ G]\!] \Longrightarrow x \in y <\#\ H$
proof (*simp add*: *l-coset-def*)
assume $\exists\, h{\in}H.\ y = x \otimes h$
and x: $x \in carrier\ G$

```
    and sb: subgroup H G
  then obtain h' where h': h' ∈ H & x ⊗ h' = y by blast
  show ∃ h∈H. x = y ⊗ h
  proof
    show x = y ⊗ inv h' using h' x sb
      by (auto simp add: m-assoc subgroup.subset [THEN subsetD])
    show inv h' ∈ H using h' sb
      by (auto simp add: subgroup.subset [THEN subsetD] subgroup.m-inv-closed)
  qed
qed

lemma (in group) l-coset-carrier:
     [| y ∈ x <# H;  x ∈ carrier G;  subgroup H G |] ==> y ∈ carrier G
by (auto simp add: l-coset-def m-assoc
                   subgroup.subset [THEN subsetD] subgroup.m-closed)

lemma (in group) l-repr-imp-subset:
  assumes y: y ∈ x <# H and x: x ∈ carrier G and sb: subgroup H G
  shows y <# H ⊆ x <# H
proof −
  from y
  obtain h' where h' ∈ H x ⊗ h' = y by (auto simp add: l-coset-def)
  thus ?thesis using x sb
    by (auto simp add: l-coset-def m-assoc
                       subgroup.subset [THEN subsetD] subgroup.m-closed)
qed

lemma (in group) l-repr-independence:
  assumes y: y ∈ x <# H and x: x ∈ carrier G and sb: subgroup H G
  shows x <# H = y <# H
proof
  show x <# H ⊆ y <# H
    by (rule l-repr-imp-subset,
        (blast intro: l-coset-swap l-coset-carrier y x sb)+)
  show y <# H ⊆ x <# H by (rule l-repr-imp-subset [OF y x sb])
qed

lemma (in group) setmult-subset-G:
     ⟦H ⊆ carrier G; K ⊆ carrier G⟧ ⟹ H <#> K ⊆ carrier G
by (auto simp add: set-mult-def subsetD)

lemma (in group) subgroup-mult-id: subgroup H G ⟹ H <#> H = H
apply (auto simp add: subgroup.m-closed set-mult-def Sigma-def image-def)
apply (rule-tac x = x in bexI)
apply (rule bexI [of - 1])
apply (auto simp add: subgroup.m-closed subgroup.one-closed
                      r-one subgroup.subset [THEN subsetD])
done
```

6.3.1 Set of inverses of an *r-coset*.

lemma (**in** *normal*) *rcos-inv*:
  assumes *x*: $x \in carrier\ G$
  shows *set-inv* $(H\ \#{>}\ x) = H\ \#{>}\ (inv\ x)$
proof (*simp add*: *r-coset-def SET-INV-def x inv-mult-group*, *safe*)
  fix *h*
  assume $h \in H$
  show $inv\ x \otimes inv\ h \in (\bigcup j{\in}H.\ \{j \otimes inv\ x\})$
  proof
    show $inv\ x \otimes inv\ h \otimes x \in H$
      by (*simp add*: *inv-op-closed1 prems*)
    show $inv\ x \otimes inv\ h \in \{inv\ x \otimes inv\ h \otimes x \otimes inv\ x\}$
      by (*simp add*: *prems m-assoc*)
  qed
next
  fix *h*
  assume $h \in H$
  show $h \otimes inv\ x \in (\bigcup j{\in}H.\ \{inv\ x \otimes inv\ j\})$
  proof
    show $x \otimes inv\ h \otimes inv\ x \in H$
      by (*simp add*: *inv-op-closed2 prems*)
    show $h \otimes inv\ x \in \{inv\ x \otimes inv\ (x \otimes inv\ h \otimes inv\ x)\}$
      by (*simp add*: *prems m-assoc* [*symmetric*] *inv-mult-group*)
  qed
qed

6.3.2 Theorems for <#> with #> or <#.

lemma (**in** *group*) *setmult-rcos-assoc*:
  $[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G;\ x \in carrier\ G]\!]$
  $\Longrightarrow H\ {<}\#{>}\ (K\ \#{>}\ x) = (H\ {<}\#{>}\ K)\ \#{>}\ x$
by (*force simp add*: *r-coset-def set-mult-def m-assoc*)

lemma (**in** *group*) *rcos-assoc-lcos*:
  $[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G;\ x \in carrier\ G]\!]$
  $\Longrightarrow (H\ \#{>}\ x)\ {<}\#{>}\ K = H\ {<}\#{>}\ (x\ {<}\#\ K)$
by (*force simp add*: *r-coset-def l-coset-def set-mult-def m-assoc*)

lemma (**in** *normal*) *rcos-mult-step1*:
  $[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
  $\Longrightarrow (H\ \#{>}\ x)\ {<}\#{>}\ (H\ \#{>}\ y) = (H\ {<}\#{>}\ (x\ {<}\#\ H))\ \#{>}\ y$
by (*simp add*: *setmult-rcos-assoc subset*
    r-coset-subset-G l-coset-subset-G rcos-assoc-lcos)

lemma (**in** *normal*) *rcos-mult-step2*:
  $[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
  $\Longrightarrow (H\ {<}\#{>}\ (x\ {<}\#\ H))\ \#{>}\ y = (H\ {<}\#{>}\ (H\ \#{>}\ x))\ \#{>}\ y$
by (*insert coset-eq*, *simp add*: *normal-def*)

```
lemma (in normal) rcos-mult-step3:
  ⟦x ∈ carrier G; y ∈ carrier G⟧
   ⟹ (H <#> (H #> x)) #> y = H #> (x ⊗ y)
by (simp add: setmult-rcos-assoc coset-mult-assoc
         subgroup-mult-id subset prems)

lemma (in normal) rcos-sum:
  ⟦x ∈ carrier G; y ∈ carrier G⟧
   ⟹ (H #> x) <#> (H #> y) = H #> (x ⊗ y)
by (simp add: rcos-mult-step1 rcos-mult-step2 rcos-mult-step3)

lemma (in normal) rcosets-mult-eq: M ∈ rcosets H ⟹ H <#> M = M
 — generalizes subgroup-mult-id
 by (auto simp add: RCOSETS-def subset
      setmult-rcos-assoc subgroup-mult-id prems)
```

6.3.3 An Equivalence Relation

```
constdefs (structure G)
 r-congruent :: [('a,'b)monoid-scheme, 'a set] ⇒ ('a∗'a)set
          (rcongı -)
  rcong H ≡ {(x,y). x ∈ carrier G & y ∈ carrier G & inv x ⊗ y ∈ H}


lemma (in subgroup) equiv-rcong:
  includes group G
  shows equiv (carrier G) (rcong H)
proof (intro equiv.intro)
 show refl (carrier G) (rcong H)
  by (auto simp add: r-congruent-def refl-def)
next
 show sym (rcong H)
 proof (simp add: r-congruent-def sym-def, clarify)
  fix x y
  assume [simp]: x ∈ carrier G y ∈ carrier G
    and inv x ⊗ y ∈ H
  hence inv (inv x ⊗ y) ∈ H by (simp add: m-inv-closed)
  thus inv y ⊗ x ∈ H by (simp add: inv-mult-group)
 qed
next
 show trans (rcong H)
 proof (simp add: r-congruent-def trans-def, clarify)
  fix x y z
  assume [simp]: x ∈ carrier G y ∈ carrier G z ∈ carrier G
    and inv x ⊗ y ∈ H and inv y ⊗ z ∈ H
  hence (inv x ⊗ y) ⊗ (inv y ⊗ z) ∈ H by simp
  hence inv x ⊗ (y ⊗ inv y) ⊗ z ∈ H by (simp add: m-assoc del: r-inv)
  thus inv x ⊗ z ∈ H by simp
 qed
```

qed

Equivalence classes of *rcong* correspond to left cosets. Was there a mistake in the definitions? I'd have expected them to correspond to right cosets.

lemma (**in** *subgroup*) *l-coset-eq-rcong*:
  includes *group G*
  assumes *a*: $a \in carrier\ G$
  shows $a <\#\ H = rcong\ H$ ‘‘ $\{a\}$
by (*force simp add*: *r-congruent-def l-coset-def m-assoc* [*symmetric*] *a*)

6.3.4 Two distinct right cosets are disjoint

lemma (**in** *group*) *rcos-equation*:
  includes *subgroup H G*
  shows
    $[\![ha \otimes a = h \otimes b;\ a \in carrier\ G;\ \ b \in carrier\ G;$
      $h \in H;\ \ ha \in H;\ \ hb \in H]\!]$
    $\Longrightarrow hb \otimes a \in (\bigcup h{\in}H.\ \{h \otimes b\})$
apply (*rule UN-I* [*of hb* $\otimes$ ((*inv ha*) $\otimes$ *h*)])
apply (*simp add*:)
apply (*simp add*: *m-assoc transpose-inv*)
done

lemma (**in** *group*) *rcos-disjoint*:
  includes *subgroup H G*
  shows $[\![a \in rcosets\ H;\ b \in rcosets\ H;\ a{\neq}b]\!] \Longrightarrow a \cap b = \{\}$
apply (*simp add*: *RCOSETS-def r-coset-def*)
apply (*blast intro*: *rcos-equation prems sym*)
done

6.4 Order of a Group and Lagrange's Theorem

constdefs
  order :: (*'a*, *'b*) *monoid-scheme* $\Rightarrow$ *nat*
  order S $\equiv$ *card* (*carrier S*)

lemma (**in** *group*) *rcos-self*:
  includes *subgroup*
  shows $x \in carrier\ G \Longrightarrow x \in H\ \#>\ x$
apply (*simp add*: *r-coset-def*)
apply (*rule-tac x*=**1** **in** *bexI*)
apply (*auto simp add*:)
done

lemma (**in** *group*) *rcosets-part-G*:
  includes *subgroup*
  shows $\bigcup(rcosets\ H) = carrier\ G$
apply (*rule equalityI*)
 apply (*force simp add*: *RCOSETS-def r-coset-def*)

apply (*auto simp add*: *RCOSETS-def intro*: *rcos-self prems*)
done

lemma (**in** *group*) *cosets-finite*:
$[\![c \in \mathit{rcosets}\ H;\ \ H \subseteq \mathit{carrier}\ G;\ \ \mathit{finite}\ (\mathit{carrier}\ G)]\!] \Longrightarrow \mathit{finite}\ c$
apply (*auto simp add*: *RCOSETS-def*)
apply (*simp add*: *r-coset-subset-G* [*THEN finite-subset*])
done

The next two lemmas support the proof of *card-cosets-equal*.

lemma (**in** *group*) *inj-on-f*:
$[\![H \subseteq \mathit{carrier}\ G;\ \ a \in \mathit{carrier}\ G]\!] \Longrightarrow \mathit{inj\text{-}on}\ (\lambda y.\ y \otimes \mathit{inv}\ a)\ (H\ \#\!>\ a)$
apply (*rule inj-onI*)
apply (*subgoal-tac* $x \in \mathit{carrier}\ G\ \&\ y \in \mathit{carrier}\ G$)
prefer *2* **apply** (*blast intro*: *r-coset-subset-G* [*THEN subsetD*])
apply (*simp add*: *subsetD*)
done

lemma (**in** *group*) *inj-on-g*:
$[\![H \subseteq \mathit{carrier}\ G;\ \ a \in \mathit{carrier}\ G]\!] \Longrightarrow \mathit{inj\text{-}on}\ (\lambda y.\ y \otimes a)\ H$
by (*force simp add*: *inj-on-def subsetD*)

lemma (**in** *group*) *card-cosets-equal*:
$[\![c \in \mathit{rcosets}\ H;\ \ H \subseteq \mathit{carrier}\ G;\ \mathit{finite}(\mathit{carrier}\ G)]\!]$
$\Longrightarrow \mathit{card}\ c = \mathit{card}\ H$
apply (*auto simp add*: *RCOSETS-def*)
apply (*rule card-bij-eq*)
apply (*rule inj-on-f*, *assumption*+)
apply (*force simp add*: *m-assoc subsetD r-coset-def*)
apply (*rule inj-on-g*, *assumption*+)
apply (*force simp add*: *m-assoc subsetD r-coset-def*)

The sets $H\ \#\!>\ a$ and H are finite.

apply (*simp add*: *r-coset-subset-G* [*THEN finite-subset*])
apply (*blast intro*: *finite-subset*)
done

lemma (**in** *group*) *rcosets-subset-PowG*:
$\mathit{subgroup}\ H\ G \Longrightarrow \mathit{rcosets}\ H \subseteq \mathit{Pow}(\mathit{carrier}\ G)$
apply (*simp add*: *RCOSETS-def*)
apply (*blast dest*: *r-coset-subset-G subgroup.subset*)
done

theorem (**in** *group*) *lagrange*:
$[\![\mathit{finite}(\mathit{carrier}\ G);\ \mathit{subgroup}\ H\ G]\!]$
$\Longrightarrow \mathit{card}(\mathit{rcosets}\ H) * \mathit{card}(H) = \mathit{order}(G)$
apply (*simp* (*no-asm-simp*) *add*: *order-def rcosets-part-G* [*symmetric*])
apply (*subst mult-commute*)

apply (*rule card-partition*)
apply (*simp add*: *rcosets-subset-PowG* [*THEN finite-subset*])
apply (*simp add*: *rcosets-part-G*)
apply (*simp add*: *card-cosets-equal subgroup.subset*)
apply (*simp add*: *rcos-disjoint*)
done

6.5 Quotient Groups: Factorization of a Group

constdefs
FactGroup :: [(′*a*,′*b*) *monoid-scheme*, ′*a set*] ⇒ (′*a set*) *monoid*
(**infixl** *Mod 65*)
— Actually defined for groups rather than monoids
FactGroup G H ≡
(| *carrier* = *rcosets*$_G$ *H*, *mult* = *set-mult G*, *one* = *H* |)

lemma (**in** *normal*) *setmult-closed*:
⟦*K1* ∈ *rcosets H*; *K2* ∈ *rcosets H*⟧ ⟹ *K1* <#> *K2* ∈ *rcosets H*
by (*auto simp add*: *rcos-sum RCOSETS-def*)

lemma (**in** *normal*) *setinv-closed*:
K ∈ *rcosets H* ⟹ *set-inv K* ∈ *rcosets H*
by (*auto simp add*: *rcos-inv RCOSETS-def*)

lemma (**in** *normal*) *rcosets-assoc*:
⟦*M1* ∈ *rcosets H*; *M2* ∈ *rcosets H*; *M3* ∈ *rcosets H*⟧
⟹ *M1* <#> *M2* <#> *M3* = *M1* <#> (*M2* <#> *M3*)
by (*auto simp add*: *RCOSETS-def rcos-sum m-assoc*)

lemma (**in** *subgroup*) *subgroup-in-rcosets*:
includes *group G*
shows *H* ∈ *rcosets H*
proof −
have *H* #> **1** = *H*
by (*rule coset-join2*, *auto*)
then show *?thesis*
by (*auto simp add*: *RCOSETS-def*)
qed

lemma (**in** *normal*) *rcosets-inv-mult-group-eq*:
M ∈ *rcosets H* ⟹ *set-inv M* <#> *M* = *H*
by (*auto simp add*: *RCOSETS-def rcos-inv rcos-sum subgroup.subset prems*)

theorem (**in** *normal*) *factorgroup-is-group*:
group (*G Mod H*)
apply (*simp add*: *FactGroup-def*)
apply (*rule groupI*)
apply (*simp add*: *setmult-closed*)
apply (*simp add*: *normal-imp-subgroup subgroup-in-rcosets* [*OF is-group*])

apply (*simp add*: *restrictI setmult-closed rcosets-assoc*)
apply (*simp add*: *normal-imp-subgroup*
subgroup-in-rcosets rcosets-mult-eq)
apply (*auto dest*: *rcosets-inv-mult-group-eq simp add*: *setinv-closed*)
done

lemma *mult-FactGroup* [*simp*]: $X \otimes_{(G\ Mod\ H)} X' = X <\#>_G X'$
by (*simp add*: *FactGroup-def*)

lemma (**in** *normal*) *inv-FactGroup*:
$X \in carrier\ (G\ Mod\ H) \Longrightarrow inv_{G\ Mod\ H}\ X = set\text{-}inv\ X$
apply (*rule group.inv-equality* [*OF factorgroup-is-group*])
apply (*simp-all add*: *FactGroup-def setinv-closed rcosets-inv-mult-group-eq*)
done

The coset map is a homomorphism from G to the quotient group $G\ Mod\ H$

lemma (**in** *normal*) *r-coset-hom-Mod*:
$(\lambda a.\ H\ \#>\ a) \in hom\ G\ (G\ Mod\ H)$
by (*auto simp add*: *FactGroup-def RCOSETS-def Pi-def hom-def rcos-sum*)

6.6 The First Isomorphism Theorem

The quotient by the kernel of a homomorphism is isomorphic to the range of that homomorphism.

constdefs
$kernel :: ('a, 'm)\ monoid\text{-}scheme \Rightarrow ('b, 'n)\ monoid\text{-}scheme \Rightarrow$
$('a \Rightarrow 'b) \Rightarrow 'a\ set$
— the kernel of a homomorphism
$kernel\ G\ H\ h \equiv \{x.\ x \in carrier\ G\ \&\ h\ x = \mathbf{1}_H\}$

lemma (**in** *group-hom*) *subgroup-kernel*: *subgroup* (*kernel G H h*) *G*
apply (*rule subgroup.intro*)
apply (*auto simp add*: *kernel-def group.intro prems*)
done

The kernel of a homomorphism is a normal subgroup

lemma (**in** *group-hom*) *normal-kernel*: $(kernel\ G\ H\ h) \lhd G$
apply (*simp add*: *group.normal-inv-iff subgroup-kernel group.intro prems*)
apply (*simp add*: *kernel-def*)
done

lemma (**in** *group-hom*) *FactGroup-nonempty*:
assumes X: $X \in carrier\ (G\ Mod\ kernel\ G\ H\ h)$
shows $X \neq \{\}$
proof −
from X
obtain g **where** $g \in carrier\ G$
and $X = kernel\ G\ H\ h\ \#>\ g$

```
    by (auto simp add: FactGroup-def RCOSETS-def)
  thus ?thesis
   by (auto simp add: kernel-def r-coset-def image-def intro: hom-one)
qed


lemma (in group-hom) FactGroup-contents-mem:
  assumes X: X ∈ carrier (G Mod (kernel G H h))
  shows contents (h`X) ∈ carrier H
proof −
  from X
  obtain g where g: g ∈ carrier G
           and X = kernel G H h #> g
    by (auto simp add: FactGroup-def RCOSETS-def)
  hence h ` X = {h g} by (auto simp add: kernel-def r-coset-def image-def g)
  thus ?thesis by (auto simp add: g)
qed

lemma (in group-hom) FactGroup-hom:
     (λX. contents (h`X)) ∈ hom (G Mod (kernel G H h)) H
apply (simp add: hom-def FactGroup-contents-mem  normal.factorgroup-is-group
[OF normal-kernel] group.axioms monoid.m-closed)
proof (simp add: hom-def funcsetI FactGroup-contents-mem, intro ballI)
  fix X and X'
  assume X:  X  ∈ carrier (G Mod kernel G H h)
     and X': X' ∈ carrier (G Mod kernel G H h)
  then
  obtain g and g'
          where g ∈ carrier G and g' ∈ carrier G
           and X = kernel G H h #> g and X' = kernel G H h #> g'
    by (auto simp add: FactGroup-def RCOSETS-def)
  hence all: ∀x∈X. h x = h g ∀x∈X'. h x = h g'
    and Xsub: X ⊆ carrier G and X'sub: X' ⊆ carrier G
    by (force simp add: kernel-def r-coset-def image-def)+
  hence h ` (X <#> X') = {h g ⊗_H h g'} using X X'
    by (auto dest!: FactGroup-nonempty
           simp add: set-mult-def image-eq-UN
                  subsetD [OF Xsub] subsetD [OF X'sub])
  thus contents (h ` (X <#> X')) = contents (h ` X) ⊗_H contents (h ` X')
    by (simp add: all image-eq-UN FactGroup-nonempty X X')
qed
```

