Equivalents of the Axiom of Choice

Krzysztof Grąbczewski

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Abstract

This development [1] proves the equivalence of seven formulations of the well-ordering theorem and twenty formulations of the axiom of choice. It formalizes the first two chapters of the monograph *Equivalents of the Axiom of Choice* by Rubin and Rubin [2]. Some of this mmaterial involves extremely complex techniques.

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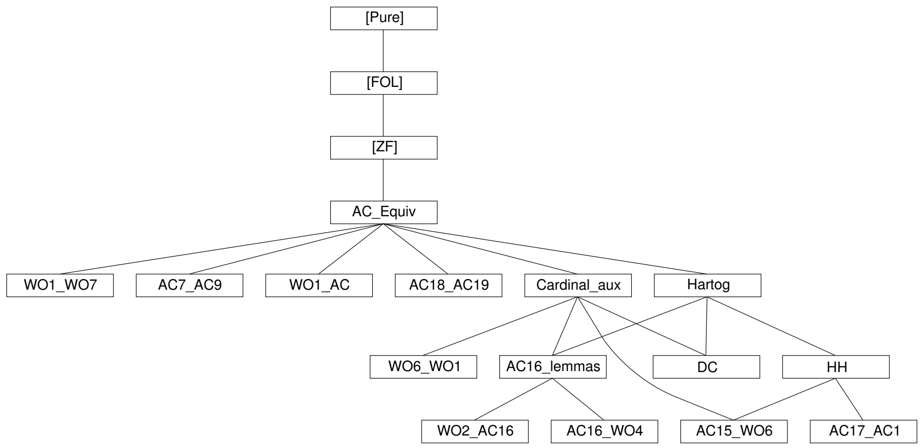

theory *AC_Equiv* **imports** *Main* **begin**

constdefs

```
WO1 :: o
  "WO1 == ∀A. ∃R. well_ord(A,R)"

WO2 :: o
  "WO2 == ∀A. ∃a. Ord(a) & A≈a"

WO3 :: o
  "WO3 == ∀A. ∃a. Ord(a) & (∃b. b ⊆ a & A≈b)"

WO4 :: "i => o"
  "WO4(m) == ∀A. ∃a f. Ord(a) & domain(f)=a &
                    (⋃b<a. f`b) = A & (∀b<a. f`b ≲ m)"

WO5 :: o
  "WO5 == ∃m ∈ nat. 1≤m & WO4(m)"

WO6 :: o
  "WO6 == ∀A. ∃m ∈ nat. 1≤m & (∃a f. Ord(a) & domain(f)=a
                        & (⋃b<a. f`b) = A & (∀b<a. f`b ≲ m))"

WO7 :: o
  "WO7 == ∀A. Finite(A) <-> (∀R. well_ord(A,R) --> well_ord(A,converse(R)))"

WO8 :: o
  "WO8 == ∀A. (∃f. f ∈ (Π X ∈ A. X)) --> (∃R. well_ord(A,R))"


pairwise_disjoint :: "i => o"
  "pairwise_disjoint(A) == ∀A1 ∈ A. ∀A2 ∈ A. A1 Int A2 ≠ 0 --> A1=A2"

sets_of_size_between :: "[i, i, i] => o"
  "sets_of_size_between(A,m,n) == ∀B ∈ A. m ≲ B & B ≲ n"


AC0 :: o
  "AC0 == ∀A. ∃f. f ∈ (Π X ∈ Pow(A)-{0}. X)"

AC1 :: o
  "AC1 == ∀A. 0∉A --> (∃f. f ∈ (Π X ∈ A. X))"
```

```
  AC2 :: o
    "AC2 == ∀A. 0∉A & pairwise_disjoint(A)
                   --> (∃C. ∀B ∈ A. ∃y. B Int C = {y})"
  AC3 :: o
    "AC3 == ∀A B. ∀f ∈ A->B. ∃g. g ∈ (Π x ∈ {a ∈ A. f`a≠0}. f`x)"

  AC4 :: o
    "AC4 == ∀R A B. (R ⊆ A*B --> (∃f. f ∈ (Π x ∈ domain(R). R``{x})))"

  AC5 :: o
    "AC5 == ∀A B. ∀f ∈ A->B. ∃g ∈ range(f)->A. ∀x ∈ domain(g). f`(g`x)
= x"

  AC6 :: o
    "AC6 == ∀A. 0∉A --> (Π B ∈ A. B)≠0"

  AC7 :: o
    "AC7 == ∀A. 0∉A & (∀B1 ∈ A. ∀B2 ∈ A. B1≈B2) --> (Π B ∈ A. B)
≠ 0"

  AC8 :: o
    "AC8 == ∀A. (∀B ∈ A. ∃B1 B2. B=<B1,B2> & B1≈B2)
                   --> (∃f. ∀B ∈ A. f`B ∈ bij(fst(B),snd(B)))"

  AC9 :: o
    "AC9 == ∀A. (∀B1 ∈ A. ∀B2 ∈ A. B1≈B2) -->
                   (∃f. ∀B1 ∈ A. ∀B2 ∈ A. f`<B1,B2> ∈ bij(B1,B2))"

  AC10 :: "i => o"
    "AC10(n) ==  ∀A. (∀B ∈ A. ~Finite(B)) -->
                   (∃f. ∀B ∈ A. (pairwise_disjoint(f`B) &
                   sets_of_size_between(f`B, 2, succ(n)) & Union(f`B)=B))"

  AC11 :: o
    "AC11 == ∃n ∈ nat. 1≤n & AC10(n)"

  AC12 :: o
    "AC12 == ∀A. (∀B ∈ A. ~Finite(B)) -->
                 (∃n ∈ nat. 1≤n & (∃f. ∀B ∈ A. (pairwise_disjoint(f`B)
&
                      sets_of_size_between(f`B, 2, succ(n)) & Union(f`B)=B)))"

  AC13 :: "i => o"
    "AC13(m) == ∀A. 0∉A --> (∃f. ∀B ∈ A. f`B≠0 & f`B ⊆ B & f`B ≲
m)"

  AC14 :: o
    "AC14 == ∃m ∈ nat. 1≤m & AC13(m)"
```

```
  AC15 :: o
    "AC15 == ∀A. 0∉A -->
                  (∃m ∈ nat. 1≤m & (∃f. ∀B ∈ A. f`B≠0 & f`B ⊆ B &
f`B ≲ m))"

  AC16 :: "[i, i] => o"
    "AC16(n, k)  ==
       ∀A. ~Finite(A) -->
           (∃T. T ⊆ {X ∈ Pow(A). X≈succ(n)} &
           (∀X ∈ {X ∈ Pow(A). X≈succ(k)}. ∃! Y. Y ∈ T & X ⊆ Y))"

  AC17 :: o
    "AC17 == ∀A. ∀g ∈ (Pow(A)-{0} -> A) -> Pow(A)-{0}.
                   ∃f ∈ Pow(A)-{0} -> A. f`(g`f) ∈ g`f"
```

locale *AC18* =
  assumes *AC18*: "A≠0 & (∀a ∈ A. B(a) ≠ 0) -->
    ((⋂a ∈ A. ⋃b ∈ B(a). X(a,b)) =
      (⋃f ∈ Π a ∈ A. B(a). ⋂a ∈ A. X(a, f`a)))"
  — AC18 cannot be expressed within the object-logic

constdefs

```
  AC19 :: o
    "AC19 == ∀A. A≠0 & 0∉A --> ((⋂a ∈ A. ⋃b ∈ a. b) =
                   (⋃f ∈ (Π B ∈ A. B). ⋂a ∈ A. f`a))"
```

lemma *rvimage_id*: "rvimage(A,id(A),r) = r Int A*A"
⟨*proof*⟩

lemma *ordertype_Int*:
      "well_ord(A,r) ==> ordertype(A, r Int A*A) = ordertype(A,r)"
⟨*proof*⟩

lemma *lam_sing_bij*: "(λx ∈ A. {x}) ∈ bij(A, {{x}. x ∈ A})"
⟨*proof*⟩

lemma *inj_strengthen_type*:
      "[| f ∈ inj(A, B); !!a. a ∈ A ==> f`a ∈ C |] ==> f ∈ inj(A,C)"
⟨*proof*⟩

lemma *nat_not_Finite*: "~ Finite(nat)"

⟨*proof*⟩

lemmas *le_imp_lepoll = le_imp_subset [THEN subset_imp_lepoll]*

lemma *ex1_two_eq: "[| ∃! x. P(x); P(x); P(y) |] ==> x=y"*
⟨*proof*⟩

lemma *surj_image_eq: "f ∈ surj(A, B) ==> f``A = B"*
⟨*proof*⟩

lemma *first_in_B:*
    "[| well_ord(Union(A),r); 0∉A; B ∈ A |] ==> (THE b. first(b,B,r)) ∈ B"
⟨*proof*⟩

lemma *ex_choice_fun: "[| well_ord(Union(A), R); 0∉A |] ==> ∃f. f:(Π X ∈ A. X)"*
⟨*proof*⟩

lemma *ex_choice_fun_Pow: "well_ord(A, R) ==> ∃f. f:(Π X ∈ Pow(A)-{0}. X)"*
⟨*proof*⟩

lemma *lepoll_m_imp_domain_lepoll_m:*
    "[| m ∈ nat; u ≲ m |] ==> domain(u) ≲ m"
⟨*proof*⟩

lemma *rel_domain_ex1:*

```
"[| succ(m) ≲ domain(r); r ≲ succ(m); m ∈ nat |] ==> function(r)"
```
⟨*proof*⟩

lemma *rel_is_fun:*
```
"[| succ(m) ≲ domain(r);  r ≲ succ(m);  m ∈ nat;
    r ⊆ A*B; A=domain(r) |] ==> r ∈ A->B"
```
⟨*proof*⟩

end

theory *Cardinal_aux* **imports** *AC_Equiv* **begin**

lemma *Diff_lepoll:* `"[| A ≲ succ(m); B ⊆ A; B≠0 |] ==> A-B ≲ m"`
⟨*proof*⟩

lemma *lepoll_imp_ex_le_eqpoll:*
```
"[| A ≲ i; Ord(i) |] ==> ∃j. j le i & A ≈ j"
```
⟨*proof*⟩

lemma *lesspoll_imp_ex_lt_eqpoll:*
```
"[| A ≺ i; Ord(i) |] ==> ∃j. j<i & A ≈ j"
```
⟨*proof*⟩

lemma *Inf_Ord_imp_InfCard_cardinal:* `"[| ~Finite(i); Ord(i) |] ==> InfCard(|i|)"`
⟨*proof*⟩

An alternative and more general proof goes like this: A and B are both well-ordered (because they are injected into an ordinal), either A lepoll B or B lepoll A. Also both are equipollent to their cardinalities, so (if A and B are infinite) then A Un B lepoll —A—+—B— = max(—A—,—B—) lepoll i. In fact, the correctly strengthened version of this theorem appears below.

lemma *Un_lepoll_Inf_Ord_weak:*
```
"[|A ≈ i; B ≈ i; ¬ Finite(i); Ord(i)|] ==> A ∪ B ≲ i"
```
⟨*proof*⟩

lemma *Un_eqpoll_Inf_Ord:*
```
"[| A ≈ i; B ≈ i; ~Finite(i); Ord(i) |] ==> A Un B ≈ i"
```
⟨*proof*⟩

lemma *paired_bij: "?f ∈ bij({{y,z}. y ∈ x}, x)"*
⟨*proof*⟩

lemma *paired_eqpoll: "{{y,z}. y ∈ x} ≈ x"*
⟨*proof*⟩

lemma *ex_eqpoll_disjoint: "∃B. B ≈ A & B Int C = 0"*
⟨*proof*⟩

lemma *Un_lepoll_Inf_Ord:*
    "[| A ≲ i; B ≲ i; ~Finite(i); Ord(i) |] ==> A Un B ≲ i"
⟨*proof*⟩

lemma *Least_in_Ord: "[| P(i); i ∈ j; Ord(j) |] ==> (LEAST i. P(i)) ∈ j"*
⟨*proof*⟩

lemma *Diff_first_lepoll:*
    "[| well_ord(x,r); y ⊆ x; y ≲ succ(n); n ∈ nat |]
    ==> y - {THE b. first(b,y,r)} ≲ n"
⟨*proof*⟩

lemma *UN_subset_split:*
    "(⋃x ∈ X. P(x)) ⊆ (⋃x ∈ X. P(x)-Q(x)) Un (⋃x ∈ X. Q(x))"
⟨*proof*⟩

lemma *UN_sing_lepoll: "Ord(a) ==> (⋃x ∈ a. {P(x)}) ≲ a"*
⟨*proof*⟩

lemma *UN_fun_lepoll_lemma [rule_format]:*
    "[| well_ord(T, R); ~Finite(a); Ord(a); n ∈ nat |]
    ==> ∀f. (∀b ∈ a. f`b ≲ n & f`b ⊆ T) --> (⋃b ∈ a. f`b) ≲ a"
⟨*proof*⟩

lemma *UN_fun_lepoll:*
    "[| ∀b ∈ a. f`b ≲ n & f`b ⊆ T; well_ord(T, R);
        ~Finite(a); Ord(a); n ∈ nat |] ==> (⋃b ∈ a. f`b) ≲ a"
⟨*proof*⟩

lemma *UN_lepoll:*
    "[| ∀b ∈ a. F(b) ≲ n & F(b) ⊆ T; well_ord(T, R);
        ~Finite(a); Ord(a); n ∈ nat |]
    ==> (⋃b ∈ a. F(b)) ≲ a"
⟨*proof*⟩

lemma *UN_eq_UN_Diffs:*
    "Ord(a) ==> (⋃b ∈ a. F(b)) = (⋃b ∈ a. F(b) - (⋃c ∈ b. F(c)))"

