Hoare Logic for Parallel Programs

Leonor Prensa Nieto

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Abstract

In the following theories a formalization of the Owicki-Gries and the rely-guarantee methods is presented. These methods are widely used for correctness proofs of parallel imperative programs with shared variables. We define syntax, semantics and proof rules in Isabelle/HOL. The proof rules also provide for programs parameterized in the number of parallel components. Their correctness w.r.t. the semantics is proven. Completeness proofs for both methods are extended to the new case of parameterized programs. (These proofs have not been formalized in Isabelle. They can be found in [?].) Using this formalizations we verify several non-trivial examples for parameterized and non-parameterized programs. For the automatic generation of verification conditions with the Owicki-Gries method we define a tactic based on the proof rules. The most involved examples are the verification of two garbage-collection algorithms, the second one parameterized in the number of mutators.

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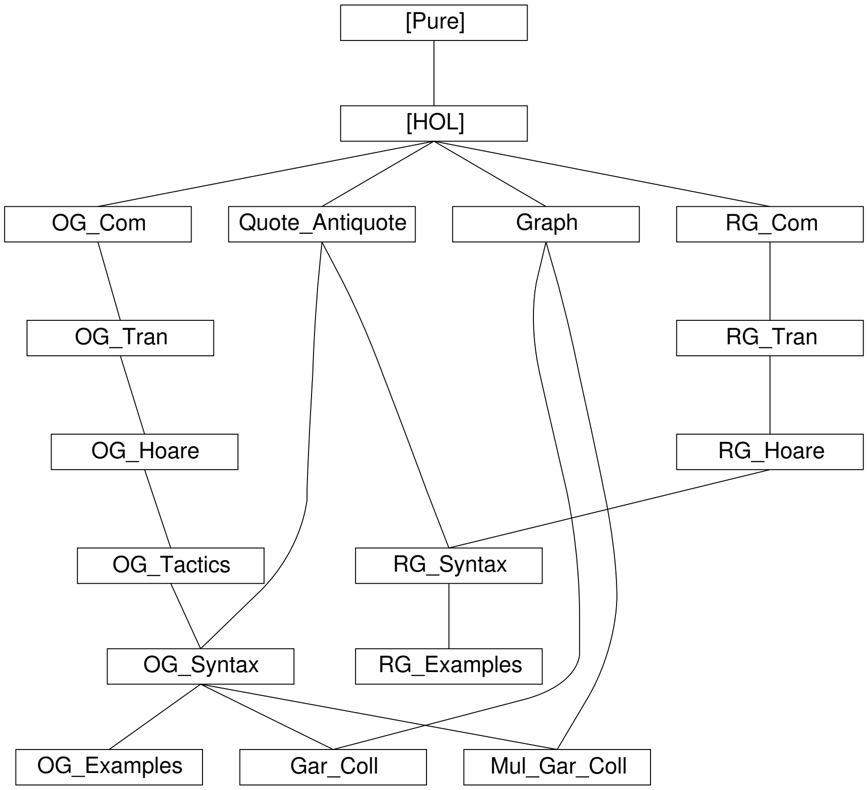
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Chapter 1

The Owicki-Gries Method

1.1 Abstract Syntax

theory *OG-Com* **imports** *Main* **begin**

Type abbreviations for boolean expressions and assertions:

types
$\quad 'a\ bexp = 'a\ set$
$\quad 'a\ assn = 'a\ set$

The syntax of commands is defined by two mutually recursive datatypes: *'a ann-com* for annotated commands and *'a com* for non-annotated commands.

datatype *'a ann-com* =
  AnnBasic (*'a assn*) (*'a* ⇒ *'a*)
  | *AnnSeq* (*'a ann-com*) (*'a ann-com*)
  | *AnnCond1* (*'a assn*) (*'a bexp*) (*'a ann-com*) (*'a ann-com*)
  | *AnnCond2* (*'a assn*) (*'a bexp*) (*'a ann-com*)
  | *AnnWhile* (*'a assn*) (*'a bexp*) (*'a assn*) (*'a ann-com*)
  | *AnnAwait* (*'a assn*) (*'a bexp*) (*'a com*)
and *'a com* =
  Parallel (*'a ann-com option* × *'a assn*) *list*
  | *Basic* (*'a* ⇒ *'a*)
  | *Seq* (*'a com*) (*'a com*)
  | *Cond* (*'a bexp*) (*'a com*) (*'a com*)
  | *While* (*'a bexp*) (*'a assn*) (*'a com*)

The function *pre* extracts the precondition of an annotated command:

consts
  pre ::*'a ann-com* ⇒ *'a assn*
primrec
  pre (*AnnBasic r f*) = *r*
  pre (*AnnSeq c1 c2*) = *pre c1*
  pre (*AnnCond1 r b c1 c2*) = *r*
  pre (*AnnCond2 r b c*) = *r*
  pre (*AnnWhile r b i c*) = *r*

$pre\ (AnnAwait\ r\ b\ c) = r$

Well-formedness predicate for atomic programs:

consts $atom\text{-}com :: 'a\ com \Rightarrow bool$
primrec
$atom\text{-}com\ (Parallel\ Ts) = False$
$atom\text{-}com\ (Basic\ f) = True$
$atom\text{-}com\ (Seq\ c1\ c2) = (atom\text{-}com\ c1 \wedge atom\text{-}com\ c2)$
$atom\text{-}com\ (Cond\ b\ c1\ c2) = (atom\text{-}com\ c1 \wedge atom\text{-}com\ c2)$
$atom\text{-}com\ (While\ b\ i\ c) = atom\text{-}com\ c$

end

1.2 Operational Semantics

theory *OG-Tran* **imports** *OG-Com* **begin**

types
$'a\ ann\text{-}com\text{-}op = ('a\ ann\text{-}com)\ option$
$'a\ ann\text{-}triple\text{-}op = ('a\ ann\text{-}com\text{-}op \times 'a\ assn)$

consts $com :: 'a\ ann\text{-}triple\text{-}op \Rightarrow 'a\ ann\text{-}com\text{-}op$
primrec $com\ (c,\ q) = c$

consts $post :: 'a\ ann\text{-}triple\text{-}op \Rightarrow 'a\ assn$
primrec $post\ (c,\ q) = q$

constdefs
$All\text{-}None :: 'a\ ann\text{-}triple\text{-}op\ list \Rightarrow bool$
$All\text{-}None\ Ts \equiv \forall (c,\ q) \in set\ Ts.\ c = None$

1.2.1 The Transition Relation

consts
$ann\text{-}transition :: (('a\ ann\text{-}com\text{-}op \times 'a) \times ('a\ ann\text{-}com\text{-}op \times 'a))\ set$
$transition :: (('a\ com \times 'a) \times ('a\ com \times 'a))\ set$

syntax
$\text{-}ann\text{-}transition :: ('a\ ann\text{-}com\text{-}op \times 'a) \Rightarrow ('a\ ann\text{-}com\text{-}op \times 'a) \Rightarrow bool$
$(\text{-}\ -1\!\rightarrow\ \text{-}[81,81]\ 100)$
$\text{-}ann\text{-}transition\text{-}n :: ('a\ ann\text{-}com\text{-}op \times 'a) \Rightarrow nat \Rightarrow ('a\ ann\text{-}com\text{-}op \times 'a)$
$\Rightarrow bool\ \ (\text{-}\ -\text{-}\!\rightarrow\ \text{-}[81,81]\ 100)$
$\text{-}ann\text{-}transition\text{-}* :: ('a\ ann\text{-}com\text{-}op \times 'a) \Rightarrow ('a\ ann\text{-}com\text{-}op \times 'a) \Rightarrow bool$
$(\text{-}\ -*\!\rightarrow\ \text{-}[81,81]\ 100)$

$\text{-}transition :: ('a\ com \times 'a) \Rightarrow ('a\ com \times 'a) \Rightarrow bool\ \ (\text{-}\ -P1\!\rightarrow\ \text{-}[81,81]\ 100)$
$\text{-}transition\text{-}n :: ('a\ com \times 'a) \Rightarrow nat \Rightarrow ('a\ com \times 'a) \Rightarrow bool$
$(\text{-}\ -P\text{-}\!\rightarrow\ \text{-}[81,81,81]\ 100)$
$\text{-}transition\text{-}* :: ('a\ com \times 'a) \Rightarrow ('a\ com \times 'a) \Rightarrow bool\ \ (\text{-}\ -P*\!\rightarrow\ \text{-}[81,81]\ 100)$

The corresponding syntax translations are:

translations

$con\text{-}0\ -1\rightarrow\ con\text{-}1 \rightleftharpoons (con\text{-}0,\ con\text{-}1) \in ann\text{-}transition$
$con\text{-}0\ -n\rightarrow\ con\text{-}1 \rightleftharpoons (con\text{-}0,\ con\text{-}1) \in ann\text{-}transition\hat{\ }n$
$con\text{-}0\ -*\rightarrow\ con\text{-}1 \rightleftharpoons (con\text{-}0,\ con\text{-}1) \in ann\text{-}transition^*$

$con\text{-}0\ -P1\rightarrow\ con\text{-}1 \rightleftharpoons (con\text{-}0,\ con\text{-}1) \in transition$
$con\text{-}0\ -Pn\rightarrow\ con\text{-}1 \rightleftharpoons (con\text{-}0,\ con\text{-}1) \in transition\hat{\ }n$
$con\text{-}0\ -P*\rightarrow\ con\text{-}1 \rightleftharpoons (con\text{-}0,\ con\text{-}1) \in transition^*$

inductive *ann-transition transition*
intros

AnnBasic: $(Some\ (AnnBasic\ r\ f),\ s)\ -1\rightarrow\ (None,\ f\ s)$

AnnSeq1: $(Some\ c0,\ s)\ -1\rightarrow\ (None,\ t) \Longrightarrow$
$(Some\ (AnnSeq\ c0\ c1),\ s)\ -1\rightarrow\ (Some\ c1,\ t)$
AnnSeq2: $(Some\ c0,\ s)\ -1\rightarrow\ (Some\ c2,\ t) \Longrightarrow$
$(Some\ (AnnSeq\ c0\ c1),\ s)\ -1\rightarrow\ (Some\ (AnnSeq\ c2\ c1),\ t)$

AnnCond1T: $s \in b \Longrightarrow (Some\ (AnnCond1\ r\ b\ c1\ c2),\ s)\ -1\rightarrow\ (Some\ c1,\ s)$
AnnCond1F: $s \notin b \Longrightarrow (Some\ (AnnCond1\ r\ b\ c1\ c2),\ s)\ -1\rightarrow\ (Some\ c2,\ s)$

AnnCond2T: $s \in b \Longrightarrow (Some\ (AnnCond2\ r\ b\ c),\ s)\ -1\rightarrow\ (Some\ c,\ s)$
AnnCond2F: $s \notin b \Longrightarrow (Some\ (AnnCond2\ r\ b\ c),\ s)\ -1\rightarrow\ (None,\ s)$

AnnWhileF: $s \notin b \Longrightarrow (Some\ (AnnWhile\ r\ b\ i\ c),\ s)\ -1\rightarrow\ (None,\ s)$
AnnWhileT: $s \in b \Longrightarrow (Some\ (AnnWhile\ r\ b\ i\ c),\ s)\ -1\rightarrow$
$(Some\ (AnnSeq\ c\ (AnnWhile\ i\ b\ i\ c)),\ s)$

AnnAwait: $[\![\ s \in b;\ atom\text{-}com\ c;\ (c,\ s)\ -P*\rightarrow\ (Parallel\ [],\ t)\ ]\!] \Longrightarrow$
$(Some\ (AnnAwait\ r\ b\ c),\ s)\ -1\rightarrow\ (None,\ t)$

Parallel: $[\![\ i<length\ Ts;\ Ts!i = (Some\ c,\ q);\ (Some\ c,\ s)\ -1\rightarrow\ (r,\ t)\ ]\!]$
$\Longrightarrow (Parallel\ Ts,\ s)\ -P1\rightarrow\ (Parallel\ (Ts\ [i:=(r,\ q)]),\ t)$

Basic: $(Basic\ f,\ s)\ -P1\rightarrow\ (Parallel\ [],\ f\ s)$

Seq1: $All\text{-}None\ Ts \Longrightarrow (Seq\ (Parallel\ Ts)\ c,\ s)\ -P1\rightarrow\ (c,\ s)$
Seq2: $(c0,\ s)\ -P1\rightarrow\ (c2,\ t) \Longrightarrow (Seq\ c0\ c1,\ s)\ -P1\rightarrow\ (Seq\ c2\ c1,\ t)$

CondT: $s \in b \Longrightarrow (Cond\ b\ c1\ c2,\ s)\ -P1\rightarrow\ (c1,\ s)$
CondF: $s \notin b \Longrightarrow (Cond\ b\ c1\ c2,\ s)\ -P1\rightarrow\ (c2,\ s)$

WhileF: $s \notin b \Longrightarrow (While\ b\ i\ c,\ s)\ -P1\rightarrow\ (Parallel\ [],\ s)$
WhileT: $s \in b \Longrightarrow (While\ b\ i\ c,\ s)\ -P1\rightarrow\ (Seq\ c\ (While\ b\ i\ c),\ s)$

monos *rtrancl-mono*

1.2.2 Definition of Semantics

constdefs
  ann-sem :: $'a$ *ann-com* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *set*
  ann-sem $c \equiv \lambda s.\ \{t.\ (Some\ c,\ s)\ -{*}{\rightarrow}\ (None,\ t)\}$

  ann-SEM :: $'a$ *ann-com* $\Rightarrow$ $'a$ *set* $\Rightarrow$ $'a$ *set*
  ann-SEM $c\ S \equiv \bigcup$ *ann-sem* c ‘ S

  sem :: $'a$ *com* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *set*
  sem $c \equiv \lambda s.\ \{t.\ \exists\ Ts.\ (c,\ s)\ -P{*}{\rightarrow}\ (Parallel\ Ts,\ t) \wedge All\text{-}None\ Ts\}$

  SEM :: $'a$ *com* $\Rightarrow$ $'a$ *set* $\Rightarrow$ $'a$ *set*
  SEM $c\ S \equiv \bigcup$ *sem* c ‘ S

syntax *-Omega* :: $'a$ *com*   (Ω *63*)
translations $\Omega \rightleftharpoons$ *While UNIV UNIV* (*Basic id*)

consts *fwhile* :: $'a$ *bexp* $\Rightarrow$ $'a$ *com* $\Rightarrow$ *nat* $\Rightarrow$ $'a$ *com*
primrec
  fwhile b c 0 = Ω
  fwhile b c (*Suc n*) = *Cond b* (*Seq c* (*fwhile b c n*)) (*Basic id*)

Proofs

declare *ann-transition-transition.intros* [*intro*]
inductive-cases *transition-cases*:
  (*Parallel T*,*s*) $-P1\rightarrow$ *t*
  (*Basic f*, *s*) $-P1\rightarrow$ *t*
  (*Seq c1 c2*, *s*) $-P1\rightarrow$ *t*
  (*Cond b c1 c2*, *s*) $-P1\rightarrow$ *t*
  (*While b i c*, *s*) $-P1\rightarrow$ *t*

lemma *Parallel-empty-lemma* [*rule-format* (*no-asm*)]:
 (*Parallel* [],*s*) $-Pn\rightarrow$ (*Parallel Ts*,*t*) $\longrightarrow$ *Ts*=[] $\wedge$ *n*=*0* $\wedge$ *s*=*t*
apply(*induct n*)
 apply(*simp* (*no-asm*))
apply *clarify*
apply(*drule rel-pow-Suc-D2*)
apply(*force elim*:*transition-cases*)
done

lemma *Parallel-AllNone-lemma* [*rule-format* (*no-asm*)]:
 All-None Ss $\longrightarrow$ (*Parallel Ss*,*s*) $-Pn\rightarrow$ (*Parallel Ts*,*t*) $\longrightarrow$ *Ts*=*Ss* $\wedge$ *n*=*0* $\wedge$ *s*=*t*
apply(*induct n*)
 apply(*simp* (*no-asm*))
apply *clarify*
apply(*drule rel-pow-Suc-D2*)
apply *clarify*
apply(*erule transition-cases*,*simp-all*)

apply(*force dest*:*nth-mem simp add*:*All-None-def*)
done

lemma *Parallel-AllNone*: *All-None Ts* $\Longrightarrow$ (*SEM* (*Parallel Ts*) *X*) = *X*
apply (*unfold SEM-def sem-def*)
apply *auto*
apply(*drule rtrancl-imp-UN-rel-pow*)
apply *clarify*
apply(*drule Parallel-AllNone-lemma*)
apply *auto*
done

lemma *Parallel-empty*: *Ts*=[] $\Longrightarrow$ (*SEM* (*Parallel Ts*) *X*) = *X*
apply(*rule Parallel-AllNone*)
apply(*simp add*:*All-None-def*)
done

Set of lemmas from Apt and Olderog "Verification of sequential and concurrent programs", page 63.

lemma *L3-5i*: $X \subseteq Y \Longrightarrow$ *SEM c X* $\subseteq$ *SEM c Y*
apply (*unfold SEM-def*)
apply *force*
done

lemma *L3-5ii-lemma1*:
$[\![$ (*c1*, *s1*) $-P{*}\rightarrow$ (*Parallel Ts*, *s2*); *All-None Ts*;
(*c2*, *s2*) $-P{*}\rightarrow$ (*Parallel Ss*, *s3*); *All-None Ss* $]\!]$
$\Longrightarrow$ (*Seq c1 c2*, *s1*) $-P{*}\rightarrow$ (*Parallel Ss*, *s3*)
apply(*erule converse-rtrancl-induct2*)
apply(*force intro*:*converse-rtrancl-into-rtrancl*)+
done

lemma *L3-5ii-lemma2* [*rule-format* (*no-asm*)]:
$\forall$ *c1 c2 s t*. (*Seq c1 c2*, *s*) $-Pn\rightarrow$ (*Parallel Ts*, *t*) $\longrightarrow$
(*All-None Ts*) $\longrightarrow$ ($\exists$ *y m Rs*. (*c1*,*s*) $-P{*}\rightarrow$ (*Parallel Rs*, *y*) $\wedge$
(*All-None Rs*) $\wedge$ (*c2*, *y*) $-Pm\rightarrow$ (*Parallel Ts*, *t*) $\wedge$ $m \leq n$)
apply(*induct n*)
apply(*force*)
apply(*safe dest*!: *rel-pow-Suc-D2*)
apply(*erule transition-cases*,*simp-all*)
apply (*fast intro*!: *le-SucI*)
apply (*fast intro*!: *le-SucI elim*!: *rel-pow-imp-rtrancl converse-rtrancl-into-rtrancl*)
done

lemma *L3-5ii-lemma3*:
$[\![$(*Seq c1 c2*,*s*) $-P{*}\rightarrow$ (*Parallel Ts*,*t*); *All-None Ts*$]\!] \Longrightarrow$
($\exists$ *y Rs*. (*c1*,*s*) $-P{*}\rightarrow$ (*Parallel Rs*,*y*) $\wedge$ *All-None Rs*
$\wedge$ (*c2*,*y*) $-P{*}\rightarrow$ (*Parallel Ts*,*t*))
apply(*drule rtrancl-imp-UN-rel-pow*)

apply(*fast dest*: *L3-5ii-lemma2 rel-pow-imp-rtrancl*)
done

lemma *L3-5ii*: *SEM* (*Seq c1 c2*) *X* = *SEM c2* (*SEM c1 X*)
apply (*unfold SEM-def sem-def*)
apply *auto*
 apply(*fast dest*: *L3-5ii-lemma3*)
apply(*fast elim*: *L3-5ii-lemma1*)
done

lemma *L3-5iii*: *SEM* (*Seq* (*Seq c1 c2*) *c3*) *X* = *SEM* (*Seq c1* (*Seq c2 c3*)) *X*
apply (*simp* (*no-asm*) *add*: *L3-5ii*)
done

lemma *L3-5iv*:
 SEM (*Cond b c1 c2*) *X* = (*SEM c1* (*X* ∩ *b*)) *Un* (*SEM c2* (*X* ∩ (−*b*)))
apply (*unfold SEM-def sem-def*)
apply *auto*
apply(*erule converse-rtranclE*)
 prefer *2*
 apply (*erule transition-cases*,*simp-all*)
  apply(*fast intro*: *converse-rtrancl-into-rtrancl elim*: *transition-cases*)+
done

lemma *L3-5v-lemma1*[*rule-format*]:
 (*S*,*s*) −*Pn*→ (*T*,*t*) ⟶ *S*=Ω ⟶ (¬(∃ *Rs*. *T*=(*Parallel Rs*) ∧ *All-None Rs*))
apply (*unfold UNIV-def*)
apply(*rule nat-less-induct*)
apply *safe*
apply(*erule rel-pow-E2*)
 apply *simp-all*
apply(*erule transition-cases*)
 apply *simp-all*
apply(*erule rel-pow-E2*)
 apply(*simp add*: *Id-def*)
apply(*erule transition-cases*,*simp-all*)
apply *clarify*
apply(*erule transition-cases*,*simp-all*)
apply(*erule rel-pow-E2*,*simp*)
apply *clarify*
apply(*erule transition-cases*)
 apply *simp*+
   apply *clarify*
   apply(*erule transition-cases*)
apply *simp-all*
done

lemma *L3-5v-lemma2*: ⟦(Ω, *s*) −*P*∗→ (*Parallel Ts*, *t*); *All-None Ts* ⟧ ⟹ *False*

apply(*fast dest*: *rtrancl-imp-UN-rel-pow L3-5v-lemma1*)
done

lemma *L3-5v-lemma3*: *SEM* (Ω) S = {}
apply (*unfold SEM-def sem-def*)
apply(*fast dest*: *L3-5v-lemma2*)
done

lemma *L3-5v-lemma4* [*rule-format*]:
$\forall s.$ (*While b i c*, *s*) $-Pn\rightarrow$ (*Parallel Ts*, *t*) $\longrightarrow$ *All-None Ts* $\longrightarrow$
($\exists k.$ (*fwhile b c k*, *s*) $-P{*}\rightarrow$ (*Parallel Ts*, *t*))
apply(*rule nat-less-induct*)
apply *safe*
apply(*erule rel-pow-E2*)
apply *safe*
apply(*erule transition-cases*,*simp-all*)
apply (*rule-tac x* = *1* **in** *exI*)
apply(*force dest*: *Parallel-empty-lemma intro*: *converse-rtrancl-into-rtrancl simp add*: *Id-def*)
apply *safe*
apply(*drule L3-5ii-lemma2*)
apply *safe*
apply(*drule le-imp-less-Suc*)
apply (*erule allE* , *erule impE*,*assumption*)
apply (*erule allE* , *erule impE*, *assumption*)
apply *safe*
apply (*rule-tac x* = *k*+*1* **in** *exI*)
apply(*simp* (*no-asm*))
apply(*rule converse-rtrancl-into-rtrancl*)
apply *fast*
apply(*fast elim*: *L3-5ii-lemma1*)
done

lemma *L3-5v-lemma5* [*rule-format*]:
$\forall s.$ (*fwhile b c k*, *s*) $-P{*}\rightarrow$ (*Parallel Ts*, *t*) $\longrightarrow$ *All-None Ts* $\longrightarrow$
(*While b i c*, *s*) $-P{*}\rightarrow$ (*Parallel Ts*,*t*)
apply(*induct k*)
apply(*force dest*: *L3-5v-lemma2*)
apply *safe*
apply(*erule converse-rtranclE*)
apply *simp-all*
apply(*erule transition-cases*,*simp-all*)
apply(*rule converse-rtrancl-into-rtrancl*)
apply(*fast*)
apply(*fast elim!*: *L3-5ii-lemma1 dest*: *L3-5ii-lemma3*)
apply(*drule rtrancl-imp-UN-rel-pow*)
apply *clarify*
apply(*erule rel-pow-E2*)
apply *simp-all*

apply(*erule transition-cases*,*simp-all*)
apply(*fast dest*: *Parallel-empty-lemma*)
done

lemma *L3-5v*: *SEM* (*While b i c*) = (λx. ($\bigcup k$. *SEM* (*fwhile b c k*) *x*))
apply(*rule ext*)
apply (*simp add*: *SEM-def sem-def*)
apply *safe*
 apply(*drule rtrancl-imp-UN-rel-pow*,*simp*)
 apply *clarify*
 apply(*fast dest*:*L3-5v-lemma4*)
apply(*fast intro*: *L3-5v-lemma5*)
done

1.3 Validity of Correctness Formulas

constdefs
  com-validity :: *'a assn* ⇒ *'a com* ⇒ *'a assn* ⇒ *bool* ((*3*‖= -// -//-) [*90*,*55*,*90*] *50*)
  ‖= *p c q* ≡ *SEM c p* ⊆ *q*

  ann-com-validity :: *'a ann-com* ⇒ *'a assn* ⇒ *bool*   (⊨ - - [*60*,*90*] *45*)
  ⊨ *c q* ≡ *ann-SEM c* (*pre c*) ⊆ *q*

end

1.4 The Proof System

theory *OG-Hoare* **imports** *OG-Tran* **begin**

consts *assertions* :: *'a ann-com* ⇒ (*'a assn*) *set*
primrec
  assertions (*AnnBasic r f*) = {*r*}
  assertions (*AnnSeq c1 c2*) = *assertions c1* ∪ *assertions c2*
  assertions (*AnnCond1 r b c1 c2*) = {*r*} ∪ *assertions c1* ∪ *assertions c2*
  assertions (*AnnCond2 r b c*) = {*r*} ∪ *assertions c*
  assertions (*AnnWhile r b i c*) = {*r*, *i*} ∪ *assertions c*
  assertions (*AnnAwait r b c*) = {*r*}

consts *atomics* :: *'a ann-com* ⇒ (*'a assn* × *'a com*) *set*
primrec
  atomics (*AnnBasic r f*) = {(*r*, *Basic f*)}
  atomics (*AnnSeq c1 c2*) = *atomics c1* ∪ *atomics c2*
  atomics (*AnnCond1 r b c1 c2*) = *atomics c1* ∪ *atomics c2*
  atomics (*AnnCond2 r b c*) = *atomics c*
  atomics (*AnnWhile r b i c*) = *atomics c*
  atomics (*AnnAwait r b c*) = {(*r* ∩ *b*, *c*)}

consts *com* :: $'a$ *ann-triple-op* $\Rightarrow$ $'a$ *ann-com-op*
primrec *com* $(c,\ q) = c$

consts *post* :: $'a$ *ann-triple-op* $\Rightarrow$ $'a$ *assn*
primrec *post* $(c,\ q) = q$

constdefs *interfree-aux* :: $('a\ \mathit{ann\text{-}com\text{-}op} \times 'a\ \mathit{assn} \times 'a\ \mathit{ann\text{-}com\text{-}op}) \Rightarrow \mathit{bool}$
$\quad \mathit{interfree\text{-}aux} \equiv \lambda(co,\ q,\ co').\ co' = \mathit{None} \lor$
$\qquad (\forall (r,a) \in \mathit{atomics}\ (\mathit{the}\ co').\ \Vert\!= (q \cap r)\ a\ q \land$
$\qquad (co = \mathit{None} \lor (\forall p \in \mathit{assertions}\ (\mathit{the}\ co).\ \Vert\!= (p \cap r)\ a\ p)))$

constdefs *interfree* :: $(('a\ \mathit{ann\text{-}triple\text{-}op})\ \mathit{list}) \Rightarrow \mathit{bool}$
$\quad \mathit{interfree}\ Ts \equiv \forall\, i\ j.\ i < \mathit{length}\ Ts \land j < \mathit{length}\ Ts \land i \neq j \longrightarrow$
$\qquad \mathit{interfree\text{-}aux}\ (\mathit{com}\ (Ts!i),\ \mathit{post}\ (Ts!i),\ \mathit{com}\ (Ts!j))$

consts *ann-hoare* :: $('a\ \mathit{ann\text{-}com} \times 'a\ \mathit{assn})\ \mathit{set}$
syntax *-ann-hoare* :: $'a\ \mathit{ann\text{-}com} \Rightarrow 'a\ \mathit{assn} \Rightarrow \mathit{bool}$ $((2\vdash\ \text{-//}\ \text{-})\ [60,90]\ 45)$
translations $\vdash c\ q \rightleftharpoons (c,\ q) \in \mathit{ann\text{-}hoare}$

consts *oghoare* :: $('a\ \mathit{assn} \times 'a\ \mathit{com} \times 'a\ \mathit{assn})\ \mathit{set}$
syntax *-oghoare* :: $'a\ \mathit{assn} \Rightarrow 'a\ \mathit{com} \Rightarrow 'a\ \mathit{assn} \Rightarrow \mathit{bool}$ $((3\Vert\!-\ \text{-//-//-})\ [90,55,90]$
$50)$
translations $\Vert\!-\ p\ c\ q \rightleftharpoons (p,\ c,\ q) \in \mathit{oghoare}$

inductive *oghoare ann-hoare*
intros
AnnBasic: $r \subseteq \{s.\ f\ s \in q\} \Longrightarrow\ \vdash (\mathit{AnnBasic}\ r\ f)\ q$