Lemma for the following injectivity result

```
lemma (in group-hom) FactGroup-subset:
     ⟦g ∈ carrier G; g' ∈ carrier G; h g = h g'⟧
      ⟹ kernel G H h #> g ⊆ kernel G H h #> g'
apply (clarsimp simp add: kernel-def r-coset-def image-def)
apply (rename-tac y)
apply (rule-tac x=y ⊗ g ⊗ inv g' in exI)
```

```
apply (simp add: G.m-assoc)
done

lemma (in group-hom) FactGroup-inj-on:
    inj-on (λX. contents (h ‘ X)) (carrier (G Mod kernel G H h))
proof (simp add: inj-on-def, clarify)
  fix X and X'
  assume X: X ∈ carrier (G Mod kernel G H h)
     and X': X' ∈ carrier (G Mod kernel G H h)
  then
  obtain g and g'
           where gX: g ∈ carrier G  g' ∈ carrier G
              X = kernel G H h #> g X' = kernel G H h #> g'
    by (auto simp add: FactGroup-def RCOSETS-def)
  hence all: ∀x∈X. h x = h g ∀x∈X'. h x = h g'
    by (force simp add: kernel-def r-coset-def image-def)+
  assume contents (h ‘ X) = contents (h ‘ X')
  hence h: h g = h g'
    by (simp add: image-eq-UN all FactGroup-nonempty X X')
  show X=X' by (rule equalityI) (simp-all add: FactGroup-subset h gX)
qed
```

If the homomorphism h is onto H, then so is the homomorphism from the quotient group

```
lemma (in group-hom) FactGroup-onto:
  assumes h: h ‘ carrier G = carrier H
  shows (λX. contents (h ‘ X)) ‘ carrier (G Mod kernel G H h) = carrier H
proof
  show (λX. contents (h ‘ X)) ‘ carrier (G Mod kernel G H h) ⊆ carrier H
    by (auto simp add: FactGroup-contents-mem)
  show carrier H ⊆ (λX. contents (h ‘ X)) ‘ carrier (G Mod kernel G H h)
  proof
    fix y
    assume y: y ∈ carrier H
    with h obtain g where g: g ∈ carrier G h g = y
      by (blast elim: equalityE)
    hence (⋃x∈kernel G H h #> g. {h x}) = {y}
      by (auto simp add: y kernel-def r-coset-def)
    with g show y ∈ (λX. contents (h ‘ X)) ‘ carrier (G Mod kernel G H h)
      by (auto intro!: bexI simp add: FactGroup-def RCOSETS-def image-eq-UN)
  qed
qed
```

If h is a homomorphism from G onto H, then the quotient group *G Mod kernel G H h* is isomorphic to H.

```
theorem (in group-hom) FactGroup-iso:
  h ‘ carrier G = carrier H
   ⟹ (λX. contents (h‘X)) ∈ (G Mod (kernel G H h)) ≅ H
by (simp add: iso-def FactGroup-hom FactGroup-inj-on bij-betw-def
```

```
    FactGroup-onto)

end
```

7 Sylow: Sylow's theorem

```
theory Sylow imports Coset begin
```

See also [3].

The combinatorial argument is in theory Exponent

```
locale sylow = group +
  fixes p and a and m and calM and RelM
  assumes prime-p:   prime p
    and order-G:   order(G) = (p ^a) * m
    and finite-G [iff]:  finite (carrier G)
  defines calM == {s. s ⊆ carrier(G) & card(s) = p ^a}
    and RelM == {(N1,N2). N1 ∈ calM & N2 ∈ calM &
                      (∃ g ∈ carrier(G). N1 = (N2 #> g) )}

lemma (in sylow) RelM-refl: refl calM RelM
apply (auto simp add: refl-def RelM-def calM-def)
apply (blast intro!: coset-mult-one [symmetric])
done

lemma (in sylow) RelM-sym: sym RelM
proof (unfold sym-def RelM-def, clarify)
  fix y g
  assume   y ∈ calM
    and g: g ∈ carrier G
  hence y = y #> g #> (inv g) by (simp add: coset-mult-assoc calM-def)
  thus ∃ g'∈carrier G. y = y #> g #> g'
   by (blast intro: g inv-closed)
qed

lemma (in sylow) RelM-trans: trans RelM
by (auto simp add: trans-def RelM-def calM-def coset-mult-assoc)

lemma (in sylow) RelM-equiv: equiv calM RelM
apply (unfold equiv-def)
apply (blast intro: RelM-refl RelM-sym RelM-trans)
done

lemma (in sylow) M-subset-calM-prep: M' ∈ calM // RelM  ==> M' ⊆ calM
apply (unfold RelM-def)
apply (blast elim!: quotientE)
done
```

7.1 Main Part of the Proof

```
locale sylow-central = sylow +
  fixes H and M1 and M
  assumes M-in-quot: M ∈ calM // RelM
      and not-dvd-M: ~(p ^ Suc(exponent p m) dvd card(M))
      and M1-in-M:   M1 ∈ M
  defines H == {g. g∈carrier G & M1 #> g = M1}

lemma (in sylow-central) M-subset-calM: M ⊆ calM
by (rule M-in-quot [THEN M-subset-calM-prep])

lemma (in sylow-central) card-M1: card(M1) = p^a
apply (cut-tac M-subset-calM M1-in-M)
apply (simp add: calM-def, blast)
done

lemma card-nonempty: 0 < card(S) ==> S ≠ {}
by force

lemma (in sylow-central) exists-x-in-M1: ∃x. x∈M1
apply (subgoal-tac 0 < card M1)
 apply (blast dest: card-nonempty)
apply (cut-tac prime-p [THEN prime-imp-one-less])
apply (simp (no-asm-simp) add: card-M1)
done

lemma (in sylow-central) M1-subset-G [simp]: M1 ⊆ carrier G
apply (rule subsetD [THEN PowD])
apply (rule-tac [2] M1-in-M)
apply (rule M-subset-calM [THEN subset-trans])
apply (auto simp add: calM-def)
done

lemma (in sylow-central) M1-inj-H: ∃f ∈ H→M1. inj-on f H
  proof −
    from exists-x-in-M1 obtain m1 where m1M: m1 ∈ M1..
   have m1G: m1 ∈ carrier G by (simp add: m1M M1-subset-G [THEN subsetD])
    show ?thesis
    proof
      show inj-on (λz∈H. m1 ⊗ z) H
        by (simp add: inj-on-def l-cancel [of m1 x y, THEN iffD1] H-def m1G)
      show restrict (op ⊗ m1) H ∈ H → M1
      proof (rule restrictI)
        fix z assume zH: z ∈ H
        show m1 ⊗ z ∈ M1
        proof −
          from zH
          have zG: z ∈ carrier G and M1zeq: M1 #> z = M1
            by (auto simp add: H-def)
```

```
        show ?thesis
          by (rule subst [OF M1zeq], simp add: m1M zG rcosI)
      qed
    qed
  qed
qed
```

7.2 Discharging the Assumptions of *sylow-central*

```
lemma (in sylow) EmptyNotInEquivSet: {} ∉ calM // RelM
by (blast elim!: quotientE dest: RelM-equiv [THEN equiv-class-self])

lemma (in sylow) existsM1inM: M ∈ calM // RelM ==> ∃ M1. M1 ∈ M
apply (subgoal-tac M ≠ {})
 apply blast
apply (cut-tac EmptyNotInEquivSet, blast)
done

lemma (in sylow) zero-less-o-G: 0 < order(G)
apply (unfold order-def)
apply (blast intro: one-closed zero-less-card-empty)
done

lemma (in sylow) zero-less-m: 0 < m
apply (cut-tac zero-less-o-G)
apply (simp add: order-G)
done

lemma (in sylow) card-calM: card(calM) = (p ^a) * m choose p ^a
by (simp add: calM-def n-subsets order-G [symmetric] order-def)

lemma (in sylow) zero-less-card-calM: 0 < card calM
by (simp add: card-calM zero-less-binomial le-extend-mult zero-less-m)

lemma (in sylow) max-p-div-calM:
     ~ (p ^ Suc(exponent p m) dvd card(calM))
apply (subgoal-tac exponent p m = exponent p (card calM) )
 apply (cut-tac zero-less-card-calM prime-p)
 apply (force dest: power-Suc-exponent-Not-dvd)
apply (simp add: card-calM zero-less-m [THEN const-p-fac])
done

lemma (in sylow) finite-calM: finite calM
apply (unfold calM-def)
apply (rule-tac B = Pow (carrier G)  in finite-subset)
apply auto
done

lemma (in sylow) lemma-A1:
```

$\exists\, M \in calM\ //\ RelM.\ {\sim}\ (p\ \hat{}\ Suc(exponent\ p\ m)\ dvd\ card(M))$
apply (*rule max-p-div-calM* [*THEN contrapos-np*])
apply (*simp add*: *finite-calM equiv-imp-dvd-card* [*OF* - *RelM-equiv*])
done

7.2.1 Introduction and Destruct Rules for *H*

lemma (**in** *sylow-central*) *H-I*: [|*g* ∈ *carrier G*; *M1* #> *g* = *M1*|] ==> *g* ∈ *H*
by (*simp add*: *H-def*)

lemma (**in** *sylow-central*) *H-into-carrier-G*: *x* ∈ *H* ==> *x* ∈ *carrier G*
by (*simp add*: *H-def*)

lemma (**in** *sylow-central*) *in-H-imp-eq*: *g* : *H* ==> *M1* #> *g* = *M1*
by (*simp add*: *H-def*)

lemma (**in** *sylow-central*) *H-m-closed*: [| *x*∈*H*; *y*∈*H*|] ==> *x* ⊗ *y* ∈ *H*
apply (*unfold H-def*)
apply (*simp add*: *coset-mult-assoc* [*symmetric*] *m-closed*)
done

lemma (**in** *sylow-central*) *H-not-empty*: *H* ≠ {}
apply (*simp add*: *H-def*)
apply (*rule exI* [*of* - **1**], *simp*)
done

lemma (**in** *sylow-central*) *H-is-subgroup*: *subgroup H G*
apply (*rule subgroupI*)
apply (*rule subsetI*)
apply (*erule H-into-carrier-G*)
apply (*rule H-not-empty*)
apply (*simp add*: *H-def*, *clarify*)
apply (*erule-tac P* = %*z*. ?*lhs*(*z*) = *M1* **in** *subst*)
apply (*simp add*: *coset-mult-assoc*)
apply (*blast intro*: *H-m-closed*)
done

lemma (**in** *sylow-central*) *rcosetGM1g-subset-G*:
  [| *g* ∈ *carrier G*; *x* ∈ *M1* #> *g* |] ==> *x* ∈ *carrier G*
by (*blast intro*: *M1-subset-G* [*THEN r-coset-subset-G*, *THEN subsetD*])

lemma (**in** *sylow-central*) *finite-M1*: *finite M1*
by (*rule finite-subset* [*OF M1-subset-G finite-G*])

lemma (**in** *sylow-central*) *finite-rcosetGM1g*: *g*∈*carrier G* ==> *finite* (*M1* #> *g*)
apply (*rule finite-subset*)
apply (*rule subsetI*)
apply (*erule rcosetGM1g-subset-G*, *assumption*)

apply (*rule finite-G*)
done

lemma (**in** *sylow-central*) *M1-cardeq-rcosetGM1g*:
  $g \in$ *carrier G* ==> *card*(*M1* #> *g*) = *card*(*M1*)
by (*simp* (*no-asm-simp*) *add*: *M1-subset-G card-cosets-equal rcosetsI*)

lemma (**in** *sylow-central*) *M1-RelM-rcosetGM1g*:
  $g \in$ *carrier G* ==> (*M1*, *M1* #> *g*) $\in$ *RelM*
apply (*simp* (*no-asm*) *add*: *RelM-def calM-def card-M1 M1-subset-G*)
apply (*rule conjI*)
 apply (*blast intro*: *rcosetGM1g-subset-G*)
apply (*simp* (*no-asm-simp*) *add*: *card-M1 M1-cardeq-rcosetGM1g*)
apply (*rule bexI* [*of* - *inv g*])
apply (*simp-all add*: *coset-mult-assoc M1-subset-G*)
done

7.3 Equal Cardinalities of M and the Set of Cosets

Injections between M and $rcosets_G\ H$ show that their cardinalities are equal.

lemma *ElemClassEquiv*:
  [| *equiv A r*; *C* $\in$ *A* // *r* |] ==> $\forall x \in C.\ \forall y \in C.\ (x,y) \in r$
by (*unfold equiv-def quotient-def sym-def trans-def*, *blast*)

lemma (**in** *sylow-central*) *M-elem-map*:
  M2 $\in$ *M* ==> $\exists g.\ g \in$ *carrier G* & *M1* #> *g* = *M2*
apply (*cut-tac M1-in-M M-in-quot* [*THEN RelM-equiv* [*THEN ElemClassEquiv*]])
apply (*simp add*: *RelM-def*)
apply (*blast dest*!: *bspec*)
done

lemmas (**in** *sylow-central*) *M-elem-map-carrier* =
  M-elem-map [*THEN someI-ex*, *THEN conjunct1*]

lemmas (**in** *sylow-central*) *M-elem-map-eq* =
  M-elem-map [*THEN someI-ex*, *THEN conjunct2*]

lemma (**in** *sylow-central*) *M-funcset-rcosets-H*:
  (%*x*:*M*. *H* #> (*SOME g*. *g* $\in$ *carrier G* & *M1* #> *g* = *x*)) $\in$ *M* $\rightarrow$ *rcosets H*
apply (*rule rcosetsI* [*THEN restrictI*])
apply (*rule H-is-subgroup* [*THEN subgroup.subset*])
apply (*erule M-elem-map-carrier*)
done

lemma (**in** *sylow-central*) *inj-M-GmodH*: $\exists f \in$ *M*$\rightarrow$*rcosets H*. *inj-on f M*
apply (*rule bexI*)
apply (*rule-tac* [2] *M-funcset-rcosets-H*)
apply (*rule inj-onI*, *simp*)
apply (*rule trans* [*OF* - *M-elem-map-eq*])

```
prefer 2 apply assumption
apply (rule M-elem-map-eq [symmetric, THEN trans], assumption)
apply (rule coset-mult-inv1)
apply (erule-tac [2] M-elem-map-carrier)+
apply (rule-tac [2] M1-subset-G)
apply (rule coset-join1 [THEN in-H-imp-eq])
apply (rule-tac [3] H-is-subgroup)
prefer 2 apply (blast intro: m-closed M-elem-map-carrier inv-closed)
apply (simp add: coset-mult-inv2 H-def M-elem-map-carrier subset-def)
done
```

7.3.1 The opposite injection

```
lemma (in sylow-central) H-elem-map:
    H1 ∈ rcosets H ==> ∃ g. g ∈ carrier G & H #> g = H1
by (auto simp add: RCOSETS-def)

lemmas (in sylow-central) H-elem-map-carrier =
      H-elem-map [THEN someI-ex, THEN conjunct1]

lemmas (in sylow-central) H-elem-map-eq =
      H-elem-map [THEN someI-ex, THEN conjunct2]


lemma EquivElemClass:
    [|equiv A r; M ∈ A//r; M1∈M; (M1,M2) ∈ r |] ==> M2 ∈ M
by (unfold equiv-def quotient-def sym-def trans-def, blast)


lemma (in sylow-central) rcosets-H-funcset-M:
  (λC ∈ rcosets H. M1 #> (@g. g ∈ carrier G ∧ H #> g = C)) ∈ rcosets H →
M
apply (simp add: RCOSETS-def)
apply (fast intro: someI2
          intro!: restrictI M1-in-M
            EquivElemClass [OF RelM-equiv M-in-quot -  M1-RelM-rcosetGM1g])
done
```

close to a duplicate of *inj-M-GmodH*

```
lemma (in sylow-central) inj-GmodH-M:
    ∃ g ∈ rcosets H→M. inj-on g (rcosets H)
apply (rule bexI)
apply (rule-tac [2] rcosets-H-funcset-M)
apply (rule inj-onI)
apply (simp)
apply (rule trans [OF - H-elem-map-eq])
prefer 2 apply assumption
apply (rule H-elem-map-eq [symmetric, THEN trans], assumption)
apply (rule coset-mult-inv1)
```

```
apply (erule-tac [2] H-elem-map-carrier)+
apply (rule-tac [2] H-is-subgroup [THEN subgroup.subset])
apply (rule coset-join2)
apply (blast intro: m-closed inv-closed H-elem-map-carrier)
apply (rule H-is-subgroup)
apply (simp add: H-I coset-mult-inv2 M1-subset-G H-elem-map-carrier)
done

lemma (in sylow-central) calM-subset-PowG: calM ⊆ Pow(carrier G)
by (auto simp add: calM-def)


lemma (in sylow-central) finite-M: finite M
apply (rule finite-subset)
apply (rule M-subset-calM [THEN subset-trans])
apply (rule calM-subset-PowG, blast)
done

lemma (in sylow-central) cardMeqIndexH: card(M) = card(rcosets H)
apply (insert inj-M-GmodH inj-GmodH-M)
apply (blast intro: card-bij finite-M H-is-subgroup
            rcosets-subset-PowG [THEN finite-subset]
            finite-Pow-iff [THEN iffD2])
done

lemma (in sylow-central) index-lem: card(M) * card(H) = order(G)
by (simp add: cardMeqIndexH lagrange H-is-subgroup)

lemma (in sylow-central) lemma-leq1: p^a ≤ card(H)
apply (rule dvd-imp-le)
 apply (rule div-combine [OF prime-p not-dvd-M])
 prefer 2 apply (blast intro: subgroup.finite-imp-card-positive H-is-subgroup)
apply (simp add: index-lem order-G power-add mult-dvd-mono power-exponent-dvd
               zero-less-m)
done

lemma (in sylow-central) lemma-leq2: card(H) ≤ p^a
apply (subst card-M1 [symmetric])
apply (cut-tac M1-inj-H)
apply (blast intro!: M1-subset-G intro:
            card-inj H-into-carrier-G finite-subset [OF - finite-G])
done

lemma (in sylow-central) card-H-eq: card(H) = p^a
by (blast intro: le-anti-sym lemma-leq1 lemma-leq2)

lemma (in sylow) sylow-thm: ∃H. subgroup H G & card(H) = p^a
apply (cut-tac lemma-A1, clarify)
apply (frule existsM1inM, clarify)
```

apply (*subgoal-tac sylow-central G p a m M1 M*)
apply (*blast dest*: *sylow-central.H-is-subgroup sylow-central.card-H-eq*)
apply (*simp add*: *sylow-central-def sylow-central-axioms-def prems*)
done

Needed because the locale's automatic definition refers to *semigroup G* and *group-axioms G* rather than simply to *group G*.