⟨proof⟩

lemma *lepoll_imp_eqpoll_subset:*
```
    "a ≲ X ==> ∃Y. Y ⊆ X & a ≈ Y"
```
⟨proof⟩

lemma *Diff_lesspoll_eqpoll_Card_lemma:*
```
    "[| A≈a; ~Finite(a); Card(a); B ≺ a; A-B ≺ a |] ==> P"
```
⟨proof⟩

lemma *Diff_lesspoll_eqpoll_Card:*
```
    "[| A ≈ a; ~Finite(a); Card(a); B ≺ a |] ==> A - B ≈ a"
```
⟨proof⟩

end

theory *WO6_WO1* **imports** *Cardinal_aux* **begin**

constdefs

```
  NN  :: "i => i"
    "NN(y) == {m ∈ nat. ∃a. ∃f. Ord(a) & domain(f)=a  &
                        (⋃b<a. f`b) = y & (∀b<a. f`b ≲ m)}"

  uu  :: "[i, i, i, i] => i"
    "uu(f, beta, gamma, delta) == (f`beta * f`gamma) Int f`delta"


  vv1 :: "[i, i, i] => i"
     "vv1(f,m,b) ==
           let g = LEAST g. (∃d. Ord(d) & (domain(uu(f,b,g,d)) ≠ 0 &

                                  domain(uu(f,b,g,d)) ≲ m));
               d = LEAST d. domain(uu(f,b,g,d)) ≠ 0 &
                            domain(uu(f,b,g,d)) ≲ m
           in  if f`b ≠ 0 then domain(uu(f,b,g,d)) else 0"

  ww1 :: "[i, i, i] => i"
     "ww1(f,m,b) == f`b - vv1(f,m,b)"

  gg1 :: "[i, i, i] => i"
     "gg1(f,a,m) == λb ∈ a++a. if b<a then vv1(f,m,b) else ww1(f,m,b--a)"
```

```
  vv2 :: "[i, i, i, i] => i"
     "vv2(f,b,g,s) ==
           if f‘g ≠ 0 then {uu(f, b, g, LEAST d. uu(f,b,g,d) ≠ 0)‘s}
else 0"

  ww2 :: "[i, i, i, i] => i"
     "ww2(f,b,g,s) == f‘g - vv2(f,b,g,s)"

  gg2 :: "[i, i, i, i] => i"
     "gg2(f,a,b,s) ==
              λg ∈ a++a. if g<a then vv2(f,b,g,s) else ww2(f,b,g--a,s)"
```

lemma *WO2_WO3: "WO2 ==> WO3"*
⟨*proof*⟩

lemma *WO3_WO1: "WO3 ==> WO1"*
⟨*proof*⟩

lemma *WO1_WO2: "WO1 ==> WO2"*
⟨*proof*⟩

lemma *lam_sets: "f ∈ A->B ==> (λx ∈ A. {f‘x}): A -> {{b}. b ∈ B}"*
⟨*proof*⟩

lemma *surj_imp_eq_: "f ∈ surj(A,B) ==> (⋃a ∈ A. {f‘a}) = B"*
⟨*proof*⟩

lemma *surj_imp_eq: "[| f ∈ surj(A,B); Ord(A) |] ==> (⋃a<A. {f‘a}) = B"*
⟨*proof*⟩

lemma *WO1_WO4: "WO1 ==> WO4(1)"*
⟨*proof*⟩

lemma *WO4_mono: "[| m≤n; WO4(m) |] ==> WO4(n)"*
⟨*proof*⟩

lemma *WO4_WO5: "[| m ∈ nat; 1≤m; WO4(m) |] ==> WO5"*
⟨*proof*⟩

lemma *WO5_WO6: "WO5 ==> WO6"*
⟨*proof*⟩

lemma *lt_oadd_odiff_disj:*
```
      "[| k < i++j;  Ord(i);  Ord(j) |]
       ==> k < i  |  (~ k<i & k = i ++ (k--i) & (k--i)<j)"
```
⟨*proof*⟩

lemma *domain_uu_subset: "domain(uu(f,b,g,d)) ⊆ f`b"*
⟨*proof*⟩

lemma *quant_domain_uu_lepoll_m:*
```
     "∀b<a. f`b ≲ m ==> ∀b<a. ∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≲ m"
```
⟨*proof*⟩

lemma *uu_subset1: "uu(f,b,g,d) ⊆ f`b * f`g"*
⟨*proof*⟩

lemma *uu_subset2: "uu(f,b,g,d) ⊆ f`d"*
⟨*proof*⟩

lemma *uu_lepoll_m: "[| ∀b<a. f`b ≲ m; d<a |] ==> uu(f,b,g,d) ≲ m"*
⟨*proof*⟩

lemma *cases:*
```
     "∀b<a. ∀g<a. ∀d<a. u(f,b,g,d) ≲ m
      ==> (∀b<a. f`b ≠ 0 -->
                  (∃g<a. ∃d<a. u(f,b,g,d) ≠ 0 & u(f,b,g,d) ≺ m))
        | (∃b<a. f`b ≠ 0 & (∀g<a. ∀d<a. u(f,b,g,d) ≠ 0 -->
```

```
                                              u(f,b,g,d) ≈ m))"
```
⟨*proof*⟩

lemma *UN_oadd*: `"Ord(a) ==> (⋃b<a++a. C(b)) = (⋃b<a. C(b) Un C(a++b))"`
⟨*proof*⟩

lemma *vv1_subset*: `"vv1(f,m,b) ⊆ f‘b"`
⟨*proof*⟩

lemma *UN_gg1_eq*:
```
  "[| Ord(a);  m ∈ nat |] ==> (⋃b<a++a. gg1(f,a,m)‘b) = (⋃b<a. f‘b)"
```
⟨*proof*⟩

lemma *domain_gg1*: `"domain(gg1(f,a,m)) = a++a"`
⟨*proof*⟩

lemma *nested_LeastI*:
```
     "[| P(a, b);  Ord(a);  Ord(b);
         Least_a = (LEAST a. ∃x. Ord(x) & P(a, x)) |]
      ==> P(Least_a, LEAST b. P(Least_a, b))"
```
⟨*proof*⟩

lemmas *nested_Least_instance* =
```
       nested_LeastI [of "%g d. domain(uu(f,b,g,d)) ≠ 0 &
                                domain(uu(f,b,g,d)) ≲ m",
                      standard]
```

lemma *gg1_lepoll_m*:
```
     "[| Ord(a);  m ∈ nat;
         ∀b<a. f‘b ≠0 -->
             (∃g<a. ∃d<a. domain(uu(f,b,g,d)) ≠ 0  &
                          domain(uu(f,b,g,d)) ≲ m);
         ∀b<a. f‘b ≲ succ(m);  b<a++a |]
      ==> gg1(f,a,m)‘b ≲ m"
```
⟨*proof*⟩

lemma *ex_d_uu_not_empty:*
```
     "[| b<a;  g<a;  f`b≠0;  f`g≠0;
         y*y ⊆ y;  (⋃b<a. f`b)=y |]
      ==> ∃d<a. uu(f,b,g,d) ≠ 0"
```
⟨*proof*⟩

lemma *uu_not_empty:*
```
     "[| b<a; g<a; f`b≠0; f`g≠0;  y*y ⊆ y; (⋃b<a. f`b)=y |]
      ==> uu(f,b,g,LEAST d. (uu(f,b,g,d) ≠ 0)) ≠ 0"
```
⟨*proof*⟩

lemma *not_empty_rel_imp_domain:* `"[| r ⊆ A*B; r≠0 |] ==> domain(r)≠0"`
⟨*proof*⟩

lemma *Least_uu_not_empty_lt_a:*
```
     "[| b<a; g<a; f`b≠0; f`g≠0; y*y ⊆ y; (⋃b<a. f`b)=y |]
      ==> (LEAST d. uu(f,b,g,d) ≠ 0) < a"
```
⟨*proof*⟩

lemma *subset_Diff_sing:* `"[| B ⊆ A; a∉B |] ==> B ⊆ A-{a}"`
⟨*proof*⟩

lemma *supset_lepoll_imp_eq:*
```
     "[| A ≲ m; m ≲ B; B ⊆ A; m ∈ nat |] ==> A=B"
```
⟨*proof*⟩

lemma *uu_Least_is_fun:*
```
     "[| ∀g<a. ∀d<a. domain(uu(f, b, g, d))≠0 -->
           domain(uu(f, b, g, d)) ≈ succ(m);
           ∀b<a. f`b ≲ succ(m);  y*y ⊆ y;
           (⋃b<a. f`b)=y;  b<a;  g<a;  d<a;
           f`b≠0;  f`g≠0;  m ∈ nat;  s ∈ f`b |]
      ==> uu(f, b, g, LEAST d. uu(f,b,g,d)≠0) ∈ f`b -> f`g"
```
⟨*proof*⟩

lemma *vv2_subset:*
```
     "[| ∀g<a. ∀d<a. domain(uu(f, b, g, d))≠0 -->
                       domain(uu(f, b, g, d)) ≈ succ(m);
         ∀b<a. f`b ≲ succ(m); y*y ⊆ y;
```

```
      (⋃b<a. f`b)=y;  b<a;  g<a;  m ∈ nat;  s ∈ f`b |]
   ==> vv2(f,b,g,s) ⊆ f`g"
```
⟨*proof*⟩

lemma *UN_gg2_eq:*
```
    "[| ∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≠ 0 -->
            domain(uu(f,b,g,d)) ≈ succ(m);
       ∀b<a. f`b ≲ succ(m); y*y ⊆ y;
       (⋃b<a. f`b)=y;  Ord(a);  m ∈ nat;  s ∈ f`b;  b<a |]
    ==> (⋃g<a++a. gg2(f,a,b,s) ` g) = y"
```
⟨*proof*⟩

lemma *domain_gg2:* `"domain(gg2(f,a,b,s)) = a++a"`
⟨*proof*⟩

lemma *vv2_lepoll:* `"[| m ∈ nat; m≠0 |] ==> vv2(f,b,g,s) ≲ m"`
⟨*proof*⟩

lemma *ww2_lepoll:*
```
   "[| ∀b<a. f`b ≲ succ(m);  g<a;  m ∈ nat;  vv2(f,b,g,d) ⊆ f`g |]

    ==> ww2(f,b,g,d) ≲ m"
```
⟨*proof*⟩

lemma *gg2_lepoll_m:*
```
    "[| ∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≠ 0 -->
                domain(uu(f,b,g,d)) ≈ succ(m);
       ∀b<a. f`b ≲ succ(m);  y*y ⊆ y;
       (⋃b<a. f`b)=y;  b<a;  s ∈ f`b;  m ∈ nat;  m≠ 0;  g<a++a |]

    ==> gg2(f,a,b,s) ` g ≲ m"
```
⟨*proof*⟩

lemma *lemma_ii:* `"[| succ(m) ∈ NN(y); y*y ⊆ y; m ∈ nat; m≠0 |] ==> m ∈ NN(y)"`
⟨*proof*⟩

lemma *z_n_subset_z_succ_n:*
"$\forall$n $\in$ nat. rec(n, x, %k r. r Un r*r) $\subseteq$ rec(succ(n), x, %k r. r Un r*r)"
$\langle proof \rangle$

lemma *le_subsets:*
"[| $\forall$n $\in$ nat. f(n)<=f(succ(n)); n$\leq$m; n $\in$ nat; m $\in$ nat |]
==> f(n)<=f(m)"
$\langle proof \rangle$

lemma *le_imp_rec_subset:*
"[| n$\leq$m; m $\in$ nat |]
==> rec(n, x, %k r. r Un r*r) $\subseteq$ rec(m, x, %k r. r Un r*r)"
$\langle proof \rangle$