AnnSeq: $[\![\ \vdash c0\ \mathit{pre}\ c1;\ \vdash c1\ q\ ]\!] \Longrightarrow\ \vdash (\mathit{AnnSeq}\ c0\ c1)\ q$

AnnCond1: $[\![\ r \cap b \subseteq \mathit{pre}\ c1;\ \vdash c1\ q;\ r \cap -b \subseteq \mathit{pre}\ c2;\ \vdash c2\ q]\!]$
$\qquad \Longrightarrow\ \vdash (\mathit{AnnCond1}\ r\ b\ c1\ c2)\ q$
AnnCond2: $[\![\ r \cap b \subseteq \mathit{pre}\ c;\ \vdash c\ q;\ r \cap -b \subseteq q\ ]\!] \Longrightarrow\ \vdash (\mathit{AnnCond2}\ r\ b\ c)\ q$

AnnWhile: $[\![\ r \subseteq i;\ i \cap b \subseteq \mathit{pre}\ c;\ \vdash c\ i;\ i \cap -b \subseteq q\ ]\!]$
$\qquad \Longrightarrow\ \vdash (\mathit{AnnWhile}\ r\ b\ i\ c)\ q$

AnnAwait: $[\![\ \mathit{atom\text{-}com}\ c;\ \Vert\!-\ (r \cap b)\ c\ q\ ]\!] \Longrightarrow\ \vdash (\mathit{AnnAwait}\ r\ b\ c)\ q$

AnnConseq: $[\![\vdash c\ q;\ q \subseteq q'\ ]\!] \Longrightarrow\ \vdash c\ q'$

Parallel: $[\![\ \forall\, i{<}\mathit{length}\ Ts.\ \exists\, c\ q.\ Ts!i = (\mathit{Some}\ c,\ q) \land \vdash c\ q;\ \mathit{interfree}\ Ts\ ]\!]$
$\qquad \Longrightarrow \Vert\!-\ (\bigcap i{\in}\{i.\ i{<}\mathit{length}\ Ts\}.\ \mathit{pre}(\mathit{the}(\mathit{com}(Ts!i))))$
$\qquad\qquad \mathit{Parallel}\ Ts$
$\qquad\qquad (\bigcap i{\in}\{i.\ i{<}\mathit{length}\ Ts\}.\ \mathit{post}(Ts!i))$

Basic: $\Vert\!-\ \{s.\ f\ s\ {\in} q\}\ (\mathit{Basic}\ f)\ q$

Seq: $[\![\parallel\!\!- \ p \ c1 \ r; \parallel\!\!- \ r \ c2 \ q \ ]\!] \Longrightarrow \parallel\!\!- \ p \ (Seq \ c1 \ c2) \ q$

Cond: $[\![\parallel\!\!- \ (p \cap b) \ c1 \ q; \parallel\!\!- \ (p \cap -b) \ c2 \ q \ ]\!] \Longrightarrow \parallel\!\!- \ p \ (Cond \ b \ c1 \ c2) \ q$

While: $[\![\parallel\!\!- \ (p \cap b) \ c \ p \ ]\!] \Longrightarrow \parallel\!\!- \ p \ (While \ b \ i \ c) \ (p \cap -b)$

Conseq: $[\![\ p' \subseteq p; \parallel\!\!- \ p \ c \ q \ ; \ q \subseteq q' \ ]\!] \Longrightarrow \parallel\!\!- \ p' \ c \ q'$

1.5 Soundness

lemmas [*cong del*] = *if-weak-cong*

lemmas *ann-hoare-induct* = *oghoare-ann-hoare.induct* [*THEN conjunct2*]
lemmas *oghoare-induct* = *oghoare-ann-hoare.induct* [*THEN conjunct1*]

lemmas *AnnBasic* = *oghoare-ann-hoare.AnnBasic*
lemmas *AnnSeq* = *oghoare-ann-hoare.AnnSeq*
lemmas *AnnCond1* = *oghoare-ann-hoare.AnnCond1*
lemmas *AnnCond2* = *oghoare-ann-hoare.AnnCond2*
lemmas *AnnWhile* = *oghoare-ann-hoare.AnnWhile*
lemmas *AnnAwait* = *oghoare-ann-hoare.AnnAwait*
lemmas *AnnConseq* = *oghoare-ann-hoare.AnnConseq*

lemmas *Parallel* = *oghoare-ann-hoare.Parallel*
lemmas *Basic* = *oghoare-ann-hoare.Basic*
lemmas *Seq* = *oghoare-ann-hoare.Seq*
lemmas *Cond* = *oghoare-ann-hoare.Cond*
lemmas *While* = *oghoare-ann-hoare.While*
lemmas *Conseq* = *oghoare-ann-hoare.Conseq*

1.5.1 Soundness of the System for Atomic Programs

lemma *Basic-ntran* [*rule-format*]:
$(Basic\ f,\ s)\ -Pn\rightarrow\ (Parallel\ Ts,\ t) \longrightarrow All\text{-}None\ Ts \longrightarrow t = f\ s$
apply(*induct n*)
 apply(*simp* (*no-asm*))
apply(*fast dest*: *rel-pow-Suc-D2 Parallel-empty-lemma elim*: *transition-cases*)
done

lemma *SEM-fwhile*: $SEM\ S\ (p \cap b) \subseteq p \Longrightarrow SEM\ (fwhile\ b\ S\ k)\ p \subseteq (p \cap -b)$
apply (*induct k*)
 apply(*simp* (*no-asm*) *add*: *L3-5v-lemma3*)
apply(*simp* (*no-asm*) *add*: *L3-5iv L3-5ii Parallel-empty*)
apply(*rule conjI*)
 apply (*blast dest*: *L3-5i*)
apply(*simp add*: *SEM-def sem-def id-def*)
apply (*blast dest*: *Basic-ntran rtrancl-imp-UN-rel-pow*)
done

lemma *atom-hoare-sound* [*rule-format*]:
$\parallel- \ p\ c\ q \longrightarrow atom\text{-}com(c) \longrightarrow \parallel= p\ c\ q$
apply (*unfold com-validity-def*)
apply(*rule oghoare-induct*)
apply *simp-all*
— Basic
apply(*simp add*: *SEM-def sem-def*)
apply(*fast dest*: *rtrancl-imp-UN-rel-pow Basic-ntran*)
— Seq
apply(*rule impI*)
apply(*rule subset-trans*)
prefer *2* **apply** *simp*
apply(*simp add*: *L3-5ii L3-5i*)
— Cond
apply(*simp add*: *L3-5iv*)
— While
apply (*force simp add*: *L3-5v dest*: *SEM-fwhile*)
— Conseq
apply(*force simp add*: *SEM-def sem-def*)
done

1.5.2 Soundness of the System for Component Programs

inductive-cases *ann-transition-cases*:
$(None,s) -1\rightarrow t$
$(Some\ (AnnBasic\ r\ f),s) -1\rightarrow t$
$(Some\ (AnnSeq\ c1\ c2),\ s) -1\rightarrow t$
$(Some\ (AnnCond1\ r\ b\ c1\ c2),\ s) -1\rightarrow t$
$(Some\ (AnnCond2\ r\ b\ c),\ s) -1\rightarrow t$
$(Some\ (AnnWhile\ r\ b\ I\ c),\ s) -1\rightarrow t$
$(Some\ (AnnAwait\ r\ b\ c),s) -1\rightarrow t$

Strong Soundness for Component Programs:

lemma *ann-hoare-case-analysis* [*rule-format*]: $\vdash C\ q' \longrightarrow$
$((\forall r\ f.\ C = AnnBasic\ r\ f \longrightarrow (\exists q.\ r \subseteq \{s.\ f\ s \in q\} \wedge q \subseteq q')) \wedge$
$(\forall c0\ c1.\ C = AnnSeq\ c0\ c1 \longrightarrow (\exists q.\ q \subseteq q' \wedge \vdash c0\ pre\ c1 \wedge \vdash c1\ q)) \wedge$
$(\forall r\ b\ c1\ c2.\ C = AnnCond1\ r\ b\ c1\ c2 \longrightarrow (\exists q.\ q \subseteq q' \wedge$
$r \cap b \subseteq pre\ c1 \wedge \vdash c1\ q \wedge r \cap -b \subseteq pre\ c2 \wedge \vdash c2\ q)) \wedge$
$(\forall r\ b\ c.\ C = AnnCond2\ r\ b\ c \longrightarrow$
$(\exists q.\ q \subseteq q' \wedge r \cap b \subseteq pre\ c \ \wedge \vdash c\ q \wedge r \cap -b \subseteq q)) \wedge$
$(\forall r\ i\ b\ c.\ C = AnnWhile\ r\ b\ i\ c \longrightarrow$
$(\exists q.\ q \subseteq q' \wedge r \subseteq i \wedge i \cap b \subseteq pre\ c \wedge \vdash c\ i \wedge i \cap -b \subseteq q)) \wedge$
$(\forall r\ b\ c.\ C = AnnAwait\ r\ b\ c \longrightarrow (\exists q.\ q \subseteq q' \wedge \parallel- (r \cap b)\ c\ q)))$
apply(*rule ann-hoare-induct*)
apply *simp-all*
apply(*rule-tac x=q* **in** *exI*,*simp*)+
apply(*rule conjI*,*clarify*,*simp*,*clarify*,*rule-tac x=qa* **in** *exI*,*fast*)+
apply(*clarify*,*simp*,*clarify*,*rule-tac x=qa* **in** *exI*,*fast*)
done

lemma *Help*: (*transition* ∩ {(*v*,*v*,*u*). *True*}) = (*transition*)
apply *force*
done

lemma *Strong-Soundness-aux-aux* [*rule-format*]:
(*co*, *s*) −*1*→ (*co′*, *t*) ⟶ (∀ *c*. *co* = *Some c* ⟶ *s*∈ *pre c* ⟶
(∀ *q*. ⊢ *c q* ⟶ (*if co′* = *None then t*∈*q else t* ∈ *pre*(*the co′*) ∧ ⊢ (*the co′*) *q*)))
apply(*rule ann-transition-transition.induct* [*THEN conjunct1*])
apply *simp-all*
— Basic
apply *clarify*
apply(*frule ann-hoare-case-analysis*)
apply *force*
— Seq
apply *clarify*
apply(*frule ann-hoare-case-analysis*,*simp*)
apply(*fast intro*: *AnnConseq*)
apply *clarify*
apply(*frule ann-hoare-case-analysis*,*simp*)
apply *clarify*
apply(*rule conjI*)
apply *force*
apply(*rule AnnSeq*,*simp*)
apply(*fast intro*: *AnnConseq*)
— Cond1
apply *clarify*
apply(*frule ann-hoare-case-analysis*,*simp*)
apply(*fast intro*: *AnnConseq*)
apply *clarify*
apply(*frule ann-hoare-case-analysis*,*simp*)
apply(*fast intro*: *AnnConseq*)
— Cond2
apply *clarify*
apply(*frule ann-hoare-case-analysis*,*simp*)
apply(*fast intro*: *AnnConseq*)
apply *clarify*
apply(*frule ann-hoare-case-analysis*,*simp*)
apply(*fast intro*: *AnnConseq*)
— While
apply *clarify*
apply(*frule ann-hoare-case-analysis*,*simp*)
apply *force*
apply *clarify*
apply(*frule ann-hoare-case-analysis*,*simp*)
apply *auto*
apply(*rule AnnSeq*)
apply *simp*
apply(*rule AnnWhile*)

apply *simp-all*
— Await
apply(*frule ann-hoare-case-analysis*,*simp*)
apply *clarify*
apply(*drule atom-hoare-sound*)
apply *simp*
apply(*simp add*: *com-validity-def SEM-def sem-def*)
apply(*simp add*: *Help All-None-def*)
apply *force*
done

lemma *Strong-Soundness-aux*: $[\![$ (*Some c*, *s*) $-*\rightarrow$ (*co*, *t*); $s \in$ *pre c*; $\vdash c\ q$ $]\!]$
$\Longrightarrow$ *if co* = *None then* $t \in q$ *else* $t \in$ *pre* (*the co*) $\wedge \vdash$ (*the co*) *q*
apply(*erule rtrancl-induct2*)
apply *simp*
apply(*case-tac a*)
apply(*fast elim*: *ann-transition-cases*)
apply(*erule Strong-Soundness-aux-aux*)
apply *simp*
apply *simp-all*
done

lemma *Strong-Soundness*: $[\![$ (*Some c*, *s*)$-*\rightarrow$(*co*, *t*); $s \in$ *pre c*; $\vdash c\ q$ $]\!]$
$\Longrightarrow$ *if co* = *None then* $t{\in}q$ *else* $t \in$ *pre* (*the co*)
apply(*force dest*:*Strong-Soundness-aux*)
done

lemma *ann-hoare-sound*: $\vdash c\ q \Longrightarrow\ \models c\ q$
apply (*unfold ann-com-validity-def ann-SEM-def ann-sem-def*)
apply *clarify*
apply(*drule Strong-Soundness*)
apply *simp-all*
done

1.5.3 Soundness of the System for Parallel Programs

lemma *Parallel-length-post-P1*: (*Parallel Ts*,*s*) $-P1\rightarrow$ (R', *t*) $\Longrightarrow$
($\exists$ *Rs*. R' = (*Parallel Rs*) $\wedge$ (*length Rs*) = (*length Ts*) $\wedge$
($\forall$ *i*. *i*<*length Ts* $\longrightarrow$ *post*(*Rs* ! *i*) = *post*(*Ts* ! *i*)))
apply(*erule transition-cases*)
apply *simp*
apply *clarify*
apply(*case-tac i*=*ia*)
apply *simp*+
done

lemma *Parallel-length-post-PStar*: (*Parallel Ts*,*s*) $-P*\rightarrow$ (R',*t*) $\Longrightarrow$
($\exists$ *Rs*. R' = (*Parallel Rs*) $\wedge$ (*length Rs*) = (*length Ts*) $\wedge$
($\forall$ *i*. *i*<*length Ts* $\longrightarrow$ *post*(*Ts* ! *i*) = *post*(*Rs* ! *i*)))

```
apply(erule rtrancl-induct2)
 apply(simp-all)
apply clarify
apply simp
apply(drule Parallel-length-post-P1)
apply auto
done

lemma assertions-lemma: pre c ∈ assertions c
apply(rule ann-com-com.induct [THEN conjunct1])
apply auto
done

lemma interfree-aux1 [rule-format]:
  (c,s) −1→ (r,t)  ⟶ (interfree-aux(c1, q1, c) ⟶ interfree-aux(c1, q1, r))
apply (rule ann-transition-transition.induct [THEN conjunct1])
apply(safe)
prefer 13
apply (rule TrueI)
apply (simp-all add:interfree-aux-def)
apply force+
done

lemma interfree-aux2 [rule-format]:
  (c,s) −1→ (r,t) ⟶ (interfree-aux(c, q, a)  ⟶ interfree-aux(r, q, a) )
apply (rule ann-transition-transition.induct [THEN conjunct1])
apply(force simp add:interfree-aux-def)+
done

lemma interfree-lemma: ⟦ (Some c, s) −1→ (r, t);interfree Ts ; i<length Ts;
           Ts!i = (Some c, q) ⟧ ⟹ interfree (Ts[i:= (r, q)])
apply(simp add: interfree-def)
apply clarify
apply(case-tac i=j)
 apply(drule-tac t = ia in not-sym)
 apply simp-all
apply(force elim: interfree-aux1)
apply(force elim: interfree-aux2 simp add:nth-list-update)
done
```

Strong Soundness Theorem for Parallel Programs:

```
lemma Parallel-Strong-Soundness-Seq-aux:
  ⟦interfree Ts; i<length Ts; com(Ts ! i) = Some(AnnSeq c0 c1) ⟧
  ⟹  interfree (Ts[i:=(Some c0, pre c1)])
apply(simp add: interfree-def)
apply clarify
apply(case-tac i=j)
 apply(force simp add: nth-list-update interfree-aux-def)
apply(case-tac i=ia)
```

```
 apply(erule-tac x=ia in allE)
 apply(force simp add:interfree-aux-def assertions-lemma)
apply simp
done

lemma Parallel-Strong-Soundness-Seq [rule-format (no-asm)]:
 ⟦ ∀ i<length Ts. (if com(Ts!i) = None then b ∈ post(Ts!i)
  else b ∈ pre(the(com(Ts!i))) ∧ ⊢ the(com(Ts!i)) post(Ts!i));
  com(Ts ! i) = Some(AnnSeq c0 c1); i<length Ts; interfree Ts ⟧ ⟹
 (∀ ia<length Ts. (if com(Ts[i:=(Some c0, pre c1)]! ia) = None
  then b ∈ post(Ts[i:=(Some c0, pre c1)]! ia)
 else b ∈ pre(the(com(Ts[i:=(Some c0, pre c1)]! ia))) ∧
 ⊢ the(com(Ts[i:=(Some c0, pre c1)]! ia)) post(Ts[i:=(Some c0, pre c1)]! ia)))
  ∧ interfree (Ts[i:= (Some c0, pre c1)])
apply(rule conjI)
 apply safe
 apply(case-tac i=ia)
  apply simp
  apply(force dest: ann-hoare-case-analysis)
 apply simp
apply(fast elim: Parallel-Strong-Soundness-Seq-aux)
done

lemma Parallel-Strong-Soundness-aux-aux [rule-format]:
 (Some c, b) −1→ (co, t) ⟶
  (∀ Ts. i<length Ts ⟶ com(Ts ! i) = Some c ⟶
  (∀ i<length Ts. (if com(Ts ! i) = None then b∈post(Ts!i)
  else b∈pre(the(com(Ts!i))) ∧ ⊢ the(com(Ts!i)) post(Ts!i))) ⟶
 interfree Ts ⟶
  (∀ j. j<length Ts ∧ i≠j ⟶ (if com(Ts!j) = None then t∈post(Ts!j)
  else t∈pre(the(com(Ts!j))) ∧ ⊢ the(com(Ts!j)) post(Ts!j))) )
apply(rule ann-transition-transition.induct [THEN conjunct1])
apply safe
prefer 11
apply(rule TrueI)
apply simp-all
— Basic
   apply(erule-tac x = i in all-dupE, erule (1) notE impE)
   apply(erule-tac x = j in allE , erule (1) notE impE)
   apply(simp add: interfree-def)
   apply(erule-tac x = j in allE,simp)
   apply(erule-tac x = i in allE,simp)
   apply(drule-tac t = i in not-sym)
   apply(case-tac com(Ts ! j)=None)
    apply(force intro: converse-rtrancl-into-rtrancl
         simp add: interfree-aux-def com-validity-def SEM-def sem-def All-None-def)
   apply(simp add:interfree-aux-def)
   apply clarify
   apply simp
```

apply(*erule-tac x=pre y* **in** *ballE*)
apply(*force intro*: *converse-rtrancl-into-rtrancl*
simp add: *com-validity-def SEM-def sem-def All-None-def*)
apply(*simp add*:*assertions-lemma*)
— Seqs
apply(*erule-tac x* = *Ts*[*i*:=(*Some c0*, *pre c1*)] **in** *allE*)
apply(*drule Parallel-Strong-Soundness-Seq*,*simp*+)
apply(*erule-tac x* = *Ts*[*i*:=(*Some c0*, *pre c1*)] **in** *allE*)
apply(*drule Parallel-Strong-Soundness-Seq*,*simp*+)
— Await
apply(*rule-tac x* = *i* **in** *allE* , *assumption* , *erule* (*1*) *notE impE*)
apply(*erule-tac x* = *j* **in** *allE* , *erule* (*1*) *notE impE*)
apply(*simp add*: *interfree-def*)
apply(*erule-tac x* = *j* **in** *allE*,*simp*)
apply(*erule-tac x* = *i* **in** *allE*,*simp*)
apply(*drule-tac t* = *i* **in** *not-sym*)
apply(*case-tac com*(*Ts* ! *j*)=*None*)
apply(*force intro*: *converse-rtrancl-into-rtrancl simp add*: *interfree-aux-def*
com-validity-def SEM-def sem-def All-None-def Help)
apply(*simp add*:*interfree-aux-def*)
apply *clarify*
apply *simp*
apply(*erule-tac x=pre y* **in** *ballE*)
apply(*force intro*: *converse-rtrancl-into-rtrancl*
simp add: *com-validity-def SEM-def sem-def All-None-def Help*)
apply(*simp add*:*assertions-lemma*)
done

lemma *Parallel-Strong-Soundness-aux* [*rule-format*]:
$[\![(Ts',s)\ {-P{*}\rightarrow}\ (Rs',t);\ \ Ts' = (Parallel\ Ts);\ interfree\ Ts;$
$\forall i.\ i{<}length\ Ts \longrightarrow (\exists c\ q.\ (Ts\ !\ i) = (Some\ c,\ q) \land s{\in}(pre\ c) \land \vdash c\ q\ )\ ]\!] \Longrightarrow$
$\forall Rs.\ Rs' = (Parallel\ Rs) \longrightarrow (\forall j.\ j{<}length\ Rs \longrightarrow$
$(if\ com(Rs\ !\ j) = None\ then\ t{\in}post(Ts\ !\ j)$
$else\ t{\in}pre(the(com(Rs\ !\ j))) \land \vdash the(com(Rs\ !\ j))\ post(Ts\ !\ j))) \land interfree\ Rs$
apply(*erule rtrancl-induct2*)
apply *clarify*
— Base
apply *force*
— Induction step
apply *clarify*
apply(*drule Parallel-length-post-PStar*)
apply *clarify*
apply (*ind-cases* (*Parallel Ts*, *s*) $-P1\rightarrow$ (*Parallel Rs*, *t*))
apply(*rule conjI*)
apply *clarify*
apply(*case-tac i=j*)
apply(*simp split del*:*split-if*)
apply(*erule Strong-Soundness-aux-aux*,*simp*+)
apply *force*

apply *force*
apply(*simp split del*: *split-if*)
apply(*erule Parallel-Strong-Soundness-aux-aux*)
apply(*simp-all add*: *split del*:*split-if*)
apply *force*
apply(*rule interfree-lemma*)
apply *simp-all*
done

lemma *Parallel-Strong-Soundness*:
$[\![(Parallel\ Ts,\ s)\ -P{*}{\longrightarrow}\ (Parallel\ Rs,\ t);\ interfree\ Ts;\ j{<}length\ Rs;$
$\forall\, i.\ i{<}length\ Ts \longrightarrow (\exists\, c\ q.\ Ts\ !\ i = (Some\ c,\ q) \wedge s{\in}pre\ c \wedge \vdash\ c\ q)\ ]\!] \Longrightarrow$
$if\ com(Rs\ !\ j) = None\ then\ t{\in}post(Ts\ !\ j)\ else\ t{\in}pre\ (the(com(Rs\ !\ j)))$
apply(*drule Parallel-Strong-Soundness-aux*)
apply *simp+*
done

lemma *oghoare-sound* [*rule-format*]: $\|{-}\ p\ c\ q \longrightarrow \|{=}\ p\ c\ q$
apply (*unfold com-validity-def*)
apply(*rule oghoare-induct*)
apply(*rule TrueI*)+
— Parallel
apply(*simp add*: *SEM-def sem-def*)
apply *clarify*
apply(*frule Parallel-length-post-PStar*)
apply *clarify*
apply(*drule-tac j=i* **in** *Parallel-Strong-Soundness*)
apply *clarify*
apply *simp*
apply *force*
apply *simp*
apply(*erule-tac* $V = \forall\, i.\ ?P\ i$ **in** *thin-rl*)
apply(*drule-tac* $s = length\ Rs$ **in** *sym*)
apply(*erule allE*, *erule impE*, *assumption*)
apply(*force dest*: *nth-mem simp add*: *All-None-def*)
— Basic
apply(*simp add*: *SEM-def sem-def*)
apply(*force dest*: *rtrancl-imp-UN-rel-pow Basic-ntran*)
— Seq
apply(*rule subset-trans*)
prefer *2* **apply** *assumption*
apply(*simp add*: *L3-5ii L3-5i*)
— Cond
apply(*simp add*: *L3-5iv*)
— While
apply(*simp add*: *L3-5v*)
apply (*blast dest*: *SEM-fwhile*)
— Conseq
apply(*auto simp add*: *SEM-def sem-def*)

done

end

1.6 Generation of Verification Conditions

theory *OG-Tactics* **imports** *OG-Hoare*
begin

lemmas *ann-hoare-intros=AnnBasic AnnSeq AnnCond1 AnnCond2 AnnWhile AnnAwait AnnConseq*
lemmas *oghoare-intros=Parallel Basic Seq Cond While Conseq*

lemma *ParallelConseqRule*:
$[\![\ p \subseteq (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ pre(the(com(Ts\ !\ i))));$
$\ \|{-}\ (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ pre(the(com(Ts\ !\ i))))$
$\quad (Parallel\ Ts)$
$\quad (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ post(Ts\ !\ i));$
$\ (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ post(Ts\ !\ i)) \subseteq q\ ]\!]$
$\ \Longrightarrow \|{-}\ p\ (Parallel\ Ts)\ q$
apply (*rule Conseq*)
prefer *2*
 apply *fast*
apply *assumption+*
done

lemma *SkipRule*: $p \subseteq q \Longrightarrow \|{-}\ p\ (Basic\ id)\ q$
apply(*rule oghoare-intros*)
  prefer *2* **apply**(*rule Basic*)
 prefer *2* **apply**(*rule subset-refl*)
apply(*simp add:Id-def*)
done

lemma *BasicRule*: $p \subseteq \{s.\ (f\ s){\in}q\} \Longrightarrow \|{-}\ p\ (Basic\ f)\ q$
apply(*rule oghoare-intros*)
  prefer *2* **apply**(*rule oghoare-intros*)
 prefer *2* **apply**(*rule subset-refl*)
apply *assumption*
done

lemma *SeqRule*: $[\![\ \|{-}\ p\ c1\ r;\ \|{-}\ r\ c2\ q\ ]\!] \Longrightarrow \|{-}\ p\ (Seq\ c1\ c2)\ q$
apply(*rule Seq*)
apply *fast+*
done

lemma *CondRule*:
$[\![\ p \subseteq \{s.\ (s{\in}b \longrightarrow s{\in}w) \wedge (s{\notin}b \longrightarrow s{\in}w')\};\ \|{-}\ w\ c1\ q;\ \|{-}\ w'\ c2\ q\ ]\!]$
$\ \Longrightarrow \|{-}\ p\ (Cond\ b\ c1\ c2)\ q$
apply(*rule Cond*)

apply(*rule Conseq*)
prefer *4* **apply**(*rule Conseq*)
apply *simp-all*
apply *force+*
done

lemma *WhileRule*: $[\![\ p \subseteq i;\ \Vert- \ (i \cap b)\ c\ i\ ;\ (i \cap (-b)) \subseteq q\ ]\!]$
$\Longrightarrow \Vert- \ p\ (\mathit{While}\ b\ i\ c)\ q$
apply(*rule Conseq*)
prefer *2* **apply**(*rule While*)
apply *assumption+*
done