lemma *sylow-eq*: *sylow G p a m* = (*group G* & *sylow-axioms G p a m*)
by (*simp add*: *sylow-def group-def*)

theorem *sylow-thm*:
[| *prime p*; *group*(*G*); *order*(*G*) = (*p*ˆ*a*) ∗ *m*; *finite* (*carrier G*)|]
==> ∃ *H*. *subgroup H G* & *card*(*H*) = *p*ˆ*a*
apply (*rule sylow.sylow-thm* [*of G p a m*])
apply (*simp add*: *sylow-eq sylow-axioms-def*)
done

end

8 Bij: Bijections of a Set, Permutation Groups, Automorphism Groups

theory *Bij* **imports** *Group* **begin**

constdefs
Bij :: ′*a set* ⇒ (′*a* ⇒ ′*a*) *set*
— Only extensional functions, since otherwise we get too many.
Bij S ≡ *extensional S* ∩ {*f*. *bij-betw f S S*}

BijGroup :: ′*a set* ⇒ (′*a* ⇒ ′*a*) *monoid*
BijGroup S ≡
(|*carrier* = *Bij S*,
mult = λ*g* ∈ *Bij S*. λ*f* ∈ *Bij S*. *compose S g f*,
one = λ*x* ∈ *S*. *x*|)

declare *Id-compose* [*simp*] *compose-Id* [*simp*]

lemma *Bij-imp-extensional*: *f* ∈ *Bij S* =⇒ *f* ∈ *extensional S*
by (*simp add*: *Bij-def*)

lemma *Bij-imp-funcset*: *f* ∈ *Bij S* =⇒ *f* ∈ *S* → *S*
by (*auto simp add*: *Bij-def bij-betw-imp-funcset*)

8.1 Bijections Form a Group

lemma *restrict-Inv-Bij*: $f \in Bij\ S \Longrightarrow (\lambda x \in S.\ (Inv\ S\ f)\ x) \in Bij\ S$
by (*simp add*: *Bij-def bij-betw-Inv*)

lemma *id-Bij*: $(\lambda x{\in}S.\ x) \in Bij\ S$
by (*auto simp add*: *Bij-def bij-betw-def inj-on-def*)

lemma *compose-Bij*: $[\![x \in Bij\ S;\ y \in Bij\ S]\!] \Longrightarrow compose\ S\ x\ y \in Bij\ S$
by (*auto simp add*: *Bij-def bij-betw-compose*)

lemma *Bij-compose-restrict-eq*:
$f \in Bij\ S \Longrightarrow compose\ S\ (restrict\ (Inv\ S\ f)\ S)\ f = (\lambda x{\in}S.\ x)$
by (*simp add*: *Bij-def compose-Inv-id*)

theorem *group-BijGroup*: *group* (*BijGroup S*)
apply (*simp add*: *BijGroup-def*)
apply (*rule groupI*)
apply (*simp add*: *compose-Bij*)
apply (*simp add*: *id-Bij*)
apply (*simp add*: *compose-Bij*)
apply (*blast intro*: *compose-assoc* [*symmetric*] *Bij-imp-funcset*)
apply (*simp add*: *id-Bij Bij-imp-funcset Bij-imp-extensional*, *simp*)
apply (*blast intro*: *Bij-compose-restrict-eq restrict-Inv-Bij*)
done

8.2 Automorphisms Form a Group

lemma *Bij-Inv-mem*: $[\![\ f \in Bij\ S;\ \ x \in S]\!] \Longrightarrow Inv\ S\ f\ x \in S$
by (*simp add*: *Bij-def bij-betw-def Inv-mem*)

lemma *Bij-Inv-lemma*:
assumes *eq*: $\bigwedge x\ y.\ [\![x \in S;\ y \in S]\!] \Longrightarrow h(g\ x\ y) = g\ (h\ x)\ (h\ y)$
shows $[\![h \in Bij\ S;\ \ g \in S \to S \to S;\ \ x \in S;\ \ y \in S]\!]$
$\Longrightarrow Inv\ S\ h\ (g\ x\ y) = g\ (Inv\ S\ h\ x)\ (Inv\ S\ h\ y)$
apply (*simp add*: *Bij-def bij-betw-def*)
apply (*subgoal-tac* $\exists\, x'{\in}S.\ \exists\, y'{\in}S.\ x = h\ x'\ \&\ y = h\ y'$, *clarify*)
apply (*simp add*: *eq* [*symmetric*] *Inv-f-f funcset-mem* [*THEN funcset-mem*], *blast*)
done

constdefs
$auto :: ('a,\ 'b)\ monoid\text{-}scheme \Rightarrow ('a \Rightarrow 'a)\ set$
$auto\ G \equiv hom\ G\ G \cap Bij\ (carrier\ G)$

$AutoGroup :: ('a,\ 'c)\ monoid\text{-}scheme \Rightarrow ('a \Rightarrow 'a)\ monoid$
$AutoGroup\ G \equiv BijGroup\ (carrier\ G)\ (\!|carrier := auto\ G|\!)$

lemma (**in** *group*) *id-in-auto*: $(\lambda x \in carrier\ G.\ x) \in auto\ G$
by (*simp add*: *auto-def hom-def restrictI group.axioms id-Bij*)

lemma (**in** *group*) *mult-funcset*: *mult* $G \in$ *carrier* $G \to$ *carrier* $G \to$ *carrier* G
  by (*simp add*: *Pi-I group.axioms*)

lemma (**in** *group*) *restrict-Inv-hom*:
  $[\![h \in$ *hom* G G; $h \in$ *Bij* (*carrier* G)$]\!]$
  $\Longrightarrow$ *restrict* (*Inv* (*carrier* G) h) (*carrier* G) $\in$ *hom* G G
  by (*simp add*: *hom-def Bij-Inv-mem restrictI mult-funcset*
      group.axioms Bij-Inv-lemma)

lemma *inv-BijGroup*:
  $f \in$ *Bij* $S \Longrightarrow$ *m-inv* (*BijGroup* S) f = ($\lambda x \in S$. (*Inv* S f) x)
apply (*rule group.inv-equality*)
apply (*rule group-BijGroup*)
apply (*simp-all add*: *BijGroup-def restrict-Inv-Bij Bij-compose-restrict-eq*)
done

lemma (**in** *group*) *subgroup-auto*:
  subgroup (*auto* G) (*BijGroup* (*carrier* G))
proof (*rule subgroup.intro*)
  show *auto* $G \subseteq$ *carrier* (*BijGroup* (*carrier* G))
    by (*force simp add*: *auto-def BijGroup-def*)
next
  fix x y
  assume $x \in$ *auto* G $y \in$ *auto* G
  thus $x \otimes_{BijGroup\ (carrier\ G)} y \in$ *auto* G
    by (*force simp add*: *BijGroup-def is-group auto-def Bij-imp-funcset*
      group.hom-compose compose-Bij)
next
  show $\mathbf{1}_{BijGroup\ (carrier\ G)} \in$ *auto* G **by** (*simp add*: *BijGroup-def id-in-auto*)
next
  fix x
  assume $x \in$ *auto* G
  thus $inv_{BijGroup\ (carrier\ G)}\ x \in$ *auto* G
    by (*simp del*: *restrict-apply*
      add: *inv-BijGroup auto-def restrict-Inv-Bij restrict-Inv-hom*)
qed

theorem (**in** *group*) *AutoGroup*: *group* (*AutoGroup* G)
by (*simp add*: *AutoGroup-def subgroup.subgroup-is-group subgroup-auto*
    group-BijGroup)

end

9 CRing: Abelian Groups

theory *CRing* **imports** *FiniteProduct*

uses (*ringsimp.ML*) **begin**

record *'a ring* = *'a monoid* +
  zero :: *'a* (**0**ı)
  add :: [*'a*, *'a*] => *'a* (**infixl** ⊕ı *65*)

Derived operations.

constdefs (**structure** *R*)
  a-inv :: [(*'a*, *'m*) *ring-scheme*, *'a*] => *'a* (⊖ı - [*81*] *80*)
  a-inv R == *m-inv* (| *carrier* = *carrier R*, *mult* = *add R*, *one* = *zero R* |)

  minus :: [(*'a*, *'m*) *ring-scheme*, *'a*, *'a*] => *'a* (**infixl** ⊖ı *65*)
  [| *x* ∈ *carrier R*; *y* ∈ *carrier R* |] ==> *x* ⊖ *y* == *x* ⊕ (⊖ *y*)

locale *abelian-monoid* = *struct G* +
  assumes *a-comm-monoid*:
    comm-monoid (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)

The following definition is redundant but simple to use.

locale *abelian-group* = *abelian-monoid* +
  assumes *a-comm-group*:
    comm-group (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)

9.1 Basic Properties

lemma *abelian-monoidI*:
  includes *struct R*
  assumes *a-closed*:
    !!*x y*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R* |] ==> *x* ⊕ *y* ∈ *carrier R*
   and *zero-closed*: **0** ∈ *carrier R*
   and *a-assoc*:
    !!*x y z*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R*; *z* ∈ *carrier R* |] ==>
    (*x* ⊕ *y*) ⊕ *z* = *x* ⊕ (*y* ⊕ *z*)
   and *l-zero*: !!*x*. *x* ∈ *carrier R* ==> **0** ⊕ *x* = *x*
   and *a-comm*:
    !!*x y*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R* |] ==> *x* ⊕ *y* = *y* ⊕ *x*
  shows *abelian-monoid R*
  by (*auto intro*!: *abelian-monoid.intro comm-monoidI intro*: *prems*)

lemma *abelian-groupI*:
  includes *struct R*
  assumes *a-closed*:
    !!*x y*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R* |] ==> *x* ⊕ *y* ∈ *carrier R*
   and *zero-closed*: *zero R* ∈ *carrier R*
   and *a-assoc*:
    !!*x y z*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R*; *z* ∈ *carrier R* |] ==>
    (*x* ⊕ *y*) ⊕ *z* = *x* ⊕ (*y* ⊕ *z*)
   and *a-comm*:
    !!*x y*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R* |] ==> *x* ⊕ *y* = *y* ⊕ *x*

and *l-zero*: !!x. $x \in$ *carrier* R ==> $\mathbf{0} \oplus x = x$
and *l-inv-ex*: !!x. $x \in$ *carrier* R ==> *EX* y : *carrier* R. $y \oplus x = \mathbf{0}$
shows *abelian-group* R
by (*auto intro*!: *abelian-group.intro abelian-monoidI*
abelian-group-axioms.intro comm-monoidI comm-groupI
intro: *prems*)

lemma (**in** *abelian-monoid*) *a-monoid*:
monoid (| *carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G |)
by (*rule comm-monoid.axioms*, *rule a-comm-monoid*)

lemma (**in** *abelian-group*) *a-group*:
group (| *carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G |)
by (*simp add*: *group-def a-monoid comm-group.axioms a-comm-group*)

lemmas *monoid-record-simps* = *partial-object.simps monoid.simps*

lemma (**in** *abelian-monoid*) *a-closed* [*intro*, *simp*]:
$[\![$ $x \in$ *carrier* G; $y \in$ *carrier* G $]\!] \Longrightarrow x \oplus y \in$ *carrier* G
by (*rule monoid.m-closed* [*OF a-monoid*, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *zero-closed* [*intro*, *simp*]:
$\mathbf{0} \in$ *carrier* G
by (*rule monoid.one-closed* [*OF a-monoid*, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-inv-closed* [*intro*, *simp*]:
$x \in$ *carrier* G ==> $\ominus$ $x \in$ *carrier* G
by (*simp add*: *a-inv-def*
group.inv-closed [*OF a-group*, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *minus-closed* [*intro*, *simp*]:
[| $x \in$ *carrier* G; $y \in$ *carrier* G |] ==> $x \ominus y \in$ *carrier* G
by (*simp add*: *minus-def*)

lemma (**in** *abelian-group*) *a-l-cancel* [*simp*]:
[| $x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* G |] ==>
$(x \oplus y = x \oplus z) = (y = z)$
by (*rule group.l-cancel* [*OF a-group*, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-r-cancel* [*simp*]:
[| $x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* G |] ==>
$(y \oplus x = z \oplus x) = (y = z)$
by (*rule group.r-cancel* [*OF a-group*, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *a-assoc*:
$[\![x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* $G]\!] \Longrightarrow$
$(x \oplus y) \oplus z = x \oplus (y \oplus z)$
by (*rule monoid.m-assoc* [*OF a-monoid*, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *l-zero* [*simp*]:
$x \in carrier\ G ==> \mathbf{0} \oplus x = x$
by (*rule monoid.l-one* [*OF a-monoid*, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *l-neg*:
$x \in carrier\ G ==> \ominus\ x \oplus x = \mathbf{0}$
by (*simp add*: *a-inv-def*
group.l-inv [*OF a-group*, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *a-comm*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!] \Longrightarrow x \oplus y = y \oplus x$
by (*rule comm-monoid.m-comm* [*OF a-comm-monoid*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *a-lcomm*:
$[\![x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G]\!] \Longrightarrow$
$x \oplus (y \oplus z) = y \oplus (x \oplus z)$
by (*rule comm-monoid.m-lcomm* [*OF a-comm-monoid*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *r-zero* [*simp*]:
$x \in carrier\ G ==> x \oplus \mathbf{0} = x$
using *monoid.r-one* [*OF a-monoid*]
by *simp*

lemma (**in** *abelian-group*) *r-neg*:
$x \in carrier\ G ==> x \oplus (\ominus\ x) = \mathbf{0}$
using *group.r-inv* [*OF a-group*]
by (*simp add*: *a-inv-def*)

lemma (**in** *abelian-group*) *minus-zero* [*simp*]:
$\ominus\ \mathbf{0} = \mathbf{0}$
by (*simp add*: *a-inv-def*
group.inv-one [*OF a-group*, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *minus-minus* [*simp*]:
$x \in carrier\ G ==> \ominus\ (\ominus\ x) = x$
using *group.inv-inv* [*OF a-group*, *simplified monoid-record-simps*]
by (*simp add*: *a-inv-def*)

lemma (**in** *abelian-group*) *a-inv-inj*:
inj-on (*a-inv G*) (*carrier G*)
using *group.inv-inj* [*OF a-group*, *simplified monoid-record-simps*]
by (*simp add*: *a-inv-def*)

lemma (**in** *abelian-group*) *minus-add*:
$[|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> \ominus\ (x \oplus y) = \ominus\ x \oplus \ominus\ y$
using *comm-group.inv-mult* [*OF a-comm-group*]
by (*simp add*: *a-inv-def*)

lemmas (**in** *abelian-monoid*) *a-ac* = *a-assoc a-comm a-lcomm*

9.2 Sums over Finite Sets

This definition makes it easy to lift lemmas from *finprod*.

constdefs
finsum :: [('b, 'm) *ring-scheme*, 'a => 'b, 'a *set*] => 'b
finsum G f A == *finprod* (| *carrier* = *carrier G*,
mult = *add G*, *one* = *zero G* |) *f A*

syntax
-finsum :: *index* => *idt* => 'a *set* => 'b => 'b
((3⨁--:-. -) [1000, 0, 51, 10] 10)
syntax (*xsymbols*)
-finsum :: *index* => *idt* => 'a *set* => 'b => 'b
((3⨁--∈-. -) [1000, 0, 51, 10] 10)
syntax (*HTML* **output**)
-finsum :: *index* => *idt* => 'a *set* => 'b => 'b
((3⨁--∈-. -) [1000, 0, 51, 10] 10)
translations
⨁ı*i*:*A*. *b* == *finsum* ⋄ı (%*i*. *b*) *A*
— Beware of argument permutation!

lemma (**in** *abelian-monoid*) *finsum-empty* [*simp*]:
finsum G f {} = **0**
by (*rule comm-monoid.finprod-empty* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-insert* [*simp*]:
[| *finite F*; *a* ∉ *F*; *f* ∈ *F* −> *carrier G*; *f a* ∈ *carrier G* |]
==> *finsum G f* (*insert a F*) = *f a* ⊕ *finsum G f F*
by (*rule comm-monoid.finprod-insert* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-zero* [*simp*]:
finite A ==> (⨁ *i*∈*A*. **0**) = **0**
by (*rule comm-monoid.finprod-one* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *finsum-closed* [*simp*]:
fixes *A*
assumes *fin*: *finite A* **and** *f*: *f* ∈ *A* −> *carrier G*
shows *finsum G f A* ∈ *carrier G*
by (*rule comm-monoid.finprod-closed* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-Un-Int*:
[| *finite A*; *finite B*; *g* ∈ *A* −> *carrier G*; *g* ∈ *B* −> *carrier G* |] ==>
finsum G g (*A Un B*) ⊕ *finsum G g* (*A Int B*) =
finsum G g A ⊕ *finsum G g B*
by (*rule comm-monoid.finprod-Un-Int* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-Un-disjoint*:
[| *finite A*; *finite B*; *A Int B* = {};
g ∈ *A* −> *carrier G*; *g* ∈ *B* −> *carrier G* |]
==> *finsum G g* (*A Un B*) = *finsum G g A* ⊕ *finsum G g B*
by (*rule comm-monoid.finprod-Un-disjoint* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-addf*:
[| *finite A*; *f* ∈ *A* −> *carrier G*; *g* ∈ *A* −> *carrier G* |] ==>
finsum G (%*x*. *f x* ⊕ *g x*) *A* = (*finsum G f A* ⊕ *finsum G g A*)
by (*rule comm-monoid.finprod-multf* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-cong'*:
[| *A* = *B*; *g* : *B* −> *carrier G*;
!!*i*. *i* : *B* ==> *f i* = *g i* |] ==> *finsum G f A* = *finsum G g B*
by (*rule comm-monoid.finprod-cong'* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*]) *auto*

lemma (**in** *abelian-monoid*) *finsum-0* [*simp*]:
f : {*0*::*nat*} −> *carrier G* ==> *finsum G f* {..*0*} = *f 0*
by (*rule comm-monoid.finprod-0* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *finsum-Suc* [*simp*]:
f : {..*Suc n*} −> *carrier G* ==>
finsum G f {..*Suc n*} = (*f* (*Suc n*) ⊕ *finsum G f* {..*n*})
by (*rule comm-monoid.finprod-Suc* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *finsum-Suc2*:
f : {..*Suc n*} −> *carrier G* ==>
finsum G f {..*Suc n*} = (*finsum G* (%*i*. *f* (*Suc i*)) {..*n*} ⊕ *f 0*)
by (*rule comm-monoid.finprod-Suc2* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *finsum-add* [*simp*]:
[| *f* : {..*n*} −> *carrier G*; *g* : {..*n*} −> *carrier G* |] ==>
finsum G (%*i*. *f i* ⊕ *g i*) {..*n*::*nat*} =
finsum G f {..*n*} ⊕ *finsum G g* {..*n*}
by (*rule comm-monoid.finprod-mult* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *finsum-cong*:
  [| $A = B$; $f : B$ −> *carrier* G;
    !!i. $i : B$ =*simp*=> $f\ i = g\ i$ |] ==> *finsum* $G\ f\ A$ = *finsum* $G\ g\ B$
  by (*rule comm-monoid.finprod-cong* [*OF a-comm-monoid*, *folded finsum-def*,
   simplified monoid-record-simps]) (*auto simp add*: *simp-implies-def*)

Usually, if this rule causes a failed congruence proof error, the reason is that the premise $g \in B$ −> *carrier* G cannot be shown. Adding *Pi-def* to the simpset is often useful.