lemma *lemma_iv:* "$\exists$y. x Un y*y $\subseteq$ y"
$\langle proof \rangle$

lemma *WO6_imp_NN_not_empty:* "WO6 ==> NN(y) $\neq$ 0"
$\langle proof \rangle$

lemma *lemma1:*

"[| (⋃b<a. f`b)=y; x ∈ y; ∀b<a. f`b ≲ 1; Ord(a) |] ==> ∃c<a. f`c = {x}"
⟨*proof*⟩

lemma *lemma2:*
"[| (⋃b<a. f`b)=y; x ∈ y; ∀b<a. f`b ≲ 1; Ord(a) |]
==> f` (LEAST i. f`i = {x}) = {x}"
⟨*proof*⟩

lemma *NN_imp_ex_inj: "1 ∈ NN(y) ==> ∃a f. Ord(a) & f ∈ inj(y, a)"*
⟨*proof*⟩

lemma *y_well_ord: "[| y*y ⊆ y; 1 ∈ NN(y) |] ==> ∃r. well_ord(y, r)"*
⟨*proof*⟩

lemma *rev_induct_lemma [rule_format]:*
"[| n ∈ nat; !!m. [| m ∈ nat; m≠0; P(succ(m)) |] ==> P(m) |]
==> n≠0 --> P(n) --> P(1)"
⟨*proof*⟩

lemma *rev_induct:*
"[| n ∈ nat; P(n); n≠0;
!!m. [| m ∈ nat; m≠0; P(succ(m)) |] ==> P(m) |]
==> P(1)"
⟨*proof*⟩

lemma *NN_into_nat: "n ∈ NN(y) ==> n ∈ nat"*
⟨*proof*⟩

lemma *lemma3: "[| n ∈ NN(y); y*y ⊆ y; n≠0 |] ==> 1 ∈ NN(y)"*
⟨*proof*⟩

lemma *NN_y_0: "0 ∈ NN(y) ==> y=0"*
⟨*proof*⟩

lemma *WO6_imp_WO1: "WO6 ==> WO1"*
⟨*proof*⟩

end

theory *WO1_WO7* **imports** *AC_Equiv* **begin**

constdefs

```
  LEMMA :: o
    "LEMMA ==
     ∀X. ~Finite(X) --> (∃R. well_ord(X,R) & ~well_ord(X,converse(R)))"
```

lemma *WO7_iff_LEMMA: "WO7 <-> LEMMA"*
⟨*proof*⟩

lemma *LEMMA_imp_WO1: "LEMMA ==> WO1"*
⟨*proof*⟩

lemma *converse_Memrel_not_wf_on:*

```
    "[| Ord(a); ~Finite(a) |] ==> ~wf[a](converse(Memrel(a)))"
```

⟨*proof*⟩

lemma *converse_Memrel_not_well_ord:*

```
    "[| Ord(a); ~Finite(a) |] ==> ~well_ord(a,converse(Memrel(a)))"
```

⟨*proof*⟩

lemma *well_ord_rvimage_ordertype:*

```
     "well_ord(A,r) ⟹
       rvimage (ordertype(A,r), converse(ordermap(A,r)),r) =
       Memrel(ordertype(A,r))"
```

⟨*proof*⟩

lemma *well_ord_converse_Memrel:*
```
"[| well_ord(A,r); well_ord(A,converse(r)) |]
 ==> well_ord(ordertype(A,r), converse(Memrel(ordertype(A,r))))"
```
⟨*proof*⟩

lemma *WO1_imp_LEMMA: "WO1 ==> LEMMA"*
⟨*proof*⟩

lemma *WO1_iff_WO7: "WO1 <-> WO7"*
⟨*proof*⟩

lemma *WO1_WO8: "WO1 ==> WO8"*
⟨*proof*⟩

lemma *WO8_WO1: "WO8 ==> WO1"*
⟨*proof*⟩

end

theory *AC7_AC9* **imports** *AC_Equiv* **begin**

lemma *Sigma_fun_space_not0: "[| 0∉A; B ∈ A |] ==> (nat->Union(A)) * B ≠ 0"*
⟨*proof*⟩

lemma *inj_lemma:*
```
"C ∈ A ==> (λg ∈ (nat->Union(A))*C.
        (λn ∈ nat. if(n=0, snd(g), fst(g)`(n #- 1))))
        ∈ inj((nat->Union(A))*C, (nat->Union(A)) ) "
```
⟨*proof*⟩

lemma *Sigma_fun_space_eqpoll:*
"[| C ∈ A; 0∉A |] ==> (nat->Union(A)) * C ≈ (nat->Union(A))"
⟨*proof*⟩

lemma *AC6_AC7: "AC6 ==> AC7"*
⟨*proof*⟩

lemma *lemma1_1: "y ∈ (Π B ∈ A. Y*B) ==> (λB ∈ A. snd(y‘B)) ∈ (Π B ∈ A. B)"*
⟨*proof*⟩

lemma *lemma1_2:*
"y ∈ (Π B ∈ {Y*C. C ∈ A}. B) ==> (λB ∈ A. y‘(Y*B)) ∈ (Π B ∈ A. Y*B)"
⟨*proof*⟩

lemma *AC7_AC6_lemma1:*
"(Π B ∈ {(nat->Union(A))*C. C ∈ A}. B) ≠ 0 ==> (Π B ∈ A. B) ≠ 0"
⟨*proof*⟩

lemma *AC7_AC6_lemma2: "0 ∉ A ==> 0 ∉ {(nat -> Union(A)) * C. C ∈ A}"*
⟨*proof*⟩

lemma *AC7_AC6: "AC7 ==> AC6"*
⟨*proof*⟩

lemma *AC1_AC8_lemma1:*
"∀B ∈ A. ∃B1 B2. B=<B1,B2> & B1 ≈ B2
==> 0 ∉ { bij(fst(B),snd(B)). B ∈ A }"
⟨*proof*⟩

lemma *AC1_AC8_lemma2:*

"[| f ∈ (Π X ∈ RepFun(A,p). X); D ∈ A |] ==> (λx ∈ A. f`p(x))`D ∈ p(D)"
⟨proof⟩

lemma *AC1_AC8: "AC1 ==> AC8"*
⟨proof⟩

lemma *AC8_AC9_lemma:*
```
    "∀B1 ∈ A. ∀B2 ∈ A. B1 ≈ B2
     ==> ∀B ∈ A*A. ∃B1 B2. B=<B1,B2> & B1 ≈ B2"
```
⟨proof⟩

lemma *AC8_AC9: "AC8 ==> AC9"*
⟨proof⟩

lemma *snd_lepoll_SigmaI: "b ∈ B ⟹ X ≲ B × X"*
⟨proof⟩

lemma *nat_lepoll_lemma:*
```
    "[|0 ∉ A; B ∈ A|] ==> nat ≲ ((nat → Union(A)) × B) × nat"
```
⟨proof⟩

lemma *AC9_AC1_lemma1:*
```
    "[| 0∉A;  A≠0;
        C = {((nat->Union(A))*B)*nat. B ∈ A}  Un
            {cons(0,((nat->Union(A))*B)*nat). B ∈ A};
        B1 ∈ C;  B2 ∈ C |]
     ==> B1 ≈ B2"
```
⟨proof⟩

lemma *AC9_AC1_lemma2:*
```
    "∀B1 ∈ {(F*B)*N. B ∈ A} Un {cons(0,(F*B)*N). B ∈ A}.
     ∀B2 ∈ {(F*B)*N. B ∈ A} Un {cons(0,(F*B)*N). B ∈ A}.
```

f‘<B1,B2> ∈ bij(B1, B2)
*==> (λB ∈ A. snd(fst((f‘<cons(0,(F*B)*N),(F*B)*N>)‘0))) ∈ (Π X ∈ A. X)"*
⟨*proof*⟩

lemma *AC9_AC1: "AC9 ==> AC1"*
⟨*proof*⟩

end

theory *WO1_AC* **imports** *AC_Equiv* **begin**

theorem *WO1_AC1: "WO1 ==> AC1"*
⟨*proof*⟩

lemma *lemma1: "[| WO1; ∀B ∈ A. ∃C ∈ D(B). P(C,B) |] ==> ∃f. ∀B ∈ A. P(f‘B,B)"*
⟨*proof*⟩

lemma *lemma2_1: "[| ~Finite(B); WO1 |] ==> |B| + |B| ≈ B"*
⟨*proof*⟩

lemma *lemma2_2:*
"f ∈ bij(D+D, B) ==> {{f‘Inl(i), f‘Inr(i)}. i ∈ D} ∈ Pow(Pow(B))"
⟨*proof*⟩

lemma *lemma2_3:*
"f ∈ bij(D+D, B) ==> pairwise_disjoint({{f‘Inl(i), f‘Inr(i)}. i ∈ D})"
⟨*proof*⟩

lemma *lemma2_4:*
"[| f ∈ bij(D+D, B); 1≤n |]
==> sets_of_size_between({{f‘Inl(i), f‘Inr(i)}. i ∈ D}, 2, succ(n))"
⟨*proof*⟩

lemma *lemma2_5:*
"f ∈ bij(D+D, B) ==> Union({{f‘Inl(i), f‘Inr(i)}. i ∈ D})=B"

⟨proof⟩

lemma *lemma2:*
```
"[| WO1; ~Finite(B); 1≤n  |]
 ==> ∃C ∈ Pow(Pow(B)). pairwise_disjoint(C) &
           sets_of_size_between(C, 2, succ(n)) &
           Union(C)=B"
```
⟨proof⟩

theorem *WO1_AC10:* `"[| WO1; 1≤n |] ==> AC10(n)"`
⟨proof⟩

end

theory *Hartog* **imports** *AC_Equiv* **begin**

constdefs
```
  Hartog :: "i => i"
   "Hartog(X) == LEAST i. ~ i ≲ X"
```

lemma *Ords_in_set:* `"∀a. Ord(a) --> a ∈ X ==> P"`
⟨proof⟩

lemma *Ord_lepoll_imp_ex_well_ord:*
```
"[| Ord(a); a ≲ X |]
 ==> ∃Y. Y ⊆ X & (∃R. well_ord(Y,R) & ordertype(Y,R)=a)"
```
⟨proof⟩

lemma *Ord_lepoll_imp_eq_ordertype:*
```
"[| Ord(a); a ≲ X |] ==> ∃Y. Y ⊆ X & (∃R. R ⊆ X*X & ordertype(Y,R)=a)"
```
⟨proof⟩

lemma *Ords_lepoll_set_lemma:*
```
"(∀a. Ord(a) --> a ≲ X) ==>
   ∀a. Ord(a) -->
    a ∈ {b. Z ∈ Pow(X)*Pow(X*X), ∃Y R. Z=<Y,R> & ordertype(Y,R)=b}"
```
⟨proof⟩

lemma *Ords_lepoll_set:* `"∀a. Ord(a) --> a ≲ X ==> P"`
⟨proof⟩

lemma *ex_Ord_not_lepoll:* `"∃a. Ord(a) & ~a ≲ X"`
⟨proof⟩

lemma *not_Hartog_lepoll_self:* `"~ Hartog(A) ≲ A"`
⟨proof⟩

lemmas *Hartog_lepoll_selfE = not_Hartog_lepoll_self [THEN notE, standard]*

lemma *Ord_Hartog: "Ord(Hartog(A))"*
⟨*proof*⟩

lemma *less_HartogE1: "[| i < Hartog(A); ~ i ≲ A |] ==> P"*
⟨*proof*⟩

lemma *less_HartogE: "[| i < Hartog(A); i ≈ Hartog(A) |] ==> P"*
⟨*proof*⟩

lemma *Card_Hartog: "Card(Hartog(A))"*
⟨*proof*⟩

end

theory *HH* **imports** *AC_Equiv Hartog* **begin**

constdefs

```
  HH :: "[i, i, i] => i"
    "HH(f,x,a) == transrec(a, %b r. let z = x - (⋃c ∈ b. r`c)
                                     in  if f`z ∈ Pow(z)-{0} then f`z else
{x})"
```