Three new proof rules for special instances of the *AnnBasic* and the *AnnAwait* commands when the transformation performed on the state is the identity, and for an *AnnAwait* command where the boolean condition is $\{s.\ \mathit{True}\}$:

lemma *AnnatomRule*:
$[\![\ \mathit{atom\text{-}com}(c);\ \Vert- \ r\ c\ q\ ]\!] \ \Longrightarrow \vdash (\mathit{AnnAwait}\ r\ \{s.\ \mathit{True}\}\ c)\ q$
apply(*rule AnnAwait*)
apply *simp-all*
done

lemma *AnnskipRule*:
$r \subseteq q \Longrightarrow \vdash (\mathit{AnnBasic}\ r\ \mathit{id})\ q$
apply(*rule AnnBasic*)
apply *simp*
done

lemma *AnnwaitRule*:
$[\![\ (r \cap b) \subseteq q\ ]\!] \Longrightarrow \vdash (\mathit{AnnAwait}\ r\ b\ (\mathit{Basic}\ \mathit{id}))\ q$
apply(*rule AnnAwait*)
apply *simp*
apply(*rule BasicRule*)
apply *simp*
done

Lemmata to avoid using the definition of *map-ann-hoare*, *interfree-aux*, *interfree-swap* and *interfree* by splitting it into different cases:

lemma *interfree-aux-rule1*: *interfree-aux*(*co*, *q*, *None*)
by(*simp add*:*interfree-aux-def*)

lemma *interfree-aux-rule2*:
$\forall (R,r)\in(\mathit{atomics}\ a).\ \Vert- \ (q \cap R)\ r\ q \Longrightarrow \mathit{interfree\text{-}aux}(\mathit{None},\ q,\ \mathit{Some}\ a)$
apply(*simp add*:*interfree-aux-def*)
apply(*force elim*:*oghoare-sound*)
done

lemma *interfree-aux-rule3*:

$(\forall (R, r)\in(atomics\ a).\ \Vert- (q \cap R)\ r\ q \wedge (\forall p\in(assertions\ c).\ \Vert- (p \cap R)\ r\ p))$
$\Longrightarrow$ *interfree-aux*(*Some c*, *q*, *Some a*)
apply(*simp add*:*interfree-aux-def*)
apply(*force elim*:*oghoare-sound*)
done

lemma *AnnBasic-assertions*:
$[\![$*interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*None*, *q*, *Some a*)$]\!] \Longrightarrow$
interfree-aux(*Some* (*AnnBasic r f*), *q*, *Some a*)
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnSeq-assertions*:
$[\![$ *interfree-aux*(*Some c1*, *q*, *Some a*); *interfree-aux*(*Some c2*, *q*, *Some a*)$]\!]\Longrightarrow$
interfree-aux(*Some* (*AnnSeq c1 c2*), *q*, *Some a*)
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnCond1-assertions*:
$[\![$ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*Some c1*, *q*, *Some a*);
interfree-aux(*Some c2*, *q*, *Some a*)$]\!]\Longrightarrow$
interfree-aux(*Some*(*AnnCond1 r b c1 c2*), *q*, *Some a*)
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnCond2-assertions*:
$[\![$ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*Some c*, *q*, *Some a*)$]\!]\Longrightarrow$
interfree-aux(*Some* (*AnnCond2 r b c*), *q*, *Some a*)
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnWhile-assertions*:
$[\![$ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*None*, *i*, *Some a*);
interfree-aux(*Some c*, *q*, *Some a*)$]\!]\Longrightarrow$
interfree-aux(*Some* (*AnnWhile r b i c*), *q*, *Some a*)
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnAwait-assertions*:
$[\![$ *interfree-aux*(*None*, *r*, *Some a*); *interfree-aux*(*None*, *q*, *Some a*)$]\!]\Longrightarrow$
interfree-aux(*Some* (*AnnAwait r b c*), *q*, *Some a*)
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnBasic-atomics*:
$\Vert- (q \cap r)\ (Basic\ f)\ q \Longrightarrow$ *interfree-aux*(*None*, *q*, *Some* (*AnnBasic r f*))
by(*simp add*: *interfree-aux-def oghoare-sound*)

lemma *AnnSeq-atomics*:

$\llbracket$ *interfree-aux*(*Any*, *q*, *Some a1*); *interfree-aux*(*Any*, *q*, *Some a2*)$\rrbracket \Longrightarrow$
interfree-aux(*Any*, *q*, *Some* (*AnnSeq a1 a2*))
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnCond1-atomics*:
$\llbracket$ *interfree-aux*(*Any*, *q*, *Some a1*); *interfree-aux*(*Any*, *q*, *Some a2*)$\rrbracket \Longrightarrow$
interfree-aux(*Any*, *q*, *Some* (*AnnCond1 r b a1 a2*))
apply(*simp add*: *interfree-aux-def*)
by *force*

lemma *AnnCond2-atomics*:
interfree-aux (*Any*, *q*, *Some a*)$\Longrightarrow$ *interfree-aux*(*Any*, *q*, *Some* (*AnnCond2 r b a*))
by(*simp add*: *interfree-aux-def*)

lemma *AnnWhile-atomics*: *interfree-aux* (*Any*, *q*, *Some a*)
$\Longrightarrow$ *interfree-aux*(*Any*, *q*, *Some* (*AnnWhile r b i a*))
by(*simp add*: *interfree-aux-def*)

lemma *Annatom-atomics*:
$\Vert-$ $(q \cap r)$ *a q* $\Longrightarrow$ *interfree-aux* (*None*, *q*, *Some* (*AnnAwait r* $\{x.$ *True*$\}$ *a*))
by(*simp add*: *interfree-aux-def oghoare-sound*)

lemma *AnnAwait-atomics*:
$\Vert-$ $(q \cap (r \cap b))$ *a q* $\Longrightarrow$ *interfree-aux* (*None*, *q*, *Some* (*AnnAwait r b a*))
by(*simp add*: *interfree-aux-def oghoare-sound*)

constdefs
interfree-swap :: ('*a ann-triple-op* $*$ ('*a ann-triple-op*) *list*) $\Rightarrow$ *bool*
interfree-swap == $\lambda(x,\ xs).\ \forall y \in set\ xs.$ *interfree-aux* (*com x*, *post x*, *com y*)
$\wedge$ *interfree-aux*(*com y*, *post y*, *com x*)

lemma *interfree-swap-Empty*: *interfree-swap* (*x*, [])
by(*simp add*:*interfree-swap-def*)

lemma *interfree-swap-List*:
$\llbracket$ *interfree-aux* (*com x*, *post x*, *com y*);
interfree-aux (*com y*, *post y* ,*com x*); *interfree-swap* (*x*, *xs*) $\rrbracket$
$\Longrightarrow$ *interfree-swap* (*x*, *y*#*xs*)
by(*simp add*:*interfree-swap-def*)

lemma *interfree-swap-Map*: $\forall k.\ i \le k \wedge k < j \longrightarrow$ *interfree-aux* (*com x*, *post x*, *c k*)
$\wedge$ *interfree-aux* (*c k*, *Q k*, *com x*)
$\Longrightarrow$ *interfree-swap* (*x*, *map* ($\lambda k.$ (*c k*, *Q k*)) $[i..<j]$)
by(*force simp add*: *interfree-swap-def less-diff-conv*)

lemma *interfree-Empty*: *interfree* []

by(*simp add*:*interfree-def*)

lemma *interfree-List*:
$[\![$ *interfree-swap*$(x,\ xs)$; *interfree* xs $]\!] \Longrightarrow$ *interfree* $(x\#xs)$
apply(*simp add*:*interfree-def interfree-swap-def*)
apply *clarify*
apply(*case-tac i*)
apply(*case-tac j*)
apply *simp-all*
apply(*case-tac j*,*simp*+)
done

lemma *interfree-Map*:
$(\forall\, i\ j.\ a{\leq}i \wedge i{<}b \wedge a{\leq}j \wedge j{<}b \wedge i{\neq}j \longrightarrow$ *interfree-aux* $(c\ i,\ Q\ i,\ c\ j))$
$\Longrightarrow$ *interfree* $(map\ (\lambda k.\ (c\ k,\ Q\ k))\ [a..{<}b])$
by(*force simp add*: *interfree-def less-diff-conv*)

constdefs *map-ann-hoare* :: $(('a\ ann\text{-}com\text{-}op * 'a\ assn)\ list) \Rightarrow bool$ $([\vdash]\ \text{-}\ [0]\ 45)$
$[\vdash]\ Ts == (\forall\, i{<}length\ Ts.\ \exists\, c\ q.\ Ts!i{=}(Some\ c,\ q) \wedge \vdash c\ q)$

lemma *MapAnnEmpty*: $[\vdash]\ []$
by(*simp add*:*map-ann-hoare-def*)

lemma *MapAnnList*: $[\![\vdash c\ q\ ;\ [\vdash]\ xs\ ]\!] \Longrightarrow [\vdash]\ (Some\ c,q)\#xs$
apply(*simp add*:*map-ann-hoare-def*)
apply *clarify*
apply(*case-tac i*,*simp*+)
done

lemma *MapAnnMap*:
$\forall\, k.\ i{\leq}k \wedge k{<}j \longrightarrow \vdash (c\ k)\ (Q\ k) \Longrightarrow [\vdash]\ map\ (\lambda k.\ (Some\ (c\ k),\ Q\ k))\ [i..{<}j]$
apply(*simp add*: *map-ann-hoare-def less-diff-conv*)
done

lemma *ParallelRule*:$[\![\ [\vdash]\ Ts\ ;\ interfree\ Ts\ ]\!]$
$\Longrightarrow \Vert\!- (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ pre(the(com(Ts!i))))$
Parallel Ts
$(\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ post(Ts!i))$
apply(*rule Parallel*)
apply(*simp add*:*map-ann-hoare-def*)
apply *simp*
done

The following are some useful lemmas and simplification tactics to control which theorems are used to simplify at each moment, so that the original input does not suffer any unexpected transformation.

lemma *Compl-Collect*: $-(Collect\ b) = \{x.\ \neg(b\ x)\}$
by *fast*
lemma *list-length*: $length\ []{=}0 \wedge length\ (x\#xs) = Suc(length\ xs)$

by *simp*
lemma *list-lemmas*: *length* []=*0* $\wedge$ *length* (*x#xs*) = *Suc*(*length xs*)
$\wedge$ (*x#xs*) ! *0*=*x* $\wedge$ (*x#xs*) ! *Suc n* = *xs* ! *n*
by *simp*
lemma *le-Suc-eq-insert*: {*i*. *i* <*Suc n*} = *insert n* {*i*. *i*< *n*}
by *auto*
lemmas *primrecdef-list* = *pre.simps assertions.simps atomics.simps atom-com.simps*
lemmas *my-simp-list* = *list-lemmas fst-conv snd-conv*
not-less0 refl le-Suc-eq-insert Suc-not-Zero Zero-not-Suc Suc-Suc-eq
Collect-mem-eq ball-simps option.simps primrecdef-list
lemmas *ParallelConseq-list* = *INTER-def Collect-conj-eq length-map length-upt*
length-append list-length

ML ⟪
val before-interfree-simp-tac = (*simp-tac* (*HOL-basic-ss addsimps* [*thm com.simps*,
thm post.simps]))

val interfree-simp-tac = (*asm-simp-tac* (*HOL-ss addsimps* [*thm split*, *thm ball-Un*,
thm ball-empty]@(*thms my-simp-list*)))

val ParallelConseq = (*simp-tac* (*HOL-basic-ss addsimps* (*thms ParallelConseq-list*)@(*thms*
my-simp-list)))
⟫

The following tactic applies *tac* to each conjunct in a subgoal of the form $A \Longrightarrow a1 \wedge a2 \wedge .. \wedge an$ returning n subgoals, one for each conjunct:

ML ⟪
```
fun conjI-Tac tac i st = st |>
      ( (EVERY [rtac conjI i,
         conjI-Tac tac (i+1),
         tac i]) ORELSE (tac i) )
```
⟫

Tactic for the generation of the verification conditions

The tactic basically uses two subtactics:

HoareRuleTac is called at the level of parallel programs, it uses the ParallelTac to solve parallel composition of programs. This verification has two parts, namely, (1) all component programs are correct and (2) they are interference free. *HoareRuleTac* is also called at the level of atomic regions, i.e. ⟨ ⟩ and *AWAIT b THEN - END*, and at each interference freedom test.

AnnHoareRuleTac is for component programs which are annotated programs and so, there are not unknown assertions (no need to use the parameter precond, see NOTE).

NOTE: precond(::bool) informs if the subgoal has the form $\|-$ *?p c q*, in this case we have precond=False and the generated verification condition would have the form *?p* $\subseteq \ldots$ which can be solved by *rtac subset-refl*, if True we proceed to simplify it using the simplification tactics above.

ML ⟪

```
 fun WlpTac i = (rtac (thm SeqRule) i) THEN (HoareRuleTac false (i+1))
and HoareRuleTac precond i st = st |>
   ( (WlpTac i THEN HoareRuleTac precond i)
     ORELSE
     (FIRST[rtac (thm SkipRule) i,
            rtac (thm BasicRule) i,
            EVERY[rtac (thm ParallelConseqRule) i,
                  ParallelConseq (i+2),
                  ParallelTac (i+1),
                  ParallelConseq i],
            EVERY[rtac (thm CondRule) i,
                  HoareRuleTac false (i+2),
                  HoareRuleTac false (i+1)],
            EVERY[rtac (thm WhileRule) i,
                  HoareRuleTac true (i+1)],
            K all-tac i ]
      THEN (if precond then (K all-tac i) else (rtac (thm subset-refl) i))))

and  AnnWlpTac i = (rtac (thm AnnSeq) i) THEN (AnnHoareRuleTac (i+1))
and AnnHoareRuleTac i st = st |>
   ( (AnnWlpTac i THEN AnnHoareRuleTac i )
     ORELSE
     (FIRST[(rtac (thm AnnskipRule) i),
            EVERY[rtac (thm AnnatomRule) i,
                  HoareRuleTac true (i+1)],
            (rtac (thm AnnwaitRule) i),
            rtac (thm AnnBasic) i,
            EVERY[rtac (thm AnnCond1) i,
                  AnnHoareRuleTac (i+3),
                  AnnHoareRuleTac (i+1)],
            EVERY[rtac (thm AnnCond2) i,
                  AnnHoareRuleTac (i+1)],
            EVERY[rtac (thm AnnWhile) i,
                  AnnHoareRuleTac (i+2)],
            EVERY[rtac (thm AnnAwait) i,
                  HoareRuleTac true (i+1)],
            K all-tac i]))

and ParallelTac i = EVERY[rtac (thm ParallelRule) i,
                          interfree-Tac (i+1),
                           MapAnn-Tac i]
```

```
and MapAnn-Tac i st = st |>
    (FIRST[rtac (thm MapAnnEmpty) i,
           EVERY[rtac (thm MapAnnList) i,
                 MapAnn-Tac (i+1),
                 AnnHoareRuleTac i],
           EVERY[rtac (thm MapAnnMap) i,
                 rtac (thm allI) i,rtac (thm impI) i,
                 AnnHoareRuleTac i]])

and interfree-swap-Tac i st = st |>
    (FIRST[rtac (thm interfree-swap-Empty) i,
           EVERY[rtac (thm interfree-swap-List) i,
                 interfree-swap-Tac (i+2),
                 interfree-aux-Tac (i+1),
                 interfree-aux-Tac i ],
           EVERY[rtac (thm interfree-swap-Map) i,
                 rtac (thm allI) i,rtac (thm impI) i,
                 conjI-Tac (interfree-aux-Tac) i]])

and interfree-Tac i st = st |>
   (FIRST[rtac (thm interfree-Empty) i,
          EVERY[rtac (thm interfree-List) i,
                interfree-Tac (i+1),
                interfree-swap-Tac i],
          EVERY[rtac (thm interfree-Map) i,
                rtac (thm allI) i,rtac (thm allI) i,rtac (thm impI) i,
                interfree-aux-Tac i ]])

and interfree-aux-Tac i = (before-interfree-simp-tac i ) THEN
       (FIRST[rtac (thm interfree-aux-rule1) i,
              dest-assertions-Tac i])

and dest-assertions-Tac i st = st |>
    (FIRST[EVERY[rtac (thm AnnBasic-assertions) i,
                 dest-atomics-Tac (i+1),
                 dest-atomics-Tac i],
           EVERY[rtac (thm AnnSeq-assertions) i,
                 dest-assertions-Tac (i+1),
                 dest-assertions-Tac i],
           EVERY[rtac (thm AnnCond1-assertions) i,
                 dest-assertions-Tac (i+2),
                 dest-assertions-Tac (i+1),
                 dest-atomics-Tac i],
           EVERY[rtac (thm AnnCond2-assertions) i,
                 dest-assertions-Tac (i+1),
                 dest-atomics-Tac i],
           EVERY[rtac (thm AnnWhile-assertions) i,
                 dest-assertions-Tac (i+2),
```

```
          dest-atomics-Tac (i+1),
          dest-atomics-Tac i],
      EVERY[rtac (thm AnnAwait-assertions) i,
          dest-atomics-Tac (i+1),
          dest-atomics-Tac i],
      dest-atomics-Tac i])

and dest-atomics-Tac i st = st |>
  (FIRST[EVERY[rtac (thm AnnBasic-atomics) i,
          HoareRuleTac true i],
      EVERY[rtac (thm AnnSeq-atomics) i,
          dest-atomics-Tac (i+1),
          dest-atomics-Tac i],
      EVERY[rtac (thm AnnCond1-atomics) i,
          dest-atomics-Tac (i+1),
          dest-atomics-Tac i],
      EVERY[rtac (thm AnnCond2-atomics) i,
          dest-atomics-Tac i],
      EVERY[rtac (thm AnnWhile-atomics) i,
          dest-atomics-Tac i],
      EVERY[rtac (thm Annatom-atomics) i,
          HoareRuleTac true i],
      EVERY[rtac (thm AnnAwait-atomics) i,
          HoareRuleTac true i],
          K all-tac i])
⟫
```

The final tactic is given the name *oghoare*:

```
ML ⟪
fun oghoare-tac i thm = SUBGOAL (fn (term, -) =>
  (HoareRuleTac true i)) i thm
⟫
```

Notice that the tactic for parallel programs *oghoare-tac* is initially invoked with the value *true* for the parameter *precond*.

Parts of the tactic can be also individually used to generate the verification conditions for annotated sequential programs and to generate verification conditions out of interference freedom tests:

```
ML ⟪ fun annhoare-tac i thm = SUBGOAL (fn (term, -) =>
 (AnnHoareRuleTac i)) i thm

fun interfree-aux-tac i thm = SUBGOAL (fn (term, -) =>
  (interfree-aux-Tac i)) i thm
⟫
```

The so defined ML tactics are then "exported" to be used in Isabelle proofs.

```
method-setup oghoare = ⟪
 Method.no-args
```

(*Method.SIMPLE-METHOD′ HEADGOAL* (*oghoare-tac*)) ⟫
verification condition generator for the oghoare logic

method-setup *annhoare* = ⟪
Method.no-args
(*Method.SIMPLE-METHOD′ HEADGOAL* (*annhoare-tac*)) ⟫
verification condition generator for the ann-hoare logic

method-setup *interfree-aux* = ⟪
Method.no-args
(*Method.SIMPLE-METHOD′ HEADGOAL* (*interfree-aux-tac*)) ⟫
verification condition generator for interference freedom tests

Tactics useful for dealing with the generated verification conditions:

method-setup *conjI-tac* = ⟪
Method.no-args
(*Method.SIMPLE-METHOD′ HEADGOAL* (*conjI-Tac* (*K all-tac*))) ⟫
verification condition generator for interference freedom tests

ML ⟪
```
fun disjE-Tac tac i st = st |>
      ( (EVERY [etac disjE i,
          disjE-Tac tac (i+1),
          tac i]) ORELSE (tac i) )
```
⟫

method-setup *disjE-tac* = ⟪
Method.no-args
(*Method.SIMPLE-METHOD′ HEADGOAL* (*disjE-Tac* (*K all-tac*))) ⟫
verification condition generator for interference freedom tests

end

1.7 Concrete Syntax

theory *Quote-Antiquote* **imports** *Main* **begin**

syntax
-quote :: ′b ⇒ (′a ⇒ ′b) ((«-») [0] 1000)
-antiquote :: (′a ⇒ ′b) ⇒ ′b (´- [1000] 1000)
-Assert :: ′a ⇒ ′a set ((.{-}.) [0] 1000)

syntax (*xsymbols*)
-Assert :: ′a ⇒ ′a set ((⦃-⦄) [0] 1000)

translations
.{b}. ⇀ *Collect* «b»

parse-translation ⟪
```
  let
    fun quote-tr [t] = Syntax.quote-tr -antiquote t
      | quote-tr ts = raise TERM (quote-tr, ts);
  in [(-quote, quote-tr)] end
```
⟫

end
theory *OG-Syntax*
imports *OG-Tactics Quote-Antiquote*
begin

Syntax for commands and for assertions and boolean expressions in commands *com* and annotated commands *ann-com*.

syntax
```
  -Assign      :: idt ⇒ 'b ⇒ 'a com    ((´- :=/ -) [70, 65] 61)
  -AnnAssign   :: 'a assn ⇒ idt ⇒ 'b ⇒ 'a com    ((- ´- :=/ -) [90,70,65] 61)
```

translations
```
  ´ x := a ⇀ Basic «´ (-update-name x a)»
  r ´ x := a ⇀ AnnBasic r «´ (-update-name x a)»
```

syntax
```
  -AnnSkip    :: 'a assn ⇒ 'a ann-com              (-//SKIP [90] 63)
  -AnnSeq     :: 'a ann-com ⇒ 'a ann-com ⇒ 'a ann-com  (-;;/ - [60,61] 60)

  -AnnCond1   :: 'a assn ⇒ 'a bexp ⇒ 'a ann-com ⇒ 'a ann-com ⇒ 'a ann-com
                 (- //IF - /THEN - /ELSE - /FI  [90,0,0,0] 61)
  -AnnCond2   :: 'a assn ⇒ 'a bexp ⇒ 'a ann-com ⇒ 'a ann-com
                 (- //IF - /THEN - /FI  [90,0,0] 61)
  -AnnWhile   :: 'a assn ⇒ 'a bexp ⇒ 'a assn ⇒ 'a ann-com ⇒ 'a ann-com
                 (- //WHILE - /INV - //DO -//OD  [90,0,0,0] 61)
  -AnnAwait   :: 'a assn ⇒ 'a bexp ⇒ 'a com ⇒ 'a ann-com
                 (- //AWAIT - /THEN /- /END  [90,0,0] 61)
  -AnnAtom    :: 'a assn ⇒ 'a com ⇒ 'a ann-com   (-//⟨-⟩ [90,0] 61)
  -AnnWait    :: 'a assn ⇒ 'a bexp ⇒ 'a ann-com   (-//WAIT - END [90,0] 61)

  -Skip       :: 'a com                   (SKIP 63)
  -Seq        :: 'a com ⇒ 'a com ⇒ 'a com (-,,/ - [55, 56] 55)
  -Cond       :: 'a bexp ⇒ 'a com ⇒ 'a com ⇒ 'a com
                          ((0IF -/ THEN -/ ELSE -/ FI) [0, 0, 0] 61)
  -Cond2      :: 'a bexp ⇒ 'a com ⇒ 'a com   (IF - THEN - FI [0,0] 56)
  -While-inv  :: 'a bexp ⇒ 'a assn ⇒ 'a com ⇒ 'a com
                 ((0WHILE -/ INV - //DO - /OD)  [0, 0, 0] 61)
  -While      :: 'a bexp ⇒ 'a com ⇒ 'a com
                 ((0WHILE - //DO - /OD)  [0, 0] 61)
```

translations
```
  SKIP ⇌ Basic id
```

```
c-1,, c-2 ⇌ Seq c-1 c-2

IF b THEN c1 ELSE c2 FI ⇀ Cond .{b}. c1 c2
IF b THEN c FI ⇌ IF b THEN c ELSE SKIP FI
WHILE b INV i DO c OD ⇀ While .{b}. i c
WHILE b DO c OD ⇌ WHILE b INV arbitrary DO c OD

r SKIP ⇌ AnnBasic r id
c-1;; c-2 ⇌ AnnSeq c-1 c-2
r IF b THEN c1 ELSE c2 FI ⇀ AnnCond1 r .{b}. c1 c2
r IF b THEN c FI ⇀ AnnCond2 r .{b}. c
r WHILE b INV i DO c OD ⇀ AnnWhile r .{b}. i c
r AWAIT b THEN c END ⇀ AnnAwait r .{b}. c
r ⟨c⟩ ⇌ r AWAIT True THEN c END
r WAIT b END ⇌ r AWAIT b THEN SKIP END
```

nonterminals

```
prgs
```

syntax

```
-PAR :: prgs ⇒ 'a                (COBEGIN//-//COEND [57] 56)
-prg :: ['a, 'a] ⇒ prgs          (-//- [60, 90] 57)
-prgs :: ['a, 'a, prgs] ⇒ prgs  (-//-//‖//- [60,90,57] 57)

-prg-scheme :: ['a, 'a, 'a, 'a, 'a] ⇒ prgs
              (SCHEME [- ≤ - < -] -// - [0,0,0,60, 90] 57)
```

translations

```
-prg c q ⇌ [(Some c, q)]
-prgs c q ps ⇌ (Some c, q) # ps
-PAR ps ⇌ Parallel ps

-prg-scheme j i k c q ⇌ map (λi. (Some c, q)) [j..<k]
```

print-translation ⟪

```
let
  fun quote-tr' f (t :: ts) =
        Term.list-comb (f $ Syntax.quote-tr' -antiquote t, ts)
    | quote-tr' - - = raise Match;

  fun annquote-tr' f (r :: t :: ts) =
        Term.list-comb (f $ r $ Syntax.quote-tr' -antiquote t, ts)
    | annquote-tr' - - = raise Match;

  val assert-tr' = quote-tr' (Syntax.const -Assert);

  fun bexp-tr' name ((Const (Collect, -) $ t) :: ts) =
        quote-tr' (Syntax.const name) (t :: ts)
    | bexp-tr' - - = raise Match;
```

```
    fun annbexp-tr' name (r :: (Const (Collect, -) $ t) :: ts) =
          annquote-tr' (Syntax.const name) (r :: t :: ts)
      | annbexp-tr' - - = raise Match;

    fun upd-tr' (x-upd, T) =
      (case try (unsuffix RecordPackage.updateN) x-upd of
        SOME x => (x, if T = dummyT then T else Term.domain-type T)
      | NONE => raise Match);

    fun update-name-tr' (Free x) = Free (upd-tr' x)
      | update-name-tr' ((c as Const (-free, -)) $ Free x) =
          c $ Free (upd-tr' x)
      | update-name-tr' (Const x) = Const (upd-tr' x)
      | update-name-tr' - = raise Match;

    fun assign-tr' (Abs (x, -, f $ t $ Bound 0) :: ts) =
          quote-tr' (Syntax.const -Assign $ update-name-tr' f)
            (Abs (x, dummyT, t) :: ts)
      | assign-tr' - = raise Match;

    fun annassign-tr' (r :: Abs (x, -, f $ t $ Bound 0) :: ts) =
          quote-tr' (Syntax.const -AnnAssign $ r $ update-name-tr' f)
            (Abs (x, dummyT, t) :: ts)
      | annassign-tr' - = raise Match;

    fun Parallel-PAR [(Const (Cons,-) $ (Const (Pair,-) $ (Const (Some,-) $ t1 )
$ t2) $ Const (Nil,-))] =
                    (Syntax.const -prg $ t1 $ t2)
      | Parallel-PAR [(Const (Cons,-) $ (Const (Pair,-) $ (Const (Some,-) $ t1) $
t2) $ ts)] =
                      (Syntax.const -prgs $ t1 $ t2 $ Parallel-PAR [ts])
      | Parallel-PAR - = raise Match;

fun Parallel-tr' ts = Syntax.const -PAR $ Parallel-PAR ts;
  in
    [(Collect, assert-tr'), (Basic, assign-tr'),
      (Cond, bexp-tr' -Cond), (While, bexp-tr' -While-inv),
      (AnnBasic, annassign-tr'),
      (AnnWhile, annbexp-tr' -AnnWhile), (AnnAwait, annbexp-tr' -AnnAwait),
     (AnnCond1, annbexp-tr' -AnnCond1), (AnnCond2, annbexp-tr' -AnnCond2)]

  end

⟫

end
```