10 The Algebraic Hierarchy of Rings

10.1 Basic Definitions

locale *ring* = *abelian-group* R + *monoid* R +
  assumes *l-distr*: [| $x \in$ *carrier* R; $y \in$ *carrier* R; $z \in$ *carrier* R |]
    ==> $(x \oplus y) \otimes z = x \otimes z \oplus y \otimes z$
   and *r-distr*: [| $x \in$ *carrier* R; $y \in$ *carrier* R; $z \in$ *carrier* R |]
    ==> $z \otimes (x \oplus y) = z \otimes x \oplus z \otimes y$

locale *cring* = *ring* + *comm-monoid* R

locale *domain* = *cring* +
  assumes *one-not-zero* [*simp*]: **1** ~= **0**
   and *integral*: [| $a \otimes b$ = **0**; $a \in$ *carrier* R; $b \in$ *carrier* R |] ==>
        a = **0** | b = **0**

locale *field* = *domain* +
  assumes *field-Units*: *Units* R = *carrier* R − {**0**}

10.2 Basic Facts of Rings

lemma *ringI*:
  includes *struct* R
  assumes *abelian-group*: *abelian-group* R
   and *monoid*: *monoid* R
   and *l-distr*: !!$x\ y\ z$. [| $x \in$ *carrier* R; $y \in$ *carrier* R; $z \in$ *carrier* R |]
    ==> $(x \oplus y) \otimes z = x \otimes z \oplus y \otimes z$
   and *r-distr*: !!$x\ y\ z$. [| $x \in$ *carrier* R; $y \in$ *carrier* R; $z \in$ *carrier* R |]
    ==> $z \otimes (x \oplus y) = z \otimes x \oplus z \otimes y$
  shows *ring* R
  by (*auto intro*: *ring.intro*
   abelian-group.axioms ring-axioms.intro prems)

lemma (**in** *ring*) *is-abelian-group*:
  abelian-group R
  by (*auto intro*!: *abelian-groupI a-assoc a-comm l-neg*)

lemma (**in** *ring*) *is-monoid*:
monoid R
by (*auto intro*!: *monoidI m-assoc*)

lemma *cringI*:
includes *struct R*
assumes *abelian-group*: *abelian-group R*
and *comm-monoid*: *comm-monoid R*
and *l-distr*: !!$x\ y\ z$. [| $x \in$ *carrier R*; $y \in$ *carrier R*; $z \in$ *carrier R* |]
==> $(x \oplus y) \otimes z = x \otimes z \oplus y \otimes z$
shows *cring R*
proof (*rule cring.intro*)
show *ring-axioms R*
— Right-distributivity follows from left-distributivity and commutativity.
proof (*rule ring-axioms.intro*)
fix $x\ y\ z$
assume *R*: $x \in$ *carrier R* $y \in$ *carrier R* $z \in$ *carrier R*
note [*simp*]= *comm-monoid.axioms* [*OF comm-monoid*]
abelian-group.axioms [*OF abelian-group*]
abelian-monoid.a-closed

from *R* **have** $z \otimes (x \oplus y) = (x \oplus y) \otimes z$
by (*simp add*: *comm-monoid.m-comm* [*OF comm-monoid.intro*])
also from *R* **have** ... $= x \otimes z \oplus y \otimes z$ **by** (*simp add*: *l-distr*)
also from *R* **have** ... $= z \otimes x \oplus z \otimes y$
by (*simp add*: *comm-monoid.m-comm* [*OF comm-monoid.intro*])
finally show $z \otimes (x \oplus y) = z \otimes x \oplus z \otimes y$ **.**
qed
qed (*auto intro*: *cring.intro*
abelian-group.axioms comm-monoid.axioms ring-axioms.intro prems)

lemma (**in** *cring*) *is-comm-monoid*:
comm-monoid R
by (*auto intro*!: *comm-monoidI m-assoc m-comm*)

10.3 Normaliser for Rings

lemma (**in** *abelian-group*) *r-neg2*:
[| $x \in$ *carrier G*; $y \in$ *carrier G* |] ==> $x \oplus (\ominus x \oplus y) = y$
proof −
assume *G*: $x \in$ *carrier G* $y \in$ *carrier G*
then have $(x \oplus \ominus x) \oplus y = y$
by (*simp only*: *r-neg l-zero*)
with *G* **show** *?thesis*
by (*simp add*: *a-ac*)
qed

lemma (**in** *abelian-group*) *r-neg1*:
[| $x \in$ *carrier G*; $y \in$ *carrier G* |] ==> $\ominus x \oplus (x \oplus y) = y$

proof –
  assume *G*: $x \in$ *carrier G* $y \in$ *carrier G*
  then have $(\ominus\ x \oplus x) \oplus y = y$
    by (*simp only*: *l-neg l-zero*)
  with *G* **show** *?thesis* **by** (*simp add*: *a-ac*)
qed

The following proofs are from Jacobson, Basic Algebra I, pp. 88–89

lemma (**in** *ring*) *l-null* [*simp*]:
  $x \in$ *carrier R* ==> $\mathbf{0} \otimes x = \mathbf{0}$
proof –
  assume *R*: $x \in$ *carrier R*
  then have $\mathbf{0} \otimes x \oplus \mathbf{0} \otimes x = (\mathbf{0} \oplus \mathbf{0}) \otimes x$
    by (*simp add*: *l-distr del*: *l-zero r-zero*)
  also from *R* **have** ... $= \mathbf{0} \otimes x \oplus \mathbf{0}$ **by** *simp*
  finally have $\mathbf{0} \otimes x \oplus \mathbf{0} \otimes x = \mathbf{0} \otimes x \oplus \mathbf{0}$ **.**
  with *R* **show** *?thesis* **by** (*simp del*: *r-zero*)
qed

lemma (**in** *ring*) *r-null* [*simp*]:
  $x \in$ *carrier R* ==> $x \otimes \mathbf{0} = \mathbf{0}$
proof –
  assume *R*: $x \in$ *carrier R*
  then have $x \otimes \mathbf{0} \oplus x \otimes \mathbf{0} = x \otimes (\mathbf{0} \oplus \mathbf{0})$
    by (*simp add*: *r-distr del*: *l-zero r-zero*)
  also from *R* **have** ... $= x \otimes \mathbf{0} \oplus \mathbf{0}$ **by** *simp*
  finally have $x \otimes \mathbf{0} \oplus x \otimes \mathbf{0} = x \otimes \mathbf{0} \oplus \mathbf{0}$ **.**
  with *R* **show** *?thesis* **by** (*simp del*: *r-zero*)
qed

lemma (**in** *ring*) *l-minus*:
  [| $x \in$ *carrier R*; $y \in$ *carrier R* |] ==> $\ominus\ x \otimes y = \ominus\ (x \otimes y)$
proof –
  assume *R*: $x \in$ *carrier R* $y \in$ *carrier R*
  then have $(\ominus\ x) \otimes y \oplus x \otimes y = (\ominus\ x \oplus x) \otimes y$ **by** (*simp add*: *l-distr*)
  also from *R* **have** ... $= \mathbf{0}$ **by** (*simp add*: *l-neg l-null*)
  finally have $(\ominus\ x) \otimes y \oplus x \otimes y = \mathbf{0}$ **.**
  with *R* **have** $(\ominus\ x) \otimes y \oplus x \otimes y \oplus \ominus\ (x \otimes y) = \mathbf{0} \oplus \ominus\ (x \otimes y)$ **by** *simp*
  with *R* **show** *?thesis* **by** (*simp add*: *a-assoc r-neg*)
qed

lemma (**in** *ring*) *r-minus*:
  [| $x \in$ *carrier R*; $y \in$ *carrier R* |] ==> $x \otimes \ominus\ y = \ominus\ (x \otimes y)$
proof –
  assume *R*: $x \in$ *carrier R* $y \in$ *carrier R*
  then have $x \otimes (\ominus\ y) \oplus x \otimes y = x \otimes (\ominus\ y \oplus y)$ **by** (*simp add*: *r-distr*)
  also from *R* **have** ... $= \mathbf{0}$ **by** (*simp add*: *l-neg r-null*)
  finally have $x \otimes (\ominus\ y) \oplus x \otimes y = \mathbf{0}$ **.**
  with *R* **have** $x \otimes (\ominus\ y) \oplus x \otimes y \oplus \ominus\ (x \otimes y) = \mathbf{0} \oplus \ominus\ (x \otimes y)$ **by** *simp*

with R **show** *?thesis* **by** (*simp add*: *a-assoc r-neg*)
qed

lemma (**in** *ring*) *minus-eq*:
[| $x \in$ *carrier* R; $y \in$ *carrier* R |] ==> $x \ominus y = x \oplus \ominus y$
by (*simp only*: *minus-def*)

lemmas (**in** *ring*) *ring-simprules* =
a-closed zero-closed a-inv-closed minus-closed m-closed one-closed
a-assoc l-zero l-neg a-comm m-assoc l-one l-distr minus-eq
r-zero r-neg r-neg2 r-neg1 minus-add minus-minus minus-zero
a-lcomm r-distr l-null r-null l-minus r-minus

lemmas (**in** *cring*) *cring-simprules* =
a-closed zero-closed a-inv-closed minus-closed m-closed one-closed
a-assoc l-zero l-neg a-comm m-assoc l-one l-distr m-comm minus-eq
r-zero r-neg r-neg2 r-neg1 minus-add minus-minus minus-zero
a-lcomm m-lcomm r-distr l-null r-null l-minus r-minus

use *ringsimp.ML*

method-setup *algebra* =
⟪ *Method.ctxt-args cring-normalise* ⟫
⟪ *computes distributive normal form in locale context cring* ⟫

lemma (**in** *cring*) *nat-pow-zero*:
(n::*nat*) ~= *0* ==> **0** (^) n = **0**
by (*induct n*) *simp-all*

Two examples for use of method algebra

lemma
includes *ring R* + *cring S*
shows [| $a \in$ *carrier* R; $b \in$ *carrier* R; $c \in$ *carrier* S; $d \in$ *carrier* S |] ==>
$a \oplus \ominus (a \oplus \ominus b) = b$ & $c \otimes_S d = d \otimes_S c$
by *algebra*

lemma
includes *cring*
shows [| $a \in$ *carrier* R; $b \in$ *carrier* R |] ==> $a \ominus (a \ominus b) = b$
by *algebra*

10.4 Sums over Finite Sets

lemma (**in** *cring*) *finsum-ldistr*:
[| *finite* A; $a \in$ *carrier* R; $f \in A$ -> *carrier* R |] ==>
finsum $R\ f\ A \otimes a$ = *finsum* R (%i. $f\ i \otimes a$) A
proof (*induct set*: *Finites*)
case *empty* **then show** *?case* **by** *simp*
next

```
  case (insert x F) then show ?case by (simp add: Pi-def l-distr)
qed

lemma (in cring) finsum-rdistr:
  [| finite A; a ∈ carrier R; f ∈ A -> carrier R |] ==>
   a ⊗ finsum R f A = finsum R (%i. a ⊗ f i) A
proof (induct set: Finites)
  case empty then show ?case by simp
next
  case (insert x F) then show ?case by (simp add: Pi-def r-distr)
qed
```

10.5 Facts of Integral Domains

```
lemma (in domain) zero-not-one [simp]:
  0 ~= 1
  by (rule not-sym) simp

lemma (in domain) integral-iff:
  [| a ∈ carrier R; b ∈ carrier R |] ==> (a ⊗ b = 0) = (a = 0 | b = 0)
proof
  assume a ∈ carrier R b ∈ carrier R a ⊗ b = 0
  then show a = 0 | b = 0 by (simp add: integral)
next
  assume a ∈ carrier R b ∈ carrier R a = 0 | b = 0
  then show a ⊗ b = 0 by auto
qed

lemma (in domain) m-lcancel:
  assumes prem: a ~= 0
    and R: a ∈ carrier R b ∈ carrier R c ∈ carrier R
  shows (a ⊗ b = a ⊗ c) = (b = c)
proof
  assume eq: a ⊗ b = a ⊗ c
  with R have a ⊗ (b ⊖ c) = 0 by algebra
  with R have a = 0 | (b ⊖ c) = 0 by (simp add: integral-iff)
  with prem and R have b ⊖ c = 0 by auto
  with R have b = b ⊖ (b ⊖ c) by algebra
  also from R have b ⊖ (b ⊖ c) = c by algebra
  finally show b = c .
next
  assume b = c then show a ⊗ b = a ⊗ c by simp
qed

lemma (in domain) m-rcancel:
  assumes prem: a ~= 0
    and R: a ∈ carrier R b ∈ carrier R c ∈ carrier R
  shows conc: (b ⊗ a = c ⊗ a) = (b = c)
proof -
```

from *prem* **and** *R* **have** $(a \otimes b = a \otimes c) = (b = c)$ **by** (*rule m-lcancel*)
with *R* **show** *?thesis* **by** *algebra*
qed

10.6 Morphisms

constdefs (**structure** *R S*)
ring-hom :: [('a, 'm) *ring-scheme*, ('b, 'n) *ring-scheme*] => ('a => 'b) *set*
ring-hom R S == $\{h.\ h \in$ *carrier R* −> *carrier S* &
(*ALL x y*. $x \in$ *carrier R* & $y \in$ *carrier R* −−>
$h\ (x \otimes y) = h\ x \otimes_S h\ y$ & $h\ (x \oplus y) = h\ x \oplus_S h\ y$) &
$h\ \mathbf{1} = \mathbf{1}_S\}$

lemma *ring-hom-memI*:
includes *struct R* + *struct S*
assumes *hom-closed*: !!*x*. $x \in$ *carrier R* ==> $h\ x \in$ *carrier S*
and *hom-mult*: !!*x y*. [| $x \in$ *carrier R*; $y \in$ *carrier R* |] ==>
$h\ (x \otimes y) = h\ x \otimes_S h\ y$
and *hom-add*: !!*x y*. [| $x \in$ *carrier R*; $y \in$ *carrier R* |] ==>
$h\ (x \oplus y) = h\ x \oplus_S h\ y$
and *hom-one*: $h\ \mathbf{1} = \mathbf{1}_S$
shows $h \in$ *ring-hom R S*
by (*auto simp add*: *ring-hom-def prems Pi-def*)

lemma *ring-hom-closed*:
[| $h \in$ *ring-hom R S*; $x \in$ *carrier R* |] ==> $h\ x \in$ *carrier S*
by (*auto simp add*: *ring-hom-def funcset-mem*)

lemma *ring-hom-mult*:
includes *struct R* + *struct S*
shows
[| $h \in$ *ring-hom R S*; $x \in$ *carrier R*; $y \in$ *carrier R* |] ==>
$h\ (x \otimes y) = h\ x \otimes_S h\ y$
by (*simp add*: *ring-hom-def*)

lemma *ring-hom-add*:
includes *struct R* + *struct S*
shows
[| $h \in$ *ring-hom R S*; $x \in$ *carrier R*; $y \in$ *carrier R* |] ==>
$h\ (x \oplus y) = h\ x \oplus_S h\ y$
by (*simp add*: *ring-hom-def*)

lemma *ring-hom-one*:
includes *struct R* + *struct S*
shows $h \in$ *ring-hom R S* ==> $h\ \mathbf{1} = \mathbf{1}_S$
by (*simp add*: *ring-hom-def*)

locale *ring-hom-cring* = *cring R* + *cring S* + *var h* +
assumes *homh* [*simp*, *intro*]: $h \in$ *ring-hom R S*

notes *hom-closed* [*simp*, *intro*] = *ring-hom-closed* [*OF homh*]
and *hom-mult* [*simp*] = *ring-hom-mult* [*OF homh*]
and *hom-add* [*simp*] = *ring-hom-add* [*OF homh*]
and *hom-one* [*simp*] = *ring-hom-one* [*OF homh*]

lemma (**in** *ring-hom-cring*) *hom-zero* [*simp*]:
$h\ \mathbf{0} = \mathbf{0}_S$
proof −
have $h\ \mathbf{0} \oplus_S h\ \mathbf{0} = h\ \mathbf{0} \oplus_S \mathbf{0}_S$
by (*simp add*: *hom-add* [*symmetric*] *del*: *hom-add*)
then show *?thesis* **by** (*simp del*: *S.r-zero*)
qed

lemma (**in** *ring-hom-cring*) *hom-a-inv* [*simp*]:
$x \in$ *carrier R* ==> $h\ (\ominus\ x) = \ominus_S\ h\ x$
proof −
assume *R*: $x \in$ *carrier R*
then have $h\ x \oplus_S h\ (\ominus\ x) = h\ x \oplus_S (\ominus_S\ h\ x)$
by (*simp add*: *hom-add* [*symmetric*] *R.r-neg S.r-neg del*: *hom-add*)
with *R* **show** *?thesis* **by** *simp*
qed

lemma (**in** *ring-hom-cring*) *hom-finsum* [*simp*]:
[| *finite A*; $f \in A$ −> *carrier R* |] ==>
h (*finsum R f A*) = *finsum S* (*h o f*) *A*
proof (*induct set*: *Finites*)
case *empty* **then show** *?case* **by** *simp*
next
case *insert* **then show** *?case* **by** (*simp add*: *Pi-def*)
qed

lemma (**in** *ring-hom-cring*) *hom-finprod*:
[| *finite A*; $f \in A$ −> *carrier R* |] ==>
h (*finprod R f A*) = *finprod S* (*h o f*) *A*
proof (*induct set*: *Finites*)
case *empty* **then show** *?case* **by** *simp*
next
case *insert* **then show** *?case* **by** (*simp add*: *Pi-def*)
qed

declare *ring-hom-cring.hom-finprod* [*simp*]

lemma *id-ring-hom* [*simp*]:
$id \in$ *ring-hom R R*
by (*auto intro*!: *ring-hom-memI*)

end

11 Module: Modules over an Abelian Group

theory *Module* **imports** *CRing* **begin**

record ('*a*, '*b*) *module* = '*b ring* +
  smult :: ['*a*, '*b*] => '*b* (**infixl** ⊙ı *70*)

locale *module* = *cring R* + *abelian-group M* +
  assumes *smult-closed* [*simp*, *intro*]:
    [| $a \in$ *carrier R*; $x \in$ *carrier M* |] ==> $a \odot_M x \in$ *carrier M*
  and *smult-l-distr*:
    [| $a \in$ *carrier R*; $b \in$ *carrier R*; $x \in$ *carrier M* |] ==>
    $(a \oplus b) \odot_M x = a \odot_M x \oplus_M b \odot_M x$
  and *smult-r-distr*:
    [| $a \in$ *carrier R*; $x \in$ *carrier M*; $y \in$ *carrier M* |] ==>
    $a \odot_M (x \oplus_M y) = a \odot_M x \oplus_M a \odot_M y$
  and *smult-assoc1*:
    [| $a \in$ *carrier R*; $b \in$ *carrier R*; $x \in$ *carrier M* |] ==>
    $(a \otimes b) \odot_M x = a \odot_M (b \odot_M x)$
  and *smult-one* [*simp*]:
    $x \in$ *carrier M* ==> $\mathbf{1} \odot_M x = x$

locale *algebra* = *module R M* + *cring M* +
  assumes *smult-assoc2*:
    [| $a \in$ *carrier R*; $x \in$ *carrier M*; $y \in$ *carrier M* |] ==>
    $(a \odot_M x) \otimes_M y = a \odot_M (x \otimes_M y)$