0.1 Lemmas useful in each of the three proofs

lemma *HH_def_satisfies_eq:*
"HH(f,x,a) = (let z = x - (⋃b ∈ a. HH(f,x,b))
in if f`z ∈ Pow(z)-{0} then f`z else {x})"
⟨*proof*⟩

lemma *HH_values: "HH(f,x,a) ∈ Pow(x)-{0} | HH(f,x,a)={x}"*
⟨*proof*⟩

lemma *subset_imp_Diff_eq:*
"B ⊆ A ==> X-(⋃a ∈ A. P(a)) = X-(⋃a ∈ A-B. P(a))-(⋃b ∈ B. P(b))"
⟨*proof*⟩

lemma *Ord_DiffE: "[| c ∈ a-b; b<a |] ==> c=b | b<c & c<a"*
⟨*proof*⟩

lemma *Diff_UN_eq_self: "(!!y. y∈A ==> P(y) = {x}) ==> x - (⋃y ∈ A. P(y)) = x"*
⟨*proof*⟩

lemma *HH_eq:* "x - (⋃b ∈ a. HH(f,x,b)) = x - (⋃b ∈ a1. HH(f,x,b))
    ==> HH(f,x,a) = HH(f,x,a1)"
⟨*proof*⟩

lemma *HH_is_x_gt_too:* "[| HH(f,x,b)={x}; b<a |] ==> HH(f,x,a)={x}"
⟨*proof*⟩

lemma *HH_subset_x_lt_too:*
    "[| HH(f,x,a) ∈ Pow(x)-{0}; b<a |] ==> HH(f,x,b) ∈ Pow(x)-{0}"
⟨*proof*⟩

lemma *HH_subset_x_imp_subset_Diff_UN:*
    "HH(f,x,a) ∈ Pow(x)-{0} ==> HH(f,x,a) ∈ Pow(x - (⋃b ∈ a. HH(f,x,b)))-{0}"
⟨*proof*⟩

lemma *HH_eq_arg_lt:*
    "[| HH(f,x,v)=HH(f,x,w); HH(f,x,v) ∈ Pow(x)-{0}; v ∈ w |] ==> P"
⟨*proof*⟩

lemma *HH_eq_imp_arg_eq:*
    "[| HH(f,x,v)=HH(f,x,w); HH(f,x,w) ∈ Pow(x)-{0}; Ord(v); Ord(w) |] ==>
v=w"
⟨*proof*⟩

lemma *HH_subset_x_imp_lepoll:*
    "[| HH(f, x, i) ∈ Pow(x)-{0}; Ord(i) |] ==> i lepoll Pow(x)-{0}"
⟨*proof*⟩

lemma *HH_Hartog_is_x:* "HH(f, x, Hartog(Pow(x)-{0})) = {x}"
⟨*proof*⟩

lemma *HH_Least_eq_x:* "HH(f, x, LEAST i. HH(f, x, i) = {x}) = {x}"
⟨*proof*⟩

lemma *less_Least_subset_x:*
    "a ∈ (LEAST i. HH(f,x,i)={x}) ==> HH(f,x,a) ∈ Pow(x)-{0}"
⟨*proof*⟩

0.2 Lemmas used in the proofs of AC1 ==¿ WO2 and AC17 ==¿ AC1

lemma *lam_Least_HH_inj_Pow:*
    "(λa ∈ (LEAST i. HH(f,x,i)={x}). HH(f,x,a))
     ∈ inj(LEAST i. HH(f,x,i)={x}, Pow(x)-{0})"
⟨*proof*⟩

lemma *lam_Least_HH_inj:*
    "∀a ∈ (LEAST i. HH(f,x,i)={x}). ∃z ∈ x. HH(f,x,a) = {z}
     ==> (λa ∈ (LEAST i. HH(f,x,i)={x}). HH(f,x,a))

```
          ∈ inj(LEAST i. HH(f,x,i)={x}, {{y}. y ∈ x})"
```
⟨*proof*⟩

lemma `lam_surj_sing:`
```
      "[| x - (⋃a ∈ A. F(a)) = 0;  ∀a ∈ A. ∃z ∈ x. F(a) = {z} |]

       ==> (λa ∈ A. F(a)) ∈ surj(A, {{y}. y ∈ x})"
```
⟨*proof*⟩

lemma `not_emptyI2: "y ∈ Pow(x)-{0} ==> x ≠ 0"`
⟨*proof*⟩

lemma `f_subset_imp_HH_subset:`
```
    "f`(x - (⋃j ∈ i. HH(f,x,j))) ∈ Pow(x - (⋃j ∈ i. HH(f,x,j)))-{0}

     ==> HH(f, x, i) ∈ Pow(x) - {0}"
```
⟨*proof*⟩

lemma `f_subsets_imp_UN_HH_eq_x:`
```
    "∀z ∈ Pow(x)-{0}. f`z ∈ Pow(z)-{0}
     ==> x - (⋃j ∈ (LEAST i. HH(f,x,i)={x}). HH(f,x,j)) = 0"
```
⟨*proof*⟩

lemma `HH_values2: "HH(f,x,i) = f`(x - (⋃j ∈ i. HH(f,x,j))) | HH(f,x,i)={x}"`
⟨*proof*⟩

lemma `HH_subset_imp_eq:`
```
    "HH(f,x,i): Pow(x)-{0} ==> HH(f,x,i)=f`(x - (⋃j ∈ i. HH(f,x,j)))"
```
⟨*proof*⟩

lemma `f_sing_imp_HH_sing:`
```
    "[| f ∈ (Pow(x)-{0}) -> {{z}. z ∈ x};
        a ∈ (LEAST i. HH(f,x,i)={x}) |] ==> ∃z ∈ x. HH(f,x,a) = {z}"
```
⟨*proof*⟩

lemma `f_sing_lam_bij:`
```
    "[| x - (⋃j ∈ (LEAST i. HH(f,x,i)={x}). HH(f,x,j)) = 0;
        f ∈ (Pow(x)-{0}) -> {{z}. z ∈ x} |]
     ==> (λa ∈ (LEAST i. HH(f,x,i)={x}). HH(f,x,a))
          ∈ bij(LEAST i. HH(f,x,i)={x}, {{y}. y ∈ x})"
```
⟨*proof*⟩

lemma `lam_singI:`
```
    "f ∈ (Π X ∈ Pow(x)-{0}. F(X))
     ==> (λX ∈ Pow(x)-{0}. {f`X}) ∈ (Π X ∈ Pow(x)-{0}. {{z}. z ∈ F(X)})"
```
⟨*proof*⟩

lemmas `bij_Least_HH_x =`

```
comp_bij [OF f_sing_lam_bij [OF _ lam_singI]
          lam_sing_bij [THEN bij_converse_bij], standard]
```

0.3 The proof of AC1 ==¿ WO2

lemma *bijection:*
```
"f ∈ (Π X ∈ Pow(x) - {0}. X)
 ==> ∃g. g ∈ bij(x, LEAST i. HH(λX ∈ Pow(x)-{0}. {f`X}, x, i) =
{x})"
```
⟨*proof*⟩

lemma *AC1_WO2: "AC1 ==> WO2"*
⟨*proof*⟩

end

theory *AC15_WO6* **imports** *HH Cardinal_aux* **begin**

lemma *lepoll_Sigma: "A≠0 ==> B ≲ A*B"*
⟨*proof*⟩

lemma *cons_times_nat_not_Finite:*
```
"0∉A ==> ∀B ∈ {cons(0,x*nat). x ∈ A}. ~Finite(B)"
```
⟨*proof*⟩

lemma *lemma1: "[| Union(C)=A; a ∈ A |] ==> ∃B ∈ C. a ∈ B & B ⊆ A"*
⟨*proof*⟩

lemma *lemma2:*
```
"[| pairwise_disjoint(A); B ∈ A; C ∈ A; a ∈ B; a ∈ C |] ==>
B=C"
```
⟨*proof*⟩

lemma *lemma3:*
```
"∀B ∈ {cons(0, x*nat). x ∈ A}. pairwise_disjoint(f`B) &
        sets_of_size_between(f`B, 2, n) & Union(f`B)=B
==> ∀B ∈ A. ∃! u. u ∈ f`cons(0, B*nat) & u ⊆ cons(0, B*nat) &
```

```
               0 ∈ u & 2 ≲ u & u ≲ n"
```
⟨*proof*⟩

lemma *lemma4:* `"[| A ≲ i; Ord(i) |] ==> {P(a). a ∈ A} ≲ i"`
⟨*proof*⟩

lemma *lemma5_1:*
```
     "[| B ∈ A; 2 ≲ u(B) |] ==> (λx ∈ A. {fst(x). x ∈ u(x)-{0}})`B ≠
0"
```
⟨*proof*⟩

lemma *lemma5_2:*
```
     "[|  B ∈ A; u(B) ⊆ cons(0, B*nat) |]
      ==> (λx ∈ A. {fst(x). x ∈ u(x)-{0}})`B ⊆ B"
```
⟨*proof*⟩

lemma *lemma5_3:*
```
     "[| n ∈ nat; B ∈ A; 0 ∈ u(B); u(B) ≲ succ(n) |]
      ==> (λx ∈ A. {fst(x). x ∈ u(x)-{0}})`B ≲ n"
```
⟨*proof*⟩

lemma *ex_fun_AC13_AC15:*
```
     "[| ∀B ∈ {cons(0, x*nat). x ∈ A}.
                pairwise_disjoint(f`B) &
                sets_of_size_between(f`B, 2, succ(n)) & Union(f`B)=B;

         n ∈ nat |]
      ==> ∃f. ∀B ∈ A. f`B ≠ 0 & f`B ⊆ B & f`B ≲ n"
```
⟨*proof*⟩

theorem *AC10_AC11:* `"[| n ∈ nat; 1≤n; AC10(n) |] ==> AC11"`
⟨*proof*⟩

theorem *AC11_AC12:* `"AC11 ==> AC12"`
⟨*proof*⟩

theorem *AC12_AC15: "AC12 ==> AC15"*
⟨*proof*⟩

lemma *OUN_eq_UN: "Ord(x) ==> (⋃a<x. F(a)) = (⋃a ∈ x. F(a))"*
⟨*proof*⟩

lemma *AC15_WO6_aux1:*
"∀x ∈ Pow(A)-{0}. f`x≠0 & f`x ⊆ x & f`x ≲ m
==> (⋃i<LEAST x. HH(f,A,x)={A}. HH(f,A,i)) = A"
⟨*proof*⟩

lemma *AC15_WO6_aux2:*
"∀x ∈ Pow(A)-{0}. f`x≠0 & f`x ⊆ x & f`x ≲ m
==> ∀x < (LEAST x. HH(f,A,x)={A}). HH(f,A,x) ≲ m"
⟨*proof*⟩

theorem *AC15_WO6: "AC15 ==> WO6"*
⟨*proof*⟩

theorem *AC10_AC13: "[| n ∈ nat; 1≤n; AC10(n) |] ==> AC13(n)"*
⟨*proof*⟩

lemma *AC1_AC13: "AC1 ==> AC13(1)"*
⟨*proof*⟩

lemma *AC13_mono: "[| m≤n; AC13(m) |] ==> AC13(n)"*
⟨*proof*⟩

theorem *AC13_AC14: "[| n ∈ nat; 1≤n; AC13(n) |] ==> AC14"*
⟨*proof*⟩

theorem *AC14_AC15: "AC14 ==> AC15"*
⟨*proof*⟩

lemma *lemma_aux: "[| A≠0; A $\lesssim$ 1 |] ==> ∃a. A={a}"*
⟨*proof*⟩

lemma *AC13_AC1_lemma:*
"∀B ∈ A. f(B)≠0 & f(B)<=B & f(B) $\lesssim$ 1
==> (λx ∈ A. THE y. f(x)={y}) ∈ (Π X ∈ A. X)"
⟨*proof*⟩

theorem *AC13_AC1: "AC13(1) ==> AC1"*
⟨*proof*⟩

theorem *AC11_AC14: "AC11 ==> AC14"*
⟨*proof*⟩

end

theory *AC16_lemmas* **imports** *AC_Equiv Hartog Cardinal_aux* **begin**

lemma *cons_Diff_eq: "a∉A ==> cons(a,A)-{a}=A"*
⟨*proof*⟩

lemma *nat_1_lepoll_iff: "1≲X <-> (∃x. x ∈ X)"*
⟨*proof*⟩

lemma *eqpoll_1_iff_singleton: "X≈1 <-> (∃x. X={x})"*
⟨*proof*⟩

lemma *cons_eqpoll_succ: "[| x≈n; y∉x |] ==> cons(y,x)≈succ(n)"*
⟨*proof*⟩

lemma *subsets_eqpoll_1_eq: "{Y ∈ Pow(X). Y≈1} = {{x}. x ∈ X}"*
⟨*proof*⟩

lemma *eqpoll_RepFun_sing: "X≈{{x}. x ∈ X}"*
⟨*proof*⟩

lemma *subsets_eqpoll_1_eqpoll: "{Y ∈ Pow(X). Y≈1}≈X"*
⟨*proof*⟩

lemma *InfCard_Least_in:*
"[| InfCard(x); y ⊆ x; y ≈ succ(z) |] ==> (LEAST i. i ∈ y) ∈ y"
⟨*proof*⟩

lemma *subsets_lepoll_lemma1:*
"[| InfCard(x); n ∈ nat |]
==> {y ∈ Pow(x). y≈succ(succ(n))} ≲ x{y ∈ Pow(x). y≈succ(n)}"*
⟨*proof*⟩