1.8 Examples

theory *OG-Examples* **imports** *OG-Syntax* **begin**

1.8.1 Mutual Exclusion

Peterson's Algorithm I

Eike Best. "Semantics of Sequential and Parallel Programs", page 217.

record *Petersons-mutex-1 =*
pr1 :: *nat*
pr2 :: *nat*
in1 :: *bool*
in2 :: *bool*
hold :: *nat*

lemma *Petersons-mutex-1*:
```
 ‖- .{´pr1=0 ∧ ¬´in1 ∧ ´pr2=0 ∧ ¬´in2 }.
 COBEGIN .{´pr1=0 ∧ ¬´in1}.
 WHILE True INV .{´pr1=0 ∧ ¬´in1}.
 DO
 .{´pr1=0 ∧ ¬´in1}. ⟨ ´in1:=True,,´pr1:=1 ⟩;;
 .{´pr1=1 ∧ ´in1}.  ⟨ ´hold:=1,,´pr1:=2 ⟩;;
 .{´pr1=2 ∧ ´in1 ∧ (´hold=1 ∨ ´hold=2 ∧ ´pr2=2)}.
 AWAIT (¬´in2 ∨ ¬(´hold=1)) THEN ´pr1:=3 END;;
 .{´pr1=3 ∧ ´in1 ∧ (´hold=1 ∨ ´hold=2 ∧ ´pr2=2)}.
  ⟨´in1:=False,,´pr1:=0⟩
 OD .{´pr1=0 ∧ ¬´in1}.
 ‖
 .{´pr2=0 ∧ ¬´in2}.
 WHILE True INV .{´pr2=0 ∧ ¬´in2}.
 DO
 .{´pr2=0 ∧ ¬´in2}. ⟨ ´in2:=True,,´pr2:=1 ⟩;;
 .{´pr2=1 ∧ ´in2}. ⟨  ´hold:=2,,´pr2:=2 ⟩;;
 .{´pr2=2 ∧ ´in2 ∧ (´hold=2 ∨ (´hold=1 ∧ ´pr1=2))}.
 AWAIT (¬´in1 ∨ ¬(´hold=2)) THEN ´pr2:=3  END;;
 .{´pr2=3 ∧ ´in2 ∧ (´hold=2 ∨ (´hold=1 ∧ ´pr1=2))}.
   ⟨´in2:=False,,´pr2:=0⟩
 OD .{´pr2=0 ∧ ¬´in2}.
 COEND
 .{´pr1=0 ∧ ¬´in1 ∧ ´pr2=0 ∧ ¬´in2}.
```
apply *oghoare*
— 104 verification conditions.
apply *auto*
done

Peterson's Algorithm II: A Busy Wait Solution

Apt and Olderog. "Verification of sequential and concurrent Programs", page 282.

record *Busy-wait-mutex* =
flag1 :: *bool*
flag2 :: *bool*
turn :: *nat*
after1 :: *bool*
after2 :: *bool*

lemma *Busy-wait-mutex*:
$\Vert-$.{*True*}.
´*flag1*:=*False*,, ´*flag2*:=*False*,,
COBEGIN .{¬´*flag1*}.
WHILE True
INV .{¬´*flag1*}.
DO .{¬´*flag1*}. ⟨ ´*flag1*:=*True*,,´*after1*:=*False* ⟩;;
.{´*flag1* ∧ ¬´*after1*}. ⟨ ´*turn*:=*1*,,´*after1*:=*True* ⟩;;
.{´*flag1* ∧ ´*after1* ∧ (´*turn*=*1* ∨ ´*turn*=*2*)}.
WHILE ¬(´*flag2* ⟶ ´*turn*=*2*)
INV .{´*flag1* ∧ ´*after1* ∧ (´*turn*=*1* ∨ ´*turn*=*2*)}.
DO .{´*flag1* ∧ ´*after1* ∧ (´*turn*=*1* ∨ ´*turn*=*2*)}. *SKIP OD*;;
.{´*flag1* ∧ ´*after1* ∧ (´*flag2* ∧ ´*after2* ⟶ ´*turn*=*2*)}.
´*flag1*:=*False*
OD
.{*False*}.
‖
.{¬´*flag2*}.
WHILE True
INV .{¬´*flag2*}.
DO .{¬´*flag2*}. ⟨ ´*flag2*:=*True*,,´*after2*:=*False* ⟩;;
.{´*flag2* ∧ ¬´*after2*}. ⟨ ´*turn*:=*2*,,´*after2*:=*True* ⟩;;
.{´*flag2* ∧ ´*after2* ∧ (´*turn*=*1* ∨ ´*turn*=*2*)}.
WHILE ¬(´*flag1* ⟶ ´*turn*=*1*)
INV .{´*flag2* ∧ ´*after2* ∧ (´*turn*=*1* ∨ ´*turn*=*2*)}.
DO .{´*flag2* ∧ ´*after2* ∧ (´*turn*=*1* ∨ ´*turn*=*2*)}. *SKIP OD*;;
.{´*flag2* ∧ ´*after2* ∧ (´*flag1* ∧ ´*after1* ⟶ ´*turn*=*1*)}.
´*flag2*:=*False*
OD
.{*False*}.
COEND
.{*False*}.
apply *oghoare*
— 122 vc
apply *auto*
done

Peterson's Algorithm III: A Solution using Semaphores

```
record Semaphores-mutex =
 out :: bool
 who :: nat

lemma Semaphores-mutex:
 ‖− .{i≠j}.
  ´out:=True ,,
  COBEGIN .{i≠j}.
      WHILE True INV .{i≠j}.
      DO .{i≠j}. AWAIT ´out THEN  ´out:=False,, ´who:=i END;;
         .{¬´out ∧ ´who=i ∧ i≠j}. ´out:=True OD
      .{False}.
  ‖
      .{i≠j}.
      WHILE True INV .{i≠j}.
      DO .{i≠j}. AWAIT ´out THEN  ´out:=False,,´who:=j END;;
         .{¬´out ∧ ´who=j ∧ i≠j}. ´out:=True OD
      .{False}.
  COEND
  .{False}.
apply oghoare
— 38 vc
apply auto
done
```

Peterson's Algorithm III: Parameterized version:

```
lemma Semaphores-parameterized-mutex:
 0<n ⟹ ‖− .{True}.
  ´out:=True ,,
 COBEGIN
  SCHEME [0≤ i< n]
    .{True}.
     WHILE True INV .{True}.
      DO .{True}. AWAIT ´out THEN  ´out:=False,, ´who:=i END;;
         .{¬´out ∧ ´who=i}. ´out:=True OD
    .{False}.
 COEND
  .{False}.
apply oghoare
— 20 vc
apply auto
done
```

The Ticket Algorithm

```
record Ticket-mutex =
 num :: nat
```

nextv :: *nat*
turn :: *nat list*
index :: *nat*

lemma *Ticket-mutex*:
$\llbracket$ *0<n*; *I=≪n=length ´turn ∧ 0<´nextv ∧ (∀ k l. k<n ∧ l<n ∧ k≠l*
  ⟶ ´turn!k < ´num ∧ (´turn!k =0 ∨ ´turn!k≠´turn!l))≫ $\rrbracket$
  ⟹ ‖− .{n=length ´turn}.
  ´index:= 0,,
  WHILE ´index < n INV .{n=length ´turn ∧ (∀ i<´index. ´turn!i=0)}.
   DO ´turn:= ´turn[´index:=0],, ´index:=´index +1 OD,,
 ´num:=1 ,, ´nextv:=1 ,,
 COBEGIN
 SCHEME [0≤ i< n]
  .{´I}.
   WHILE True INV .{´I}.
   DO .{´I}. ⟨ ´turn :=´turn[i:=´num],, ´num:=´num+1 ⟩;;
     .{´I}. WAIT ´turn!i=´nextv END;;
     .{´I ∧ ´turn!i=´nextv}. ´nextv:=´nextv+1
   OD
  .{False}.
 COEND
 .{False}.
apply *oghoare*
— 35 vc
apply *simp-all*
— 21 vc
apply(*tactic* ⟪ *ALLGOALS Clarify-tac* ⟫)
— 11 vc
apply *simp-all*
apply(*tactic* ⟪ *ALLGOALS Clarify-tac* ⟫)
— 10 subgoals left
apply(*erule less-SucE*)
 apply *simp*
apply *simp*
— 9 subgoals left
apply(*case-tac i=k*)
 apply *force*
apply *simp*
apply(*case-tac i=l*)
 apply *force*
apply *force*
— 8 subgoals left
prefer *8*
apply *force*
apply *force*
— 6 subgoals left
prefer *6*
apply(*erule-tac x=i* **in** *allE*)

```
apply fastsimp
— 5 subgoals left
prefer 5
apply(tactic ⟪ ALLGOALS (case-tac j=k) ⟫)
— 10 subgoals left
apply simp-all
apply(erule-tac x=k in allE)
apply force
— 9 subgoals left
apply(case-tac j=l)
 apply simp
 apply(erule-tac x=k in allE)
 apply(erule-tac x=k in allE)
 apply(erule-tac x=l in allE)
 apply force
apply(erule-tac x=k in allE)
apply(erule-tac x=k in allE)
apply(erule-tac x=l in allE)
apply force
— 8 subgoals left
apply force
apply(case-tac j=l)
 apply simp
apply(erule-tac x=k in allE)
apply(erule-tac x=l in allE)
apply force
apply force
apply force
— 5 subgoals left
apply(erule-tac x=k in allE)
apply(erule-tac x=l in allE)
apply(case-tac j=l)
 apply force
apply force
apply force
— 3 subgoals left
apply(erule-tac x=k in allE)
apply(erule-tac x=l in allE)
apply(case-tac j=l)
 apply force
apply force
apply force
— 1 subgoals left
apply(erule-tac x=k in allE)
apply(erule-tac x=l in allE)
apply(case-tac j=l)
 apply force
apply force
done
```

1.8.2 Parallel Zero Search

Synchronized Zero Search. Zero-6

Apt and Olderog. "Verification of sequential and concurrent Programs" page 294:

```
record Zero-search =
  turn :: nat
  found :: bool
  x :: nat
  y :: nat

lemma Zero-search:
 [[I1= ≪ a≤´x ∧ (´found ⟶ (a<´x ∧ f(´x)=0) ∨ (´y≤a ∧ f(´y)=0))
    ∧ (¬´found ∧ a<´ x ⟶ f(´x)≠0) ≫ ;
   I2= ≪´y≤a+1 ∧ (´found ⟶ (a<´x ∧ f(´x)=0) ∨ (´y≤a ∧ f(´y)=0))
    ∧ (¬´found ∧ ´y≤a ⟶ f(´y)≠0) ≫ ]] ⟹
 ||− .{∃ u. f(u)=0}.
 ´turn:=1,, ´found:= False,,
 ´x:=a,, ´y:=a+1 ,,
 COBEGIN .{´I1}.
     WHILE ¬´found
     INV .{´I1}.
     DO .{a≤´x ∧ (´found ⟶ ´y≤a ∧ f(´y)=0) ∧ (a<´x ⟶ f(´x)≠0)}.
        WAIT ´turn=1 END;;
        .{a≤´x ∧ (´found ⟶ ´y≤a ∧ f(´y)=0) ∧ (a<´x ⟶ f(´x)≠0)}.
        ´turn:=2;;
        .{a≤´x ∧ (´found ⟶ ´y≤a ∧ f(´y)=0) ∧ (a<´x ⟶ f(´x)≠0)}.
        ⟨ ´x:=´x+1,,
          IF f(´x)=0 THEN ´found:=True ELSE SKIP FI⟩
     OD;;
     .{´I1 ∧ ´found}.
     ´turn:=2
     .{´I1 ∧ ´found}.
 ||
    .{´I2}.
     WHILE ¬´found
     INV .{´I2}.
     DO .{´y≤a+1 ∧ (´found ⟶ a<´x ∧ f(´x)=0) ∧ (´y≤a ⟶ f(´y)≠0)}.
        WAIT ´turn=2 END;;
        .{´y≤a+1 ∧ (´found ⟶ a<´x ∧ f(´x)=0) ∧ (´y≤a ⟶ f(´y)≠0)}.
        ´turn:=1;;
        .{´y≤a+1 ∧ (´found ⟶ a<´x ∧ f(´x)=0) ∧ (´y≤a ⟶ f(´y)≠0)}.
        ⟨ ´y:=(´y − 1),,
          IF f(´y)=0 THEN ´found:=True ELSE SKIP FI⟩
     OD;;
     .{´I2 ∧ ´found}.
     ´turn:=1
     .{´I2 ∧ ´found}.
```

COEND
.{*f*(´*x*)=*0* ∨ *f*(´*y*)=*0*}.
apply *oghoare*
— 98 verification conditions
apply *auto*
— auto takes about 3 minutes !!
apply *arith+*
done

Easier Version: without AWAIT. Apt and Olderog. page 256:

lemma *Zero-Search-2*:
⟦*I1*=≪ *a*≤´*x* ∧ (´*found* ⟶ (*a*<´*x* ∧ *f*(´*x*)=*0*) ∨ (´*y*≤*a* ∧ *f*(´*y*)=*0*))
∧ (¬´*found* ∧ *a*<´*x* ⟶ *f*(´*x*)≠*0*)≫;
I2= ≪´*y*≤*a*+*1* ∧ (´*found* ⟶ (*a*<´*x* ∧ *f*(´*x*)=*0*) ∨ (´*y*≤*a* ∧ *f*(´*y*)=*0*))
∧ (¬´*found* ∧ ´*y*≤*a* ⟶ *f*(´*y*)≠*0*)≫⟧ ⟹
‖− .{∃ *u*. *f*(*u*)=*0*}.
´*found*:= *False*,,
´*x*:=*a*,, ´*y*:=*a*+*1*,,
COBEGIN .{´*I1*}.
WHILE ¬´*found*
INV .{´*I1*}.
DO .{*a*≤´*x* ∧ (´*found* ⟶ ´*y*≤*a* ∧ *f*(´*y*)=*0*) ∧ (*a*<´*x* ⟶ *f*(´*x*)≠*0*)}.
⟨ ´*x*:=´*x*+*1*,,*IF* *f*(´*x*)=*0* *THEN* ´*found*:=*True* *ELSE* *SKIP* *FI*⟩
OD
.{´*I1* ∧ ´*found*}.
‖
.{´*I2*}.
WHILE ¬´*found*
INV .{´*I2*}.
DO .{´*y*≤*a*+*1* ∧ (´*found* ⟶ *a*<´*x* ∧ *f*(´*x*)=*0*) ∧ (´*y*≤*a* ⟶ *f*(´*y*)≠*0*)}.
⟨ ´*y*:=(´*y* − *1*),,*IF* *f*(´*y*)=*0* *THEN* ´*found*:=*True* *ELSE* *SKIP* *FI*⟩
OD
.{´*I2* ∧ ´*found*}.
COEND
.{*f*(´*x*)=*0* ∨ *f*(´*y*)=*0*}.
apply *oghoare*
— 20 vc
apply *auto*
— auto takes aprox. 2 minutes.
apply *arith+*
done

1.8.3 Producer/Consumer

Previous lemmas

lemma *nat-lemma2*: ⟦ *b* = *m*∗(*n*::*nat*) + *t*; *a* = *s*∗*n* + *u*; *t*=*u*; *b*−*a* < *n* ⟧ ⟹ *m* ≤ *s*
proof −
assume *b* = *m*∗(*n*::*nat*) + *t* *a* = *s*∗*n* + *u* *t*=*u*

hence $(m - s) * n = b - a$ **by** (*simp add*: *diff-mult-distrib*)
also assume $\ldots < n$
finally have $m - s < 1$ **by** *simp*
thus *?thesis* **by** *arith*
qed

lemma *mod-lemma*: $[\![\ (c::nat) \leq a;\ a < b;\ b - c < n\ ]\!] \Longrightarrow b\ mod\ n \neq a\ mod\ n$
apply(*subgoal-tac* $b=b\ div\ n*n + b\ mod\ n$)
prefer *2* **apply** (*simp add*: *mod-div-equality* [*symmetric*])
apply(*subgoal-tac* $a=a\ div\ n*n + a\ mod\ n$)
prefer *2*
apply(*simp add*: *mod-div-equality* [*symmetric*])
apply(*subgoal-tac* $b - a \leq b - c$)
prefer *2* **apply** *arith*
apply(*drule le-less-trans*)
back
apply *assumption*
apply(*frule less-not-refl2*)
apply(*drule less-imp-le*)
apply (*drule-tac* $m = a$ **and** $k = n$ **in** *div-le-mono*)
apply(*safe*)
apply(*frule-tac* $b = b$ **and** $a = a$ **and** $n = n$ **in** *nat-lemma2*, *assumption*, *assumption*)
apply *assumption*
apply(*drule order-antisym*, *assumption*)
apply(*rotate-tac* -3)
apply(*simp*)
done

Producer/Consumer Algorithm

record *Producer-consumer* =
ins :: *nat*
outs :: *nat*
li :: *nat*
lj :: *nat*
vx :: *nat*
vy :: *nat*
buffer :: *nat list*
b :: *nat list*

The whole proof takes aprox. 4 minutes.

lemma *Producer-consumer*:
$[\![INIT= \ll 0{<}length\ a \wedge 0{<}length\ ´buffer \wedge length\ ´b{=}length\ a\gg ;$
$I= \ll(\forall k{<}´ins.\ ´outs{\leq}k \longrightarrow (a\ !\ k) = ´buffer\ !\ (k\ mod\ (length\ ´buffer))) \wedge$
$´outs{\leq}´ins \wedge ´ins{-}´outs{\leq}length\ ´buffer\gg ;$
$I1= \ll´I \wedge ´li{\leq}length\ a\gg ;$
$p1= \ll´I1 \wedge ´li{=}´ins\gg ;$
$I2 = \ll´I \wedge (\forall k{<}´lj.\ (a\ !\ k){=}(´b\ !\ k)) \wedge ´lj{\leq}length\ a\gg ;$

```
  p2 = ≪´I2 ∧ ´lj=´outs≫ ⟧ =⇒
 ‖− .{´INIT}.
´ins:=0,, ´outs:=0,, ´li:=0,, ´lj:=0,,
COBEGIN .{´p1 ∧ ´INIT}.
  WHILE ´li <length a
    INV .{´p1 ∧ ´INIT}.
  DO .{´p1 ∧ ´INIT ∧ ´li<length a}.
      ´vx:= (a ! ´li);;
     .{´p1 ∧ ´INIT ∧ ´li<length a ∧ ´vx=(a ! ´li)}.
       WAIT ´ins−´outs < length ´buffer END;;
     .{´p1 ∧ ´INIT ∧ ´li<length a ∧ ´vx=(a ! ´li)
       ∧ ´ins−´outs < length ´buffer}.
      ´buffer:=(list-update ´buffer (´ins mod (length ´buffer)) ´vx);;
     .{´p1 ∧ ´INIT ∧ ´li<length a
       ∧ (a ! ´li)=(´buffer ! (´ins mod (length ´buffer)))
       ∧ ´ins−´outs <length ´buffer}.
      ´ins:=´ins+1;;
     .{´I1 ∧ ´INIT ∧ (´li+1)=´ins ∧ ´li<length a}.
      ´li:=´li+1
  OD
 .{´p1 ∧ ´INIT ∧ ´li=length a}.
 ‖
 .{´p2 ∧ ´INIT}.
  WHILE ´lj < length a
    INV .{´p2 ∧ ´INIT}.
  DO .{´p2 ∧ ´lj<length a ∧ ´INIT}.
       WAIT ´outs<´ins END;;
     .{´p2 ∧ ´lj<length a ∧ ´outs<´ins ∧ ´INIT}.
      ´vy:=(´buffer ! (´outs mod (length ´buffer)));;
     .{´p2 ∧ ´lj<length a ∧ ´outs<´ins ∧ ´vy=(a ! ´lj) ∧ ´INIT}.
      ´outs:=´outs+1;;
     .{´I2 ∧ (´lj+1)=´outs ∧ ´lj<length a ∧ ´vy=(a ! ´lj) ∧ ´INIT}.
      ´b:=(list-update ´b ´lj ´vy);;
     .{´I2 ∧ (´lj+1)=´outs ∧ ´lj<length a ∧ (a ! ´lj)=(´b ! ´lj) ∧ ´INIT}.
      ´lj:=´lj+1
  OD
 .{´p2 ∧ ´lj=length a ∧ ´INIT}.
COEND
.{ ∀k<length a. (a ! k)=(´b ! k)}.
apply oghoare
— 138 vc
apply(tactic ⟪ ALLGOALS Clarify-tac ⟫)
— 112 subgoals left
apply(simp-all (no-asm))
apply(tactic ⟪ALLGOALS (conjI-Tac (K all-tac)) ⟫)
— 930 subgoals left
apply(tactic ⟪ ALLGOALS Clarify-tac ⟫)
apply(simp-all (asm-lr) only:length-0-conv [THEN sym])
— 44 subgoals left
```

apply (*simp-all* (*asm-lr*) *del*:*length-0-conv* *add*: *nth-list-update mod-less-divisor mod-lemma*)
— 32 subgoals left
apply(*tactic* ⟪ *ALLGOALS Clarify-tac* ⟫)

apply(*tactic* ⟪ *TRYALL simple-arith-tac* ⟫)
— 9 subgoals left
apply (*force simp add*:*less-Suc-eq*)
apply(*drule sym*)
apply (*force simp add*:*less-Suc-eq*)+
done

1.8.4 Parameterized Examples

Set Elements of an Array to Zero

record *Example1* =
  a :: *nat* ⇒ *nat*

lemma *Example1*:
 $\|-$.{*True*}.
   COBEGIN SCHEME [$0\leq i<n$] .{*True*}. ´a:=´a (i:=0) .{´a i=0}. *COEND*
  .{$\forall\, i < n.$ ´a i = 0}.
apply *oghoare*
apply *simp-all*
done

Same example with lists as auxiliary variables.

record *Example1-list* =
  A :: *nat list*
lemma *Example1-list*:
 $\|-$.{n < *length* ´A}.
   COBEGIN
     SCHEME [$0\leq i<n$] .{n < *length* ´A}. ´A:=´A[i:=0] .{´A!i=0}.
   COEND
    .{$\forall\, i < n.$ ´A!i = 0}.
apply *oghoare*
apply *force*+
done

Increment a Variable in Parallel

First some lemmas about summation properties.

lemma *Example2-lemma2-aux*: !!b. $j<n \Longrightarrow$
 ($\sum i$=0..<n. (b i::*nat*)) =
 ($\sum i$=0..<j. b i) + b j + ($\sum i$=0..<$n-$(*Suc* j) . b (*Suc* j + i))
apply(*induct* n)
 apply *simp-all*
apply(*simp add*:*less-Suc-eq*)

apply(*auto*)
apply(*subgoal-tac* $n - j = Suc(n - Suc\ j)$)
apply *simp*
apply *arith*
done

lemma *Example2-lemma2-aux2*:
$!!b.\ j \le s \Longrightarrow (\sum i{::}nat{=}0..{<}j.\ (b\ (s{:=}t))\ i) = (\sum i{=}0..{<}j.\ b\ i)$
apply(*induct j*)
apply (*simp-all cong:setsum-cong*)
done

lemma *Example2-lemma2*:
$!!b.\ [\![j{<}n;\ b\ j{=}0]\!] \Longrightarrow Suc\ (\sum i{::}nat{=}0..{<}n.\ b\ i){=}(\sum i{=}0..{<}n.\ (b\ (j := Suc\ 0))\ i)$
apply(*frule-tac* $b{=}(b\ (j{:=}(Suc\ 0)))$ **in** *Example2-lemma2-aux*)
apply(*erule-tac* $t{=}setsum\ (b(j := (Suc\ 0)))\ \{0..{<}n\}$ **in** *ssubst*)
apply(*frule-tac* $b{=}b$ **in** *Example2-lemma2-aux*)
apply(*erule-tac* $t{=}setsum\ b\ \{0..{<}n\}$ **in** *ssubst*)
apply(*subgoal-tac* $Suc\ (setsum\ b\ \{0..{<}j\} + b\ j + (\sum i{=}0..{<}n - Suc\ j.\ b\ (Suc\ j + i))){=}(setsum\ b\ \{0..{<}j\} + Suc\ (b\ j) + (\sum i{=}0..{<}n - Suc\ j.\ b\ (Suc\ j + i)))$)
apply(*rotate-tac* -1)
apply(*erule ssubst*)
apply(*subgoal-tac* $j{\le}j$)
apply(*drule-tac* $b{=}b$ **and** $t{=}(Suc\ 0)$ **in** *Example2-lemma2-aux2*)
apply(*rotate-tac* -1)
apply(*erule ssubst*)
apply *simp-all*
done

record *Example2* =
$c :: nat \Rightarrow nat$
$x :: nat$

lemma *Example-2*: $0{<}n \Longrightarrow$
$\Vert{-}\ .\{\acute{}x{=}0 \wedge (\sum i{=}0..{<}n.\ \acute{}c\ i){=}0\}.$
COBEGIN
SCHEME $[0{\le}i{<}n]$
$.\{\acute{}x{=}(\sum i{=}0..{<}n.\ \acute{}c\ i) \wedge \acute{}c\ i{=}0\}.$
$\langle\ \acute{}x{:=}\acute{}x{+}(Suc\ 0),,\ \acute{}c{:=}\acute{}c\ (i{:=}(Suc\ 0))\ \rangle$
$.\{\acute{}x{=}(\sum i{=}0..{<}n.\ \acute{}c\ i) \wedge \acute{}c\ i{=}(Suc\ 0)\}.$
COEND
$.\{\acute{}x{=}n\}.$
apply *oghoare*
apply (*simp-all cong del*: *strong-setsum-cong*)
apply (*tactic* ⟪ *ALLGOALS Clarify-tac* ⟫)
apply (*simp-all cong del*: *strong-setsum-cong*)
apply(*erule* (*1*) *Example2-lemma2*)

apply(*erule* (*1*) *Example2-lemma2*)
apply(*erule* (*1*) *Example2-lemma2*)
apply(*simp*)
done

end

Chapter 2

Case Study: Single and Multi-Mutator Garbage Collection Algorithms

2.1 Formalization of the Memory

theory *Graph* **imports** *Main* **begin**

datatype *node* = *Black* | *White*

types
  $nodes = node\ list$
  $edge\ = nat \times nat$
  $edges = edge\ list$

consts $Roots :: nat\ set$

constdefs
  $Proper\text{-}Roots :: nodes \Rightarrow bool$
  $Proper\text{-}Roots\ M \equiv Roots{\neq}\{\} \land Roots \subseteq \{i.\ i{<}length\ M\}$

  $Proper\text{-}Edges :: (nodes \times edges) \Rightarrow bool$
  $Proper\text{-}Edges \equiv (\lambda(M,E).\ \forall i{<}length\ E.\ fst(E!i){<}length\ M \land snd(E!i){<}length\ M)$

  $BtoW :: (edge \times nodes) \Rightarrow bool$
  $BtoW \equiv (\lambda(e,M).\ (M!fst\ e){=}Black \land (M!snd\ e){\neq}Black)$

  $Blacks :: nodes \Rightarrow nat\ set$
  $Blacks\ M \equiv \{i.\ i{<}length\ M \land M!i{=}Black\}$

  $Reach :: edges \Rightarrow nat\ set$
  $Reach\ E \equiv \{x.\ (\exists path.\ 1{<}length\ path \land path!(length\ path - 1){\in}Roots \land x{=}path!0$

```
∧ (∀ i<length path − 1. (∃ j<length E. E!j=(path!(i+1), path!i))))
∨ x∈Roots}
```

Reach: the set of reachable nodes is the set of Roots together with the nodes reachable from some Root by a path represented by a list of nodes (at least two since we traverse at least one edge), where two consecutive nodes correspond to an edge in E.