lemma *moduleI*:
  includes *struct R* + *struct M*
  assumes *cring*: *cring R*
  and *abelian-group*: *abelian-group M*
  and *smult-closed*:
    !!*a x*. [| $a \in$ *carrier R*; $x \in$ *carrier M* |] ==> $a \odot_M x \in$ *carrier M*
  and *smult-l-distr*:
    !!*a b x*. [| $a \in$ *carrier R*; $b \in$ *carrier R*; $x \in$ *carrier M* |] ==>
    $(a \oplus b) \odot_M x = (a \odot_M x) \oplus_M (b \odot_M x)$
  and *smult-r-distr*:
    !!*a x y*. [| $a \in$ *carrier R*; $x \in$ *carrier M*; $y \in$ *carrier M* |] ==>
    $a \odot_M (x \oplus_M y) = (a \odot_M x) \oplus_M (a \odot_M y)$
  and *smult-assoc1*:
    !!*a b x*. [| $a \in$ *carrier R*; $b \in$ *carrier R*; $x \in$ *carrier M* |] ==>
    $(a \otimes b) \odot_M x = a \odot_M (b \odot_M x)$
  and *smult-one*:
    !!*x*. $x \in$ *carrier M* ==> $\mathbf{1} \odot_M x = x$
  shows *module R M*
  by (*auto intro*: *module.intro cring.axioms abelian-group.axioms*
    module-axioms.intro prems)

lemma *algebraI*:

includes *struct R* + *struct M*
assumes *R-cring*: *cring R*
and *M-cring*: *cring M*
and *smult-closed*:
$!!a\ x.\ [|\ a \in carrier\ R;\ x \in carrier\ M\ |] ==> a \odot_M x \in carrier\ M$
and *smult-l-distr*:
$!!a\ b\ x.\ [|\ a \in carrier\ R;\ b \in carrier\ R;\ x \in carrier\ M\ |] ==>$
$(a \oplus b) \odot_M x = (a \odot_M x) \oplus_M (b \odot_M x)$
and *smult-r-distr*:
$!!a\ x\ y.\ [|\ a \in carrier\ R;\ x \in carrier\ M;\ y \in carrier\ M\ |] ==>$
$a \odot_M (x \oplus_M y) = (a \odot_M x) \oplus_M (a \odot_M y)$
and *smult-assoc1*:
$!!a\ b\ x.\ [|\ a \in carrier\ R;\ b \in carrier\ R;\ x \in carrier\ M\ |] ==>$
$(a \otimes b) \odot_M x = a \odot_M (b \odot_M x)$
and *smult-one*:
$!!x.\ x \in carrier\ M ==> (one\ R) \odot_M x = x$
and *smult-assoc2*:
$!!a\ x\ y.\ [|\ a \in carrier\ R;\ x \in carrier\ M;\ y \in carrier\ M\ |] ==>$
$(a \odot_M x) \otimes_M y = a \odot_M (x \otimes_M y)$
shows *algebra R M*
by (*auto intro*!: *algebra.intro algebra-axioms.intro cring.axioms*
module-axioms.intro prems)

lemma (**in** *algebra*) *R-cring*:
cring R
by (*rule cring.intro*)

lemma (**in** *algebra*) *M-cring*:
cring M
by (*rule cring.intro*)

lemma (**in** *algebra*) *module*:
module R M
by (*auto intro*: *moduleI R-cring is-abelian-group*
smult-l-distr smult-r-distr smult-assoc1)

11.1 Basic Properties of Algebras

lemma (**in** *algebra*) *smult-l-null* [*simp*]:
$x \in carrier\ M ==> \mathbf{0} \odot_M x = \mathbf{0}_M$
proof −
assume *M*: $x \in carrier\ M$
note *facts* = *M smult-closed*
from *facts* **have** $\mathbf{0} \odot_M x = (\mathbf{0} \odot_M x \oplus_M \mathbf{0} \odot_M x) \oplus_M \ominus_M (\mathbf{0} \odot_M x)$ **by** *algebra*
also from *M* **have** $... = (\mathbf{0} \oplus \mathbf{0}) \odot_M x \oplus_M \ominus_M (\mathbf{0} \odot_M x)$
by (*simp add*: *smult-l-distr del*: *R.l-zero R.r-zero*)
also from *facts* **have** $... = \mathbf{0}_M$ **by** *algebra*
finally show *?thesis* **.**
qed

lemma (**in** *algebra*) *smult-r-null* [*simp*]:
  $a \in carrier\ R \Longrightarrow a \odot_M \mathbf{0}_M = \mathbf{0}_M$
proof −
  assume R: $a \in carrier\ R$
  note *facts* = *R smult-closed*
  from *facts* **have** $a \odot_M \mathbf{0}_M = (a \odot_M \mathbf{0}_M \oplus_M a \odot_M \mathbf{0}_M) \oplus_M \ominus_M (a \odot_M \mathbf{0}_M)$
    by *algebra*
  also from *R* **have** ... $= a \odot_M (\mathbf{0}_M \oplus_M \mathbf{0}_M) \oplus_M \ominus_M (a \odot_M \mathbf{0}_M)$
    by (*simp add*: *smult-r-distr del*: *M.l-zero M.r-zero*)
  also from *facts* **have** ... $= \mathbf{0}_M$ **by** *algebra*
  finally show *?thesis* **.**
qed

lemma (**in** *algebra*) *smult-l-minus*:
  $[\mid a \in carrier\ R;\ x \in carrier\ M \mid] \Longrightarrow (\ominus a) \odot_M x = \ominus_M (a \odot_M x)$
proof −
  assume RM: $a \in carrier\ R\ x \in carrier\ M$
  note *facts* = *RM smult-closed*
  from *facts* **have** $(\ominus a) \odot_M x = (\ominus a \odot_M x \oplus_M a \odot_M x) \oplus_M \ominus_M (a \odot_M x)$
    by *algebra*
  also from *RM* **have** ... $= (\ominus a \oplus a) \odot_M x \oplus_M \ominus_M (a \odot_M x)$
    by (*simp add*: *smult-l-distr*)
  also from *facts smult-l-null* **have** ... $= \ominus_M (a \odot_M x)$ **by** *algebra*
  finally show *?thesis* **.**
qed

lemma (**in** *algebra*) *smult-r-minus*:
  $[\mid a \in carrier\ R;\ x \in carrier\ M \mid] \Longrightarrow a \odot_M (\ominus_M x) = \ominus_M (a \odot_M x)$
proof −
  assume RM: $a \in carrier\ R\ x \in carrier\ M$
  note *facts* = *RM smult-closed*
  from *facts* **have** $a \odot_M (\ominus_M x) = (a \odot_M \ominus_M x \oplus_M a \odot_M x) \oplus_M \ominus_M (a \odot_M x)$
    by *algebra*
  also from *RM* **have** ... $= a \odot_M (\ominus_M x \oplus_M x) \oplus_M \ominus_M (a \odot_M x)$
    by (*simp add*: *smult-r-distr*)
  also from *facts smult-r-null* **have** ... $= \ominus_M (a \odot_M x)$ **by** *algebra*
  finally show *?thesis* **.**
qed

end

12 UnivPoly: Univariate Polynomials

theory *UnivPoly* **imports** *Module* **begin**

Polynomials are formalised as modules with additional operations for extracting coefficients from polynomials and for obtaining monomials from coefficients and exponents (record *up-ring*). The carrier set is a set of bounded functions from Nat to the coefficient domain. Bounded means that these functions return zero above a certain bound (the degree). There is a chapter on the formalisation of polynomials in the PhD thesis [1], which was implemented with axiomatic type classes. This was later ported to Locales.

12.1 The Constructor for Univariate Polynomials

Functions with finite support.

locale *bound* =
  fixes $z :: 'a$
    and $n :: nat$
    and $f :: nat => 'a$
  assumes *bound*: $!!m.\ n < m \Longrightarrow f\ m = z$

declare *bound.intro* [*intro*!]
  and *bound.bound* [*dest*]

lemma *bound-below*:
  assumes *bound*: *bound z m f* **and** *nonzero*: $f\ n \neq z$ **shows** $n \leq m$
proof (*rule classical*)
  assume ~ *?thesis*
  then have $m < n$ **by** *arith*
  with *bound* **have** $f\ n = z$ **..**
  with *nonzero* **show** *?thesis* **by** *contradiction*
qed

record ('a, 'p) *up-ring* = ('a, 'p) *module* +
  monom :: ['a, *nat*] => 'p
  coeff :: ['p, *nat*] => 'a

constdefs (**structure** *R*)
  up :: ('a, 'm) *ring-scheme* => (*nat* => 'a) *set*
  $up\ R == \{f.\ f \in UNIV\ -> carrier\ R\ \&\ (EX\ n.\ bound\ \mathbf{0}\ n\ f)\}$
  UP :: ('a, 'm) *ring-scheme* => ('a, *nat* => 'a) *up-ring*
  UP R == (|
    carrier = *up R*,
    $mult = (\%p{:}up\ R.\ \%q{:}up\ R.\ \%n.\ \bigoplus i \in \{..n\}.\ p\ i \otimes q\ (n-i))$,
    $one = (\%i.\ if\ i{=}0\ then\ \mathbf{1}\ else\ \mathbf{0})$,
    $zero = (\%i.\ \mathbf{0})$,
    $add = (\%p{:}up\ R.\ \%q{:}up\ R.\ \%i.\ p\ i \oplus q\ i)$,
    $smult = (\%a{:}carrier\ R.\ \%p{:}up\ R.\ \%i.\ a \otimes p\ i)$,
    $monom = (\%a{:}carrier\ R.\ \%n\ i.\ if\ i{=}n\ then\ a\ else\ \mathbf{0})$,
    $coeff = (\%p{:}up\ R.\ \%n.\ p\ n)$ |)

Properties of the set of polynomials *up*.

lemma *mem-upI* [*intro*]:
  $[|\ !!n.\ f\ n \in carrier\ R;\ EX\ n.\ bound\ (zero\ R)\ n\ f\ |] ==> f \in up\ R$

```
  by (simp add: up-def Pi-def)

lemma mem-upD [dest]:
  f ∈ up R ==> f n ∈ carrier R
  by (simp add: up-def Pi-def)

lemma (in cring) bound-upD [dest]:
  f ∈ up R ==> EX n. bound 0 n f
  by (simp add: up-def)

lemma (in cring) up-one-closed:
   (%n. if n = 0 then 1 else 0) ∈ up R
  using up-def by force

lemma (in cring) up-smult-closed:
  [| a ∈ carrier R; p ∈ up R |] ==> (%i. a ⊗ p i) ∈ up R
  by force

lemma (in cring) up-add-closed:
  [| p ∈ up R; q ∈ up R |] ==> (%i. p i ⊕ q i) ∈ up R
proof
  fix n
  assume p ∈ up R and q ∈ up R
  then show p n ⊕ q n ∈ carrier R
    by auto
next
  assume UP: p ∈ up R q ∈ up R
  show EX n. bound 0 n (%i. p i ⊕ q i)
  proof -
    from UP obtain n where boundn: bound 0 n p by fast
    from UP obtain m where boundm: bound 0 m q by fast
    have bound 0 (max n m) (%i. p i ⊕ q i)
    proof
      fix i
      assume max n m < i
      with boundn and boundm and UP show p i ⊕ q i = 0 by fastsimp
    qed
    then show ?thesis ..
  qed
qed

lemma (in cring) up-a-inv-closed:
  p ∈ up R ==> (%i. ⊖ (p i)) ∈ up R
proof
  assume R: p ∈ up R
  then obtain n where bound 0 n p by auto
  then have bound 0 n (%i. ⊖ p i) by auto
  then show EX n. bound 0 n (%i. ⊖ p i) by auto
qed auto
```

```
lemma (in cring) up-mult-closed:
  [| p ∈ up R; q ∈ up R |] ==>
  (%n. ⨁ i ∈ {..n}. p i ⊗ q (n−i)) ∈ up R
proof
  fix n
  assume p ∈ up R q ∈ up R
  then show (⨁ i ∈ {..n}. p i ⊗ q (n−i)) ∈ carrier R
    by (simp add: mem-upD funcsetI)
next
  assume UP: p ∈ up R q ∈ up R
  show EX n. bound 0 n (%n. ⨁ i ∈ {..n}. p i ⊗ q (n−i))
  proof −
    from UP obtain n where boundn: bound 0 n p by fast
    from UP obtain m where boundm: bound 0 m q by fast
    have bound 0 (n + m) (%n. ⨁ i ∈ {..n}. p i ⊗ q (n − i))
    proof
      fix k assume bound: n + m < k
      {
        fix i
        have p i ⊗ q (k−i) = 0
        proof (cases n < i)
          case True
          with boundn have p i = 0 by auto
          moreover from UP have q (k−i) ∈ carrier R by auto
          ultimately show ?thesis by simp
        next
          case False
          with bound have m < k−i by arith
          with boundm have q (k−i) = 0 by auto
          moreover from UP have p i ∈ carrier R by auto
          ultimately show ?thesis by simp
        qed
      }
      then show (⨁ i ∈ {..k}. p i ⊗ q (k−i)) = 0
        by (simp add: Pi-def)
    qed
    then show ?thesis by fast
  qed
qed
```

12.2 Effect of operations on coefficients

```
locale UP = struct R + struct P +
  defines P-def: P == UP R

locale UP-cring = UP + cring R

locale UP-domain = UP-cring + domain R
```

Temporarily declare $P \equiv UP\ R$ as simp rule.

```
declare (in UP) P-def [simp]

lemma (in UP-cring) coeff-monom [simp]:
  a ∈ carrier R ==>
  coeff P (monom P a m) n = (if m=n then a else 0)
proof -
  assume R: a ∈ carrier R
  then have (%n. if n = m then a else 0) ∈ up R
    using up-def by force
  with R show ?thesis by (simp add: UP-def)
qed

lemma (in UP-cring) coeff-zero [simp]:
  coeff P 0_P n = 0
  by (auto simp add: UP-def)

lemma (in UP-cring) coeff-one [simp]:
  coeff P 1_P n = (if n=0 then 1 else 0)
  using up-one-closed by (simp add: UP-def)

lemma (in UP-cring) coeff-smult [simp]:
  [| a ∈ carrier R; p ∈ carrier P |] ==>
  coeff P (a ⊙_P p) n = a ⊗ coeff P p n
  by (simp add: UP-def up-smult-closed)

lemma (in UP-cring) coeff-add [simp]:
  [| p ∈ carrier P; q ∈ carrier P |] ==>
  coeff P (p ⊕_P q) n = coeff P p n ⊕ coeff P q n
  by (simp add: UP-def up-add-closed)

lemma (in UP-cring) coeff-mult [simp]:
  [| p ∈ carrier P; q ∈ carrier P |] ==>
  coeff P (p ⊗_P q) n = (⨁ i ∈ {..n}. coeff P p i ⊗ coeff P q (n−i))
  by (simp add: UP-def up-mult-closed)

lemma (in UP) up-eqI:
  assumes prem: !!n. coeff P p n = coeff P q n
    and R: p ∈ carrier P q ∈ carrier P
  shows p = q
proof
  fix x
  from prem and R show p x = q x by (simp add: UP-def)
qed
```

12.3 Polynomials form a commutative ring.

Operations are closed over P.

lemma (**in** *UP-cring*) *UP-mult-closed* [*simp*]:
[| $p \in$ *carrier P*; $q \in$ *carrier P* |] ==> $p \otimes_P q \in$ *carrier P*
by (*simp add*: *UP-def up-mult-closed*)

lemma (**in** *UP-cring*) *UP-one-closed* [*simp*]:
$\mathbf{1}_P \in$ *carrier P*
by (*simp add*: *UP-def up-one-closed*)

lemma (**in** *UP-cring*) *UP-zero-closed* [*intro*, *simp*]:
$\mathbf{0}_P \in$ *carrier P*
by (*auto simp add*: *UP-def*)

lemma (**in** *UP-cring*) *UP-a-closed* [*intro*, *simp*]:
[| $p \in$ *carrier P*; $q \in$ *carrier P* |] ==> $p \oplus_P q \in$ *carrier P*
by (*simp add*: *UP-def up-add-closed*)

lemma (**in** *UP-cring*) *monom-closed* [*simp*]:
$a \in$ *carrier R* ==> *monom P a n* $\in$ *carrier P*
by (*auto simp add*: *UP-def up-def Pi-def*)

lemma (**in** *UP-cring*) *UP-smult-closed* [*simp*]:
[| $a \in$ *carrier R*; $p \in$ *carrier P* |] ==> $a \odot_P p \in$ *carrier P*
by (*simp add*: *UP-def up-smult-closed*)

lemma (**in** *UP*) *coeff-closed* [*simp*]:
$p \in$ *carrier P* ==> *coeff P p n* $\in$ *carrier R*
by (*auto simp add*: *UP-def*)

declare (**in** *UP*) *P-def* [*simp del*]