lemma *set_of_Ord_succ_Union: "(∀y ∈ z. Ord(y)) ==> z ⊆ succ(Union(z))"*
⟨*proof*⟩

lemma *subset_not_mem: "j ⊆ i ==> i ∉ j"*

⟨proof⟩

lemma *succ_Union_not_mem:*
    "(!!y. y $\in$ z ==> Ord(y)) ==> succ(Union(z)) $\notin$ z"
⟨proof⟩

lemma *Union_cons_eq_succ_Union:*
    "Union(cons(succ(Union(z)),z)) = succ(Union(z))"
⟨proof⟩

lemma *Un_Ord_disj:* "[| Ord(i); Ord(j) |] ==> i Un j = i | i Un j = j"
⟨proof⟩

lemma *Union_eq_Un:* "x $\in$ X ==> Union(X) = x Un Union(X-{x})"
⟨proof⟩

lemma *Union_in_lemma [rule_format]:*
    "n $\in$ nat ==> $\forall$z. ($\forall$y $\in$ z. Ord(y)) & z$\approx$n & z$\neq$0 --> Union(z) $\in$
z"
⟨proof⟩

lemma *Union_in:* "[| $\forall$x $\in$ z. Ord(x); z$\approx$n; z$\neq$0; n $\in$ nat |] ==> Union(z)
$\in$ z"
⟨proof⟩

lemma *succ_Union_in_x:*
    "[| InfCard(x); z $\in$ Pow(x); z$\approx$n; n $\in$ nat |] ==> succ(Union(z))
$\in$ x"
⟨proof⟩

lemma *succ_lepoll_succ_succ:*
    "[| InfCard(x); n $\in$ nat |]
    ==> {y $\in$ Pow(x). y$\approx$succ(n)} $\lesssim$ {y $\in$ Pow(x). y$\approx$succ(succ(n))}"
⟨proof⟩

lemma *subsets_eqpoll_X:*
    "[| InfCard(X); n $\in$ nat |] ==> {Y $\in$ Pow(X). Y$\approx$succ(n)} $\approx$ X"
⟨proof⟩

lemma *image_vimage_eq:*
    "[| f $\in$ surj(A,B); y $\subseteq$ B |] ==> f``(converse(f)``y) = y"
⟨proof⟩

lemma *vimage_image_eq:* "[| f $\in$ inj(A,B); y $\subseteq$ A |] ==> converse(f)``(f``y)
= y"
⟨proof⟩

lemma *subsets_eqpoll:*
    "A$\approx$B ==> {Y $\in$ Pow(A). Y$\approx$n}$\approx${Y $\in$ Pow(B). Y$\approx$n}"

⟨proof⟩

lemma *WO2_imp_ex_Card: "WO2 ==> ∃ a. Card(a) & X≈a"*
⟨proof⟩

lemma *lepoll_infinite: "[| X≲Y; ~Finite(X) |] ==> ~Finite(Y)"*
⟨proof⟩

lemma *infinite_Card_is_InfCard: "[| ~Finite(X); Card(X) |] ==> InfCard(X)"*
⟨proof⟩

lemma *WO2_infinite_subsets_eqpoll_X: "[| WO2; n ∈ nat; ~Finite(X) |]*

==> {Y ∈ Pow(X). Y≈succ(n)}≈X"
⟨proof⟩

lemma *well_ord_imp_ex_Card: "well_ord(X,R) ==> ∃ a. Card(a) & X≈a"*
⟨proof⟩

lemma *well_ord_infinite_subsets_eqpoll_X:*
"[| well_ord(X,R); n ∈ nat; ~Finite(X) |] ==> {Y ∈ Pow(X). Y≈succ(n)}≈X"
⟨proof⟩

end

theory *WO2_AC16* **imports** *AC_Equiv AC16_lemmas Cardinal_aux* **begin**

constdefs
```
  recfunAC16 :: "[i,i,i,i] => i"
    "recfunAC16(f,h,i,a) ==
         transrec2(i, 0,
              %g r. if (∃ y ∈ r. h`g ⊆ y) then r
                    else r Un {f`(LEAST i. h`g ⊆ f`i &
                         (∀ b<a. (h`b ⊆ f`i --> (∀ t ∈ r. ~ h`b ⊆ t))))})"
```

lemma *recfunAC16_0: "recfunAC16(f,h,0,a) = 0"*
⟨proof⟩

lemma *recfunAC16_succ:*
```
     "recfunAC16(f,h,succ(i),a) =
      (if (∃ y ∈ recfunAC16(f,h,i,a). h ` i ⊆ y) then recfunAC16(f,h,i,a)
```

```
        else recfunAC16(f,h,i,a) Un
             {f ` (LEAST j. h ` i ⊆ f ` j &
              (∀b<a. (h`b ⊆ f`j
               --> (∀t ∈ recfunAC16(f,h,i,a). ~ h`b ⊆ t))))})"
```
⟨proof⟩

lemma *recfunAC16_Limit: "Limit(i)*
```
        ==> recfunAC16(f,h,i,a) = (⋃j<i. recfunAC16(f,h,j,a))"
```
⟨proof⟩

lemma *transrec2_mono_lemma [rule_format]:*
```
     "[| !!g r. r ⊆ B(g,r);  Ord(i) |]
      ==> j<i --> transrec2(j, 0, B) ⊆ transrec2(i, 0, B)"
```
⟨proof⟩

lemma *transrec2_mono:*
```
     "[| !!g r. r ⊆ B(g,r); j≤i |]
      ==> transrec2(j, 0, B) ⊆ transrec2(i, 0, B)"
```
⟨proof⟩

lemma *recfunAC16_mono:*
```
       "i≤j ==> recfunAC16(f, g, i, a) ⊆ recfunAC16(f, g, j, a)"
```
⟨proof⟩

lemma *lemma3_1:*
```
    "[| ∀y<x. ∀z<a. z<y | (∃Y ∈ F(y). f(z)<=Y) --> (∃! Y. Y ∈ F(y)
& f(z)<=Y);
        ∀i j. i≤j --> F(i) ⊆ F(j); j≤i; i<x; z<a;
        V ∈ F(i); f(z)<=V; W ∈ F(j); f(z)<=W |]
     ==> V = W"
```
⟨proof⟩

lemma *lemma3:*
```
    "[| ∀y<x. ∀z<a. z<y | (∃Y ∈ F(y). f(z)<=Y) --> (∃! Y. Y ∈ F(y)
```

```
& f(z)<=Y);
        ∀i j. i≤j --> F(i) ⊆ F(j); i<x; j<x; z<a;
        V ∈ F(i); f(z)<=V; W ∈ F(j); f(z)<=W |]
     ==> V = W"
```
⟨*proof*⟩

lemma *lemma4:*
```
     "[| ∀y<x. F(y) ⊆ X &
               (∀x<a. x < y | (∃Y ∈ F(y). h(x) ⊆ Y) -->
                (∃! Y. Y ∈ F(y) & h(x) ⊆ Y));
         x < a |]
      ==> ∀y<x. ∀z<a. z < y | (∃Y ∈ F(y). h(z) ⊆ Y) -->
                      (∃! Y. Y ∈ F(y) & h(z) ⊆ Y)"
```
⟨*proof*⟩

lemma *lemma5:*
```
     "[| ∀y<x. F(y) ⊆ X &
              (∀x<a. x < y | (∃Y ∈ F(y). h(x) ⊆ Y) -->
               (∃! Y. Y ∈ F(y) & h(x) ⊆ Y));
         x < a; Limit(x); ∀i j. i≤j --> F(i) ⊆ F(j) |]
      ==> (⋃x<x. F(x)) ⊆ X &
          (∀xa<a. xa < x | (∃x ∈ ⋃x<x. F(x). h(xa) ⊆ x)
                --> (∃! Y. Y ∈ (⋃x<x. F(x)) & h(xa) ⊆ Y))"
```
⟨*proof*⟩

lemma *dbl_Diff_eqpoll_Card:*
```
      "[| A≈a; Card(a); ~Finite(a); B≺a; C≺a |] ==> A - B - C≈a"
```
⟨*proof*⟩

lemma *Finite_lesspoll_infinite_Ord:*
```
      "[| Finite(X); ~Finite(a); Ord(a) |] ==> X≺a"
```
⟨*proof*⟩

lemma *Union_lesspoll:*
```
"[| ∀x ∈ X. x lepoll n & x ⊆ T; well_ord(T, R); X lepoll b;
    b<a; ~Finite(a); Card(a); n ∈ nat |]
 ==> Union(X)≺a"
```
⟨*proof*⟩

lemma *Un_sing_eq_cons:* `"A Un {a} = cons(a, A)"`
⟨*proof*⟩

lemma *Un_lepoll_succ:* `"A lepoll B ==> A Un {a} lepoll succ(B)"`
⟨*proof*⟩

lemma *Diff_UN_succ_empty:* `"Ord(a) ==> F(a) - (⋃b<succ(a). F(b)) = 0"`
⟨*proof*⟩

lemma *Diff_UN_succ_subset:* `"Ord(a) ==> F(a) Un X - (⋃b<succ(a). F(b)) ⊆ X"`
⟨*proof*⟩

lemma *recfunAC16_Diff_lepoll_1:*
```
"Ord(x)
 ==> recfunAC16(f, g, x, a) - (⋃i<x. recfunAC16(f, g, i, a)) lepoll 1"
```
⟨*proof*⟩

lemma *in_Least_Diff:*
```
"[| z ∈ F(x); Ord(x) |]
 ==> z ∈ F(LEAST i. z ∈ F(i)) - (⋃j<(LEAST i. z ∈ F(i)). F(j))"
```
⟨*proof*⟩

lemma *Least_eq_imp_ex:*
```
"[| (LEAST i. w ∈ F(i)) = (LEAST i. z ∈ F(i));
    w ∈ (⋃i<a. F(i)); z ∈ (⋃i<a. F(i)) |]
 ==> ∃b<a. w ∈ (F(b) - (⋃c<b. F(c))) & z ∈ (F(b) - (⋃c<b. F(c)))"
```
⟨*proof*⟩

lemma *two_in_lepoll_1:* `"[| A lepoll 1; a ∈ A; b ∈ A |] ==> a=b"`
⟨*proof*⟩

lemma *UN_lepoll_index:*
```
"[| ∀i<a. F(i)-(⋃j<i. F(j)) lepoll 1; Limit(a) |]
 ==> (⋃x<a. F(x)) lepoll a"
```
⟨*proof*⟩

lemma *recfunAC16_lepoll_index: "Ord(y) ==> recfunAC16(f, h, y, a) lepoll y"*
⟨*proof*⟩

lemma *Union_recfunAC16_lesspoll:*

```
"[| recfunAC16(f,g,y,a) ⊆ {X ∈ Pow(A). X≈n};
    A≈a;  y<a;  ~Finite(a);  Card(a);  n ∈ nat |]
 ==> Union(recfunAC16(f,g,y,a))≺a"
```

⟨*proof*⟩

lemma *dbl_Diff_eqpoll:*

```
"[| recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
    Card(a); ~ Finite(a); A≈a;
    k ∈ nat;  y<a;
    h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)}) |]
 ==> A - Union(recfunAC16(f, h, y, a)) - h`y≈a"
```

⟨*proof*⟩

lemmas *disj_Un_eqpoll_nat_sum =*

```
eqpoll_trans [THEN eqpoll_trans,
              OF disj_Un_eqpoll_sum sum_eqpoll_cong nat_sum_eqpoll_sum,
              standard]
```

lemma *Un_in_Collect:*

```
"[| x ∈ Pow(A - B - h`i); x≈m;
    h ∈ bij(a, {x ∈ Pow(A) . x≈k}); i<a; k ∈ nat; m ∈ nat |]
    ==> h ` i Un x ∈ {x ∈ Pow(A) . x≈k #+ m}"
```

⟨*proof*⟩

lemma *lemma6:*

```
"[| ∀y<succ(j). F(y)<=X & (∀x<a. x<y | P(x,y) --> Q(x,y)); succ(j)<a |]
 ==> F(j)<=X & (∀x<a. x<j | P(x,j) --> Q(x,j))"
```

⟨*proof*⟩

lemma *lemma7:*

```
"[| ∀x<a. x<j | P(x,j) --> Q(x,j);  succ(j)<a |]
 ==> P(j,j) --> (∀x<a. x≤j | P(x,j) --> Q(x,j))"
```