2.1.1 Proofs about Graphs

```
lemmas Graph-defs= Blacks-def Proper-Roots-def Proper-Edges-def BtoW-def
declare Graph-defs [simp]
```

Graph 1

```
lemma Graph1-aux [rule-format]:
  ⟦ Roots⊆Blacks M; ∀ i<length E. ¬BtoW(E!i,M)⟧
  ⟹ 1< length path ⟶ (path!(length path − 1))∈Roots ⟶
  (∀ i<length path − 1. (∃ j. j < length E ∧ E!j=(path!(Suc i), path!i)))
  ⟶ M!(path!0) = Black
apply(induct-tac path)
 apply force
apply clarify
apply simp
apply(case-tac list)
 apply force
apply simp
apply(rotate-tac −2)
apply(erule-tac x = 0 in all-dupE)
apply simp
apply clarify
apply(erule allE , erule (1) notE impE)
apply simp
apply(erule mp)
apply(case-tac lista)
 apply force
apply simp
apply(erule mp)
apply clarify
apply(erule-tac x = Suc i in allE)
apply force
done
```

```
lemma Graph1:
  ⟦Roots⊆Blacks M; Proper-Edges(M, E); ∀ i<length E. ¬BtoW(E!i,M) ⟧
  ⟹ Reach E⊆Blacks M
apply (unfold Reach-def)
apply simp
apply clarify
```

```
apply(erule disjE)
 apply clarify
 apply(rule conjI)
  apply(subgoal-tac 0< length path - Suc 0)
   apply(erule allE , erule (1) notE impE)
   apply force
  apply simp
 apply(rule Graph1-aux)
apply auto
done
```

Graph 2

```
lemma Ex-first-occurrence [rule-format]:
  P (n::nat) ⟶ (∃ m. P m ∧ (∀ i. i<m ⟶ ¬ P i))
apply(rule nat-less-induct)
apply clarify
apply(case-tac ∀ m. m<n ⟶ ¬ P m)
apply auto
done

lemma Compl-lemma: (n::nat)≤l ⟹ (∃ m. m≤l ∧ n=l - m)
apply(rule-tac x = l - n in exI)
apply arith
done

lemma Ex-last-occurrence:
  ⟦P (n::nat); n≤l⟧ ⟹ (∃ m. P (l - m) ∧ (∀ i. i<m ⟶ ¬P (l - i)))
apply(drule Compl-lemma)
apply clarify
apply(erule Ex-first-occurrence)
done

lemma Graph2:
  ⟦T ∈ Reach E; R<length E⟧ ⟹ T ∈ Reach (E[R:=(fst(E!R), T)])
apply (unfold Reach-def)
apply clarify
apply simp
apply(case-tac ∀ z<length path. fst(E!R)≠path!z)
 apply(rule-tac x = path in exI)
 apply simp
 apply clarify
 apply(erule allE , erule (1) notE impE)
 apply clarify
 apply(rule-tac x = j in exI)
 apply(case-tac j=R)
  apply(erule-tac x = Suc i in allE)
  apply simp
  apply arith
```

apply (*force simp add*:*nth-list-update*)
apply *simp*
apply(*erule exE*)
apply(*subgoal-tac z* $\leq$ *length path* $-$ *Suc 0*)
prefer *2* **apply** *arith*
apply(*drule-tac P* = λm. *m*<*length path* $\wedge$ *fst*(*E*!*R*)=*path*!*m* **in** *Ex-last-occurrence*)
apply *assumption*
apply *clarify*
apply *simp*
apply(*rule-tac x* = (*path*!*0*)#(*drop* (*length path* $-$ *Suc m*) *path*) **in** *exI*)
apply *simp*
apply(*case-tac length path* $-$ (*length path* $-$ *Suc m*))
apply *arith*
apply *simp*
apply(*subgoal-tac* (*length path* $-$ *Suc m*) + *nat* $\leq$ *length path*)
prefer *2* **apply** *arith*
apply(*drule nth-drop*)
apply *simp*
apply(*subgoal-tac length path* $-$ *Suc m* + *nat* = *length path* $-$ *Suc 0*)
prefer *2* **apply** *arith*
apply *simp*
apply *clarify*
apply(*case-tac i*)
apply(*force simp add*: *nth-list-update*)
apply *simp*
apply(*subgoal-tac* (*length path* $-$ *Suc m*) + *nata* $\leq$ *length path*)
prefer *2* **apply** *arith*
apply *simp*
apply(*subgoal-tac* (*length path* $-$ *Suc m*) + (*Suc nata*) $\leq$ *length path*)
prefer *2* **apply** *arith*
apply *simp*
apply(*erule-tac x* = *length path* $-$ *Suc m* + *nata* **in** *allE*)
apply *simp*
apply *clarify*
apply(*rule-tac x* = *j* **in** *exI*)
apply(*case-tac R*=*j*)
prefer *2* **apply** *force*
apply *simp*
apply(*drule-tac t* = *path* ! (*length path* $-$ *Suc m*) **in** *sym*)
apply *simp*
apply(*case-tac length path* $-$ *Suc 0* < *m*)
apply(*subgoal-tac* (*length path* $-$ *Suc m*)=*0*)
prefer *2* **apply** *arith*
apply(*simp del*: *diff-is-0-eq*)
apply(*subgoal-tac Suc nata*$\leq$*nat*)
prefer *2* **apply** *arith*
apply(*drule-tac n* = *Suc nata* **in** *Compl-lemma*)
apply *clarify*
apply *force*

apply(*drule leI*)
apply(*subgoal-tac Suc* (*length path* $-$ *Suc m* $+$ *nata*)=(*length path* $-$ *Suc 0*) $-$ (*m* $-$ *Suc nata*))
apply(*erule-tac x* $=$ *m* $-$ (*Suc nata*) **in** *allE*)
apply(*case-tac m*)
apply *simp*
apply *simp*
apply(*subgoal-tac natb* $-$ *nata* $<$ *Suc natb*)
prefer *2* **apply**(*erule thin-rl*)+ **apply** *arith*
apply *simp*
apply(*case-tac length path*)
apply *force*
apply (*erule-tac V* $=$ *Suc natb* $\leq$ *length path* $-$ *Suc 0* **in** *thin-rl*)
apply *simp*
apply(*frule-tac i1* $=$ *length path* **and** *j1* $=$ *length path* $-$ *Suc 0* **and** *k1* $=$ *m* **in** *diff-diff-right* [*THEN mp*])
apply(*erule-tac V* $=$ *length path* $-$ *Suc m* $+$ *nat* $=$ *length path* $-$ *Suc 0* **in** *thin-rl*)
apply *simp*
apply *arith*
done

Graph 3

lemma *Graph3*:
⟦ *T*∈*Reach E*; *R*<*length E* ⟧ $\Longrightarrow$ *Reach*(*E*[*R*:=(*fst*(*E*!*R*),*T*)]) $\subseteq$ *Reach E*
apply (*unfold Reach-def*)
apply *clarify*
apply *simp*
apply(*case-tac* ∃ *i*<*length path* $-$ *1*. (*fst*(*E*!*R*),*T*)=(*path*!(*Suc i*),*path*!*i*))
— the changed edge is part of the path
apply(*erule exE*)
apply(*drule-tac P* $=$ λ*i*. *i*<*length path* $-$ *1* ∧ (*fst*(*E*!*R*),*T*)=(*path*!*Suc i*,*path*!*i*) **in** *Ex-first-occurrence*)
apply *clarify*
apply(*erule disjE*)
— T is NOT a root
apply *clarify*
apply(*rule-tac x* $=$ (*take m path*)@*patha* **in** *exI*)
apply(*subgoal-tac* ¬(*length path*≤*m*))
prefer *2* **apply** *arith*
apply(*simp add*: *min-def*)
apply(*rule conjI*)
apply(*subgoal-tac* ¬(*m* $+$ *length patha* $-$ *1* $<$ *m*))
prefer *2* **apply** *arith*
apply(*simp add*: *nth-append min-def*)
apply(*rule conjI*)
apply(*case-tac m*)
apply *force*
apply(*case-tac path*)

apply *force*
apply *force*
apply *clarify*
apply(*case-tac Suc i≤m*)
apply(*erule-tac x = i* **in** *allE*)
apply *simp*
apply *clarify*
apply(*rule-tac x = j* **in** *exI*)
apply(*case-tac Suc i<m*)
apply(*simp add*: *nth-append min-def*)
apply(*case-tac R=j*)
apply(*simp add*: *nth-list-update*)
apply(*case-tac i=m*)
apply *force*
apply(*erule-tac x = i* **in** *allE*)
apply *force*
apply(*force simp add*: *nth-list-update*)
apply(*simp add*: *nth-append min-def*)
apply(*subgoal-tac i=m − 1*)
prefer *2* **apply** *arith*
apply(*case-tac R=j*)
apply(*erule-tac x = m − 1* **in** *allE*)
apply(*simp add*: *nth-list-update*)
apply(*force simp add*: *nth-list-update*)
apply(*simp add*: *nth-append min-def*)
apply(*rotate-tac −4*)
apply(*erule-tac x = i − m* **in** *allE*)
apply(*subgoal-tac Suc* (*i − m*)=(*Suc i − m*))
prefer *2* **apply** *arith*
apply *simp*
apply(*erule mp*)
apply *arith*
— T is a root
apply(*case-tac m=0*)
apply *force*
apply(*rule-tac x = take* (*Suc m*) *path* **in** *exI*)
apply(*subgoal-tac* ¬(*length path≤Suc m*))
prefer *2* **apply** *arith*
apply(*simp add*: *min-def*)
apply *clarify*
apply(*erule-tac x = i* **in** *allE*)
apply *simp*
apply *clarify*
apply(*case-tac R=j*)
apply(*force simp add*: *nth-list-update*)
apply(*force simp add*: *nth-list-update*)
— the changed edge is not part of the path
apply(*rule-tac x = path* **in** *exI*)
apply *simp*

apply *clarify*
apply(*erule-tac* $x = i$ **in** *allE*)
apply *clarify*
apply(*case-tac* $R{=}j$)
apply(*erule-tac* $x = i$ **in** *allE*)
apply *simp*
apply(*force simp add*: *nth-list-update*)
done

Graph 4

lemma *Graph4*:
$[\![T \in Reach\ E;\ Roots \subseteq Blacks\ M;\ I \leq length\ E;\ T < length\ M;\ R < length\ E;$
$\forall\, i<I.\ \neg BtoW(E!i,M);\ R<I;\ M!fst(E!R)=Black;\ M!T \neq Black]\!] \Longrightarrow$
$(\exists\, r.\ I \leq r \wedge r < length\ E \wedge BtoW(E[R:=(fst(E!R),T)]!r,M))$
apply (*unfold Reach-def*)
apply *simp*
apply(*erule disjE*)
prefer *2* **apply** *force*
apply *clarify*
— there exist a black node in the path to T
apply(*case-tac* $\exists\, m<length\ path.\ M!(path!m)=Black$)
apply(*erule exE*)
apply(*drule-tac* $P = \lambda m.\ m<length\ path \wedge M!(path!m)=Black$ **in** *Ex-first-occurrence*)
apply *clarify*
apply(*case-tac ma*)
apply *force*
apply *simp*
apply(*case-tac length path*)
apply *force*
apply *simp*
apply(*erule-tac* $P = \lambda i.\ i < nata \longrightarrow ?P\ i$ **and** $x = nat$ **in** *allE*)
apply *simp*
apply *clarify*
apply(*erule-tac* $P = \lambda i.\ i < Suc\ nat \longrightarrow ?P\ i$ **and** $x = nat$ **in** *allE*)
apply *simp*
apply(*case-tac* $j<I$)
apply(*erule-tac* $x = j$ **in** *allE*)
apply *force*
apply(*rule-tac* $x = j$ **in** *exI*)
apply(*force simp add*: *nth-list-update*)
apply *simp*
apply(*rotate-tac* -1)
apply(*erule-tac* $x = length\ path - 1$ **in** *allE*)
apply(*case-tac length path*)
apply *force*
apply *force*
done

Graph 5

lemma *Graph5*:
$[\![\ T \in \mathit{Reach}\ E\ ;\ \mathit{Roots} \subseteq \mathit{Blacks}\ M;\ \forall\, i<R.\ \neg \mathit{BtoW}(E!i,M);\ T<\mathit{length}\ M;$
$R<\mathit{length}\ E;\ M!\mathit{fst}(E!R)=\mathit{Black};\ M!\mathit{snd}(E!R)=\mathit{Black};\ M!T \neq \mathit{Black}]\!]$
$\Longrightarrow (\exists\, r.\ R<r \wedge r<\mathit{length}\ E \wedge \mathit{BtoW}(E[R:=(\mathit{fst}(E!R),T)]!r,M))$
apply (*unfold Reach-def*)
apply *simp*
apply(*erule disjE*)
prefer *2* **apply** *force*
apply *clarify*
— there exist a black node in the path to T
apply(*case-tac* $\exists\, m<\mathit{length\ path}.\ M!(\mathit{path}!m)=\mathit{Black}$)
apply(*erule exE*)
apply(*drule-tac* $P = \lambda m.\ m<\mathit{length\ path} \wedge M!(\mathit{path}!m)=\mathit{Black}$ **in** *Ex-first-occurrence*)
apply *clarify*
apply(*case-tac ma*)
apply *force*
apply *simp*
apply(*case-tac length path*)
apply *force*
apply *simp*
apply(*erule-tac* $P = \lambda i.\ i < \mathit{nata} \longrightarrow ?P\ i$ **and** $x = \mathit{nat}$ **in** *allE*)
apply *simp*
apply *clarify*
apply(*erule-tac* $P = \lambda i.\ i < \mathit{Suc\ nat} \longrightarrow ?P\ i$ **and** $x = \mathit{nat}$ **in** *allE*)
apply *simp*
apply(*case-tac* $j \le R$)
apply(*drule le-imp-less-or-eq*)
apply(*erule disjE*)
apply(*erule allE* , *erule* (*1*) *notE impE*)
apply *force*
apply *force*
apply(*rule-tac* $x = j$ **in** *exI*)
apply(*force simp add*: *nth-list-update*)
apply *simp*
apply(*rotate-tac* -1)
apply(*erule-tac* $x = \mathit{length\ path} - 1$ **in** *allE*)
apply(*case-tac length path*)
apply *force*
apply *force*
done

Other lemmas about graphs

lemma *Graph6*:
$[\![\mathit{Proper\text{-}Edges}(M,E);\ R<\mathit{length}\ E\ ;\ T<\mathit{length}\ M]\!] \Longrightarrow \mathit{Proper\text{-}Edges}(M,E[R:=(\mathit{fst}(E!R),T)])$
apply (*unfold Proper-Edges-def*)
apply(*force simp add*: *nth-list-update*)
done

lemma *Graph7*:
$[\![Proper\text{-}Edges(M,E)]\!] \Longrightarrow Proper\text{-}Edges(M[T:=a],E)$
apply (*unfold Proper-Edges-def*)
apply *force*
done

lemma *Graph8*:
$[\![Proper\text{-}Roots(M)]\!] \Longrightarrow Proper\text{-}Roots(M[T:=a])$
apply (*unfold Proper-Roots-def*)
apply *force*
done

Some specific lemmata for the verification of garbage collection algorithms.

lemma *Graph9*: $j<length\ M \Longrightarrow Blacks\ M \subseteq Blacks\ (M[j := Black])$
apply (*unfold Blacks-def*)
apply(*force simp add*: *nth-list-update*)
done

lemma *Graph10* [*rule-format* (*no-asm*)]: $\forall\, i.\ M!i=a \longrightarrow M[i:=a]=M$
apply(*induct-tac M*)
apply *auto*
apply(*case-tac i*)
apply *auto*
done

lemma *Graph11* [*rule-format* (*no-asm*)]:
$[\![\ M!j \neq Black;j<length\ M]\!] \Longrightarrow Blacks\ M \subset Blacks\ (M[j := Black])$
apply (*unfold Blacks-def*)
apply(*rule psubsetI*)
apply(*force simp add*: *nth-list-update*)
apply *safe*
apply(*erule-tac c = j* **in** *equalityCE*)
apply *auto*
done

lemma *Graph12*: $[\![a \subseteq Blacks\ M;j<length\ M]\!] \Longrightarrow a \subseteq Blacks\ (M[j := Black])$
apply (*unfold Blacks-def*)
apply(*force simp add*: *nth-list-update*)
done

lemma *Graph13*: $[\![a \subset Blacks\ M;j<length\ M]\!] \Longrightarrow a \subset Blacks\ (M[j := Black])$
apply (*unfold Blacks-def*)
apply(*erule psubset-subset-trans*)
apply(*force simp add*: *nth-list-update*)
done

declare *Graph-defs* [*simp del*]

end

2.2 The Single Mutator Case

theory *Gar-Coll* **imports** *Graph OG-Syntax* **begin**

declare *psubsetE* [*rule del*]

Declaration of variables:

record *gar-coll-state* =
  M :: *nodes*
  E :: *edges*
  bc :: *nat set*
  obc :: *nat set*
  Ma :: *nodes*
  ind :: *nat*
  k :: *nat*
  z :: *bool*

2.2.1 The Mutator

The mutator first redirects an arbitrary edge R from an arbitrary accessible node towards an arbitrary accessible node T. It then colors the new target T black.

We declare the arbitrarily selected node and edge as constants:

consts R :: *nat* T :: *nat*

The following predicate states, given a list of nodes m and a list of edges e, the conditions under which the selected edge R and node T are valid:

constdefs
  $\mathit{Mut\text{-}init} :: \mathit{gar\text{-}coll\text{-}state} \Rightarrow \mathit{bool}$
  $\mathit{Mut\text{-}init} \equiv \ll T \in \mathit{Reach}\ {}'E \wedge R < \mathit{length}\ {}'E \wedge T < \mathit{length}\ {}'M \gg$

For the mutator we consider two modules, one for each action. An auxiliary variable $'z$ is set to false if the mutator has already redirected an edge but has not yet colored the new target.

constdefs
  $\mathit{Redirect\text{-}Edge} :: \mathit{gar\text{-}coll\text{-}state}\ \mathit{ann\text{-}com}$
  $\mathit{Redirect\text{-}Edge} \equiv .\{{}'\mathit{Mut\text{-}init} \wedge {}'z\}.\ \langle {}'E{:=}{}'E[R{:=}(\mathit{fst}({}'E!R),\ T)],,\ {}'z{:=}\ (\neg {}'z)\rangle$

  $\mathit{Color\text{-}Target} :: \mathit{gar\text{-}coll\text{-}state}\ \mathit{ann\text{-}com}$
  $\mathit{Color\text{-}Target} \equiv .\{{}'\mathit{Mut\text{-}init} \wedge \neg {}'z\}.\ \langle {}'M{:=}{}'M[T{:=}\mathit{Black}],,\ {}'z{:=}\ (\neg {}'z)\rangle$

  $\mathit{Mutator} :: \mathit{gar\text{-}coll\text{-}state}\ \mathit{ann\text{-}com}$
  $\mathit{Mutator} \equiv$

.{´*Mut-init* ∧ ´*z*}.
WHILE True INV .{´*Mut-init* ∧ ´*z*}.
DO Redirect-Edge ;; *Color-Target OD*

Correctness of the mutator

lemmas *mutator-defs* = *Mut-init-def Redirect-Edge-def Color-Target-def*

lemma *Redirect-Edge*:
⊢ *Redirect-Edge pre*(*Color-Target*)
apply (*unfold mutator-defs*)
apply *annhoare*
apply(*simp-all*)
apply(*force elim*:*Graph2*)
done

lemma *Color-Target*:
⊢ *Color-Target* .{´*Mut-init* ∧ ´*z*}.
apply (*unfold mutator-defs*)
apply *annhoare*
apply(*simp-all*)
done

lemma *Mutator*:
⊢ *Mutator* .{*False*}.
apply(*unfold Mutator-def*)
apply *annhoare*
apply(*simp-all add*:*Redirect-Edge Color-Target*)
apply(*simp add*:*mutator-defs Redirect-Edge-def*)
done

2.2.2 The Collector

A constant *M-init* is used to give ´*Ma* a suitable first value, defined as a list of nodes where only the *Roots* are black.

consts *M-init* :: *nodes*

constdefs
Proper-M-init :: *gar-coll-state* ⇒ *bool*
Proper-M-init ≡ ≪ *Blacks M-init*=*Roots* ∧ *length M-init*=*length* ´*M* ≫

Proper :: *gar-coll-state* ⇒ *bool*
Proper ≡ ≪ *Proper-Roots* ´*M* ∧ *Proper-Edges*(´*M*, ´*E*) ∧ ´*Proper-M-init* ≫

Safe :: *gar-coll-state* ⇒ *bool*
Safe ≡ ≪ *Reach* ´*E* ⊆ *Blacks* ´*M* ≫

lemmas *collector-defs* = *Proper-M-init-def Proper-def Safe-def*

Blackening the roots

constdefs
Blacken-Roots :: *gar-coll-state ann-com*
Blacken-Roots ≡
.{´*Proper*}.
´*ind*:=*0*;;
.{´*Proper* ∧ ´*ind*=*0*}.
WHILE ´*ind*<*length* ´*M*
INV .{´*Proper* ∧ (∀ *i*<´*ind*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*≤*length* ´*M*}.
DO .{´*Proper* ∧ (∀ *i*<´*ind*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M*}.
IF ´*ind*∈*Roots* *THEN*
.{´*Proper* ∧ (∀ *i*<´*ind*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M* ∧ ´*ind*∈*Roots*}.
´*M*:=´*M*[´*ind*:=*Black*] *FI*;;
.{´*Proper* ∧ (∀ *i*<´*ind*+*1*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M*}.
´*ind*:=´*ind*+*1*
OD

lemma *Blacken-Roots*:
⊢ *Blacken-Roots* .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M*}.
apply (*unfold Blacken-Roots-def*)
apply *annhoare*
apply(*simp-all add*:*collector-defs Graph-defs*)
apply *safe*
apply(*simp-all add*:*nth-list-update*)
apply (*erule less-SucE*)
apply *simp*+
apply *force*
apply *force*
done

Propagating black

constdefs
PBInv :: *gar-coll-state* ⇒ *nat* ⇒ *bool*
PBInv ≡ ≪ λ*ind*. ´*obc* < *Blacks* ´*M* ∨ (∀ *i* <*ind*. ¬*BtoW* (´*E*!*i*, ´*M*) ∨ (¬´*z* ∧ *i*=*R* ∧ (*snd*(´*E*!*R*)) = *T* ∧ (∃ *r*. *ind* ≤ *r* ∧ *r* < *length* ´*E* ∧ *BtoW*(´*E*!*r*,´*M*))))≫

constdefs
Propagate-Black-aux :: *gar-coll-state ann-com*
Propagate-Black-aux ≡
.{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*}.
´*ind*:=*0*;;
.{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M* ∧ ´*ind*=*0*}.

WHILE ´*ind*<*length* ´*E*
INV .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ ´*PBInv* ´*ind* ∧ ´*ind*≤*length* ´*E*}.
DO .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*

```
      ∧ ´PBInv ´ind ∧ ´ind<length ´E}.
   IF ´M!(fst (´E!´ind)) = Black THEN
    .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
      ∧ ´PBInv ´ind ∧ ´ind<length ´E ∧ ´M!fst(´E!´ind)=Black}.
     ´M:=´M[snd(´E!´ind):=Black];;
    .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
      ∧ ´PBInv (´ind + 1) ∧ ´ind<length ´E}.
     ´ind:=´ind+1
   FI
  OD

lemma Propagate-Black-aux:
 ⊢ Propagate-Black-aux
  .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ ( ´obc < Blacks ´M ∨ ´Safe)}.
apply (unfold Propagate-Black-aux-def  PBInv-def collector-defs)
apply annhoare
apply(simp-all add:Graph6 Graph7 Graph8 Graph12)
      apply force
     apply force
    apply force
— 4 subgoals left
apply clarify
apply(simp add:Proper-Edges-def Proper-Roots-def Graph6 Graph7 Graph8 Graph12)
apply (erule disjE)
 apply(rule disjI1)
 apply(erule Graph13)
 apply force
apply (case-tac M x ! snd (E x ! ind x)=Black)
 apply (simp add: Graph10 BtoW-def)
 apply (rule disjI2)
 apply clarify
 apply (erule less-SucE)
  apply (erule-tac x=i in allE , erule (1) notE impE)
  apply simp
  apply clarify
  apply (drule le-imp-less-or-eq)
  apply (erule disjE)
   apply (subgoal-tac Suc (ind x)≤r)
    apply fast
   apply arith
  apply fast
 apply fast
apply(rule disjI1)
apply(erule subset-psubset-trans)
apply(erule Graph11)
apply fast
— 3 subgoals left
apply force
```

apply *force*
— last
apply *clarify*
apply *simp*
apply(*subgoal-tac ind x = length (E x)*)
apply (*rotate-tac −1*)
apply (*simp* (*asm-lr*))
apply(*drule Graph1*)
apply *simp*
apply *clarify*
apply(*erule allE*, *erule impE*, *assumption*)
apply *force*
apply *force*
apply *arith*
done

Refining propagating black

constdefs

$Auxk :: gar\text{-}coll\text{-}state \Rightarrow bool$

$Auxk \equiv \ll ´k<length\ ´M \wedge (´M!´k \neq Black \vee \neg BtoW(´E!´ind,\ ´M) \vee$
$´obc<Blacks\ ´M \vee (\neg ´z \wedge ´ind=R \wedge snd(´E!R)=T$
$\wedge (\exists\, r.\ ´ind<r \wedge r<length\ ´E \wedge BtoW(´E!r,\ ´M))))\gg$

constdefs

$Propagate\text{-}Black ::\ gar\text{-}coll\text{-}state\ ann\text{-}com$
$Propagate\text{-}Black \equiv$
$.\{´Proper \wedge Roots \subseteq Blacks\ ´M \wedge ´obc \subseteq Blacks\ ´M \wedge ´bc \subseteq Blacks\ ´M\}.$
$´ind:=0;;$
$.\{´Proper \wedge Roots \subseteq Blacks\ ´M \wedge ´obc \subseteq Blacks\ ´M \wedge ´bc \subseteq Blacks\ ´M \wedge ´ind=0\}.$
$WHILE\ ´ind<length\ ´E$
$INV\ .\{´Proper \wedge Roots \subseteq Blacks\ ´M \wedge ´obc \subseteq Blacks\ ´M \wedge ´bc \subseteq Blacks\ ´M$
$\wedge ´PBInv\ ´ind \wedge ´ind \le length\ ´E\}.$
$DO\ .\{´Proper \wedge Roots \subseteq Blacks\ ´M \wedge ´obc \subseteq Blacks\ ´M \wedge ´bc \subseteq Blacks\ ´M$
$\wedge ´PBInv\ ´ind \wedge ´ind<length\ ´E\}.$
$IF\ (´M!(fst\ (´E!´ind)))=Black\ THEN$
$.\{´Proper \wedge Roots \subseteq Blacks\ ´M \wedge ´obc \subseteq Blacks\ ´M \wedge ´bc \subseteq Blacks\ ´M$
$\wedge ´PBInv\ ´ind \wedge ´ind<length\ ´E \wedge (´M!fst(´E!´ind))=Black\}.$
$´k:=(snd(´E!´ind));;$
$.\{´Proper \wedge Roots \subseteq Blacks\ ´M \wedge ´obc \subseteq Blacks\ ´M \wedge ´bc \subseteq Blacks\ ´M$
$\wedge ´PBInv\ ´ind \wedge ´ind<length\ ´E \wedge (´M!fst(´E!´ind))=Black$
$\wedge ´Auxk\}.$
$\langle ´M:=´M[´k:=Black],,\ ´ind:=´ind+1\rangle$
$ELSE\ .\{´Proper \wedge Roots \subseteq Blacks\ ´M \wedge ´obc \subseteq Blacks\ ´M \wedge ´bc \subseteq Blacks\ ´M$
$\wedge ´PBInv\ ´ind \wedge ´ind<length\ ´E\}.$
$\langle IF\ (´M!(fst\ (´E!´ind))) \neq Black\ THEN\ ´ind:=´ind+1\ FI\rangle$
FI
OD

```
lemma Propagate-Black:
  ⊢ Propagate-Black
  .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ ( ´obc < Blacks ´M ∨ ´Safe)}.
apply (unfold Propagate-Black-def  PBInv-def Auxk-def collector-defs)
apply annhoare
apply(simp-all add:Graph6 Graph7 Graph8 Graph12)
       apply force
      apply force
     apply force
— 5 subgoals left
apply clarify
apply(simp add:BtoW-def Proper-Edges-def)
— 4 subgoals left
apply clarify
apply(simp add:Proper-Edges-def Graph6 Graph7 Graph8 Graph12)
apply (erule disjE)
 apply (rule disjI1)
 apply (erule psubset-subset-trans)
 apply (erule Graph9)
apply (case-tac M x!k x=Black)
 apply (case-tac M x ! snd (E x ! ind x)=Black)
  apply (simp add: Graph10 BtoW-def)
  apply (rule disjI2)
  apply clarify
  apply (erule less-SucE)
   apply (erule-tac x=i in allE , erule (1) notE impE)
   apply simp
   apply clarify
   apply (drule le-imp-less-or-eq)
   apply (erule disjE)
    apply (subgoal-tac Suc (ind x)≤r)
     apply fast
    apply arith
   apply fast
  apply fast
 apply (simp add: Graph10 BtoW-def)
 apply (erule disjE)
  apply (erule disjI1)
 apply clarify
 apply (erule less-SucE)
  apply force
 apply simp
 apply (subgoal-tac Suc R≤r)
  apply fast
 apply arith
apply(rule disjI1)
apply(erule subset-psubset-trans)
apply(erule Graph11)
```

apply *fast*
— 3 subgoals left
apply *force*
— 2 subgoals left
apply *clarify*
apply(*simp add*:*Proper-Edges-def Graph6 Graph7 Graph8 Graph12*)
apply (*erule disjE*)
 apply *fast*
apply *clarify*
apply (*erule less-SucE*)
 apply (*erule-tac x*=*i* **in** *allE* , *erule* (*1*) *notE impE*)
 apply *simp*
 apply *clarify*
 apply (*drule le-imp-less-or-eq*)
 apply (*erule disjE*)
  apply (*subgoal-tac Suc* (*ind x*)≤*r*)
   apply *fast*
  apply *arith*
 apply (*simp add*: *BtoW-def*)
apply (*simp add*: *BtoW-def*)
— last
apply *clarify*
apply *simp*
apply(*subgoal-tac ind x* = *length* (*E x*))
 apply (*rotate-tac* −*1*)
 apply (*simp* (*asm-lr*))
 apply(*drule Graph1*)
   apply *simp*
  apply *clarify*
 apply(*erule allE*, *erule impE*, *assumption*)
  apply *force*
 apply *force*
apply *arith*
done