Algebraic ring properties

lemma (**in** *UP-cring*) *UP-a-assoc*:
assumes *R*: $p \in$ *carrier P* $q \in$ *carrier P* $r \in$ *carrier P*
shows $(p \oplus_P q) \oplus_P r = p \oplus_P (q \oplus_P r)$
by (*rule up-eqI*, *simp add*: *a-assoc R*, *simp-all add*: *R*)

lemma (**in** *UP-cring*) *UP-l-zero* [*simp*]:
assumes *R*: $p \in$ *carrier P*
shows $\mathbf{0}_P \oplus_P p = p$
by (*rule up-eqI*, *simp-all add*: *R*)

lemma (**in** *UP-cring*) *UP-l-neg-ex*:
assumes *R*: $p \in$ *carrier P*
shows *EX q* : *carrier P*. $q \oplus_P p = \mathbf{0}_P$
proof −
let *?q* = %*i*. $\ominus$ (*p i*)
from *R* **have** *closed*: *?q* $\in$ *carrier P*
by (*simp add*: *UP-def P-def up-a-inv-closed*)
from *R* **have** *coeff*: !!*n*. *coeff P ?q n* = $\ominus$ (*coeff P p n*)

```
    by (simp add: UP-def P-def up-a-inv-closed)
  show ?thesis
  proof
    show ?q ⊕_P p = 0_P
      by (auto intro!: up-eqI simp add: R closed coeff R.l-neg)
  qed (rule closed)
qed

lemma (in UP-cring) UP-a-comm:
  assumes R: p ∈ carrier P q ∈ carrier P
  shows p ⊕_P q = q ⊕_P p
  by (rule up-eqI, simp add: a-comm R, simp-all add: R)

lemma (in UP-cring) UP-m-assoc:
  assumes R: p ∈ carrier P q ∈ carrier P r ∈ carrier P
  shows (p ⊗_P q) ⊗_P r = p ⊗_P (q ⊗_P r)
proof (rule up-eqI)
  fix n
  {
    fix k and a b c :: nat=>'a
    assume R: a ∈ UNIV -> carrier R b ∈ UNIV -> carrier R
      c ∈ UNIV -> carrier R
    then have k <= n ==>
      (⨁j ∈ {..k}. (⨁i ∈ {..j}. a i ⊗ b (j-i)) ⊗ c (n-j)) =
      (⨁j ∈ {..k}. a j ⊗ (⨁i ∈ {..k-j}. b i ⊗ c (n-j-i)))
      (concl is ?eq k)
    proof (induct k)
      case 0 then show ?case by (simp add: Pi-def m-assoc)
    next
      case (Suc k)
      then have k <= n by arith
      then have ?eq k by (rule Suc)
      with R show ?case
        by (simp cong: finsum-cong
             add: Suc-diff-le Pi-def l-distr r-distr m-assoc)
          (simp cong: finsum-cong add: Pi-def a-ac finsum-ldistr m-assoc)
    qed
  }
  with R show coeff P ((p ⊗_P q) ⊗_P r) n = coeff P (p ⊗_P (q ⊗_P r)) n
    by (simp add: Pi-def)
qed (simp-all add: R)

lemma (in UP-cring) UP-l-one [simp]:
  assumes R: p ∈ carrier P
  shows 1_P ⊗_P p = p
proof (rule up-eqI)
  fix n
  show coeff P (1_P ⊗_P p) n = coeff P p n
  proof (cases n)
```

```
    case 0 with R show ?thesis by simp
  next
    case Suc with R show ?thesis
      by (simp del: finsum-Suc add: finsum-Suc2 Pi-def)
  qed
qed (simp-all add: R)

lemma (in UP-cring) UP-l-distr:
  assumes R: p ∈ carrier P q ∈ carrier P r ∈ carrier P
  shows (p ⊕_P q) ⊗_P r = (p ⊗_P r) ⊕_P (q ⊗_P r)
  by (rule up-eqI) (simp add: l-distr R Pi-def, simp-all add: R)

lemma (in UP-cring) UP-m-comm:
  assumes R: p ∈ carrier P q ∈ carrier P
  shows p ⊗_P q = q ⊗_P p
proof (rule up-eqI)
  fix n
  {
    fix k and a b :: nat=>'a
    assume R: a ∈ UNIV −> carrier R b ∈ UNIV −> carrier R
    then have k <= n ==>
      (⨁ i ∈ {..k}. a i ⊗ b (n−i)) =
      (⨁ i ∈ {..k}. a (k−i) ⊗ b (i+n−k))
      (concl is ?eq k)
    proof (induct k)
      case 0 then show ?case by (simp add: Pi-def)
    next
      case (Suc k) then show ?case
        by (subst (2) finsum-Suc2) (simp add: Pi-def a-comm)+
    qed
  }
  note l = this
  from R show coeff P (p ⊗_P q) n =  coeff P (q ⊗_P p) n
    apply (simp add: Pi-def)
    apply (subst l)
    apply (auto simp add: Pi-def)
    apply (simp add: m-comm)
    done
qed (simp-all add: R)

theorem (in UP-cring) UP-cring:
  cring P
  by (auto intro!: cringI abelian-groupI comm-monoidI UP-a-assoc UP-l-zero
    UP-l-neg-ex UP-a-comm UP-m-assoc UP-l-one UP-m-comm UP-l-distr)

lemma (in UP-cring) UP-ring:

  ring P
  by (auto intro: ring.intro cring.axioms UP-cring)
```

lemma (**in** *UP-cring*) *UP-a-inv-closed* [*intro*, *simp*]:
$p \in carrier\ P$ ==> $\ominus_P\ p \in carrier\ P$
by (*rule abelian-group.a-inv-closed*
[*OF ring.is-abelian-group* [*OF UP-ring*]])

lemma (**in** *UP-cring*) *coeff-a-inv* [*simp*]:
assumes *R*: $p \in carrier\ P$
shows $coeff\ P\ (\ominus_P\ p)\ n = \ominus\ (coeff\ P\ p\ n)$
proof −
from *R coeff-closed UP-a-inv-closed* **have**
$coeff\ P\ (\ominus_P\ p)\ n = \ominus\ coeff\ P\ p\ n \oplus (coeff\ P\ p\ n \oplus coeff\ P\ (\ominus_P\ p)\ n)$
by *algebra*
also from *R* **have** ... = $\ominus\ (coeff\ P\ p\ n)$
by (*simp del*: *coeff-add add*: *coeff-add* [*THEN sym*]
abelian-group.r-neg [*OF ring.is-abelian-group* [*OF UP-ring*]])
finally show *?thesis* **.**
qed

Interpretation of lemmas from *cring*. Saves lifting 43 lemmas manually.

interpretation *UP-cring* < *cring P*
using *UP-cring*
by − (*erule cring.axioms*)+

12.4 Polynomials form an Algebra

lemma (**in** *UP-cring*) *UP-smult-l-distr*:
[| $a \in carrier\ R$; $b \in carrier\ R$; $p \in carrier\ P$ |] ==>
$(a \oplus b) \odot_P p = a \odot_P p \oplus_P b \odot_P p$
by (*rule up-eqI*) (*simp-all add*: *R.l-distr*)

lemma (**in** *UP-cring*) *UP-smult-r-distr*:
[| $a \in carrier\ R$; $p \in carrier\ P$; $q \in carrier\ P$ |] ==>
$a \odot_P (p \oplus_P q) = a \odot_P p \oplus_P a \odot_P q$
by (*rule up-eqI*) (*simp-all add*: *R.r-distr*)

lemma (**in** *UP-cring*) *UP-smult-assoc1*:
[| $a \in carrier\ R$; $b \in carrier\ R$; $p \in carrier\ P$ |] ==>
$(a \otimes b) \odot_P p = a \odot_P (b \odot_P p)$
by (*rule up-eqI*) (*simp-all add*: *R.m-assoc*)

lemma (**in** *UP-cring*) *UP-smult-one* [*simp*]:
$p \in carrier\ P$ ==> $\mathbf{1} \odot_P p = p$
by (*rule up-eqI*) *simp-all*

lemma (**in** *UP-cring*) *UP-smult-assoc2*:
[| $a \in carrier\ R$; $p \in carrier\ P$; $q \in carrier\ P$ |] ==>
$(a \odot_P p) \otimes_P q = a \odot_P (p \otimes_P q)$
by (*rule up-eqI*) (*simp-all add*: *R.finsum-rdistr R.m-assoc Pi-def*)

Interpretation of lemmas from *algebra*.

lemma (**in** *cring*) *cring*:
cring R
by (*fast intro*: *cring.intro prems*)

lemma (**in** *UP-cring*) *UP-algebra*:
algebra R P
by (*auto intro*: *algebraI R.cring UP-cring UP-smult-l-distr UP-smult-r-distr UP-smult-assoc1 UP-smult-assoc2*)

interpretation *UP-cring* $<$ *algebra R P*
using *UP-algebra*
by $-$ (*erule algebra.axioms*)+

12.5 Further lemmas involving monomials

lemma (**in** *UP-cring*) *monom-zero* [*simp*]:
monom P $\mathbf{0}$ $n = \mathbf{0}_P$
by (*simp add*: *UP-def P-def*)

lemma (**in** *UP-cring*) *monom-mult-is-smult*:
assumes *R*: $a \in$ *carrier R* $p \in$ *carrier P*
shows *monom P a 0* $\otimes_P p = a \odot_P p$
proof (*rule up-eqI*)
fix *n*
have *coeff P* ($p \otimes_P$ *monom P a 0*) $n =$ *coeff P* ($a \odot_P p$) n
proof (*cases n*)
case *0* **with** *R* **show** *?thesis* **by** (*simp add*: *R.m-comm*)
next
case *Suc* **with** *R* **show** *?thesis*
by (*simp cong*: *R.finsum-cong add*: *R.r-null Pi-def*)
(*simp add*: *R.m-comm*)
qed
with *R* **show** *coeff P* (*monom P a 0* $\otimes_P p$) $n =$ *coeff P* ($a \odot_P p$) n
by (*simp add*: *UP-m-comm*)
qed (*simp-all add*: *R*)

lemma (**in** *UP-cring*) *monom-add* [*simp*]:
[| $a \in$ *carrier R*; $b \in$ *carrier R* |] ==>
monom P ($a \oplus b$) $n =$ *monom P a n* $\oplus_P$ *monom P b n*
by (*rule up-eqI*) *simp-all*

lemma (**in** *UP-cring*) *monom-one-Suc*:
monom P $\mathbf{1}$ (*Suc n*) $=$ *monom P* $\mathbf{1}$ $n \otimes_P$ *monom P* $\mathbf{1}$ *1*
proof (*rule up-eqI*)
fix *k*
show *coeff P* (*monom P* $\mathbf{1}$ (*Suc n*)) $k =$ *coeff P* (*monom P* $\mathbf{1}$ $n \otimes_P$ *monom P*
$\mathbf{1}$ *1*) k
proof (*cases* $k =$ *Suc n*)

```
  case True show ?thesis
  proof -
    from True have less-add-diff:
      !!i. [| n < i; i <= n + m |] ==> n + m - i < m by arith
    from True have coeff P (monom P 1 (Suc n)) k = 1 by simp
    also from True
    have ... = (⊕ i ∈ {..<n} ∪ {n}. coeff P (monom P 1 n) i ⊗
      coeff P (monom P 1 1) (k - i))
      by (simp cong: R.finsum-cong add: Pi-def)
    also have ... = (⊕ i ∈  {..n}. coeff P (monom P 1 n) i ⊗
      coeff P (monom P 1 1) (k - i))
      by (simp only: ivl-disj-un-singleton)
    also from True
    have ... = (⊕ i ∈ {..n} ∪ {n<..k}. coeff P (monom P 1 n) i ⊗
      coeff P (monom P 1 1) (k - i))
      by (simp cong: R.finsum-cong add: R.finsum-Un-disjoint ivl-disj-int-one
        order-less-imp-not-eq Pi-def)
    also from True have ... = coeff P (monom P 1 n ⊗_P monom P 1 1) k
      by (simp add: ivl-disj-un-one)
    finally show ?thesis .
  qed
next
  case False
  note neq = False
  let ?s =
    λi. (if n = i then 1 else 0) ⊗ (if Suc 0 = k - i then 1 else 0)
  from neq have coeff P (monom P 1 (Suc n)) k = 0 by simp
  also have ... = (⊕ i ∈ {..k}. ?s i)
  proof -
    have f1: (⊕ i ∈ {..<n}. ?s i) = 0
      by (simp cong: R.finsum-cong add: Pi-def)
    from neq have f2: (⊕ i ∈ {n}. ?s i) = 0
      by (simp cong: R.finsum-cong add: Pi-def) arith
    have f3: n < k ==> (⊕ i ∈ {n<..k}. ?s i) = 0
      by (simp cong: R.finsum-cong add: order-less-imp-not-eq Pi-def)
    show ?thesis
    proof (cases k < n)
      case True then show ?thesis by (simp cong: R.finsum-cong add: Pi-def)
    next
      case False then have n-le-k: n <= k by arith
      show ?thesis
      proof (cases n = k)
        case True
        then have 0 = (⊕ i ∈ {..<n} ∪ {n}. ?s i)
          by (simp cong: R.finsum-cong add: ivl-disj-int-singleton Pi-def)
        also from True have ... = (⊕ i ∈ {..k}. ?s i)
          by (simp only: ivl-disj-un-singleton)
        finally show ?thesis .
      next
```

case *False* **with** *n-le-k* **have** *n-less-k*: $n < k$ **by** *arith*
with *neq* **have** $\mathbf{0} = (\bigoplus i \in \{..<n\} \cup \{n\}.\ ?s\ i)$
by (*simp add*: *R.finsum-Un-disjoint f1 f2*
ivl-disj-int-singleton Pi-def del: *Un-insert-right*)
also have ... $= (\bigoplus i \in \{..n\}.\ ?s\ i)$
by (*simp only*: *ivl-disj-un-singleton*)
also from *n-less-k neq* **have** ... $= (\bigoplus i \in \{..n\} \cup \{n<..k\}.\ ?s\ i)$
by (*simp add*: *R.finsum-Un-disjoint f3 ivl-disj-int-one Pi-def*)
also from *n-less-k* **have** ... $= (\bigoplus i \in \{..k\}.\ ?s\ i)$
by (*simp only*: *ivl-disj-un-one*)
finally show *?thesis* **.**
qed
qed
qed
also have ... $=$ *coeff P* (*monom P* $\mathbf{1}$ *n* $\otimes_P$ *monom P* $\mathbf{1}$ *1*) *k* **by** *simp*
finally show *?thesis* **.**
qed
qed (*simp-all*)

lemma (**in** *UP-cring*) *monom-mult-smult*:
[| $a \in$ *carrier R*; $b \in$ *carrier R* |] ==> *monom P* $(a \otimes b)$ $n = a \odot_P$ *monom P* *b n*
by (*rule up-eqI*) *simp-all*

lemma (**in** *UP-cring*) *monom-one* [*simp*]:
monom P $\mathbf{1}$ *0* $= \mathbf{1}_P$
by (*rule up-eqI*) *simp-all*

lemma (**in** *UP-cring*) *monom-one-mult*:
monom P $\mathbf{1}$ $(n + m) =$ *monom P* $\mathbf{1}$ *n* $\otimes_P$ *monom P* $\mathbf{1}$ *m*
proof (*induct n*)
case *0* **show** *?case* **by** *simp*
next
case *Suc* **then show** *?case*
by (*simp only*: *add-Suc monom-one-Suc*) (*simp add*: *P.m-ac*)
qed

lemma (**in** *UP-cring*) *monom-mult* [*simp*]:
assumes *R*: $a \in$ *carrier R* $b \in$ *carrier R*
shows *monom P* $(a \otimes b)$ $(n + m) =$ *monom P a n* $\otimes_P$ *monom P b m*
proof −
from *R* **have** *monom P* $(a \otimes b)$ $(n + m) =$ *monom P* $(a \otimes b \otimes \mathbf{1})$ $(n + m)$ **by** *simp*
also from *R* **have** ... $= a \otimes b \odot_P$ *monom P* $\mathbf{1}$ $(n + m)$
by (*simp add*: *monom-mult-smult del*: *R.r-one*)
also have ... $= a \otimes b \odot_P$ (*monom P* $\mathbf{1}$ *n* $\otimes_P$ *monom P* $\mathbf{1}$ *m*)
by (*simp only*: *monom-one-mult*)
also from *R* **have** ... $= a \odot_P (b \odot_P$ (*monom P* $\mathbf{1}$ *n* $\otimes_P$ *monom P* $\mathbf{1}$ *m*))
by (*simp add*: *UP-smult-assoc1*)

also from *R* **have** ... = $a \odot_P (b \odot_P (\mathit{monom}\ P\ \mathbf{1}\ m \otimes_P \mathit{monom}\ P\ \mathbf{1}\ n))$
  by (*simp add*: *P.m-comm*)
also from *R* **have** ... = $a \odot_P ((b \odot_P \mathit{monom}\ P\ \mathbf{1}\ m) \otimes_P \mathit{monom}\ P\ \mathbf{1}\ n)$
  by (*simp add*: *UP-smult-assoc2*)
also from *R* **have** ... = $a \odot_P (\mathit{monom}\ P\ \mathbf{1}\ n \otimes_P (b \odot_P \mathit{monom}\ P\ \mathbf{1}\ m))$
  by (*simp add*: *P.m-comm*)
also from *R* **have** ... = $(a \odot_P \mathit{monom}\ P\ \mathbf{1}\ n) \otimes_P (b \odot_P \mathit{monom}\ P\ \mathbf{1}\ m)$
  by (*simp add*: *UP-smult-assoc2*)
also from *R* **have** ... = $\mathit{monom}\ P\ (a \otimes \mathbf{1})\ n \otimes_P \mathit{monom}\ P\ (b \otimes \mathbf{1})\ m$
  by (*simp add*: *monom-mult-smult del*: *R.r-one*)
also from *R* **have** ... = $\mathit{monom}\ P\ a\ n \otimes_P \mathit{monom}\ P\ b\ m$ **by** *simp*
finally show *?thesis* **.**
qed

lemma (**in** *UP-cring*) *monom-a-inv* [*simp*]:
  $a \in \mathit{carrier}\ R$ ==> $\mathit{monom}\ P\ (\ominus\ a)\ n = \ominus_P\ \mathit{monom}\ P\ a\ n$
  by (*rule up-eqI*) *simp-all*

lemma (**in** *UP-cring*) *monom-inj*:
  inj-on (%*a. monom P a n*) (*carrier R*)
proof (*rule inj-onI*)
  fix *x y*
  assume *R*: $x \in \mathit{carrier}\ R$ $y \in \mathit{carrier}\ R$ **and** *eq*: $\mathit{monom}\ P\ x\ n = \mathit{monom}\ P\ y\ n$
  then have $\mathit{coeff}\ P\ (\mathit{monom}\ P\ x\ n)\ n = \mathit{coeff}\ P\ (\mathit{monom}\ P\ y\ n)\ n$ **by** *simp*
  with *R* **show** $x = y$ **by** *simp*
qed

12.6 The degree function

constdefs (**structure** *R*)
  deg :: [('*a*, '*m*) *ring-scheme*, *nat* => '*a*] => *nat*
  deg R p == *LEAST n. bound* **0** *n* (*coeff* (*UP R*) *p*)

lemma (**in** *UP-cring*) *deg-aboveI*:
  [| (!!*m*. *n* < *m* ==> *coeff P p m* = **0**); $p \in \mathit{carrier}\ P$ |] ==> *deg R p* <= *n*
  by (*unfold deg-def P-def*) (*fast intro*: *Least-le*)

lemma (**in** *UP-cring*) *deg-aboveD*:
  [| *deg R p* < *m*; $p \in \mathit{carrier}\ P$ |] ==> *coeff P p m* = **0**
proof −
  assume *R*: $p \in \mathit{carrier}\ P$ **and** *deg R p* < *m*
  from *R* **obtain** *n* **where** *bound* **0** *n* (*coeff P p*)
    by (*auto simp add*: *UP-def P-def*)
  then have *bound* **0** (*deg R p*) (*coeff P p*)
    by (*auto simp*: *deg-def P-def dest*: *LeastI*)
  then show *?thesis* **..**
qed

lemma (**in** *UP-cring*) *deg-belowI*:
  assumes *non-zero*: *n ~= 0 ==> coeff P p n ~=* **0**
    and *R*: *p ∈ carrier P*
  shows *n <= deg R p*
— Logically, this is a slightly stronger version of *deg-aboveD*
proof (*cases n=0*)
  case *True* **then show** *?thesis* **by** *simp*
next
  case *False* **then have** *coeff P p n ~=* **0** **by** (*rule non-zero*)
  then have ~ *deg R p < n* **by** (*fast dest*: *deg-aboveD intro*: *R*)
  then show *?thesis* **by** *arith*
qed

lemma (**in** *UP-cring*) *lcoeff-nonzero-deg*:
  assumes *deg*: *deg R p ~= 0* **and** *R*: *p ∈ carrier P*
  shows *coeff P p* (*deg R p*) *~=* **0**
proof −
  from *R* **obtain** *m* **where** *deg R p <= m* **and** *m-coeff*: *coeff P p m ~=* **0**
  proof −
    have *minus*: !!(*n*::*nat*) *m*. *n ~= 0 ==>* (*n − Suc 0 < m*) = (*n <= m*)
      by *arith*

    from *deg* **have** *deg R p − 1 <* (*LEAST n*. *bound* **0** *n* (*coeff P p*))
      by (*unfold deg-def P-def*) *arith*
    then have ~ *bound* **0** (*deg R p − 1*) (*coeff P p*) **by** (*rule not-less-Least*)
    then have *EX m*. *deg R p − 1 < m & coeff P p m ~=* **0**
      by (*unfold bound-def*) *fast*
    then have *EX m*. *deg R p <= m & coeff P p m ~=* **0** **by** (*simp add*: *deg minus*)
    then show *?thesis* **by** *auto*
  qed
  with *deg-belowI R* **have** *deg R p = m* **by** *fastsimp*
  with *m-coeff* **show** *?thesis* **by** *simp*
qed

lemma (**in** *UP-cring*) *lcoeff-nonzero-nonzero*:
  assumes *deg*: *deg R p = 0* **and** *nonzero*: *p ~=* $\mathbf{0}_P$ **and** *R*: *p ∈ carrier P*
  shows *coeff P p 0 ~=* **0**
proof −
  have *EX m*. *coeff P p m ~=* **0**
  proof (*rule classical*)
    assume ~ *?thesis*
    with *R* **have** *p =* $\mathbf{0}_P$ **by** (*auto intro*: *up-eqI*)
    with *nonzero* **show** *?thesis* **by** *contradiction*
  qed
  then obtain *m* **where** *coeff*: *coeff P p m ~=* **0** **..**
  then have *m <= deg R p* **by** (*rule deg-belowI*)
  then have *m = 0* **by** (*simp add*: *deg*)

with *coeff* **show** *?thesis* **by** *simp*
qed

lemma (**in** *UP-cring*) *lcoeff-nonzero*:
assumes *neq*: *p* ~= $\mathbf{0}_P$ **and** *R*: *p* ∈ *carrier P*
shows *coeff P p* (*deg R p*) ~= **0**
proof (*cases deg R p* = *0*)
case *True* **with** *neq R* **show** *?thesis* **by** (*simp add*: *lcoeff-nonzero-nonzero*)
next
case *False* **with** *neq R* **show** *?thesis* **by** (*simp add*: *lcoeff-nonzero-deg*)
qed

lemma (**in** *UP-cring*) *deg-eqI*:
[| !!*m*. *n* < *m* ==> *coeff P p m* = **0**;
!!*n*. *n* ~= *0* ==> *coeff P p n* ~= **0**; *p* ∈ *carrier P* |] ==> *deg R p* = *n*
by (*fast intro*: *le-anti-sym deg-aboveI deg-belowI*)