⟨proof⟩

lemma *ex_subset_eqpoll:*
```
"[| A≈a; ~ Finite(a); Ord(a); m ∈ nat |] ==> ∃X ∈ Pow(A). X≈m"
```
⟨proof⟩

lemma *subset_Un_disjoint:* `"[| A ⊆ B Un C; A Int C = 0 |] ==> A ⊆ B"`
⟨proof⟩

lemma *Int_empty:*
```
"[| X ∈ Pow(A - Union(B) -C); T ∈ B; F ⊆ T |] ==> F Int X = 0"
```
⟨proof⟩

lemma *subset_imp_eq_lemma:*
```
"m ∈ nat ==> ∀A B. A ⊆ B & m lepoll A & B lepoll m --> A=B"
```
⟨proof⟩

lemma *subset_imp_eq:* `"[| A ⊆ B; m lepoll A; B lepoll m; m ∈ nat |] ==> A=B"`
⟨proof⟩

lemma *bij_imp_arg_eq:*
```
"[| f ∈ bij(a, {Y ∈ X. Y≈succ(k)}); k ∈ nat; f`b ⊆ f`y; b<a; y<a
|]
 ==> b=y"
```
⟨proof⟩

lemma *ex_next_set:*
```
"[| recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
    Card(a); ~ Finite(a); A≈a;
    k ∈ nat; m ∈ nat; y<a;
    h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)});
    ~ (∃Y ∈ recfunAC16(f, h, y, a). h`y ⊆ Y) |]
 ==> ∃X ∈ {Y ∈ Pow(A). Y≈succ(k #+ m)}. h`y ⊆ X &
```

```
                (∀b<a. h`b ⊆ X -->
                (∀T ∈ recfunAC16(f, h, y, a). ~ h`b ⊆ T))"
```
⟨proof⟩

lemma *ex_next_Ord:*
```
  "[| recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
      Card(a); ~ Finite(a); A≈a;
      k ∈ nat; m ∈ nat; y<a;
      h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)});
      f ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k #+ m)});
      ~ (∃Y ∈ recfunAC16(f, h, y, a). h`y ⊆ Y) |]
   ==> ∃c<a. h`y ⊆ f`c &
             (∀b<a. h`b ⊆ f`c -->
             (∀T ∈ recfunAC16(f, h, y, a). ~ h`b ⊆ T))"
```
⟨proof⟩

lemma *lemma8:*
```
  "[| ∀x<a. x<j | (∃xa ∈ F(j). P(x, xa))
      --> (∃! Y. Y ∈ F(j) & P(x, Y)); F(j) ⊆ X;
      L ∈ X; P(j, L) & (∀x<a. P(x, L) --> (∀xa ∈ F(j). ~P(x, xa)))
|]
   ==> F(j) Un {L} ⊆ X &
       (∀x<a. x≤j | (∃xa ∈ (F(j) Un {L}). P(x, xa)) -->
              (∃! Y. Y ∈ (F(j) Un {L}) & P(x, Y)))"
```
⟨proof⟩

lemma *main_induct:*
```
  "[| b < a; f ∈ bij(a, {Y ∈ Pow(A) . Y≈succ(k #+ m)});
      h ∈ bij(a, {Y ∈ Pow(A) . Y≈succ(k)});
      ~Finite(a); Card(a); A≈a; k ∈ nat; m ∈ nat |]
   ==> recfunAC16(f, h, b, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)} &
       (∀x<a. x < b | (∃Y ∈ recfunAC16(f, h, b, a). h ` x ⊆ Y) -->

       (∃! Y. Y ∈ recfunAC16(f, h, b, a) & h ` x ⊆ Y))"
```

⟨proof⟩

lemma *lemma_simp_induct:*
```
"[| ∀b. b<a --> F(b) ⊆ S & (∀x<a. (x<b | (∃Y ∈ F(b). f`x ⊆ Y))

                            --> (∃! Y. Y ∈ F(b) & f`x ⊆ Y));
    f ∈ a->f``(a); Limit(a);
    ∀i j. i≤j --> F(i) ⊆ F(j) |]
   ==> (⋃j<a. F(j)) ⊆ S &
       (∀x ∈ f``a. ∃! Y. Y ∈ (⋃j<a. F(j)) & x ⊆ Y)"
```
⟨proof⟩

theorem *WO2_AC16:* `"[| WO2; 0<m; k ∈ nat; m ∈ nat |] ==> AC16(k #+ m,k)"`
⟨proof⟩

end

theory *AC16_WO4* **imports** *AC16_lemmas* **begin**

lemma *lemma1:*
```
"[| Finite(A); 0<m; m ∈ nat |]
 ==> ∃a f. Ord(a) & domain(f) = a &
         (⋃b<a. f`b) = A & (∀b<a. f`b ≲ m)"
```
⟨proof⟩

lemmas *well_ord_paired* = `paired_bij [THEN bij_is_inj, THEN well_ord_rvimage]`

lemma *lepoll_trans1:* `"[| A ≲ B; ~ A ≲ C |] ==> ~ B ≲ C"`
⟨proof⟩

lemmas *lepoll_paired = paired_eqpoll [THEN eqpoll_sym, THEN eqpoll_imp_lepoll]*

lemma *lemma2: "$\exists$y R. well_ord(y,R) & x Int y = 0 & ~y $\lesssim$ z & ~Finite(y)"*
⟨*proof*⟩

lemma *infinite_Un: "~Finite(B) ==> ~Finite(A Un B)"*
⟨*proof*⟩

lemma *succ_not_lepoll_lemma:*
"[| ~($\exists$x $\in$ A. f'x=y); f $\in$ inj(A, B); y $\in$ B |]
==> (λa $\in$ succ(A). if(a=A, y, f'a)) $\in$ inj(succ(A), B)"
⟨*proof*⟩

lemma *succ_not_lepoll_imp_eqpoll: "[| ~A $\approx$ B; A $\lesssim$ B |] ==> succ(A) $\lesssim$ B"*
⟨*proof*⟩

lemmas *ordertype_eqpoll =*
ordermap_bij [THEN exI [THEN eqpoll_def [THEN def_imp_iff, THEN iffD2]]]

lemma *cons_cons_subset:*
"[| a $\subseteq$ y; b $\in$ y-a; u $\in$ x |] ==> cons(b, cons(u, a)) $\in$ Pow(x Un y)"
⟨*proof*⟩

lemma *cons_cons_eqpoll:*
```
"[| a ≈ k; a ⊆ y; b ∈ y-a; u ∈ x; x Int y = 0 |]
 ==> cons(b, cons(u, a)) ≈ succ(succ(k))"
```
⟨*proof*⟩

lemma *set_eq_cons:*
```
"[| succ(k) ≈ A; k ≈ B; B ⊆ A; a ∈ A-B; k ∈ nat |] ==> A = cons(a,
B)"
```
⟨*proof*⟩

lemma *cons_eqE:* `"[| cons(x,a) = cons(y,a); x ∉ a |] ==> x = y "`
⟨*proof*⟩

lemma *eq_imp_Int_eq:* `"A = B ==> A Int C = B Int C"`
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_lepoll_lemma [rule_format]:*
```
"[| k ∈ nat; m ∈ nat |]
 ==> ∀A B. A ≈ k #+ m & k ≲ B & B ⊆ A --> A-B ≲ m"
```
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_lepoll:*
```
"[| A ≈ succ(k #+ m); B ⊆ A; succ(k) ≲ B;  k ∈ nat; m ∈ nat |]

 ==> A-B ≲ m"
```
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_eqpoll_lemma [rule_format]:*
```
"[| k ∈ nat; m ∈ nat |]
 ==> ∀A B. A ≈ k #+ m & k ≈ B & B ⊆ A --> A-B ≈ m"
```
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_eqpoll:*
```
"[| A ≈ succ(k #+ m); B ⊆ A; succ(k) ≈ B; k ∈ nat; m ∈ nat |]

 ==> A-B ≈ m"
```
⟨*proof*⟩

```
lemma subsets_lepoll_0_eq_unit: "{x ∈ Pow(X). x ≲ 0} = {0}"
⟨proof⟩
```

```
lemma subsets_lepoll_succ:
     "n ∈ nat ==> {z ∈ Pow(y). z ≲ succ(n)} =
                  {z ∈ Pow(y). z ≲ n} Un {z ∈ Pow(y). z ≈ succ(n)}"
⟨proof⟩
```

```
lemma Int_empty:
     "n ∈ nat ==> {z ∈ Pow(y). z ≲ n} Int {z ∈ Pow(y). z ≈ succ(n)}
= 0"
⟨proof⟩
```

```
locale (open) AC16 =
  fixes x and y and k and l and m and t_n and R and MM and LL and
GG and s
  defines k_def:     "k   == succ(l)"
      and MM_def:     "MM  == {v ∈ t_n. succ(k) ≲ v Int y}"
      and LL_def:     "LL  == {v Int y. v ∈ MM}"
      and GG_def:     "GG  == λv ∈ LL. (THE w. w ∈ MM & v ⊆ w) - v"
      and s_def:      "s(u) == {v ∈ t_n. u ∈ v & k ≲ v Int y}"
  assumes all_ex:     "∀z ∈ {z ∈ Pow(x Un y) . z ≈ succ(k)}.
                        ∃! w. w ∈ t_n & z ⊆ w "
    and disjoint[iff]:  "x Int y = 0"
    and "includes":  "t_n ⊆ {v ∈ Pow(x Un y). v ≈ succ(k #+ m)}"
    and WO_R[iff]:      "well_ord(y,R)"
    and lnat[iff]:      "l ∈ nat"
    and mnat[iff]:      "m ∈ nat"
    and mpos[iff]:      "0<m"
    and Infinite[iff]:  "~ Finite(y)"
    and noLepoll:  "~ y ≲ {v ∈ Pow(x). v ≈ m}"
```

```
lemma (in AC16) knat [iff]: "k ∈ nat"
⟨proof⟩
```

```
lemma (in AC16) Diff_Finite_eqpoll: "[| l ≈ a; a ⊆ y |] ==> y - a ≈
y"
⟨proof⟩
```

```
lemma (in AC16) s_subset: "s(u) ⊆ t_n"
```
⟨proof⟩

```
lemma (in AC16) sI:
      "[| w ∈ t_n; cons(b,cons(u,a)) ⊆ w; a ⊆ y; b ∈ y-a; l ≈ a |]

       ==> w ∈ s(u)"
```
⟨proof⟩

```
lemma (in AC16) in_s_imp_u_in: "v ∈ s(u) ==> u ∈ v"
```
⟨proof⟩

```
lemma (in AC16) ex1_superset_a:
     "[| l ≈ a;  a ⊆ y;  b ∈ y - a;  u ∈ x |]
      ==> ∃! c. c ∈ s(u) & a ⊆ c & b ∈ c"
```
⟨proof⟩

```
lemma (in AC16) the_eq_cons:
     "[| ∀v ∈ s(u). succ(l) ≈ v Int y;
         l ≈ a;  a ⊆ y;  b ∈ y - a;  u ∈ x |]
      ==> (THE c. c ∈ s(u) & a ⊆ c & b ∈ c) Int y = cons(b, a)"
```
⟨proof⟩

```
lemma (in AC16) y_lepoll_subset_s:
     "[| ∀v ∈ s(u). succ(l) ≈ v Int y;
         l ≈ a;  a ⊆ y;  u ∈ x |]
      ==> y ≲ {v ∈ s(u). a ⊆ v}"
```
⟨proof⟩

```
lemma (in AC16) x_imp_not_y [dest]: "a ∈ x ==> a ∉ y"
```
⟨proof⟩

```
lemma (in AC16) w_Int_eq_w_Diff:
     "w ⊆ x Un y ==> w Int (x - {u}) = w - cons(u, w Int y)"
```
⟨proof⟩

```
lemma (in AC16) w_Int_eqpoll_m:
     "[| w ∈ {v ∈ s(u). a ⊆ v};
         l ≈ a;  u ∈ x;
         ∀v ∈ s(u). succ(l) ≈ v Int y |]
      ==> w Int (x - {u}) ≈ m"
```