Counting black nodes

constdefs
  CountInv :: *gar-coll-state* ⇒ *nat* ⇒ *bool*
  CountInv ≡ « λ*ind*. {*i*. *i*<*ind* ∧ ´*Ma*!*i*=*Black*}⊆´*bc* »

constdefs
  Count :: *gar-coll-state ann-com*
  Count ≡
  .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M*
   ∧ ´*obc*⊆*Blacks* ´*Ma* ∧ *Blacks* ´*Ma*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
   ∧ *length* ´*Ma*=*length* ´*M* ∧ (´*obc* < *Blacks* ´*Ma* ∨ ´*Safe*) ∧ ´*bc*={}}.
  ´*ind*:=*0*;;
  .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M*

$\wedge$ *´obc⊆Blacks ´Ma* $\wedge$ *Blacks ´Ma⊆Blacks ´M* $\wedge$ *´bc⊆Blacks ´M*
$\wedge$ *length ´Ma=length ´M* $\wedge$ (*´obc* < *Blacks ´Ma* $\vee$ *´Safe*) $\wedge$ *´bc*={}
$\wedge$ *´ind=0*}.
WHILE ´ind<length ´M
INV .{*´Proper* $\wedge$ *Roots⊆Blacks ´M*
$\wedge$ *´obc⊆Blacks ´Ma* $\wedge$ *Blacks ´Ma⊆Blacks ´M* $\wedge$ *´bc⊆Blacks ´M*
$\wedge$ *length ´Ma=length ´M* $\wedge$ *´CountInv ´ind*
$\wedge$ (*´obc* < *Blacks ´Ma* $\vee$ *´Safe*) $\wedge$ *´ind≤length ´M*}.
DO .{*´Proper* $\wedge$ *Roots⊆Blacks ´M*
$\wedge$ *´obc⊆Blacks ´Ma* $\wedge$ *Blacks ´Ma⊆Blacks ´M* $\wedge$ *´bc⊆Blacks ´M*
$\wedge$ *length ´Ma=length ´M* $\wedge$ *´CountInv ´ind*
$\wedge$ (*´obc* < *Blacks ´Ma* $\vee$ *´Safe*) $\wedge$ *´ind<length ´M*}.
IF ´M!*´ind=Black*
THEN .{*´Proper* $\wedge$ *Roots⊆Blacks ´M*
$\wedge$ *´obc⊆Blacks ´Ma* $\wedge$ *Blacks ´Ma⊆Blacks ´M* $\wedge$ *´bc⊆Blacks ´M*
$\wedge$ *length ´Ma=length ´M* $\wedge$ *´CountInv ´ind*
$\wedge$ (*´obc* < *Blacks ´Ma* $\vee$ *´Safe*) $\wedge$ *´ind<length ´M* $\wedge$ *´M*!*´ind=Black*}.
´bc:=*insert ´ind ´bc*
FI;;
.{*´Proper* $\wedge$ *Roots⊆Blacks ´M*
$\wedge$ *´obc⊆Blacks ´Ma* $\wedge$ *Blacks ´Ma⊆Blacks ´M* $\wedge$ *´bc⊆Blacks ´M*
$\wedge$ *length ´Ma=length ´M* $\wedge$ *´CountInv* (*´ind+1*)
$\wedge$ (*´obc* < *Blacks ´Ma* $\vee$ *´Safe*) $\wedge$ *´ind<length ´M*}.
´ind:=*´ind+1*
OD

lemma *Count*:
$\vdash$ *Count*
.{*´Proper* $\wedge$ *Roots⊆Blacks ´M*
$\wedge$ *´obc⊆Blacks ´Ma* $\wedge$ *Blacks ´Ma⊆´bc* $\wedge$ *´bc⊆Blacks ´M* $\wedge$ *length ´Ma=length*
´M
$\wedge$ (*´obc* < *Blacks ´Ma* $\vee$ *´Safe*)}.
apply(*unfold Count-def*)
apply *annhoare*
apply(*simp-all add*:*CountInv-def Graph6 Graph7 Graph8 Graph12 Blacks-def collector-defs*)
apply *force*
apply *force*
apply *force*
apply *clarify*
apply *simp*
apply(*fast elim*:*less-SucE*)
apply *clarify*
apply *simp*
apply(*fast elim*:*less-SucE*)
apply *force*
apply *force*
done

Appending garbage nodes to the free list

consts *Append-to-free* :: *nat* × *edges* ⇒ *edges*

axioms
Append-to-free0: *length* (*Append-to-free* (*i*, *e*)) = *length e*
Append-to-free1: *Proper-Edges* (*m*, *e*)
⟹ *Proper-Edges* (*m*, *Append-to-free*(*i*, *e*))
Append-to-free2: *i* ∉ *Reach e*
⟹ *n* ∈ *Reach* (*Append-to-free*(*i*, *e*)) = (*n* = *i* ∨ *n* ∈ *Reach e*)

constdefs
AppendInv :: *gar-coll-state* ⇒ *nat* ⇒ *bool*
AppendInv ≡ «λ*ind*. ∀ *i*<*length* ´*M*. *ind*≤*i* ⟶ *i*∈*Reach* ´*E* ⟶ ´*M*!*i*=*Black*»

constdefs
Append :: *gar-coll-state ann-com*
Append ≡
.{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*Safe*}.
´*ind*:=*0*;;
.{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*Safe* ∧ ´*ind*=*0*}.
WHILE ´*ind*<*length* ´*M*
INV .{´*Proper* ∧ ´*AppendInv* ´*ind* ∧ ´*ind*≤*length* ´*M*}.
DO .{´*Proper* ∧ ´*AppendInv* ´*ind* ∧ ´*ind*<*length* ´*M*}.
IF ´*M*!´*ind*=*Black THEN*
.{´*Proper* ∧ ´*AppendInv* ´*ind* ∧ ´*ind*<*length* ´*M* ∧ ´*M*!´*ind*=*Black*}.
´*M*:=´*M*[´*ind*:=*White*]
ELSE .{´*Proper* ∧ ´*AppendInv* ´*ind* ∧ ´*ind*<*length* ´*M* ∧ ´*ind*∉*Reach* ´*E*}.
´*E*:=*Append-to-free*(´*ind*,´*E*)
FI;;
.{´*Proper* ∧ ´*AppendInv* (´*ind*+*1*) ∧ ´*ind*<*length* ´*M*}.
´*ind*:=´*ind*+*1*
OD

lemma *Append*:
⊢ *Append* .{´*Proper*}.
apply(*unfold Append-def AppendInv-def*)
apply *annhoare*
apply(*simp-all add*:*collector-defs Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12*)
apply(*force simp*:*Blacks-def nth-list-update*)
apply *force*
apply *force*
apply(*force simp add*:*Graph-defs*)
apply *force*
apply *clarify*
apply *simp*
apply(*rule conjI*)
apply (*erule Append-to-free1*)
apply *clarify*

```
  apply (drule-tac n = i in Append-to-free2)
  apply force
 apply force
apply force
done
```

Correctness of the Collector

```
constdefs
  Collector :: gar-coll-state ann-com
  Collector ≡
.{´Proper}.
 WHILE True INV .{´Proper}.
 DO
  Blacken-Roots;;
  .{´Proper ∧ Roots⊆Blacks ´M}.
   ´obc:={};;
  .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc={}}.
   ´bc:=Roots;;
  .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc={} ∧ ´bc=Roots}.
   ´Ma:=M-init;;
  .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc={} ∧ ´bc=Roots ∧ ´Ma=M-init}.
   WHILE ´obc≠´bc
     INV .{´Proper ∧ Roots⊆Blacks ´M
          ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆´bc ∧ ´bc⊆Blacks ´M
          ∧ length ´Ma=length ´M ∧ (´obc < Blacks ´Ma ∨ ´Safe)}.
   DO .{´Proper ∧ Roots⊆Blacks ´M ∧ ´bc⊆Blacks ´M}.
       ´obc:=´bc;;
       Propagate-Black;;
      .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
       ∧ (´obc < Blacks ´M ∨ ´Safe)}.
       ´Ma:=´M;;
      .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´Ma
       ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M ∧ length ´Ma=length ´M
       ∧ ( ´obc < Blacks ´Ma ∨ ´Safe)}.
       ´bc:={};;
       Count
   OD;;
  Append
 OD

lemma Collector:
 ⊢ Collector .{False}.
apply(unfold Collector-def)
apply annhoare
apply(simp-all add: Blacken-Roots Propagate-Black Count Append)
apply(simp-all add:Blacken-Roots-def Propagate-Black-def Count-def Append-def
collector-defs)
   apply (force simp add: Proper-Roots-def)
```

apply *force*
apply *force*
apply *clarify*
apply (*erule disjE*)
apply(*simp add*:*psubsetI*)
apply(*force dest*:*subset-antisym*)
done

2.2.3 Interference Freedom

lemmas *modules* = *Redirect-Edge-def Color-Target-def Blacken-Roots-def Propagate-Black-def Count-def Append-def*
lemmas *Invariants* = *PBInv-def Auxk-def CountInv-def AppendInv-def*
lemmas *abbrev* = *collector-defs mutator-defs Invariants*

lemma *interfree-Blacken-Roots-Redirect-Edge*:
interfree-aux (*Some Blacken-Roots*, {}, *Some Redirect-Edge*)
apply (*unfold modules*)
apply *interfree-aux*
apply *safe*
apply (*simp-all add*:*Graph6 Graph12 abbrev*)
done

lemma *interfree-Redirect-Edge-Blacken-Roots*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Blacken-Roots*)
apply (*unfold modules*)
apply *interfree-aux*
apply *safe*
apply(*simp add*:*abbrev*)+
done

lemma *interfree-Blacken-Roots-Color-Target*:
interfree-aux (*Some Blacken-Roots*, {}, *Some Color-Target*)
apply (*unfold modules*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*Graph7 Graph8 nth-list-update abbrev*)
done

lemma *interfree-Color-Target-Blacken-Roots*:
interfree-aux (*Some Color-Target*, {}, *Some Blacken-Roots*)
apply (*unfold modules*)
apply *interfree-aux*
apply *safe*
apply(*simp add*:*abbrev*)+
done

lemma *interfree-Propagate-Black-Redirect-Edge*:
interfree-aux (*Some Propagate-Black*, {}, *Some Redirect-Edge*)

apply (*unfold modules*)
apply *interfree-aux*
— 11 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph6 Graph12*)
apply(*clarify*, *simp add*:*abbrev Graph6 Graph12*)
apply(*clarify*, *simp add*:*abbrev Graph6 Graph12*)
apply(*clarify*, *simp add*:*abbrev Graph6 Graph12*)
apply(*erule conjE*)+
apply(*erule disjE*, *erule disjI1*, *rule disjI2*, *rule allI*, (*rule impI*)+, *case-tac R*=*i*, *rule conjI*, *erule sym*)
apply(*erule Graph4*)
apply(*simp*)+
apply (*simp add*:*BtoW-def*)
apply (*simp add*:*BtoW-def*)
apply(*rule conjI*)
apply (*force simp add*:*BtoW-def*)
apply(*erule Graph4*)
apply *simp*+
— 7 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph6 Graph12*)
apply(*erule conjE*)+
apply(*erule disjE*, *erule disjI1*, *rule disjI2*, *rule allI*, (*rule impI*)+, *case-tac R*=*i*, *rule conjI*, *erule sym*)
apply(*erule Graph4*)
apply(*simp*)+
apply (*simp add*:*BtoW-def*)
apply (*simp add*:*BtoW-def*)
apply(*rule conjI*)
apply (*force simp add*:*BtoW-def*)
apply(*erule Graph4*)
apply *simp*+
— 6 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph6 Graph12*)
apply(*erule conjE*)+
apply(*rule conjI*)
apply(*erule disjE*, *erule disjI1*, *rule disjI2*, *rule allI*, (*rule impI*)+, *case-tac R*=*i*, *rule conjI*, *erule sym*)
apply(*erule Graph4*)
apply(*simp*)+
apply (*simp add*:*BtoW-def*)
apply (*simp add*:*BtoW-def*)
apply(*rule conjI*)
apply (*force simp add*:*BtoW-def*)
apply(*erule Graph4*)
apply *simp*+
apply(*simp add*:*BtoW-def nth-list-update*)
apply *force*
— 5 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph6 Graph12*)

```
— 4 subgoals left
apply(clarify, simp add:abbrev Graph6 Graph12)
apply(rule conjI)
 apply(erule disjE, erule disjI1, rule disjI2, rule allI, (rule impI)+, case-tac R=i,
rule conjI, erule sym)
  apply(erule Graph4)
    apply(simp)+
   apply (simp add:BtoW-def)
  apply (simp add:BtoW-def)
 apply(rule conjI)
  apply (force simp add:BtoW-def)
 apply(erule Graph4)
    apply simp+
apply(rule conjI)
 apply(simp add:nth-list-update)
 apply force
apply(rule impI, rule impI, erule disjE, erule disjI1, case-tac R = (ind x) ,case-tac
M x ! T = Black)
  apply(force simp add:BtoW-def)
 apply(case-tac M x !snd (E x ! ind x)=Black)
  apply(rule disjI2)
  apply simp
  apply (erule Graph5)
  apply simp+
 apply(force simp add:BtoW-def)
apply(force simp add:BtoW-def)
— 3 subgoals left
apply(clarify, simp add:abbrev Graph6 Graph12)
— 2 subgoals left
apply(clarify, simp add:abbrev Graph6 Graph12)
apply(erule disjE, erule disjI1, rule disjI2, rule allI, (rule impI)+, case-tac R=i,
rule conjI, erule sym)
 apply clarify
 apply(erule Graph4)
   apply(simp)+
  apply (simp add:BtoW-def)
 apply (simp add:BtoW-def)
apply(rule conjI)
 apply (force simp add:BtoW-def)
apply(erule Graph4)
   apply simp+
done

lemma interfree-Redirect-Edge-Propagate-Black:
  interfree-aux (Some Redirect-Edge, {}, Some Propagate-Black)
apply (unfold modules )
apply interfree-aux
apply(clarify, simp add:abbrev)+
done
```

lemma *interfree-Propagate-Black-Color-Target*:
interfree-aux (*Some Propagate-Black*, {}, *Some Color-Target*)
apply (*unfold modules*)
apply *interfree-aux*
— 11 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph7 Graph8 Graph12*)+
apply(*erule conjE*)+
apply(*erule disjE*,*rule disjI1*,*erule psubset-subset-trans*,*erule Graph9*,
case-tac M x!*T*=*Black*, *rule disjI2*,*rotate-tac* −*1*, *simp add*: *Graph10*, *clarify*,
erule allE, *erule impE*, *assumption*, *erule impE*, *assumption*,
simp add:*BtoW-def*, *rule disjI1*, *erule subset-psubset-trans*, *erule Graph11*,
force)
— 7 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph7 Graph8 Graph12*)
apply(*erule conjE*)+
apply(*erule disjE*,*rule disjI1*,*erule psubset-subset-trans*,*erule Graph9*,
case-tac M x!*T*=*Black*, *rule disjI2*,*rotate-tac* −*1*, *simp add*: *Graph10*, *clarify*,
erule allE, *erule impE*, *assumption*, *erule impE*, *assumption*,
simp add:*BtoW-def*, *rule disjI1*, *erule subset-psubset-trans*, *erule Graph11*,
force)
— 6 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph7 Graph8 Graph12*)
apply *clarify*
apply (*rule conjI*)
apply(*erule disjE*,*rule disjI1*,*erule psubset-subset-trans*,*erule Graph9*,
case-tac M x!*T*=*Black*, *rule disjI2*,*rotate-tac* −*1*, *simp add*: *Graph10*, *clarify*,
erule allE, *erule impE*, *assumption*, *erule impE*, *assumption*,
simp add:*BtoW-def*, *rule disjI1*, *erule subset-psubset-trans*, *erule Graph11*,
force)
apply(*simp add*:*nth-list-update*)
— 5 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph7 Graph8 Graph12*)
— 4 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph7 Graph8 Graph12*)
apply (*rule conjI*)
apply(*erule disjE*,*rule disjI1*,*erule psubset-subset-trans*,*erule Graph9*,
case-tac M x!*T*=*Black*, *rule disjI2*,*rotate-tac* −*1*, *simp add*: *Graph10*, *clarify*,
erule allE, *erule impE*, *assumption*, *erule impE*, *assumption*,
simp add:*BtoW-def*, *rule disjI1*, *erule subset-psubset-trans*, *erule Graph11*,
force)
apply(*rule conjI*)
apply(*simp add*:*nth-list-update*)
apply(*rule impI*,*rule impI*, *case-tac M x*!*T*=*Black*,*rotate-tac* −*1*, *force simp add*:
BtoW-def Graph10,
erule subset-psubset-trans, *erule Graph11*, *force*)
— 3 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph7 Graph8 Graph12*)
— 2 subgoals left

apply(*clarify*, *simp add*:*abbrev Graph7 Graph8 Graph12*)
apply(*erule disjE*,*rule disjI1*,*erule psubset-subset-trans*,*erule Graph9*,
case-tac M x! *T*=*Black*, *rule disjI2*,*rotate-tac* −*1*, *simp add*: *Graph10*, *clarify*,
erule allE, *erule impE*, *assumption*, *erule impE*, *assumption*,
simp add:*BtoW-def*, *rule disjI1*, *erule subset-psubset-trans*, *erule Graph11*,
force)
— 3 subgoals left
apply(*simp add*:*abbrev*)
done

lemma *interfree-Color-Target-Propagate-Black*:
interfree-aux (*Some Color-Target*, {}, *Some Propagate-Black*)
apply (*unfold modules*)
apply *interfree-aux*
apply(*clarify*, *simp add*:*abbrev*)+
done

lemma *interfree-Count-Redirect-Edge*:
interfree-aux (*Some Count*, {}, *Some Redirect-Edge*)
apply (*unfold modules*)
apply *interfree-aux*
— 9 subgoals left
apply(*simp-all add*:*abbrev Graph6 Graph12*)
— 6 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph6 Graph12*,
erule disjE,*erule disjI1*,*rule disjI2*,*rule subset-trans*, *erule Graph3*,*force*,*force*)+
done

lemma *interfree-Redirect-Edge-Count*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Count*)
apply (*unfold modules*)
apply *interfree-aux*
apply(*clarify*,*simp add*:*abbrev*)+
apply(*simp add*:*abbrev*)
done

lemma *interfree-Count-Color-Target*:
interfree-aux (*Some Count*, {}, *Some Color-Target*)
apply (*unfold modules*)
apply *interfree-aux*
— 9 subgoals left
apply(*simp-all add*:*abbrev Graph7 Graph8 Graph12*)
— 6 subgoals left
apply(*clarify*,*simp add*:*abbrev Graph7 Graph8 Graph12*,
erule disjE, *erule disjI1*, *rule disjI2*,*erule subset-trans*, *erule Graph9*)+
— 2 subgoals left
apply(*clarify*, *simp add*:*abbrev Graph7 Graph8 Graph12*)
apply(*rule conjI*)
apply(*erule disjE*, *erule disjI1*, *rule disjI2*,*erule subset-trans*, *erule Graph9*)

apply(*simp add*:*nth-list-update*)
— 1 subgoal left
apply(*clarify*, *simp add*:*abbrev Graph7 Graph8 Graph12*,
erule disjE, *erule disjI1*, *rule disjI2*,*erule subset-trans*, *erule Graph9*)
done

lemma *interfree-Color-Target-Count*:
interfree-aux (*Some Color-Target*, {}, *Some Count*)
apply (*unfold modules*)
apply *interfree-aux*
apply(*clarify*, *simp add*:*abbrev*)+
apply(*simp add*:*abbrev*)
done

lemma *interfree-Append-Redirect-Edge*:
interfree-aux (*Some Append*, {}, *Some Redirect-Edge*)
apply (*unfold modules*)
apply *interfree-aux*
apply(*simp-all add*:*abbrev Graph6 Append-to-free0 Append-to-free1 Graph12*)
apply(*clarify*, *simp add*:*abbrev Graph6 Append-to-free0 Append-to-free1 Graph12*, *force dest*:*Graph3*)+
done

lemma *interfree-Redirect-Edge-Append*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Append*)
apply (*unfold modules*)
apply *interfree-aux*
apply(*clarify*, *simp add*:*abbrev Append-to-free0*)+
apply (*force simp add*: *Append-to-free2*)
apply(*clarify*, *simp add*:*abbrev Append-to-free0*)+
done

lemma *interfree-Append-Color-Target*:
interfree-aux (*Some Append*, {}, *Some Color-Target*)
apply (*unfold modules*)
apply *interfree-aux*
apply(*clarify*, *simp add*:*abbrev Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12 nth-list-update*)+
apply(*simp add*:*abbrev Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12 nth-list-update*)
done

lemma *interfree-Color-Target-Append*:
interfree-aux (*Some Color-Target*, {}, *Some Append*)
apply (*unfold modules*)
apply *interfree-aux*
apply(*clarify*, *simp add*:*abbrev Append-to-free0*)+
apply (*force simp add*: *Append-to-free2*)
apply(*clarify*,*simp add*:*abbrev Append-to-free0*)+

done

lemmas *collector-mutator-interfree* =
interfree-Blacken-Roots-Redirect-Edge interfree-Blacken-Roots-Color-Target
interfree-Propagate-Black-Redirect-Edge interfree-Propagate-Black-Color-Target
interfree-Count-Redirect-Edge interfree-Count-Color-Target
interfree-Append-Redirect-Edge interfree-Append-Color-Target
interfree-Redirect-Edge-Blacken-Roots interfree-Color-Target-Blacken-Roots
interfree-Redirect-Edge-Propagate-Black interfree-Color-Target-Propagate-Black
interfree-Redirect-Edge-Count interfree-Color-Target-Count
interfree-Redirect-Edge-Append interfree-Color-Target-Append

Interference freedom Collector-Mutator

lemma *interfree-Collector-Mutator*:
interfree-aux (*Some Collector*, {}, *Some Mutator*)
apply(*unfold Collector-def Mutator-def*)
apply *interfree-aux*
apply(*simp-all add*:*collector-mutator-interfree*)
apply(*unfold modules collector-defs mutator-defs*)
apply(*tactic* ⟪ *TRYALL* (*interfree-aux-tac*) ⟫)
— 32 subgoals left
apply(*simp-all add*:*Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12*)
— 20 subgoals left
apply(*tactic*⟪ *TRYALL Clarify-tac* ⟫)
apply(*simp-all add*:*Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12*)
apply(*tactic* ⟪ *TRYALL* (*etac disjE*) ⟫)
apply *simp-all*
apply(*tactic* ⟪ *TRYALL*(*EVERY'*[*rtac disjI2*,*rtac subset-trans*,*etac* (*thm Graph3*),*Force-tac*, *assume-tac*]) ⟫)
apply(*tactic* ⟪ *TRYALL*(*EVERY'*[*rtac disjI2*,*etac subset-trans*,*rtac* (*thm Graph9*),*Force-tac*]) ⟫)
apply(*tactic* ⟪ *TRYALL*(*EVERY'*[*rtac disjI1*,*etac psubset-subset-trans*,*rtac* (*thm Graph9*),*Force-tac*]) ⟫)
done

Interference freedom Mutator-Collector

lemma *interfree-Mutator-Collector*:
interfree-aux (*Some Mutator*, {}, *Some Collector*)
apply(*unfold Collector-def Mutator-def*)
apply *interfree-aux*
apply(*simp-all add*:*collector-mutator-interfree*)
apply(*unfold modules collector-defs mutator-defs*)
apply(*tactic* ⟪ *TRYALL* (*interfree-aux-tac*) ⟫)
— 64 subgoals left
apply(*simp-all add*:*nth-list-update Invariants Append-to-free0*)+
apply(*tactic*⟪ *TRYALL Clarify-tac* ⟫)
— 4 subgoals left
apply *force*

apply(*simp add*:*Append-to-free2*)
apply *force*
apply(*simp add*:*Append-to-free2*)
done

The Garbage Collection algorithm

In total there are 289 verification conditions.

lemma *Gar-Coll*:
  ‖− .{´*Proper* ∧ ´*Mut-init* ∧ ´*z*}.
  COBEGIN
   Collector
  .{*False*}.
 ‖
   Mutator
  .{*False*}.
 COEND
  .{*False*}.
apply *oghoare*
apply(*force simp add*: *Mutator-def Collector-def modules*)
apply(*rule Collector*)
apply(*rule Mutator*)
apply(*simp add*:*interfree-Collector-Mutator*)
apply(*simp add*:*interfree-Mutator-Collector*)
apply *force*
done

end

2.3 The Multi-Mutator Case

theory *Mul-Gar-Coll* **imports** *Graph OG-Syntax* **begin**

The full theory takes aprox. 18 minutes.

record *mut* =
  Z :: *bool*
  R :: *nat*
  T :: *nat*

Declaration of variables:

record *mul-gar-coll-state* =
  M :: *nodes*
  E :: *edges*
  bc :: *nat set*
  obc :: *nat set*
  Ma :: *nodes*
  ind :: *nat*

k :: *nat*
q :: *nat*
l :: *nat*
Muts :: *mut list*

2.3.1 The Mutators

constdefs
Mul-mut-init :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool*
Mul-mut-init ≡ ≪ λ*n*. *n*=*length* ´*Muts* ∧ (∀ *i*<*n*. *R* (´*Muts*!*i*)<*length* ´*E*
∧ *T* (´*Muts*!*i*)<*length* ´*M*) ≫