Degree and polynomial operations

lemma (**in** *UP-cring*) *deg-add* [*simp*]:
assumes *R*: *p* ∈ *carrier P q* ∈ *carrier P*
shows *deg R* (*p* $\oplus_P$ *q*) <= *max* (*deg R p*) (*deg R q*)
proof (*cases deg R p* <= *deg R q*)
case *True* **show** *?thesis*
by (*rule deg-aboveI*) (*simp-all add*: *True R deg-aboveD*)
next
case *False* **show** *?thesis*
by (*rule deg-aboveI*) (*simp-all add*: *False R deg-aboveD*)
qed

lemma (**in** *UP-cring*) *deg-monom-le*:
a ∈ *carrier R* ==> *deg R* (*monom P a n*) <= *n*
by (*intro deg-aboveI*) *simp-all*

lemma (**in** *UP-cring*) *deg-monom* [*simp*]:
[| *a* ~= **0**; *a* ∈ *carrier R* |] ==> *deg R* (*monom P a n*) = *n*
by (*fastsimp intro*: *le-anti-sym deg-aboveI deg-belowI*)

lemma (**in** *UP-cring*) *deg-const* [*simp*]:
assumes *R*: *a* ∈ *carrier R* **shows** *deg R* (*monom P a 0*) = *0*
proof (*rule le-anti-sym*)
show *deg R* (*monom P a 0*) <= *0* **by** (*rule deg-aboveI*) (*simp-all add*: *R*)
next
show *0* <= *deg R* (*monom P a 0*) **by** (*rule deg-belowI*) (*simp-all add*: *R*)
qed

lemma (**in** *UP-cring*) *deg-zero* [*simp*]:
deg R $\mathbf{0}_P$ = *0*
proof (*rule le-anti-sym*)
show *deg R* $\mathbf{0}_P$ <= *0* **by** (*rule deg-aboveI*) *simp-all*

```
next
  show 0 <= deg R 0_P by (rule deg-belowI) simp-all
qed

lemma (in UP-cring) deg-one [simp]:
  deg R 1_P = 0
proof (rule le-anti-sym)
  show deg R 1_P <= 0 by (rule deg-aboveI) simp-all
next
  show 0 <= deg R 1_P by (rule deg-belowI) simp-all
qed

lemma (in UP-cring) deg-uminus [simp]:
  assumes R: p ∈ carrier P shows deg R (⊖_P p) = deg R p
proof (rule le-anti-sym)
  show deg R (⊖_P p) <= deg R p by (simp add: deg-aboveI deg-aboveD R)
next
  show deg R p <= deg R (⊖_P p)
    by (simp add: deg-belowI lcoeff-nonzero-deg
      inj-on-iff [OF R.a-inv-inj, of - 0, simplified] R)
qed

lemma (in UP-domain) deg-smult-ring:
  [| a ∈ carrier R; p ∈ carrier P |] ==>
  deg R (a ⊙_P p) <= (if a = 0 then 0 else deg R p)
  by (cases a = 0) (simp add: deg-aboveI deg-aboveD)+

lemma (in UP-domain) deg-smult [simp]:
  assumes R: a ∈ carrier R p ∈ carrier P
  shows deg R (a ⊙_P p) = (if a = 0 then 0 else deg R p)
proof (rule le-anti-sym)
  show deg R (a ⊙_P p) <= (if a = 0 then 0 else deg R p)
    by (rule deg-smult-ring)
next
  show (if a = 0 then 0 else deg R p) <= deg R (a ⊙_P p)
  proof (cases a = 0)
  qed (simp, simp add: deg-belowI lcoeff-nonzero-deg integral-iff R)
qed

lemma (in UP-cring) deg-mult-cring:
  assumes R: p ∈ carrier P q ∈ carrier P
  shows deg R (p ⊗_P q) <= deg R p + deg R q
proof (rule deg-aboveI)
  fix m
  assume boundm: deg R p + deg R q < m
  {
    fix k i
    assume boundk: deg R p + deg R q < k
    then have coeff P p i ⊗ coeff P q (k - i) = 0
```

proof (*cases deg R p* < *i*)
case *True* **then show** *?thesis* **by** (*simp add*: *deg-aboveD R*)
next
case *False* **with** *boundk* **have** *deg R q* < *k* − *i* **by** *arith*
then show *?thesis* **by** (*simp add*: *deg-aboveD R*)
qed
}
with *boundm R* **show** *coeff P* (*p* ⊗*P* *q*) *m* = **0** **by** *simp*
qed (*simp add*: *R*)

lemma (**in** *UP-domain*) *deg-mult* [*simp*]:
[| *p* ~= **0***P*; *q* ~= **0***P*; *p* ∈ *carrier P*; *q* ∈ *carrier P* |] ==>
deg R (*p* ⊗*P* *q*) = *deg R p* + *deg R q*
proof (*rule le-anti-sym*)
assume *p* ∈ *carrier P* *q* ∈ *carrier P*
show *deg R* (*p* ⊗*P* *q*) <= *deg R p* + *deg R q* **by** (*rule deg-mult-cring*)
next
let *?s* = (%*i*. *coeff P p i* ⊗ *coeff P q* (*deg R p* + *deg R q* − *i*))
assume *R*: *p* ∈ *carrier P q* ∈ *carrier P* **and** *nz*: *p* ~= **0***P* *q* ~= **0***P*
have *less-add-diff*: !!(*k*::*nat*) *n m*. *k* < *n* ==> *m* < *n* + *m* − *k* **by** *arith*
show *deg R p* + *deg R q* <= *deg R* (*p* ⊗*P* *q*)
proof (*rule deg-belowI*, *simp add*: *R*)
have (⨁ *i* ∈ {.. *deg R p* + *deg R q*}. *?s i*)
= (⨁ *i* ∈ {..< *deg R p*} ∪ {*deg R p* .. *deg R p* + *deg R q*}. *?s i*)
by (*simp only*: *ivl-disj-un-one*)
also have ... = (⨁ *i* ∈ {*deg R p* .. *deg R p* + *deg R q*}. *?s i*)
by (*simp cong*: *R.finsum-cong add*: *R.finsum-Un-disjoint ivl-disj-int-one*
deg-aboveD less-add-diff R Pi-def)
also have ...= (⨁ *i* ∈ {*deg R p*} ∪ {*deg R p* <.. *deg R p* + *deg R q*}. *?s i*)
by (*simp only*: *ivl-disj-un-singleton*)
also have ... = *coeff P p* (*deg R p*) ⊗ *coeff P q* (*deg R q*)
by (*simp cong*: *R.finsum-cong*
add: *ivl-disj-int-singleton deg-aboveD R Pi-def*)
finally have (⨁ *i* ∈ {.. *deg R p* + *deg R q*}. *?s i*)
= *coeff P p* (*deg R p*) ⊗ *coeff P q* (*deg R q*) **.**
with *nz* **show** (⨁ *i* ∈ {.. *deg R p* + *deg R q*}. *?s i*) ~= **0**
by (*simp add*: *integral-iff lcoeff-nonzero R*)
qed (*simp add*: *R*)
qed

lemma (**in** *UP-cring*) *coeff-finsum*:
assumes *fin*: *finite A*
shows *p* ∈ *A* −> *carrier P* ==>
coeff P (*finsum P p A*) *k* = (⨁ *i* ∈ *A*. *coeff P* (*p i*) *k*)
using *fin* **by** *induct* (*auto simp*: *Pi-def*)

lemma (**in** *UP-cring*) *up-repr*:
assumes *R*: *p* ∈ *carrier P*
shows (⨁*P* *i* ∈ {..*deg R p*}. *monom P* (*coeff P p i*) *i*) = *p*

```
proof (rule up-eqI)
  let ?s = (%i. monom P (coeff P p i) i)
  fix k
  from R have RR: !!i. (if i = k then coeff P p i else 0) ∈ carrier R
    by simp
  show coeff P (⨁P i ∈ {..deg R p}. ?s i) k = coeff P p k
  proof (cases k <= deg R p)
    case True
    hence coeff P (⨁P i ∈ {..deg R p}. ?s i) k =
          coeff P (⨁P i ∈ {..k} ∪ {k<..deg R p}. ?s i) k
      by (simp only: ivl-disj-un-one)
    also from True
    have ... = coeff P (⨁P i ∈ {..k}. ?s i) k
      by (simp cong: R.finsum-cong add: R.finsum-Un-disjoint
        ivl-disj-int-one order-less-imp-not-eq2 coeff-finsum R RR Pi-def)
    also
    have ... = coeff P (⨁P i ∈ {..<k} ∪ {k}. ?s i) k
      by (simp only: ivl-disj-un-singleton)
    also have ... = coeff P p k
      by (simp cong: R.finsum-cong
        add: ivl-disj-int-singleton coeff-finsum deg-aboveD R RR Pi-def)
    finally show ?thesis .
  next
    case False
    hence coeff P (⨁P i ∈ {..deg R p}. ?s i) k =
          coeff P (⨁P i ∈ {..<deg R p} ∪ {deg R p}. ?s i) k
      by (simp only: ivl-disj-un-singleton)
    also from False have ... = coeff P p k
      by (simp cong: R.finsum-cong
        add: ivl-disj-int-singleton coeff-finsum deg-aboveD R Pi-def)
    finally show ?thesis .
  qed
qed (simp-all add: R Pi-def)

lemma (in UP-cring) up-repr-le:
  [| deg R p <= n; p ∈ carrier P |] ==>
  (⨁P i ∈ {..n}. monom P (coeff P p i) i) = p
proof −
  let ?s = (%i. monom P (coeff P p i) i)
  assume R: p ∈ carrier P and deg R p <= n
  then have finsum P ?s {..n} = finsum P ?s ({..deg R p} ∪ {deg R p<..n})
    by (simp only: ivl-disj-un-one)
  also have ... = finsum P ?s {..deg R p}
    by (simp cong: P.finsum-cong add: P.finsum-Un-disjoint ivl-disj-int-one
      deg-aboveD R Pi-def)
  also have ... = p by (rule up-repr)
  finally show ?thesis .
qed
```

12.7 Polynomials over an integral domain form an integral domain

lemma *domainI*:
  assumes *cring*: *cring R*
    and *one-not-zero*: *one R* ~= *zero R*
    and *integral*: !!*a b*. [| *mult R a b* = *zero R*; *a* ∈ *carrier R*;
      b ∈ *carrier R* |] ==> *a* = *zero R* | *b* = *zero R*
  shows *domain R*
  by (*auto intro*!: *domain.intro domain-axioms.intro cring.axioms prems*
    del: *disjCI*)

lemma (**in** *UP-domain*) *UP-one-not-zero*:
  $\mathbf{1}_P$ ~= $\mathbf{0}_P$
proof
  assume $\mathbf{1}_P = \mathbf{0}_P$
  hence *coeff P* $\mathbf{1}_P$ *0* = (*coeff P* $\mathbf{0}_P$ *0*) **by** *simp*
  hence $\mathbf{1} = \mathbf{0}$ **by** *simp*
  with *one-not-zero* **show** *False* **by** *contradiction*
qed

lemma (**in** *UP-domain*) *UP-integral*:
  [| $p \otimes_P q = \mathbf{0}_P$; *p* ∈ *carrier P*; *q* ∈ *carrier P* |] ==> $p = \mathbf{0}_P$ | $q = \mathbf{0}_P$
proof −
  fix *p q*
  assume *pq*: $p \otimes_P q = \mathbf{0}_P$ **and** *R*: *p* ∈ *carrier P q* ∈ *carrier P*
  show $p = \mathbf{0}_P$ | $q = \mathbf{0}_P$
  proof (*rule classical*)
    assume *c*: ~ ($p = \mathbf{0}_P$ | $q = \mathbf{0}_P$)
    with *R* **have** *deg R p* + *deg R q* = *deg R* ($p \otimes_P q$) **by** *simp*
    also from *pq* **have** ... = *0* **by** *simp*
    finally have *deg R p* + *deg R q* = *0* **.**
    then have *f1*: *deg R p* = *0* & *deg R q* = *0* **by** *simp*
    from *f1 R* **have** *p* = ($\bigoplus_P$ *i* ∈ {..*0*}. *monom P* (*coeff P p i*) *i*)
      by (*simp only*: *up-repr-le*)
    also from *R* **have** ... = *monom P* (*coeff P p 0*) *0* **by** *simp*
    finally have *p*: *p* = *monom P* (*coeff P p 0*) *0* **.**
    from *f1 R* **have** *q* = ($\bigoplus_P$ *i* ∈ {..*0*}. *monom P* (*coeff P q i*) *i*)
      by (*simp only*: *up-repr-le*)
    also from *R* **have** ... = *monom P* (*coeff P q 0*) *0* **by** *simp*
    finally have *q*: *q* = *monom P* (*coeff P q 0*) *0* **.**
    from *R* **have** *coeff P p 0* ⊗ *coeff P q 0* = *coeff P* ($p \otimes_P q$) *0* **by** *simp*
    also from *pq* **have** ... = $\mathbf{0}$ **by** *simp*
    finally have *coeff P p 0* ⊗ *coeff P q 0* = $\mathbf{0}$ **.**
    with *R* **have** *coeff P p 0* = $\mathbf{0}$ | *coeff P q 0* = $\mathbf{0}$
      by (*simp add*: *R.integral-iff*)
    with *p q* **show** $p = \mathbf{0}_P$ | $q = \mathbf{0}_P$ **by** *fastsimp*
  qed
qed

```
theorem (in UP-domain) UP-domain:
  domain P
  by (auto intro!: domainI UP-cring UP-one-not-zero UP-integral del: disjCI)
```

Interpretation of theorems from *domain*.

```
interpretation UP-domain < domain P
  using UP-domain
  by (rule domain.axioms)
```

12.8 Evaluation Homomorphism and Universal Property

```
theorem (in cring) diagonal-sum:
  [| f ∈ {..n + m::nat} -> carrier R; g ∈ {..n + m} -> carrier R |] ==>
  (⨁ k ∈ {..n + m}. ⨁ i ∈ {..k}. f i ⊗ g (k − i)) =
  (⨁ k ∈ {..n + m}. ⨁ i ∈ {..n + m − k}. f k ⊗ g i)
proof −
 assume Rf: f ∈ {..n + m} -> carrier R and Rg: g ∈ {..n + m} -> carrier R
 {
  fix j
  have j <= n + m ==>
    (⨁ k ∈ {..j}. ⨁ i ∈ {..k}. f i ⊗ g (k − i)) =
    (⨁ k ∈ {..j}. ⨁ i ∈ {..j − k}. f k ⊗ g i)
  proof (induct j)
   case 0 from Rf Rg show ?case by (simp add: Pi-def)
  next
   case (Suc j)
   have R6: !!i k. [| k <= j; i <= Suc j − k |] ==> g i ∈ carrier R
     using Suc by (auto intro!: funcset-mem [OF Rg]) arith
   have R8: !!i k. [| k <= Suc j; i <= k |] ==> g (k − i) ∈ carrier R
     using Suc by (auto intro!: funcset-mem [OF Rg]) arith
   have R9: !!i k. [| k <= Suc j |] ==> f k ∈ carrier R
     using Suc by (auto intro!: funcset-mem [OF Rf])
   have R10: !!i k. [| k <= Suc j; i <= Suc j − k |] ==> g i ∈ carrier R
     using Suc by (auto intro!: funcset-mem [OF Rg]) arith
   have R11: g 0 ∈ carrier R
     using Suc by (auto intro!: funcset-mem [OF Rg])
   from Suc show ?case
     by (simp cong: finsum-cong add: Suc-diff-le a-ac
       Pi-def R6 R8 R9 R10 R11)
  qed
 }
 then show ?thesis by fast
qed

lemma (in abelian-monoid) boundD-carrier:
  [| bound 0 n f; n < m |] ==> f m ∈ carrier G
  by auto

theorem (in cring) cauchy-product:
```

```
  assumes bf: bound 0 n f and bg: bound 0 m g
    and Rf: f ∈ {..n} -> carrier R and Rg: g ∈ {..m} -> carrier R
  shows (⨁ k ∈ {..n + m}. ⨁ i ∈ {..k}. f i ⊗ g (k - i)) =
    (⨁ i ∈ {..n}. f i) ⊗ (⨁ i ∈ {..m}. g i)
proof -
  have f: !!x. f x ∈ carrier R
  proof -
    fix x
    show f x ∈ carrier R
      using Rf bf boundD-carrier by (cases x <= n) (auto simp: Pi-def)
  qed
  have g: !!x. g x ∈ carrier R
  proof -
    fix x
    show g x ∈ carrier R
      using Rg bg boundD-carrier by (cases x <= m) (auto simp: Pi-def)
  qed
  from f g have (⨁ k ∈ {..n + m}. ⨁ i ∈ {..k}. f i ⊗ g (k - i)) =
      (⨁ k ∈ {..n + m}. ⨁ i ∈ {..n + m - k}. f k ⊗ g i)
    by (simp add: diagonal-sum Pi-def)
  also have ... = (⨁ k ∈ {..n} ∪ {n<..n + m}. ⨁ i ∈ {..n + m - k}. f k ⊗ g i)
    by (simp only: ivl-disj-un-one)
  also from f g have ... = (⨁ k ∈ {..n}. ⨁ i ∈ {..n + m - k}. f k ⊗ g i)
    by (simp cong: finsum-cong
      add: bound.bound [OF bf] finsum-Un-disjoint ivl-disj-int-one Pi-def)
  also from f g
  have ... = (⨁ k ∈ {..n}. ⨁ i ∈ {..m} ∪ {m<..n + m - k}. f k ⊗ g i)
    by (simp cong: finsum-cong add: ivl-disj-un-one le-add-diff Pi-def)
  also from f g have ... = (⨁ k ∈ {..n}. ⨁ i ∈ {..m}. f k ⊗ g i)
    by (simp cong: finsum-cong
      add: bound.bound [OF bg] finsum-Un-disjoint ivl-disj-int-one Pi-def)
  also from f g have ... = (⨁ i ∈ {..n}. f i) ⊗ (⨁ i ∈ {..m}. g i)
    by (simp add: finsum-ldistr diagonal-sum Pi-def,
      simp cong: finsum-cong add: finsum-rdistr Pi-def)
  finally show ?thesis .
qed

lemma (in UP-cring) const-ring-hom:
  (%a. monom P a 0) ∈ ring-hom R P
  by (auto intro!: ring-hom-memI intro: up-eqI simp: monom-mult-is-smult)

constdefs (structure S)
  eval :: [('a, 'm) ring-scheme, ('b, 'n) ring-scheme,
          'a => 'b, 'b, nat => 'a] => 'b
  eval R S phi s == λp ∈ carrier (UP R).
    ⨁ i ∈ {..deg R p}. phi (coeff (UP R) p i) ⊗ s (^) i


lemma (in UP) eval-on-carrier:
```

includes *struct S*
shows $p \in carrier\ P ==>$
$eval\ R\ S\ phi\ s\ p = (\bigoplus_S i \in \{..deg\ R\ p\}.\ phi\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{\ })_S\ i)$
by (*unfold eval-def*, *fold P-def*) *simp*

lemma (**in** *UP*) *eval-extensional*:
$eval\ R\ S\ phi\ p \in extensional\ (carrier\ P)$
by (*unfold eval-def*, *fold P-def*) *simp*