⟨*proof*⟩

lemma *(***in** *AC16) eqpoll_m_not_empty: "a ≈ m ==> a ≠ 0"*
⟨*proof*⟩

lemma *(***in** *AC16) cons_cons_in:*
"[| z ∈ xa Int (x - {u}); l ≈ a; a ⊆ y; u ∈ x |]
==> ∃! w. w ∈ t_n & cons(z, cons(u, a)) ⊆ w"
⟨*proof*⟩

lemma *(***in** *AC16) subset_s_lepoll_w:*
"[| ∀v ∈ s(u). succ(l) ≈ v Int y; a ⊆ y; l ≈ a; u ∈ x |]
==> {v ∈ s(u). a ⊆ v} ≲ {v ∈ Pow(x). v ≈ m}"
⟨*proof*⟩

lemma *(***in** *AC16) well_ord_subsets_eqpoll_n:*
"n ∈ nat ==> ∃S. well_ord({z ∈ Pow(y) . z ≈ succ(n)}, S)"
⟨*proof*⟩

lemma *(***in** *AC16) well_ord_subsets_lepoll_n:*
"n ∈ nat ==> ∃R. well_ord({z ∈ Pow(y). z ≲ n}, R)"
⟨*proof*⟩

lemma *(***in** *AC16) LL_subset: "LL ⊆ {z ∈ Pow(y). z ≲ succ(k #+ m)}"*
⟨*proof*⟩

lemma *(***in** *AC16) well_ord_LL: "∃S. well_ord(LL,S)"*
⟨*proof*⟩

lemma *(***in** *AC16) unique_superset_in_MM:*
"v ∈ LL ==> ∃! w. w ∈ MM & v ⊆ w"
⟨*proof*⟩

lemma *(in AC16) Int_in_LL:* "$w \in MM$ ==> w Int $y \in LL$"
⟨*proof*⟩

lemma *(in AC16) in_LL_eq_Int:*
"$v \in LL$ ==> v = (THE x. $x \in MM$ & $v \subseteq x$) Int y"
⟨*proof*⟩

lemma *(in AC16) unique_superset1:* "$a \in LL \Longrightarrow$ (THE x. $x \in MM \wedge a \subseteq x) \in MM$"
⟨*proof*⟩

lemma *(in AC16) the_in_MM_subset:*
"$v \in LL$ ==> (THE x. $x \in MM$ & $v \subseteq x) \subseteq x$ Un y"
⟨*proof*⟩

lemma *(in AC16) GG_subset:* "$v \in LL$ ==> GG ‘ $v \subseteq x$"
⟨*proof*⟩

lemma *(in AC16) nat_lepoll_ordertype:* "nat $\lesssim$ ordertype(y, R)"
⟨*proof*⟩

lemma *(in AC16) ex_subset_eqpoll_n:* "$n \in$ nat ==> $\exists z$. $z \subseteq y$ & $n \approx z$"
⟨*proof*⟩

lemma *(in AC16) exists_proper_in_s:* "$u \in x$ ==> $\exists v \in s(u)$. succ(k) $\lesssim$ v Int y"
⟨*proof*⟩

lemma *(in AC16) exists_in_MM:* "$u \in x$ ==> $\exists w \in MM$. $u \in w$"
⟨*proof*⟩

lemma *(in AC16) exists_in_LL:* "$u \in x$ ==> $\exists w \in LL$. $u \in$ GG‘w"
⟨*proof*⟩

lemma *(in AC16) OUN_eq_x:* "well_ord(LL,S) ==>
($\bigcup$b<ordertype(LL,S). GG ‘ (converse(ordermap(LL,S)) ‘ b)) = x"
⟨*proof*⟩

lemma *(***in** *AC16) in_MM_eqpoll_n: "w ∈ MM ==> w ≈ succ(k #+ m)"*
⟨*proof*⟩

lemma *(***in** *AC16) in_LL_eqpoll_n: "w ∈ LL ==> succ(k) ≲ w"*
⟨*proof*⟩

lemma *(***in** *AC16) in_LL: "w ∈ LL ==> w ⊆ (THE x. x ∈ MM ∧ w ⊆ x)"*
⟨*proof*⟩

lemma *(***in** *AC16) all_in_lepoll_m:*
"well_ord(LL,S) ==>
∀ b<ordertype(LL,S). GG ‘ (converse(ordermap(LL,S)) ‘ b) ≲ m"
⟨*proof*⟩

lemma *(***in** *AC16) conclusion:*
"∃ a f. Ord(a) & domain(f) = a & (⋃ b<a. f ‘ b) = x & (∀ b<a. f ‘ b ≲ m)"
⟨*proof*⟩

theorem *AC16_WO4:*
"[| AC16(k #+ m, k); 0 < k; 0 < m; k ∈ nat; m ∈ nat |] ==> WO4(m)"
⟨*proof*⟩

end

theory *AC17_AC1* **imports** *HH* **begin**

lemma *AC0_AC1_lemma: "[| f:(Π X ∈ A. X); D ⊆ A |] ==> ∃g. g:(Π X ∈ D. X)"*
⟨*proof*⟩

lemma *AC0_AC1: "AC0 ==> AC1"*
⟨*proof*⟩

lemma *AC1_AC0: "AC1 ==> AC0"*
⟨*proof*⟩

lemma *AC1_AC17_lemma: "f ∈ (Π X ∈ Pow(A) - {0}. X) ==> f ∈ (Pow(A) - {0} -> A)"*
⟨*proof*⟩

lemma *AC1_AC17: "AC1 ==> AC17"*
⟨*proof*⟩

lemma *UN_eq_imp_well_ord:*
"[| x - (⋃j ∈ LEAST i. HH(λX ∈ Pow(x)-{0}. {f‘X}, x, i) = {x}.

HH(λX ∈ Pow(x)-{0}. {f‘X}, x, j)) = 0;
f ∈ Pow(x)-{0} -> x |]
==> ∃r. well_ord(x,r)"
⟨*proof*⟩

lemma *not_AC1_imp_ex:*
"~AC1 ==> ∃A. ∀f ∈ Pow(A)-{0} -> A. ∃u ∈ Pow(A)-{0}. f‘u ∉ u"
⟨*proof*⟩

lemma *AC17_AC1_aux1:*
"[| ∀f ∈ Pow(x) - {0} -> x. ∃u ∈ Pow(x) - {0}. f‘u∉u;
∃f ∈ Pow(x)-{0}->x.
x - (⋃a ∈ (LEAST i. HH(λX ∈ Pow(x)-{0}. {f‘X},x,i)={x}).

HH(λX ∈ Pow(x)-{0}. {f‘X},x,a)) = 0 |]
==> P"
⟨*proof*⟩

lemma *AC17_AC1_aux2:*
"~ (∃f ∈ Pow(x)-{0}->x. x - F(f) = 0)
==> (λf ∈ Pow(x)-{0}->x . x - F(f))
∈ (Pow(x) -{0} -> x) -> Pow(x) - {0}"
⟨*proof*⟩

lemma *AC17_AC1_aux3:*
"[| f'Z $\in$ Z; Z $\in$ Pow(x)-{0} |]
==> (λX $\in$ Pow(x)-{0}. {f'X})'Z $\in$ Pow(Z)-{0}"
$\langle$*proof*$\rangle$

lemma *AC17_AC1_aux4:*
"$\exists$f $\in$ F. f'((λf $\in$ F. Q(f))'f) $\in$ (λf $\in$ F. Q(f))'f
==> $\exists$f $\in$ F. f'Q(f) $\in$ Q(f)"
$\langle$*proof*$\rangle$

lemma *AC17_AC1: "AC17 ==> AC1"*
$\langle$*proof*$\rangle$

lemma *AC1_AC2_aux1:*
"[| f:(Π X $\in$ A. X); B $\in$ A; 0$\notin$A |] ==> {f'B} $\subseteq$ B Int {f'C. C $\in$ A}"
$\langle$*proof*$\rangle$

lemma *AC1_AC2_aux2:*
"[| pairwise_disjoint(A); B $\in$ A; C $\in$ A; D $\in$ B; D $\in$ C |] ==> f'B = f'C"
$\langle$*proof*$\rangle$

lemma *AC1_AC2: "AC1 ==> AC2"*
$\langle$*proof*$\rangle$

lemma *AC2_AC1_aux1: "0$\notin$A ==> 0 $\notin$ {B*{B}. B $\in$ A}"*
$\langle$*proof*$\rangle$

lemma *AC2_AC1_aux2: "[| X*{X} Int C = {y}; X $\in$ A |]*
==> (THE y. X{X} Int C = {y}): X*A"*
$\langle$*proof*$\rangle$

lemma *AC2_AC1_aux3:*
"$\forall$D $\in$ {E{E}. E $\in$ A}. $\exists$y. D Int C = {y}*
==> (λx $\in$ A. fst(THE z. (x{x} Int C = {z}))) $\in$ (Π X $\in$ A. X)"*
$\langle$*proof*$\rangle$

lemma *AC2_AC1: "AC2 ==> AC1"*
$\langle proof \rangle$

lemma *empty_notin_images: "0 $\notin$ {R``{x}. x $\in$ domain(R)}"*
$\langle proof \rangle$

lemma *AC1_AC4: "AC1 ==> AC4"*
$\langle proof \rangle$

lemma *AC4_AC3_aux1: "f $\in$ A->B ==> ($\bigcup$z $\in$ A. {z}*f`z) $\subseteq$ A*Union(B)"*
$\langle proof \rangle$

lemma *AC4_AC3_aux2: "domain($\bigcup$z $\in$ A. {z}*f(z)) = {a $\in$ A. f(a)$\neq$0}"*
$\langle proof \rangle$

lemma *AC4_AC3_aux3: "x $\in$ A ==> ($\bigcup$z $\in$ A. {z}*f(z))``{x} = f(x)"*
$\langle proof \rangle$

lemma *AC4_AC3: "AC4 ==> AC3"*
$\langle proof \rangle$

lemma *AC3_AC1_lemma:*
"b$\notin$A ==> (Π x $\in$ {a $\in$ A. id(A)`a$\neq$b}. id(A)`x) = (Π x $\in$ A. x)"
$\langle proof \rangle$

lemma *AC3_AC1: "AC3 ==> AC1"*
$\langle proof \rangle$

lemma *AC4_AC5: "AC4 ==> AC5"*
$\langle proof \rangle$

lemma *AC5_AC4_aux1: "R ⊆ A*B ==> (λx ∈ R. fst(x)) ∈ R -> A"*
⟨*proof*⟩

lemma *AC5_AC4_aux2: "R ⊆ A*B ==> range(λx ∈ R. fst(x)) = domain(R)"*
⟨*proof*⟩

lemma *AC5_AC4_aux3: "[| ∃f ∈ A->C. P(f,domain(f)); A=B |] ==> ∃f ∈ B->C. P(f,B)"*
⟨*proof*⟩

lemma *AC5_AC4_aux4: "[| R ⊆ A*B; g ∈ C->R; ∀x ∈ C. (λz ∈ R. fst(z))‘(g‘x) = x |]*
==> (λx ∈ C. snd(g‘x)): (Π x ∈ C. R‘‘{x})"
⟨*proof*⟩

lemma *AC5_AC4: "AC5 ==> AC4"*
⟨*proof*⟩

lemma *AC1_iff_AC6: "AC1 <-> AC6"*
⟨*proof*⟩

end

theory *AC18_AC19* **imports** *AC_Equiv* **begin**

constdefs
uu :: "i => i"
"uu(a) == {c Un {0}. c ∈ a}"

lemma *PROD_subsets:*
"[| f ∈ (Π b ∈ {P(a). a ∈ A}. b); ∀a ∈ A. P(a)<=Q(a) |]

```
      ==> (λa ∈ A. f`P(a)) ∈ (Π a ∈ A. Q(a))"
```
⟨*proof*⟩

lemma *lemma_AC18:*
```
     "[| ∀A. 0 ∉ A --> (∃f. f ∈ (Π X ∈ A. X)); A ≠ 0 |]
      ==> (⋂a ∈ A. ⋃b ∈ B(a). X(a, b)) ⊆
          (⋃f ∈ Π a ∈ A. B(a). ⋂a ∈ A. X(a, f`a))"
```
⟨*proof*⟩

lemma *AC1_AC18: "AC1 ==> PROP AC18"*
⟨*proof*⟩

theorem **(in** *AC18)* *AC19*
⟨*proof*⟩

lemma *RepRep_conj:*
```
        "[| A ≠ 0; 0 ∉ A |] ==> {uu(a). a ∈ A} ≠ 0 & 0 ∉ {uu(a). a
∈ A}"
```
⟨*proof*⟩

lemma *lemma1_1: "[|c ∈ a; x = c Un {0}; x ∉ a |] ==> x - {0} ∈ a"*
⟨*proof*⟩

lemma *lemma1_2:*
```
        "[| f`(uu(a)) ∉ a; f ∈ (Π B ∈ {uu(a). a ∈ A}. B); a ∈ A |]

                ==> f`(uu(a))-{0} ∈ a"
```
⟨*proof*⟩

lemma *lemma1: "∃f. f ∈ (Π B ∈ {uu(a). a ∈ A}. B) ==> ∃f. f ∈ (Π B ∈ A. B)"*
⟨*proof*⟩

lemma *lemma2_1: "a≠0 ==> 0 ∈ (⋃b ∈ uu(a). b)"*
⟨*proof*⟩

lemma *lemma2: "[| A≠0; 0∉A |] ==> (⋂x ∈ {uu(a). a ∈ A}. ⋃b ∈ x. b) ≠ 0"*
⟨*proof*⟩