Mul-Redirect-Edge :: *nat* ⇒ *nat* ⇒ *mul-gar-coll-state ann-com*
Mul-Redirect-Edge j n ≡
.{´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*)}.
⟨*IF T*(´*Muts*!*j*) ∈ *Reach* ´*E THEN*
´*E*:= ´*E*[*R* (´*Muts*!*j*):= (*fst* (´*E*!*R*(´*Muts*!*j*)), *T* (´*Muts*!*j*))] *FI*,,
´*Muts*:= ´*Muts*[*j*:= (´*Muts*!*j*) (|*Z*:=*False*|)]⟩

Mul-Color-Target :: *nat* ⇒ *nat* ⇒ *mul-gar-coll-state ann-com*
Mul-Color-Target j n ≡
.{´*Mul-mut-init n* ∧ ¬ *Z* (´*Muts*!*j*)}.
⟨´*M*:=´*M*[*T* (´*Muts*!*j*):=*Black*],, ´*Muts*:=´*Muts*[*j*:= (´*Muts*!*j*) (|*Z*:=*True*|)]⟩

Mul-Mutator :: *nat* ⇒ *nat* ⇒ *mul-gar-coll-state ann-com*
Mul-Mutator j n ≡
.{´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*)}.
WHILE True
  INV .{´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*)}.
DO Mul-Redirect-Edge j n ;;
   Mul-Color-Target j n
OD

lemmas *mul-mutator-defs* = *Mul-mut-init-def Mul-Redirect-Edge-def Mul-Color-Target-def*

Correctness of the proof outline of one mutator

lemma *Mul-Redirect-Edge*: *0*≤*j* ∧ *j*<*n* ⟹
 ⊢ *Mul-Redirect-Edge j n*
   pre(*Mul-Color-Target j n*)
apply (*unfold mul-mutator-defs*)
apply *annhoare*
apply(*simp-all*)
apply *clarify*
apply(*simp add*:*nth-list-update*)
done

lemma *Mul-Color-Target*: *0*≤*j* ∧ *j*<*n* ⟹
 ⊢ *Mul-Color-Target j n*
  .{´*Mul-mut-init n* ∧ *Z* (´*Muts*!*j*)}.

apply (*unfold mul-mutator-defs*)
apply *annhoare*
apply(*simp-all*)
apply *clarify*
apply(*simp add*:*nth-list-update*)
done

lemma *Mul-Mutator*: $0 \leq j \wedge j<n \Longrightarrow$
$\vdash$ *Mul-Mutator j n* .{*False*}.
apply(*unfold Mul-Mutator-def*)
apply *annhoare*
apply(*simp-all add*:*Mul-Redirect-Edge Mul-Color-Target*)
apply(*simp add*:*mul-mutator-defs Mul-Redirect-Edge-def*)
done

Interference freedom between mutators

lemma *Mul-interfree-Redirect-Edge-Redirect-Edge*:
$[\![0 \leq i;\ i<n;\ 0 \leq j;\ j<n;\ i \neq j]\!] \Longrightarrow$
interfree-aux (*Some* (*Mul-Redirect-Edge i n*),{}, *Some*(*Mul-Redirect-Edge j n*))
apply (*unfold mul-mutator-defs*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*: *nth-list-update*)
done

lemma *Mul-interfree-Redirect-Edge-Color-Target*:
$[\![0 \leq i;\ i<n;\ 0 \leq j;\ j<n;\ i \neq j]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Redirect-Edge i n*),{},*Some*(*Mul-Color-Target j n*))
apply (*unfold mul-mutator-defs*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*: *nth-list-update*)
done

lemma *Mul-interfree-Color-Target-Redirect-Edge*:
$[\![0 \leq i;\ i<n;\ 0 \leq j;\ j<n;\ i \neq j]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Color-Target i n*),{},*Some*(*Mul-Redirect-Edge j n*))
apply (*unfold mul-mutator-defs*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*nth-list-update*)
done

lemma *Mul-interfree-Color-Target-Color-Target*:
$[\![0 \leq i;\ i<n;\ 0 \leq j;\ j<n;\ i \neq j]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Color-Target i n*),{},*Some*(*Mul-Color-Target j n*))
apply (*unfold mul-mutator-defs*)
apply *interfree-aux*

apply *safe*
apply(*simp-all add*: *nth-list-update*)
done

lemmas *mul-mutator-interfree* =
Mul-interfree-Redirect-Edge-Redirect-Edge Mul-interfree-Redirect-Edge-Color-Target
Mul-interfree-Color-Target-Redirect-Edge Mul-interfree-Color-Target-Color-Target

lemma *Mul-interfree-Mutator-Mutator*: $[\![i < n; j < n; i \neq j]\!] \Longrightarrow$
interfree-aux (*Some* (*Mul-Mutator i n*), {}, *Some* (*Mul-Mutator j n*))
apply(*unfold Mul-Mutator-def*)
apply(*interfree-aux*)
apply(*simp-all add*:*mul-mutator-interfree*)
apply(*simp-all add*: *mul-mutator-defs*)
apply(*tactic* ⟪ *TRYALL* (*interfree-aux-tac*) ⟫)
apply(*tactic* ⟪ *ALLGOALS Clarify-tac* ⟫)
apply (*simp-all add*:*nth-list-update*)
done

Modular Parameterized Mutators

lemma *Mul-Parameterized-Mutators*: $0<n \Longrightarrow$
‖− .{´*Mul-mut-init n* ∧ (∀ *i*<*n*. *Z* (´*Muts*!*i*))}.
COBEGIN
SCHEME [$0\leq j< n$]
Mul-Mutator j n
.{*False*}.
COEND
.{*False*}.
apply *oghoare*
apply(*force simp add*:*Mul-Mutator-def mul-mutator-defs nth-list-update*)
apply(*erule Mul-Mutator*)
apply(*simp add*:*Mul-interfree-Mutator-Mutator*)
apply(*force simp add*:*Mul-Mutator-def mul-mutator-defs nth-list-update*)
done

2.3.2 The Collector

constdefs
Queue :: *mul-gar-coll-state* ⇒ *nat*
Queue ≡ ≪ *length* (*filter* (λ*i*. ¬ *Z i* ∧ ´*M*!(*T i*) ≠ *Black*) ´*Muts*) ≫

consts *M-init* :: *nodes*

constdefs
Proper-M-init :: *mul-gar-coll-state* ⇒ *bool*
Proper-M-init ≡ ≪ *Blacks M-init*=*Roots* ∧ *length M-init*=*length* ´*M* ≫

Mul-Proper :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool*

Mul-Proper $\equiv \ll \lambda n.$ *Proper-Roots* ´*M* $\wedge$ *Proper-Edges* (´*M*, ´*E*) $\wedge$ ´*Proper-M-init* $\wedge$ *n=length* ´*Muts* $\gg$

Safe :: *mul-gar-coll-state* $\Rightarrow$ *bool*
Safe $\equiv \ll$ *Reach* ´*E* $\subseteq$ *Blacks* ´*M* $\gg$

lemmas *mul-collector-defs* = *Proper-M-init-def Mul-Proper-def Safe-def*

Blackening Roots

constdefs
Mul-Blacken-Roots :: *nat* $\Rightarrow$ *mul-gar-coll-state ann-com*
Mul-Blacken-Roots n $\equiv$
.{´*Mul-Proper n*}.
´*ind*:=*0*;;
.{´*Mul-Proper n* $\wedge$ ´*ind=0*}.
WHILE ´*ind*<*length* ´*M*
INV .{´*Mul-Proper n* $\wedge$ ($\forall$ *i*<´*ind*. *i*$\in$*Roots* $\longrightarrow$ ´*M*!*i*=*Black*) $\wedge$ ´*ind*$\leq$*length* ´*M*}.
DO .{´*Mul-Proper n* $\wedge$ ($\forall$ *i*<´*ind*. *i*$\in$*Roots* $\longrightarrow$ ´*M*!*i*=*Black*) $\wedge$ ´*ind*<*length* ´*M*}.
IF ´*ind*$\in$*Roots THEN*
.{´*Mul-Proper n* $\wedge$ ($\forall$ *i*<´*ind*. *i*$\in$*Roots* $\longrightarrow$ ´*M*!*i*=*Black*) $\wedge$ ´*ind*<*length* ´*M* $\wedge$ ´*ind*$\in$*Roots*}.
´*M*:=´*M*[´*ind*:=*Black*] *FI*;;
.{´*Mul-Proper n* $\wedge$ ($\forall$ *i*<´*ind*+*1*. *i*$\in$*Roots* $\longrightarrow$ ´*M*!*i*=*Black*) $\wedge$ ´*ind*<*length* ´*M*}.
´*ind*:=´*ind*+*1*
OD

lemma *Mul-Blacken-Roots*:
$\vdash$ *Mul-Blacken-Roots n*
.{´*Mul-Proper n* $\wedge$ *Roots* $\subseteq$ *Blacks* ´*M*}.
apply (*unfold Mul-Blacken-Roots-def*)
apply *annhoare*
apply(*simp-all add*:*mul-collector-defs Graph-defs*)
apply *safe*
apply(*simp-all add*:*nth-list-update*)
apply (*erule less-SucE*)
apply *simp+*
apply *force*
apply *force*
done

Propagating Black

constdefs
Mul-PBInv :: *mul-gar-coll-state* $\Rightarrow$ *bool*
Mul-PBInv $\equiv$ $\ll$´*Safe* $\vee$ ´*obc*$\subset$*Blacks* ´*M* $\vee$ ´*l*<´*Queue*
$\vee$ ($\forall$ *i*<´*ind*. $\neg$*BtoW*(´*E*!*i*,´*M*)) $\wedge$ ´*l*$\leq$´*Queue*$\gg$

Mul-Auxk :: *mul-gar-coll-state* $\Rightarrow$ *bool*
Mul-Auxk $\equiv \ll$ ´l<´*Queue* $\vee$ ´M!´$k\neq$*Black* $\vee$ $\neg$*BtoW*(´E!´*ind*, ´M) $\vee$ ´*obc*$\subset$*Blacks* ´$M\gg$

constdefs
Mul-Propagate-Black :: *nat* $\Rightarrow$ *mul-gar-coll-state ann-com*
Mul-Propagate-Black n $\equiv$
.{´*Mul-Proper n* $\wedge$ *Roots*$\subseteq$*Blacks* ´M $\wedge$ ´*obc*$\subseteq$*Blacks* ´M $\wedge$ ´*bc*$\subseteq$*Blacks* ´M
$\wedge$ (´*Safe* $\vee$ ´$l\leq$´*Queue* $\vee$ ´*obc*$\subset$*Blacks* ´M)}.
´*ind*:=*0*;;
.{´*Mul-Proper n* $\wedge$ *Roots*$\subseteq$*Blacks* ´M
$\wedge$ ´*obc*$\subseteq$*Blacks* ´M $\wedge$ *Blacks* ´$M\subseteq$*Blacks* ´M $\wedge$ ´*bc*$\subseteq$*Blacks* ´M
$\wedge$ (´*Safe* $\vee$ ´$l\leq$´*Queue* $\vee$ ´*obc*$\subset$*Blacks* ´M) $\wedge$ ´*ind*=*0*}.
WHILE ´*ind*<*length* ´E
INV .{´*Mul-Proper n* $\wedge$ *Roots*$\subseteq$*Blacks* ´M
$\wedge$ ´*obc*$\subseteq$*Blacks* ´M $\wedge$ ´*bc*$\subseteq$*Blacks* ´M
$\wedge$ ´*Mul-PBInv* $\wedge$ ´*ind*$\leq$*length* ´E}.
DO .{´*Mul-Proper n* $\wedge$ *Roots*$\subseteq$*Blacks* ´M
$\wedge$ ´*obc*$\subseteq$*Blacks* ´M $\wedge$ ´*bc*$\subseteq$*Blacks* ´M
$\wedge$ ´*Mul-PBInv* $\wedge$ ´*ind*<*length* ´E}.
IF ´M!(*fst* (´E!´*ind*))=*Black THEN*
.{´*Mul-Proper n* $\wedge$ *Roots*$\subseteq$*Blacks* ´M
$\wedge$ ´*obc*$\subseteq$*Blacks* ´M $\wedge$ ´*bc*$\subseteq$*Blacks* ´M
$\wedge$ ´*Mul-PBInv* $\wedge$ (´M!*fst*(´E!´*ind*))=*Black* $\wedge$ ´*ind*<*length* ´E}.
´k:=*snd*(´E!´*ind*);;
.{´*Mul-Proper n* $\wedge$ *Roots*$\subseteq$*Blacks* ´M
$\wedge$ ´*obc*$\subseteq$*Blacks* ´M $\wedge$ ´*bc*$\subseteq$*Blacks* ´M
$\wedge$ (´*Safe* $\vee$ ´*obc*$\subset$*Blacks* ´M $\vee$ ´l<´*Queue* $\vee$ ($\forall$ i<´*ind*. $\neg$*BtoW*(´E!i,´M))
$\wedge$ ´$l\leq$´*Queue* $\wedge$ ´*Mul-Auxk*) $\wedge$ ´k<*length* ´M $\wedge$ ´M!*fst*(´E!´*ind*)=*Black*
$\wedge$ ´*ind*<*length* ´E}.
$\langle$´M:=´M[´k:=*Black*],,´*ind*:=´*ind*+*1*$\rangle$
ELSE .{´*Mul-Proper n* $\wedge$ *Roots*$\subseteq$*Blacks* ´M
$\wedge$ ´*obc*$\subseteq$*Blacks* ´M $\wedge$ ´*bc*$\subseteq$*Blacks* ´M
$\wedge$ ´*Mul-PBInv* $\wedge$ ´*ind*<*length* ´E}.
$\langle$*IF* ´M!(*fst* (´E!´*ind*))$\neq$*Black THEN* ´*ind*:=´*ind*+*1 FI*$\rangle$ *FI*
OD

lemma *Mul-Propagate-Black*:
$\vdash$ *Mul-Propagate-Black n*
.{´*Mul-Proper n* $\wedge$ *Roots*$\subseteq$*Blacks* ´M $\wedge$ ´*obc*$\subseteq$*Blacks* ´M $\wedge$ ´*bc*$\subseteq$*Blacks* ´M
$\wedge$ (´*Safe* $\vee$ ´*obc*$\subset$*Blacks* ´M $\vee$ ´l<´*Queue* $\wedge$ (´$l\leq$´*Queue* $\vee$ ´*obc*$\subset$*Blacks* ´M))}.
apply(*unfold Mul-Propagate-Black-def*)
apply *annhoare*
apply(*simp-all add*:*Mul-PBInv-def mul-collector-defs Mul-Auxk-def Graph6 Graph7 Graph8 Graph12 mul-collector-defs Queue-def*)
— 8 subgoals left
apply *force*

apply *force*
apply *force*
apply(*force simp add*:*BtoW-def Graph-defs*)
— 4 subgoals left
apply *clarify*
apply(*simp add*: *mul-collector-defs Graph12 Graph6 Graph7 Graph8*)
apply(*disjE-tac*)
 apply(*simp-all add*:*Graph12 Graph13*)
 apply(*case-tac M x*! *k x*=*Black*)
  apply(*simp add*: *Graph10*)
 apply(*rule disjI2*, *rule disjI1*, *erule subset-psubset-trans*, *erule Graph11*, *force*)
apply(*case-tac M x*! *k x*=*Black*)
 apply(*simp add*: *Graph10 BtoW-def*)
 apply(*rule disjI2*, *clarify*, *erule less-SucE*, *force*)
 apply(*case-tac M x*!*snd*(*E x*! *ind x*)=*Black*)
  apply(*force*)
 apply(*force*)
apply(*rule disjI2*, *rule disjI1*, *erule subset-psubset-trans*, *erule Graph11*, *force*)
— 3 subgoals left
apply *force*
— 2 subgoals left
apply *clarify*
apply(*conjI-tac*)
apply(*disjE-tac*)
 apply (*simp-all*)
apply *clarify*
apply(*erule less-SucE*)
 apply *force*
apply (*simp add*:*BtoW-def*)
— 1 subgoal left
apply *clarify*
apply *simp*
apply(*disjE-tac*)
apply (*simp-all*)
apply(*rule disjI1* , *rule Graph1*)
 apply *simp-all*
done

Counting Black Nodes

constdefs
 Mul-CountInv :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool*
 Mul-CountInv ≡ ≪ λ*ind*. {*i*. *i*<*ind* ∧ ´*Ma*!*i*=*Black*}⊆´*bc* ≫

 Mul-Count :: *nat* ⇒ *mul-gar-coll-state ann-com*
 Mul-Count n ≡
 .{´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M*
  ∧ ´*obc*⊆*Blacks* ´*Ma* ∧ *Blacks* ´*Ma*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
  ∧ *length* ´*Ma*=*length* ´*M*

```
    ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M) )
    ∧ ´q<n+1 ∧ ´bc={}}.
  ´ind:=0;;
  .{´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ length ´Ma=length ´M
    ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M) )
    ∧ ´q<n+1 ∧ ´bc={} ∧ ´ind=0}.
  WHILE ´ind<length ´M
     INV .{´Mul-Proper n ∧ Roots⊆Blacks ´M
          ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
          ∧ length ´Ma=length ´M ∧ ´Mul-CountInv ´ind
            ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks
´M))
          ∧ ´q<n+1 ∧ ´ind≤length ´M}.
  DO .{´Mul-Proper n ∧ Roots⊆Blacks ´M
       ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
       ∧ length ´Ma=length ´M ∧ ´Mul-CountInv ´ind
       ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
       ∧ ´q<n+1 ∧ ´ind<length ´M}.
     IF ´M!´ind=Black
     THEN .{´Mul-Proper n ∧ Roots⊆Blacks ´M
           ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
           ∧ length ´Ma=length ´M ∧ ´Mul-CountInv ´ind
            ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks
´M))
           ∧ ´q<n+1 ∧ ´ind<length ´M ∧ ´M!´ind=Black}.
          ´bc:=insert ´ind ´bc
     FI;;
  .{´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ length ´Ma=length ´M ∧ ´Mul-CountInv (´ind+1)
    ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
    ∧ ´q<n+1 ∧ ´ind<length ´M}.
  ´ind:=´ind+1
  OD

lemma Mul-Count:
 ⊢ Mul-Count n
  .{´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ length ´Ma=length ´M ∧ Blacks ´Ma⊆´bc
    ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
    ∧ ´q<n+1}.
apply (unfold Mul-Count-def)
apply annhoare
apply(simp-all add:Mul-CountInv-def mul-collector-defs Mul-Auxk-def Graph6 Graph7
Graph8 Graph12 mul-collector-defs Queue-def)
— 7 subgoals left
```

apply *force*
apply *force*
apply *force*
— 4 subgoals left
apply *clarify*
apply(*conjI-tac*)
apply(*disjE-tac*)
 apply *simp-all*
apply(*simp add*:*Blacks-def*)
apply *clarify*
apply(*erule less-SucE*)
 back
 apply *force*
apply *force*
— 3 subgoals left
apply *clarify*
apply(*conjI-tac*)
apply(*disjE-tac*)
 apply *simp-all*
apply *clarify*
apply(*erule less-SucE*)
 back
 apply *force*
apply *simp*
apply(*rotate-tac* -1)
apply (*force simp add*:*Blacks-def*)
— 2 subgoals left
apply *force*
— 1 subgoal left
apply *clarify*
apply(*drule le-imp-less-or-eq*)
apply(*disjE-tac*)
apply (*simp-all add*:*Blacks-def*)
done

Appending garbage nodes to the free list

consts *Append-to-free* :: *nat* $\times$ *edges* $\Rightarrow$ *edges*

axioms
 Append-to-free0: *length* (*Append-to-free* (i, e)) = *length* e
 Append-to-free1: *Proper-Edges* (m, e)
 $\Longrightarrow$ *Proper-Edges* (m, *Append-to-free*(i, e))
 Append-to-free2: $i \notin$ *Reach* e
 $\Longrightarrow$ $n \in$ *Reach* (*Append-to-free*(i, e)) = ($n = i \vee n \in$ *Reach* e)

constdefs
 Mul-AppendInv :: *mul-gar-coll-state* $\Rightarrow$ *nat* $\Rightarrow$ *bool*
 Mul-AppendInv $\equiv$ « λ*ind*. ($\forall\, i$. *ind*$\leq$*i* $\longrightarrow$ *i*<*length* ´*M* $\longrightarrow$ *i*$\in$*Reach* ´*E* $\longrightarrow$

$\acute{}M!i{=}Black)\gg$

Mul-Append :: *nat* ⇒ *mul-gar-coll-state ann-com*
Mul-Append n ≡
.{´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*Safe*}.
´*ind*:=*0*;;
.{´*Mul-Proper n* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*Safe* ∧ ´*ind*=*0*}.
WHILE ´*ind*<*length* ´*M*
INV .{´*Mul-Proper n* ∧ ´*Mul-AppendInv* ´*ind* ∧ ´*ind*≤*length* ´*M*}.
DO .{´*Mul-Proper n* ∧ ´*Mul-AppendInv* ´*ind* ∧ ´*ind*<*length* ´*M*}.
IF ´*M*!´*ind*=*Black THEN*
.{´*Mul-Proper n* ∧ ´*Mul-AppendInv* ´*ind* ∧ ´*ind*<*length* ´*M* ∧ ´*M*!´*ind*=*Black*}.

´*M*:=´*M*[´*ind*:=*White*]
ELSE
.{´*Mul-Proper n* ∧ ´*Mul-AppendInv* ´*ind* ∧ ´*ind*<*length* ´*M* ∧ ´*ind*∉*Reach* ´*E*}.
´*E*:=*Append-to-free*(´*ind*,´*E*)
FI;;
.{´*Mul-Proper n* ∧ ´*Mul-AppendInv* (´*ind*+*1*) ∧ ´*ind*<*length* ´*M*}.
´*ind*:=´*ind*+*1*
OD

lemma *Mul-Append*:
⊢ *Mul-Append n*
.{´*Mul-Proper n*}.
apply(*unfold Mul-Append-def*)
apply *annhoare*
apply(*simp-all add*: *mul-collector-defs Mul-AppendInv-def*
Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12)
apply(*force simp add*:*Blacks-def*)
apply(*force simp add*:*Blacks-def*)
apply(*force simp add*:*Blacks-def*)
apply(*force simp add*:*Graph-defs*)
apply *force*
apply(*force simp add*:*Append-to-free1 Append-to-free2*)
apply *force*
apply *force*
done

Collector

constdefs
Mul-Collector :: *nat* ⇒ *mul-gar-coll-state ann-com*
Mul-Collector n ≡
.{´*Mul-Proper n*}.
WHILE True INV .{´*Mul-Proper n*}.
DO
Mul-Blacken-Roots n ;;

.{´Mul-Proper n ∧ Roots⊆Blacks ´M}.
´obc:={};;
.{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc={}}.
´bc:=Roots;;
.{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc={} ∧ ´bc=Roots}.
´l:=0;;
.{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc={} ∧ ´bc=Roots ∧ ´l=0}.
WHILE ´l<n+1
  INV .{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´bc⊆Blacks ´M ∧
      (´Safe ∨ (´l≤´Queue ∨ ´bc⊂Blacks ´M) ∧ ´l<n+1)}.
DO .{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ (´Safe ∨ ´l≤´Queue ∨ ´bc⊂Blacks ´M)}.
  ´obc:=´bc;;
  Mul-Propagate-Black n;;
  .{´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ (´Safe ∨ ´obc⊂Blacks ´M ∨ ´l<´Queue
    ∧ (´l≤´Queue ∨ ´obc⊂Blacks ´M))}.
  ´bc:={};;
  .{´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ (´Safe ∨ ´obc⊂Blacks ´M ∨ ´l<´Queue
    ∧ (´l≤´Queue ∨ ´obc⊂Blacks ´M)) ∧ ´bc={}}.
    ⟨ ´Ma:=´M,, ´q:=´Queue ⟩;;
  Mul-Count n;;
  .{´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ length ´Ma=length ´M ∧ Blacks ´Ma⊆´bc
    ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
    ∧ ´q<n+1}.
  IF ´obc=´bc THEN
  .{´Mul-Proper n ∧ Roots⊆Blacks ´M
    ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
    ∧ length ´Ma=length ´M ∧ Blacks ´Ma⊆´bc
    ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
    ∧ ´q<n+1 ∧ ´obc=´bc}.
  ´l:=´l+1
  ELSE .{´Mul-Proper n ∧ Roots⊆Blacks ´M
      ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
      ∧ length ´Ma=length ´M ∧ Blacks ´Ma⊆´bc
        ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks
´M))
      ∧ ´q<n+1 ∧ ´obc≠´bc}.
    ´l:=0 FI
OD;;
Mul-Append n
OD

lemmas *mul-modules = Mul-Redirect-Edge-def Mul-Color-Target-def*

Mul-Blacken-Roots-def Mul-Propagate-Black-def
Mul-Count-def Mul-Append-def

lemma *Mul-Collector*:
$\vdash$ *Mul-Collector n*
.{*False*}.
apply(*unfold Mul-Collector-def*)
apply *annhoare*
apply(*simp-all only*:*pre.simps Mul-Blacken-Roots*
Mul-Propagate-Black Mul-Count Mul-Append)
apply(*simp-all add*:*mul-modules*)
apply(*simp-all add*:*mul-collector-defs Queue-def*)
apply *force*
apply *force*
apply *force*
apply (*force simp add*: *less-Suc-eq-le*)
apply *force*
apply (*force dest*:*subset-antisym*)
apply *force*
apply *force*
apply *force*
done