The universal property of the polynomial ring

locale *UP-pre-univ-prop* = *ring-hom-cring R S h* + *UP-cring R P*

locale *UP-univ-prop* = *UP-pre-univ-prop* + *var s* + *var Eval* +
assumes *indet-img-carrier* [*simp*, *intro*]: $s \in carrier\ S$
defines *Eval-def*: *Eval* == *eval R S h s*

theorem (**in** *UP-pre-univ-prop*) *eval-ring-hom*:
assumes *S*: $s \in carrier\ S$
shows $eval\ R\ S\ h\ s \in ring\text{-}hom\ P\ S$
proof (*rule ring-hom-memI*)
fix *p*
assume *R*: $p \in carrier\ P$
then show $eval\ R\ S\ h\ s\ p \in carrier\ S$
by (*simp only*: *eval-on-carrier*) (*simp add*: *S Pi-def*)
next
fix *p q*
assume *R*: $p \in carrier\ P\ q \in carrier\ P$
then show $eval\ R\ S\ h\ s\ (p \otimes_P q) = eval\ R\ S\ h\ s\ p \otimes_S eval\ R\ S\ h\ s\ q$
proof (*simp only*: *eval-on-carrier UP-mult-closed*)
from *R S* **have**
$(\bigoplus_S i \in \{..deg\ R\ (p \otimes_P q)\}.\ h\ (coeff\ P\ (p \otimes_P q)\ i) \otimes_S s\ (\hat{\ })_S\ i) =$
$(\bigoplus_S i \in \{..deg\ R\ (p \otimes_P q)\} \cup \{deg\ R\ (p \otimes_P q)<..deg\ R\ p + deg\ R\ q\}.$
$h\ (coeff\ P\ (p \otimes_P q)\ i) \otimes_S s\ (\hat{\ })_S\ i)$
by (*simp cong*: *S.finsum-cong*
add: *deg-aboveD S.finsum-Un-disjoint ivl-disj-int-one Pi-def*
del: *coeff-mult*)
also from *R* **have** ... =
$(\bigoplus_S i \in \{..deg\ R\ p + deg\ R\ q\}.\ h\ (coeff\ P\ (p \otimes_P q)\ i) \otimes_S s\ (\hat{\ })_S\ i)$
by (*simp only*: *ivl-disj-un-one deg-mult-cring*)
also from *R S* **have** ... =
$(\bigoplus_S i \in \{..deg\ R\ p + deg\ R\ q\}.$
$\bigoplus_S k \in \{..i\}.$
$h\ (coeff\ P\ p\ k) \otimes_S h\ (coeff\ P\ q\ (i - k)) \otimes_S$
$(s\ (\hat{\ })_S\ k \otimes_S s\ (\hat{\ })_S\ (i - k)))$
by (*simp cong*: *S.finsum-cong add*: *S.nat-pow-mult Pi-def*
S.m-ac S.finsum-rdistr)
also from *R S* **have** ... =
$(\bigoplus_S i{\in}\{..deg\ R\ p\}.\ h\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{\ })_S\ i) \otimes_S$
$(\bigoplus_S i{\in}\{..deg\ R\ q\}.\ h\ (coeff\ P\ q\ i) \otimes_S s\ (\hat{\ })_S\ i)$

by (*simp add*: *S.cauchy-product* [*THEN sym*] *bound.intro deg-aboveD S.m-ac Pi-def*)
finally show
$(\bigoplus_S\ i \in \{..deg\ R\ (p \otimes_P q)\}.\ h\ (coeff\ P\ (p \otimes_P q)\ i) \otimes_S s\ (\hat{\ })_S\ i) =$
$(\bigoplus_S\ i \in \{..deg\ R\ p\}.\ h\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{\ })_S\ i) \otimes_S$
$(\bigoplus_S\ i \in \{..deg\ R\ q\}.\ h\ (coeff\ P\ q\ i) \otimes_S s\ (\hat{\ })_S\ i)$ **.**
qed
next
fix $p\ q$
assume R: $p \in carrier\ P\ q \in carrier\ P$
then show $eval\ R\ S\ h\ s\ (p \oplus_P q) = eval\ R\ S\ h\ s\ p \oplus_S eval\ R\ S\ h\ s\ q$
proof (*simp only*: *eval-on-carrier P.a-closed*)
from $S\ R$ **have**
$(\bigoplus_S\ i{\in}\{..deg\ R\ (p \oplus_P q)\}.\ h\ (coeff\ P\ (p \oplus_P q)\ i) \otimes_S s\ (\hat{\ })_S\ i) =$
$(\bigoplus_S\ i{\in}\{..deg\ R\ (p \oplus_P q)\} \cup \{deg\ R\ (p \oplus_P q){<}..max\ (deg\ R\ p)\ (deg\ R\ q)\}.$
$h\ (coeff\ P\ (p \oplus_P q)\ i) \otimes_S s\ (\hat{\ })_S\ i)$
by (*simp cong*: *S.finsum-cong*
add: *deg-aboveD S.finsum-Un-disjoint ivl-disj-int-one Pi-def*
del: *coeff-add*)
also from R **have** ... =
$(\bigoplus_S\ i \in \{..max\ (deg\ R\ p)\ (deg\ R\ q)\}.$
$h\ (coeff\ P\ (p \oplus_P q)\ i) \otimes_S s\ (\hat{\ })_S\ i)$
by (*simp add*: *ivl-disj-un-one*)
also from $R\ S$ **have** ... =
$(\bigoplus_S i{\in}\{..max\ (deg\ R\ p)\ (deg\ R\ q)\}.\ h\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{\ })_S\ i) \oplus_S$
$(\bigoplus_S i{\in}\{..max\ (deg\ R\ p)\ (deg\ R\ q)\}.\ h\ (coeff\ P\ q\ i) \otimes_S s\ (\hat{\ })_S\ i)$
by (*simp cong*: *S.finsum-cong*
add: *S.l-distr deg-aboveD ivl-disj-int-one Pi-def*)
also have ... =
$(\bigoplus_S\ i \in \{..deg\ R\ p\} \cup \{deg\ R\ p{<}..max\ (deg\ R\ p)\ (deg\ R\ q)\}.$
$h\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{\ })_S\ i) \oplus_S$
$(\bigoplus_S\ i \in \{..deg\ R\ q\} \cup \{deg\ R\ q{<}..max\ (deg\ R\ p)\ (deg\ R\ q)\}.$
$h\ (coeff\ P\ q\ i) \otimes_S s\ (\hat{\ })_S\ i)$
by (*simp only*: *ivl-disj-un-one le-maxI1 le-maxI2*)
also from $R\ S$ **have** ... =
$(\bigoplus_S\ i \in \{..deg\ R\ p\}.\ h\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{\ })_S\ i) \oplus_S$
$(\bigoplus_S\ i \in \{..deg\ R\ q\}.\ h\ (coeff\ P\ q\ i) \otimes_S s\ (\hat{\ })_S\ i)$
by (*simp cong*: *S.finsum-cong*
add: *deg-aboveD S.finsum-Un-disjoint ivl-disj-int-one Pi-def*)
finally show
$(\bigoplus_S i \in \{..deg\ R\ (p \oplus_P q)\}.\ h\ (coeff\ P\ (p \oplus_P q)\ i) \otimes_S s\ (\hat{\ })_S\ i) =$
$(\bigoplus_S i \in \{..deg\ R\ p\}.\ h\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{\ })_S\ i) \oplus_S$
$(\bigoplus_S i \in \{..deg\ R\ q\}.\ h\ (coeff\ P\ q\ i) \otimes_S s\ (\hat{\ })_S\ i)$ **.**
qed
next
show $eval\ R\ S\ h\ s\ \mathbf{1}_P = \mathbf{1}_S$
by (*simp only*: *eval-on-carrier UP-one-closed*) *simp*
qed

Interpretation of ring homomorphism lemmas.

interpretation *UP-univ-prop* < *ring-hom-cring P S Eval*
 by (*unfold Eval-def*)
  (*fast intro*!: *ring-hom-cring.intro UP-cring cring.axioms prems*
   intro: *ring-hom-cring-axioms.intro eval-ring-hom*)

Further properties of the evaluation homomorphism.

lemma (**in** *UP-pre-univ-prop*) *eval-const*:
 [| $s \in$ *carrier S*; $r \in$ *carrier R* |] ==> *eval R S h s* (*monom P r 0*) = *h r*
 by (*simp only*: *eval-on-carrier monom-closed*) *simp*

The following proof is complicated by the fact that in arbitrary rings one might have $\mathbf{1}_R = \mathbf{0}_R$.

lemma (**in** *UP-pre-univ-prop*) *eval-monom1*:
 assumes *S*: $s \in$ *carrier S*
 shows *eval R S h s* (*monom P* $\mathbf{1}$ *1*) = *s*
proof (*simp only*: *eval-on-carrier monom-closed R.one-closed*)
  from *S* **have**
  ($\bigoplus_S$ $i\in$\{..*deg R* (*monom P* $\mathbf{1}$ *1*)\}. *h* (*coeff P* (*monom P* $\mathbf{1}$ *1*) *i*) $\otimes_S$ *s* (^)$_S$ *i*)
=
  ($\bigoplus_S$ $i\in$\{..*deg R* (*monom P* $\mathbf{1}$ *1*)\} $\cup$ \{*deg R* (*monom P* $\mathbf{1}$ *1*)<..*1*\}.
   h (*coeff P* (*monom P* $\mathbf{1}$ *1*) *i*) $\otimes_S$ *s* (^)$_S$ *i*)
  by (*simp cong*: *S.finsum-cong del*: *coeff-monom*
   add: *deg-aboveD S.finsum-Un-disjoint ivl-disj-int-one Pi-def*)
 also have ... =
  ($\bigoplus_S$ $i \in$ \{..*1*\}. *h* (*coeff P* (*monom P* $\mathbf{1}$ *1*) *i*) $\otimes_S$ *s* (^)$_S$ *i*)
  by (*simp only*: *ivl-disj-un-one deg-monom-le R.one-closed*)
 also have ... = *s*
 proof (*cases* $s = \mathbf{0}_S$)
  case *True* **then show** *?thesis* **by** (*simp add*: *Pi-def*)
 next
  case *False* **then show** *?thesis* **by** (*simp add*: *S Pi-def*)
 qed
 finally show ($\bigoplus_S$ $i \in$ \{..*deg R* (*monom P* $\mathbf{1}$ *1*)\}.
  h (*coeff P* (*monom P* $\mathbf{1}$ *1*) *i*) $\otimes_S$ *s* (^)$_S$ *i*) = *s* **.**
qed

lemma (**in** *UP-cring*) *monom-pow*:
 assumes *R*: $a \in$ *carrier R*
 shows (*monom P a n*) (^)$_P$ *m* = *monom P* (*a* (^) *m*) (*n* $*$ *m*)
proof (*induct m*)
 case *0* **from** *R* **show** *?case* **by** *simp*
next
 case *Suc* **with** *R* **show** *?case*
  by (*simp del*: *monom-mult add*: *monom-mult* [*THEN sym*] *add-commute*)
qed

lemma (**in** *ring-hom-cring*) *hom-pow* [*simp*]:
 $x \in$ *carrier R* ==> *h* (*x* (^) *n*) = *h x* (^)$_S$ (*n*::*nat*)
 by (*induct n*) *simp-all*

```
lemma (in UP-univ-prop) Eval-monom:
  r ∈ carrier R ==> Eval (monom P r n) = h r ⊗_S s (^)_S n
proof -
  assume R: r ∈ carrier R
  from R have Eval (monom P r n) = Eval (monom P r 0 ⊗_P (monom P 1 1)
(^)_P n)
    by (simp del: monom-mult add: monom-mult [THEN sym] monom-pow)
  also
  from R eval-monom1 [where s = s, folded Eval-def]
  have ... = h r ⊗_S s (^)_S n
    by (simp add: eval-const [where s = s, folded Eval-def])
  finally show ?thesis .
qed

lemma (in UP-pre-univ-prop) eval-monom:
  assumes R: r ∈ carrier R and S: s ∈ carrier S
  shows eval R S h s (monom P r n) = h r ⊗_S s (^)_S n
proof -
  from S interpret UP-univ-prop [R S h P s -]
    by (auto intro!: UP-univ-prop-axioms.intro)
  from R
  show ?thesis by (rule Eval-monom)
qed

lemma (in UP-univ-prop) Eval-smult:
  [| r ∈ carrier R; p ∈ carrier P |] ==> Eval (r ⊙_P p) = h r ⊗_S Eval p
proof -
  assume R: r ∈ carrier R and P: p ∈ carrier P
  then show ?thesis
    by (simp add: monom-mult-is-smult [THEN sym]
      eval-const [where s = s, folded Eval-def])
qed

lemma ring-hom-cringI:
  assumes cring R
    and cring S
    and h ∈ ring-hom R S
  shows ring-hom-cring R S h
  by (fast intro: ring-hom-cring.intro ring-hom-cring-axioms.intro
    cring.axioms prems)

lemma (in UP-pre-univ-prop) UP-hom-unique:
  includes ring-hom-cring P S Phi
  assumes Phi: Phi (monom P 1 (Suc 0)) = s
      !!r. r ∈ carrier R ==> Phi (monom P r 0) = h r
  includes ring-hom-cring P S Psi
  assumes Psi: Psi (monom P 1 (Suc 0)) = s
      !!r. r ∈ carrier R ==> Psi (monom P r 0) = h r
```

  and *P*: $p \in$ *carrier P* **and** *S*: $s \in$ *carrier S*
 shows *Phi p* = *Psi p*
proof –
 have *Phi p* =
   Phi ($\bigoplus_P i \in \{..deg\ R\ p\}$. *monom P* (*coeff P p i*) *0* $\otimes_P$ *monom P* **1** *1* (^)$_P$ *i*)
  by (*simp add*: *up-repr P monom-mult* [*THEN sym*] *monom-pow del*: *monom-mult*)
 also
 have ... =
   Psi ($\bigoplus_P i \in \{..deg\ R\ p\}$. *monom P* (*coeff P p i*) *0* $\otimes_P$ *monom P* **1** *1* (^)$_P$ *i*)
  by (*simp add*: *Phi Psi P Pi-def comp-def*)
 also have ... = *Psi p*
  by (*simp add*: *up-repr P monom-mult* [*THEN sym*] *monom-pow del*: *monom-mult*)
 finally show *?thesis* **.**
qed

lemma (**in** *UP-pre-univ-prop*) *ring-homD*:
 assumes *Phi*: *Phi* $\in$ *ring-hom P S*
 shows *ring-hom-cring P S Phi*
proof (*rule ring-hom-cring.intro*)
 show *ring-hom-cring-axioms P S Phi*
 by (*rule ring-hom-cring-axioms.intro*) (*rule Phi*)
qed (*auto intro*: *P.cring cring.axioms*)

theorem (**in** *UP-pre-univ-prop*) *UP-universal-property*:
 assumes *S*: $s \in$ *carrier S*
 shows *EX*! *Phi*. *Phi* $\in$ *ring-hom P S* $\cap$ *extensional* (*carrier P*) &
  Phi (*monom P* **1** *1*) = *s* &
  (*ALL r* : *carrier R*. *Phi* (*monom P r 0*) = *h r*)
 using *S eval-monom1*
 apply (*auto intro*: *eval-ring-hom eval-const eval-extensional*)
 apply (*rule extensionalityI*)
 apply (*auto intro*: *UP-hom-unique ring-homD*)
 done

12.9 Sample application of evaluation homomorphism

lemma *UP-pre-univ-propI*:
 assumes *cring R*
  and *cring S*
  and $h \in$ *ring-hom R S*
 shows *UP-pre-univ-prop R S h*
 by (*fast intro*: *UP-pre-univ-prop.intro ring-hom-cring-axioms.intro*
  cring.axioms prems)

constdefs
 INTEG :: *int ring*
 INTEG == (| *carrier* = *UNIV*, *mult* = *op* ∗, *one* = *1*, *zero* = *0*, *add* = *op* +
|)

lemma *INTEG-cring*:
cring INTEG
by (*unfold INTEG-def*) (*auto intro*!: *cringI abelian-groupI comm-monoidI zadd-zminus-inverse2 zadd-zmult-distrib*)

lemma *INTEG-id-eval*:
UP-pre-univ-prop INTEG INTEG id
by (*fast intro*: *UP-pre-univ-propI INTEG-cring id-ring-hom*)

Interpretation now enables to import all theorems and lemmas valid in the context of homomorphisms between *INTEG* and *UP INTEG* globally.

interpretation *INTEG*: *UP-pre-univ-prop* [*INTEG INTEG id*]
using *INTEG-id-eval*
by − (*erule UP-pre-univ-prop.axioms*)+

lemma *INTEG-closed* [*intro*, *simp*]:
$z \in$ *carrier INTEG*
by (*unfold INTEG-def*) *simp*

lemma *INTEG-mult* [*simp*]:
mult INTEG $z\ w = z * w$
by (*unfold INTEG-def*) *simp*

lemma *INTEG-pow* [*simp*]:
pow INTEG $z\ n = z\ \hat{}\ n$
by (*induct n*) (*simp-all add*: *INTEG-def nat-pow-def*)

lemma *eval INTEG INTEG id 10* (*monom* (*UP INTEG*) *5 2*) = *500*
by (*simp add*: *INTEG.eval-monom*)

end

References

[1] C. Ballarin. *Computer Algebra and Theorem Proving*. PhD thesis, University of Cambridge, 1999. http://www4.in.tum.de/~ballarin/publications.html.

[2] N. Jacobson. *Basic Algebra I*. Freeman, 1985.

[3] F. Kammüller and L. C. Paulson. A formal proof of sylow's theorem: An experiment in abstract algebra with Isabelle HOL. *J. Automated Reasoning*, (23):235–264, 1999.