lemma *AC19_AC1: "AC19 ==> AC1"*

⟨*proof*⟩

end

theory *DC* **imports** *AC_Equiv Hartog Cardinal_aux* **begin**

lemma *RepFun_lepoll: "Ord(a) ==> {P(b). b ∈ a} ≲ a"*
⟨*proof*⟩

Trivial in the presence of AC, but here we need a wellordering of X

lemma *image_Ord_lepoll: "[| f ∈ X->Y; Ord(X) |] ==> f``X ≲ X"*
⟨*proof*⟩

lemma *range_subset_domain:*
*"[| R ⊆ X*X;   !!g. g ∈ X ==> ∃u. <g,u> ∈ R |]*
==> range(R) ⊆ domain(R)"
⟨*proof*⟩

lemma *cons_fun_type: "g ∈ n->X ==> cons(<n,x>, g) ∈ succ(n) -> cons(x, X)"*
⟨*proof*⟩

lemma *cons_fun_type2:*
"[| g ∈ n->X; x ∈ X |] ==> cons(<n,x>, g) ∈ succ(n) -> X"
⟨*proof*⟩

lemma *cons_image_n: "n ∈ nat ==> cons(<n,x>, g)``n = g``n"*
⟨*proof*⟩

lemma *cons_val_n: "g ∈ n->X ==> cons(<n,x>, g)`n = x"*
⟨*proof*⟩

lemma *cons_image_k: "k ∈ n ==> cons(<n,x>, g)``k = g``k"*
⟨*proof*⟩

lemma *cons_val_k: "[| k ∈ n; g ∈ n->X |] ==> cons(<n,x>, g)`k = g`k"*
⟨*proof*⟩

lemma *domain_cons_eq_succ: "domain(f)=x ==> domain(cons(<x,y>, f)) = succ(x)"*
⟨*proof*⟩

lemma *restrict_cons_eq: "g ∈ n->X ==> restrict(cons(<n,x>, g), n) = g"*
⟨*proof*⟩

lemma *succ_in_succ: "[| Ord(k); i ∈ k |] ==> succ(i) ∈ succ(k)"*

⟨proof⟩

```
lemma restrict_eq_imp_val_eq:
      "[|restrict(f, domain(g)) = g; x ∈ domain(g)|]
       ==> f`x = g`x"
```
⟨proof⟩

```
lemma domain_eq_imp_fun_type: "[| domain(f)=A; f ∈ B->C |] ==> f ∈ A->C"
```
⟨proof⟩

```
lemma ex_in_domain: "[| R ⊆ A * B; R ≠ 0 |] ==> ∃x. x ∈ domain(R)"
```
⟨proof⟩

```
constdefs

  DC  :: "i => o"
    "DC(a) == ∀X R. R ⊆ Pow(X)*X  &
                    (∀Y ∈ Pow(X). Y ≺ a --> (∃x ∈ X. <Y,x> ∈ R))
                    --> (∃f ∈ a->X. ∀b<a. <f``b,f`b> ∈ R)"

  DC0 :: o
    "DC0 == ∀A B R. R ⊆ A*B & R≠0 & range(R) ⊆ domain(R)
                    --> (∃f ∈ nat->domain(R). ∀n ∈ nat. <f`n,f`succ(n)>:R)"

  ff  :: "[i, i, i, i] => i"
    "ff(b, X, Q, R) ==
           transrec(b, %c r. THE x. first(x, {x ∈ X. <r``c, x> ∈ R},
Q))"
```

```
locale (open) DC0_imp =
  fixes XX and RR and X and R

  assumes all_ex: "∀Y ∈ Pow(X). Y ≺ nat --> (∃x ∈ X. <Y, x> ∈ R)"

  defines XX_def: "XX == (⋃n ∈ nat. {f ∈ n->X. ∀k ∈ n. <f``k, f`k>
∈ R})"
     and RR_def:  "RR == {<z1,z2>:XX*XX. domain(z2)=succ(domain(z1))
                                  & restrict(z2, domain(z1)) = z1}"
```

lemma (**in** DC0_imp) lemma1_1: "RR ⊆ XX*XX"
⟨*proof*⟩

lemma (**in** DC0_imp) lemma1_2: "RR ≠ 0"
⟨*proof*⟩

lemma (**in** DC0_imp) lemma1_3: "range(RR) ⊆ domain(RR)"
⟨*proof*⟩

lemma (**in** DC0_imp) lemma2:
```
"[| ∀n ∈ nat. <f‘n, f‘succ(n)> ∈ RR;  f ∈ nat -> XX;  n ∈ nat |]

 ==> ∃k ∈ nat. f‘succ(n) ∈ k -> X & n ∈ k
             & <f‘succ(n)‘‘n, f‘succ(n)‘n> ∈ R"
```
⟨*proof*⟩

lemma (**in** DC0_imp) lemma3_1:
```
"[| ∀n ∈ nat. <f‘n, f‘succ(n)> ∈ RR;  f ∈ nat -> XX;  m ∈ nat |]

 ==>  {f‘succ(x)‘x. x ∈ m} = {f‘succ(m)‘x. x ∈ m}"
```
⟨*proof*⟩

lemma (**in** DC0_imp) lemma3:
```
"[| ∀n ∈ nat. <f‘n, f‘succ(n)> ∈ RR;  f ∈ nat -> XX;  m ∈ nat |]

 ==> (λx ∈ nat. f‘succ(x)‘x) ‘‘ m = f‘succ(m)‘‘m"
```
⟨*proof*⟩

theorem DC0_imp_DC_nat: "DC0 ==> DC(nat)"

⟨proof⟩

lemma *singleton_in_funs:*
```
 "x ∈ X ==> {<0,x>} ∈
           (⋃n ∈ nat. {f ∈ succ(n)->X. ∀k ∈ n. <f`k, f`succ(k)> ∈
R})"
```
⟨proof⟩

locale (**open**) *imp_DC0* =
  fixes *XX* **and** *RR* **and** *x* **and** *R* **and** *f* **and** *allRR*
  defines *XX_def:*
```
"XX == (⋃n ∈ nat.
                      {f ∈ succ(n)->domain(R). ∀k ∈ n. <f`k, f`succ(k)>
∈ R})"
```
      and *RR_def:*
```
         "RR == {<z1,z2>:Fin(XX)*XX.
                  (domain(z2)=succ(⋃f ∈ z1. domain(f))
                    & (∀f ∈ z1. restrict(z2, domain(f)) = f))
                  | (~ (∃g ∈ XX. domain(g)=succ(⋃f ∈ z1. domain(f))

                     & (∀f ∈ z1. restrict(g, domain(f)) = f)) & z2={<0,x>})}"
```
      and *allRR_def:*
```
        "allRR == ∀b<nat.
                   <f``b, f`b> ∈
                    {<z1,z2>∈Fin(XX)*XX. (domain(z2)=succ(⋃f ∈ z1. domain(f))
                                   & (⋃f ∈ z1. domain(f)) = b
                                   & (∀f ∈ z1. restrict(z2,domain(f))
= f))}"
```

lemma (**in** *imp_DC0*) *lemma4:*
```
     "[| range(R) ⊆ domain(R);  x ∈ domain(R) |]
      ==> RR ⊆ Pow(XX)*XX &
             (∀Y ∈ Pow(XX). Y ≺ nat --> (∃x ∈ XX. <Y,x>:RR))"
```
⟨proof⟩

lemma (**in** *imp_DC0*) *UN_image_succ_eq:*
```
     "[| f ∈ nat->X; n ∈ nat |]
      ==> (⋃x ∈ f``succ(n). P(x)) =  P(f`n) Un (⋃x ∈ f``n. P(x))"
```
⟨proof⟩

lemma (**in** *imp_DC0*) *UN_image_succ_eq_succ:*
```
     "[| (⋃x ∈ f``n. P(x)) = y; P(f`n) = succ(y);
         f ∈ nat -> X; n ∈ nat |] ==> (⋃x ∈ f``succ(n). P(x)) = succ(y)"
```
⟨proof⟩

lemma (**in** *imp_DC0*) *apply_domain_type:*

```
     "[| h ∈ succ(n) -> D;  n ∈ nat; domain(h)=succ(y) |] ==> h`y ∈ D"
⟨proof⟩
```

```
lemma (in imp_DC0) image_fun_succ:
     "[| h ∈ nat -> X; n ∈ nat |] ==> h``succ(n) = cons(h`n, h``n)"
⟨proof⟩
```

```
lemma (in imp_DC0) f_n_type:
     "[| domain(f`n) = succ(k); f ∈ nat -> XX;  n ∈ nat |]
      ==> f`n ∈ succ(k) -> domain(R)"
⟨proof⟩
```

```
lemma (in imp_DC0) f_n_pairs_in_R [rule_format]:
     "[| h ∈ nat -> XX;  domain(h`n) = succ(k);  n ∈ nat |]
      ==> ∀i ∈ k. <h`n`i, h`n`succ(i)> ∈ R"
⟨proof⟩
```

```
lemma (in imp_DC0) restrict_cons_eq_restrict:
     "[| restrict(h, domain(u))=u;  h ∈ n->X;  domain(u) ⊆ n |]
      ==> restrict(cons(<n, y>, h), domain(u)) = u"
⟨proof⟩
```

```
lemma (in imp_DC0) all_in_image_restrict_eq:
     "[| ∀x ∈ f``n. restrict(f`n, domain(x))=x;
         f ∈ nat -> XX;
         n ∈ nat;  domain(f`n) = succ(n);
         (⋃x ∈ f``n. domain(x)) ⊆ n |]
      ==> ∀x ∈ f``succ(n). restrict(cons(<succ(n),y>, f`n), domain(x))
= x"
⟨proof⟩
```

```
lemma (in imp_DC0) simplify_recursion:
     "[| ∀b<nat. <f``b, f`b> ∈ RR;
         f ∈ nat -> XX; range(R) ⊆ domain(R); x ∈ domain(R)|]
      ==> allRR"
⟨proof⟩
```

```
lemma (in imp_DC0) lemma2:
     "[| allRR; f ∈ nat->XX; range(R) ⊆ domain(R); x ∈ domain(R); n
∈ nat |]
      ==> f`n ∈ succ(n) -> domain(R) & (∀i ∈ n. <f`n`i, f`n`succ(i)>:R)"
⟨proof⟩
```

```
lemma (in imp_DC0) lemma3:
     "[| allRR; f ∈ nat->XX; n∈nat; range(R) ⊆ domain(R);  x ∈ domain(R)
|]
      ==> f`n`n = f`succ(n)`n"
⟨proof⟩
```

```
theorem DC_nat_imp_DC0: "DC(nat) ==> DC0"
⟨proof⟩
```

```
lemma fun_Ord_inj:
      "[| f ∈ a->X;  Ord(a);
          !!b c. [| b<c; c ∈ a |] ==> f`b≠f`c |]
       ==> f ∈ inj(a, X)"
⟨proof⟩
```

```
lemma value_in_image: "[| f ∈ X->Y; A ⊆ X; a ∈ A |] ==> f`a ∈ f``A"
⟨proof⟩
```

```
theorem DC_WO3: "(∀K. Card(K) --> DC(K)) ==> WO3"
⟨proof⟩
```

```
lemma images_eq:
     "[| ∀x ∈ A. f`x=g`x; f ∈ Df->Cf; g ∈ Dg->Cg; A ⊆ Df; A ⊆ Dg |]

      ==> f``A = g``A"
⟨proof⟩
```

```
lemma lam_images_eq:
     "[| Ord(a); b ∈ a |] ==> (λx ∈ a. h(x))``b = (λx ∈ b. h(x))``b"
⟨proof⟩
```

```
lemma lam_type_RepFun: "(λb ∈ a. h(b)) ∈ a -> {h(b). b ∈ a}"
⟨proof⟩
```

```
lemma lemmaX:
     "[| ∀Y ∈ Pow(X). Y ≺ K --> (∃x ∈ X. <Y, x> ∈ R);
         b ∈ K; Z ∈ Pow(X); Z ≺ K |]
      ==> {x ∈ X. <Z,x> ∈ R} ≠ 0"
⟨proof⟩
```

```
lemma WO1_DC_lemma:
     "[| Card(K); well_ord(X,Q);
         ∀Y ∈ Pow(X). Y ≺ K --> (∃x ∈ X. <Y, x> ∈ R); b ∈ K |]
      ==> ff(b, X, Q, R) ∈ {x ∈ X. <(λc ∈ b. ff(c, X, Q, R))``b, x>
```

```
∈ R}"
⟨proof⟩

theorem WO1_DC_Card: "WO1 ==> ∀K. Card(K) --> DC(K)"
⟨proof⟩

end
```

References

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[2] Herman Rubin and Jean E. Rubin. *Equivalents of the Axiom of Choice, II.* North-Holland, 1985.