2.3.3 Interference Freedom

lemma *le-length-filter-update*[*rule-format*]:
$\forall i.\ (\neg P\ (list!i) \vee P\ j) \wedge i<length\ list$
$\longrightarrow length(filter\ P\ list) \leq length(filter\ P\ (list[i:=j]))$
apply(*induct-tac list*)
apply(*simp*)
apply(*clarify*)
apply(*case-tac i*)
apply(*simp*)
apply(*simp*)
done

lemma *less-length-filter-update* [*rule-format*]:
$\forall i.\ P\ j \wedge \neg(P\ (list!i)) \wedge i<length\ list$
$\longrightarrow length(filter\ P\ list) < length(filter\ P\ (list[i:=j]))$
apply(*induct-tac list*)
apply(*simp*)
apply(*clarify*)
apply(*case-tac i*)
apply(*simp*)
apply(*simp*)
done

lemma *Mul-interfree-Blacken-Roots-Redirect-Edge*: $[\![0\leq j;\ j<n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Blacken-Roots n*),{},*Some*(*Mul-Redirect-Edge j n*))

apply (*unfold mul-modules*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*Graph6 Graph9 Graph12 nth-list-update mul-mutator-defs mul-collector-defs*)
done

lemma *Mul-interfree-Redirect-Edge-Blacken-Roots*: $[\![0 \le j;\ j < n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Redirect-Edge j n*),{},*Some* (*Mul-Blacken-Roots n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*mul-mutator-defs nth-list-update*)
done

lemma *Mul-interfree-Blacken-Roots-Color-Target*: $[\![0 \le j;\ j < n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Blacken-Roots n*),{},*Some* (*Mul-Color-Target j n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*mul-mutator-defs mul-collector-defs nth-list-update Graph7 Graph8 Graph9 Graph12*)
done

lemma *Mul-interfree-Color-Target-Blacken-Roots*: $[\![0 \le j;\ j < n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Color-Target j n*),{},*Some* (*Mul-Blacken-Roots n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*mul-mutator-defs nth-list-update*)
done

lemma *Mul-interfree-Propagate-Black-Redirect-Edge*: $[\![0 \le j;\ j < n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Propagate-Black n*),{},*Some* (*Mul-Redirect-Edge j n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply(*simp-all add*:*mul-mutator-defs mul-collector-defs Mul-PBInv-def nth-list-update Graph6*)
— 7 subgoals left
apply *clarify*
apply(*disjE-tac*)
apply(*simp-all add*:*Graph6*)
apply(*rule impI*,*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
— 6 subgoals left
apply *clarify*

apply(*disjE-tac*)
apply(*simp-all add*:*Graph6*)
apply(*rule impI*,*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
— 5 subgoals left
apply *clarify*
apply(*disjE-tac*)
apply(*simp-all add*:*Graph6*)
apply(*rule impI*,*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule less-le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule less-le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*erule conjE*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*rule conjI*)
apply(*rule impI*,(*rule disjI2*)+,*rule conjI*)
apply *clarify*
apply(*case-tac R* (*Muts x*! *j*)=*i*)
apply (*force simp add*: *nth-list-update BtoW-def*)
apply (*force simp add*: *nth-list-update*)
apply(*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,(*rule disjI2*)+, *erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*, *erule le-less-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le less-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*, *erule le-less-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le less-length-filter-update*)
— 4 subgoals left
apply *clarify*
apply(*disjE-tac*)
apply(*simp-all add*:*Graph6*)
apply(*rule impI*,*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule less-le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule less-le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*erule conjE*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*rule conjI*)
apply(*rule impI*,(*rule disjI2*)+,*rule conjI*)
apply *clarify*

apply(*case-tac R* (*Muts x*! *j*)=*i*)
apply (*force simp add*: *nth-list-update BtoW-def*)
apply (*force simp add*: *nth-list-update*)
apply(*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,(*rule disjI2*)+, *erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*, *erule le-less-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le less-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*, *erule le-less-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le less-length-filter-update*)
— 3 subgoals left
apply *clarify*
apply(*disjE-tac*)
apply(*simp-all add*:*Graph6*)
apply (*rule impI*)
apply(*rule conjI*)
apply(*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*case-tac R* (*Muts x* ! *j*)= *ind x*)
apply(*simp add*:*nth-list-update*)
apply(*simp add*:*nth-list-update*)
apply(*case-tac R* (*Muts x* ! *j*)= *ind x*)
apply(*simp add*:*nth-list-update*)
apply(*simp add*:*nth-list-update*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*rule conjI*)
apply(*rule impI*)
apply(*rule conjI*)
apply(*rule disjI2*,*rule disjI2*,*rule disjI1*, *erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*case-tac R* (*Muts x* ! *j*)= *ind x*)
apply(*simp add*:*nth-list-update*)
apply(*simp add*:*nth-list-update*)
apply(*rule impI*)
apply(*rule disjI2*,*rule disjI2*,*rule disjI1*, *erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule conjI*)
apply(*rule impI*)
apply(*rule conjI*)
apply(*rule disjI2*,*rule disjI2*,*rule disjI1*, *erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*case-tac R* (*Muts x* ! *j*)= *ind x*)
apply(*simp add*:*nth-list-update*)
apply(*simp add*:*nth-list-update*)
apply(*rule impI*)
apply(*rule disjI2*,*rule disjI2*,*rule disjI1*, *erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*erule conjE*)
apply(*rule conjI*)

```
  apply(case-tac M x!(T (Muts x!j))=Black)
   apply(rule impI,rule conjI,(rule disjI2)+,rule conjI)
    apply clarify
    apply(case-tac R (Muts x! j)=i)
     apply (force simp add: nth-list-update BtoW-def)
    apply (force simp add: nth-list-update)
  apply(erule le-trans,force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
  apply(case-tac R (Muts x ! j)= ind x)
   apply(simp add:nth-list-update)
  apply(simp add:nth-list-update)
 apply(rule impI,rule conjI)
  apply(rule disjI2,rule disjI2,rule disjI1, erule le-less-trans)
  apply(force simp add:Queue-def less-Suc-eq-le less-length-filter-update)
 apply(case-tac R (Muts x! j)=ind x)
  apply (force simp add: nth-list-update)
 apply (force simp add: nth-list-update)
apply(rule impI, (rule disjI2)+, erule le-trans)
apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
— 2 subgoals left
apply clarify
apply(rule conjI)
 apply(disjE-tac)
  apply(simp-all add:Mul-Auxk-def Graph6)
  apply (rule impI)
   apply(rule conjI)
    apply(rule disjI1,rule subset-trans,erule Graph3,simp,simp)
   apply(case-tac R (Muts x ! j)= ind x)
    apply(simp add:nth-list-update)
   apply(simp add:nth-list-update)
  apply(case-tac R (Muts x ! j)= ind x)
   apply(simp add:nth-list-update)
  apply(simp add:nth-list-update)
 apply(case-tac M x!(T (Muts x!j))=Black)
  apply(rule impI)
  apply(rule conjI)
   apply(rule disjI2,rule disjI2,rule disjI1, erule less-le-trans)
   apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
  apply(case-tac R (Muts x ! j)= ind x)
   apply(simp add:nth-list-update)
  apply(simp add:nth-list-update)
 apply(rule impI)
 apply(rule conjI)
  apply(rule disjI2,rule disjI2,rule disjI1, erule less-le-trans)
  apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
 apply(case-tac R (Muts x ! j)= ind x)
  apply(simp add:nth-list-update)
 apply(simp add:nth-list-update)
apply(rule impI)
apply(rule conjI)
```

apply(*erule conjE*)+
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply((*rule disjI2*)+,*rule conjI*)
apply *clarify*
apply(*case-tac R* (*Muts x*! *j*)=*i*)
apply (*force simp add*: *nth-list-update BtoW-def*)
apply (*force simp add*: *nth-list-update*)
apply(*rule conjI*)
apply(*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*)
apply(*case-tac R* (*Muts x* ! *j*)= *ind x*)
apply(*simp add*:*nth-list-update BtoW-def*)
apply (*simp add*:*nth-list-update*)
apply(*rule impI*)
apply *simp*
apply(*disjE-tac*)
apply(*rule disjI1*, *erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply *force*
apply(*rule disjI2*,*rule disjI2*,*rule disjI1*, *erule le-less-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le less-length-filter-update*)
apply(*case-tac R* (*Muts x* ! *j*)= *ind x*)
apply(*simp add*:*nth-list-update*)
apply(*simp add*:*nth-list-update*)
apply(*disjE-tac*)
apply *simp-all*
apply(*conjI-tac*)
apply(*rule impI*)
apply(*rule disjI2*,*rule disjI2*,*rule disjI1*, *erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*erule conjE*)+
apply(*rule impI*,(*rule disjI2*)+,*rule conjI*)
apply(*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*)+
apply *simp*
apply(*disjE-tac*)
apply(*rule disjI1*, *erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply *force*
— 1 subgoal left
apply *clarify*
apply(*disjE-tac*)
apply(*simp-all add*:*Graph6*)
apply(*rule impI*,*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule less-le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule less-le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)

```
apply(erule conjE)
apply(case-tac M x!(T (Muts x!j))=Black)
 apply(rule conjI)
  apply(rule impI,(rule disjI2)+,rule conjI)
   apply clarify
   apply(case-tac R (Muts x! j)=i)
    apply (force simp add: nth-list-update BtoW-def)
   apply (force simp add: nth-list-update)
  apply(erule le-trans,force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
 apply(rule impI,(rule disjI2)+, erule le-trans)
 apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update)
apply(rule conjI)
 apply(rule impI,rule disjI2,rule disjI2,rule disjI1, erule le-less-trans)
 apply(force simp add:Queue-def less-Suc-eq-le less-length-filter-update)
apply(rule impI,rule disjI2,rule disjI2,rule disjI1, erule le-less-trans)
apply(force simp add:Queue-def less-Suc-eq-le less-length-filter-update)
done

lemma Mul-interfree-Redirect-Edge-Propagate-Black: ⟦0≤j; j<n⟧⟹
  interfree-aux (Some(Mul-Redirect-Edge j n ),{},Some (Mul-Propagate-Black n))
apply (unfold mul-modules)
apply interfree-aux
apply safe
apply(simp-all add:mul-mutator-defs nth-list-update)
done

lemma Mul-interfree-Propagate-Black-Color-Target: ⟦0≤j; j<n⟧⟹
  interfree-aux (Some(Mul-Propagate-Black n),{},Some (Mul-Color-Target j n ))
apply (unfold mul-modules)
apply interfree-aux
apply(simp-all add: mul-collector-defs mul-mutator-defs)
— 7 subgoals left
apply clarify
apply (simp add:Graph7 Graph8 Graph12)
apply(disjE-tac)
  apply(simp add:Graph7 Graph8 Graph12)
 apply(case-tac M x!(T (Muts x!j))=Black)
  apply(rule disjI2,rule disjI1, erule le-trans)
  apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update Graph10)
 apply((rule disjI2)+,erule subset-psubset-trans, erule Graph11, simp)
apply((rule disjI2)+,erule psubset-subset-trans, simp add: Graph9)
— 6 subgoals left
apply clarify
apply (simp add:Graph7 Graph8 Graph12)
apply(disjE-tac)
  apply(simp add:Graph7 Graph8 Graph12)
 apply(case-tac M x!(T (Muts x!j))=Black)
  apply(rule disjI2,rule disjI1, erule le-trans)
  apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update Graph10)
```

apply((*rule disjI2*)+,*erule subset-psubset-trans*, *erule Graph11*, *simp*)
apply((*rule disjI2*)+,*erule psubset-subset-trans*, *simp add*: *Graph9*)
— 5 subgoals left
apply *clarify*
apply (*simp add*:*mul-collector-defs Mul-PBInv-def Graph7 Graph8 Graph12*)
apply(*disjE-tac*)
apply(*simp add*:*Graph7 Graph8 Graph12*)
apply(*rule disjI2*,*rule disjI1*, *erule psubset-subset-trans*,*simp add*:*Graph9*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*rule disjI2*,*rule disjI2*,*rule disjI1*, *erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply(*rule disjI2*,*rule disjI1*,*erule subset-psubset-trans*, *erule Graph11*, *simp*)
apply(*erule conjE*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply((*rule disjI2*)+)
apply (*rule conjI*)
apply(*simp add*:*Graph10*)
apply(*erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply(*rule disjI2*,*rule disjI1*,*erule subset-psubset-trans*, *erule Graph11*, *simp*)
— 4 subgoals left
apply *clarify*
apply (*simp add*:*mul-collector-defs Mul-PBInv-def Graph7 Graph8 Graph12*)
apply(*disjE-tac*)
apply(*simp add*:*Graph7 Graph8 Graph12*)
apply(*rule disjI2*,*rule disjI1*, *erule psubset-subset-trans*,*simp add*:*Graph9*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*rule disjI2*,*rule disjI2*,*rule disjI1*, *erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply(*rule disjI2*,*rule disjI1*,*erule subset-psubset-trans*, *erule Graph11*, *simp*)
apply(*erule conjE*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply((*rule disjI2*)+)
apply (*rule conjI*)
apply(*simp add*:*Graph10*)
apply(*erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply(*rule disjI2*,*rule disjI1*,*erule subset-psubset-trans*, *erule Graph11*, *simp*)
— 3 subgoals left
apply *clarify*
apply (*simp add*:*mul-collector-defs Mul-PBInv-def Graph7 Graph8 Graph12*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*simp add*:*Graph10*)
apply(*disjE-tac*)
apply *simp-all*
apply(*rule disjI2*, *rule disjI2*, *rule disjI1*,*erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply(*erule conjE*)
apply((*rule disjI2*)+,*erule le-trans*)

apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply(*rule conjI*)
apply(*rule disjI2*,*rule disjI1*, *erule subset-psubset-trans*,*simp add*:*Graph11*)
apply (*force simp add*:*nth-list-update*)
— 2 subgoals left
apply *clarify*
apply(*simp add*:*Mul-Auxk-def Graph7 Graph8 Graph12*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*simp add*:*Graph10*)
apply(*disjE-tac*)
apply *simp-all*
apply(*rule disjI2*, *rule disjI2*, *rule disjI1*,*erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply(*erule conjE*)+
apply((*rule disjI2*)+,*rule conjI*, *erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply((*rule impI*)+)
apply *simp*
apply(*erule disjE*)
apply(*rule disjI1*, *erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply *force*
apply(*rule conjI*)
apply(*rule disjI2*,*rule disjI1*, *erule subset-psubset-trans*,*simp add*:*Graph11*)
apply (*force simp add*:*nth-list-update*)
— 1 subgoal left
apply *clarify*
apply (*simp add*:*mul-collector-defs Mul-PBInv-def Graph7 Graph8 Graph12*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*simp add*:*Graph10*)
apply(*disjE-tac*)
apply *simp-all*
apply(*rule disjI2*, *rule disjI2*, *rule disjI1*,*erule less-le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply(*erule conjE*)
apply((*rule disjI2*)+,*erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply(*rule disjI2*,*rule disjI1*, *erule subset-psubset-trans*,*simp add*:*Graph11*)
done

lemma *Mul-interfree-Color-Target-Propagate-Black*: $[\![0\leq j;\ j<n]\!]\Longrightarrow$
interfree-aux (*Some*(*Mul-Color-Target j n*),{},*Some*(*Mul-Propagate-Black n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*mul-mutator-defs nth-list-update*)
done

lemma *Mul-interfree-Count-Redirect-Edge*: $[\![0\leq j;\ j<n]\!]\Longrightarrow$

interfree-aux (*Some*(*Mul-Count n*),{},*Some*(*Mul-Redirect-Edge j n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
— 9 subgoals left
apply(*simp add*:*mul-mutator-defs mul-collector-defs Mul-CountInv-def Graph6*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule impI*,*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*simp add*:*Graph6*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*simp add*:*Graph6*)
— 8 subgoals left
apply(*simp add*:*mul-mutator-defs nth-list-update*)
— 7 subgoals left
apply(*simp add*:*mul-mutator-defs mul-collector-defs*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule impI*,*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*simp add*:*Graph6*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*simp add*:*Graph6*)
— 6 subgoals left
apply(*simp add*:*mul-mutator-defs mul-collector-defs Mul-CountInv-def*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6 Queue-def*)
apply(*rule impI*,*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*simp add*:*Graph6*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def*

less-Suc-eq-le le-length-filter-update)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*simp add*:*Graph6*)
— 5 subgoals left
apply(*simp add*:*mul-mutator-defs mul-collector-defs Mul-CountInv-def*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule impI*,*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*simp add*:*Graph6*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*simp add*:*Graph6*)
— 4 subgoals left
apply(*simp add*:*mul-mutator-defs mul-collector-defs Mul-CountInv-def*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule impI*,*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*simp add*:*Graph6*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule conjI*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*simp add*:*Graph6*)
— 3 subgoals left
apply(*simp add*:*mul-mutator-defs nth-list-update*)
— 2 subgoals left
apply(*simp add*:*mul-mutator-defs mul-collector-defs Mul-CountInv-def*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule impI*,*rule disjI1*,*rule subset-trans*,*erule Graph3*,*simp*,*simp*)
apply(*simp add*:*Graph6*)
apply *clarify*
apply *disjE-tac*
apply(*simp add*:*Graph6*)
apply(*rule conjI*)

apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule impI*,*rule disjI2*,*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*simp add*:*Graph6*)
— 1 subgoal left
apply(*simp add*:*mul-mutator-defs nth-list-update*)
done

lemma *Mul-interfree-Redirect-Edge-Count*: $[\![0 \le j;\ j < n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Redirect-Edge j n*),{},*Some*(*Mul-Count n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*mul-mutator-defs nth-list-update*)
done

lemma *Mul-interfree-Count-Color-Target*: $[\![0 \le j;\ j < n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Count n*),{},*Some*(*Mul-Color-Target j n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply(*simp-all add*:*mul-collector-defs mul-mutator-defs Mul-CountInv-def*)
— 6 subgoals left
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*rule disjI2*,*rule disjI2*, *rule disjI1*, *erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply((*rule disjI2*)+,(*erule subset-psubset-trans*)+, *simp add*: *Graph11*)
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply((*rule disjI2*)+,*erule psubset-subset-trans*, *simp add*: *Graph9*)
— 5 subgoals left
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply *clarify*
apply *disjE-tac*
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
apply(*rule disjI2*,*rule disjI2*, *rule disjI1*, *erule le-trans*)
apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
apply((*rule disjI2*)+,(*erule subset-psubset-trans*)+, *simp add*: *Graph11*)
apply (*simp add*: *Graph7 Graph8 Graph12*)

```
apply((rule disjI2)+,erule psubset-subset-trans, simp add: Graph9)
— 4 subgoals left
apply clarify
apply disjE-tac
  apply (simp add: Graph7 Graph8 Graph12)
 apply (simp add: Graph7 Graph8 Graph12)
apply clarify
apply disjE-tac
 apply (simp add: Graph7 Graph8 Graph12)
 apply(case-tac M x!(T (Muts x!j))=Black)
  apply(rule disjI2,rule disjI2, rule disjI1, erule le-trans)
  apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update Graph10)
 apply((rule disjI2)+,(erule subset-psubset-trans)+, simp add: Graph11)
apply (simp add: Graph7 Graph8 Graph12)
apply((rule disjI2)+,erule psubset-subset-trans, simp add: Graph9)
— 3 subgoals left
apply clarify
apply disjE-tac
  apply (simp add: Graph7 Graph8 Graph12)
 apply (simp add: Graph7 Graph8 Graph12)
apply clarify
apply disjE-tac
 apply (simp add: Graph7 Graph8 Graph12)
 apply(case-tac M x!(T (Muts x!j))=Black)
  apply(rule disjI2,rule disjI2, rule disjI1, erule le-trans)
  apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update Graph10)
 apply((rule disjI2)+,(erule subset-psubset-trans)+, simp add: Graph11)
apply (simp add: Graph7 Graph8 Graph12)
apply((rule disjI2)+,erule psubset-subset-trans, simp add: Graph9)
— 2 subgoals left
apply clarify
apply disjE-tac
  apply (simp add: Graph7 Graph8 Graph12 nth-list-update)
 apply (simp add: Graph7 Graph8 Graph12 nth-list-update)
apply clarify
apply disjE-tac
 apply (simp add: Graph7 Graph8 Graph12)
 apply(rule conjI)
  apply(case-tac M x!(T (Muts x!j))=Black)
   apply(rule disjI2,rule disjI2, rule disjI1, erule le-trans)
   apply(force simp add:Queue-def less-Suc-eq-le le-length-filter-update Graph10)
  apply((rule disjI2)+,(erule subset-psubset-trans)+, simp add: Graph11)
 apply (simp add: nth-list-update)
apply (simp add: Graph7 Graph8 Graph12)
apply(rule conjI)
 apply((rule disjI2)+,erule psubset-subset-trans, simp add: Graph9)
apply (simp add: nth-list-update)
— 1 subgoal left
apply clarify
```

apply *disjE-tac*
 apply (*simp add*: *Graph7 Graph8 Graph12*)
 apply (*simp add*: *Graph7 Graph8 Graph12*)
apply *clarify*
apply *disjE-tac*
 apply (*simp add*: *Graph7 Graph8 Graph12*)
 apply(*case-tac M x*!(*T* (*Muts x*!*j*))=*Black*)
 apply(*rule disjI2*,*rule disjI2*, *rule disjI1*, *erule le-trans*)
 apply(*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update Graph10*)
 apply((*rule disjI2*)+,(*erule subset-psubset-trans*)+, *simp add*: *Graph11*)
apply (*simp add*: *Graph7 Graph8 Graph12*)
apply((*rule disjI2*)+,*erule psubset-subset-trans*, *simp add*: *Graph9*)
done

lemma *Mul-interfree-Color-Target-Count*: $[\![0{\le}j;\ j{<}n]\!] \Longrightarrow$
 interfree-aux (*Some*(*Mul-Color-Target j n*),{}, *Some*(*Mul-Count n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply *safe*
apply(*simp-all add*:*mul-mutator-defs nth-list-update*)
done

lemma *Mul-interfree-Append-Redirect-Edge*: $[\![0{\le}j;\ j{<}n]\!] \Longrightarrow$
 interfree-aux (*Some*(*Mul-Append n*),{}, *Some*(*Mul-Redirect-Edge j n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply(*tactic* ⟪ *ALLGOALS Clarify-tac* ⟫)
apply(*simp-all add*:*Graph6 Append-to-free0 Append-to-free1 mul-collector-defs mul-mutator-defs Mul-AppendInv-def*)
apply(*erule-tac x*=*j* **in** *allE*, *force dest*:*Graph3*)+
done

lemma *Mul-interfree-Redirect-Edge-Append*: $[\![0{\le}j;\ j{<}n]\!] \Longrightarrow$
 interfree-aux (*Some*(*Mul-Redirect-Edge j n*),{},*Some*(*Mul-Append n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply(*tactic* ⟪ *ALLGOALS Clarify-tac* ⟫)
apply(*simp-all add*:*mul-collector-defs Append-to-free0 Mul-AppendInv-def mul-mutator-defs nth-list-update*)
done

lemma *Mul-interfree-Append-Color-Target*: $[\![0{\le}j;\ j{<}n]\!] \Longrightarrow$
 interfree-aux (*Some*(*Mul-Append n*),{}, *Some*(*Mul-Color-Target j n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply(*tactic* ⟪ *ALLGOALS Clarify-tac* ⟫)
apply(*simp-all add*:*mul-mutator-defs mul-collector-defs Mul-AppendInv-def Graph7 Graph8 Append-to-free0 Append-to-free1*
 Graph12 nth-list-update)

done

lemma *Mul-interfree-Color-Target-Append*: $[\![0\leq j;\ j<n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Color-Target j n*),{}, *Some*(*Mul-Append n*))
apply (*unfold mul-modules*)
apply *interfree-aux*
apply(*tactic* ⟪ *ALLGOALS Clarify-tac* ⟫)
apply(*simp-all add*: *mul-mutator-defs nth-list-update*)
apply(*simp add*:*Mul-AppendInv-def Append-to-free0*)
done

Interference freedom Collector-Mutator

lemmas *mul-collector-mutator-interfree* =
Mul-interfree-Blacken-Roots-Redirect-Edge Mul-interfree-Blacken-Roots-Color-Target

Mul-interfree-Propagate-Black-Redirect-Edge Mul-interfree-Propagate-Black-Color-Target

Mul-interfree-Count-Redirect-Edge Mul-interfree-Count-Color-Target
Mul-interfree-Append-Redirect-Edge Mul-interfree-Append-Color-Target
Mul-interfree-Redirect-Edge-Blacken-Roots Mul-interfree-Color-Target-Blacken-Roots

Mul-interfree-Redirect-Edge-Propagate-Black Mul-interfree-Color-Target-Propagate-Black

Mul-interfree-Redirect-Edge-Count Mul-interfree-Color-Target-Count
Mul-interfree-Redirect-Edge-Append Mul-interfree-Color-Target-Append

lemma *Mul-interfree-Collector-Mutator*: $j<n \Longrightarrow$
interfree-aux (*Some* (*Mul-Collector n*), {}, *Some* (*Mul-Mutator j n*))
apply(*unfold Mul-Collector-def Mul-Mutator-def*)
apply *interfree-aux*
apply(*simp-all add*:*mul-collector-mutator-interfree*)
apply(*unfold mul-modules mul-collector-defs mul-mutator-defs*)
apply(*tactic* ⟪ *TRYALL* (*interfree-aux-tac*) ⟫)
— 42 subgoals left
apply (*clarify*,*simp add*:*Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12*)+
— 24 subgoals left
apply(*simp-all add*:*Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12*)
— 14 subgoals left
apply(*tactic* ⟪ *TRYALL Clarify-tac* ⟫)
apply(*simp-all add*:*Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12*)
apply(*tactic* ⟪ *TRYALL* (*rtac conjI*) ⟫)
apply(*tactic* ⟪ *TRYALL* (*rtac impI*) ⟫)
apply(*tactic* ⟪ *TRYALL* (*etac disjE*) ⟫)
apply(*tactic* ⟪ *TRYALL* (*etac conjE*) ⟫)
apply(*tactic* ⟪ *TRYALL* (*etac disjE*) ⟫)
apply(*tactic* ⟪ *TRYALL* (*etac disjE*) ⟫)
— 72 subgoals left

apply(*simp-all add*:*Graph6 Graph7 Graph8 Append-to-free0 Append-to-free1 Graph12*)
— 35 subgoals left
apply(*tactic* ⟪ *TRYALL*(*EVERY'*[*rtac disjI1*,*rtac subset-trans*,*etac* (*thm Graph3*),*Force-tac*, *assume-tac*]) ⟫)
— 28 subgoals left
apply(*tactic* ⟪ *TRYALL* (*etac conjE*) ⟫)
apply(*tactic* ⟪ *TRYALL* (*etac disjE*) ⟫)
— 34 subgoals left
apply(*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*rule disjI2*,*rule disjI1*,*erule le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)
apply(*tactic* ⟪ *ALLGOALS*(*case-tac M x*!(*T* (*Muts x* ! *j*))=*Black*) ⟫)
apply(*simp-all add*:*Graph10*)
— 47 subgoals left
apply(*tactic* ⟪ *TRYALL*(*EVERY'*[*REPEAT o* (*rtac disjI2*),*etac subset-psubset-trans*,*etac* (*thm Graph11*),*Force-tac*]) ⟫)
— 41 subgoals left
apply(*tactic* ⟪ *TRYALL*(*EVERY'*[*rtac disjI2*, *rtac disjI1*, *etac le-trans*, *force-tac* (*claset*(),*simpset*() *addsimps* [*thm Queue-def*, *less-Suc-eq-le*, *thm le-length-filter-update*])]) ⟫)
— 35 subgoals left
apply(*tactic* ⟪ *TRYALL*(*EVERY'*[*rtac disjI2*,*rtac disjI1*,*etac psubset-subset-trans*,*rtac* (*thm Graph9*),*Force-tac*]) ⟫)
— 31 subgoals left
apply(*tactic* ⟪ *TRYALL*(*EVERY'*[*rtac disjI2*,*rtac disjI1*,*etac subset-psubset-trans*,*etac* (*thm Graph11*),*Force-tac*]) ⟫)
— 29 subgoals left
apply(*tactic* ⟪ *TRYALL*(*EVERY'*[*REPEAT o* (*rtac disjI2*),*etac subset-psubset-trans*,*etac subset-psubset-trans*,*etac* (*thm Graph11*),*Force-tac*]) ⟫)
— 25 subgoals left
apply(*tactic* ⟪ *TRYALL*(*EVERY'*[*rtac disjI2*, *rtac disjI2*, *rtac disjI1*, *etac le-trans*, *force-tac* (*claset*(),*simpset*() *addsimps* [*thm Queue-def*, *less-Suc-eq-le*, *thm le-length-filter-update*])]) ⟫)
— 10 subgoals left
apply(*rule disjI2*,*rule disjI2*,*rule conjI*,*erule less-le-trans*,*force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*, *rule disjI1*, *rule less-imp-le*, *erule less-le-trans*, *force simp add*:*Queue-def less-Suc-eq-le le-length-filter-update*)+
done

Interference freedom Mutator-Collector

lemma *Mul-interfree-Mutator-Collector*: $j < n \Longrightarrow$
interfree-aux (*Some* (*Mul-Mutator j n*), {}, *Some* (*Mul-Collector n*))
apply(*unfold Mul-Collector-def Mul-Mutator-def*)
apply *interfree-aux*
apply(*simp-all add*:*mul-collector-mutator-interfree*)
apply(*unfold mul-modules mul-collector-defs mul-mutator-defs*)
apply(*tactic* ⟪ *TRYALL* (*interfree-aux-tac*) ⟫)

— 76 subgoals left
apply (*clarify*,*simp add*: *nth-list-update*)+
— 56 subgoals left
apply(*clarify*,*simp add*:*Mul-AppendInv-def Append-to-free0 nth-list-update*)+
done

The Multi-Mutator Garbage Collection Algorithm

The total number of verification conditions is 328

lemma *Mul-Gar-Coll*:
 $\Vert$− .{´*Mul-Proper n* ∧ ´*Mul-mut-init n* ∧ (∀ *i*<*n*. *Z* (´*Muts*!*i*))}.
 COBEGIN
  Mul-Collector n
 .{*False*}.
 $\Vert$
 SCHEME [*0*≤ *j*< *n*]
  Mul-Mutator j n
 .{*False*}.
 COEND
 .{*False*}.
apply *oghoare*
— Strengthening the precondition
apply(*rule Int-greatest*)
 apply (*case-tac n*)
  apply(*force simp add*: *Mul-Collector-def mul-mutator-defs mul-collector-defs nth-append*)
 apply(*simp add*: *Mul-Mutator-def mul-collector-defs mul-mutator-defs nth-append*)
 apply *force*
apply *clarify*
apply(*case-tac xa*)
 apply(*simp add*:*Mul-Collector-def mul-mutator-defs mul-collector-defs nth-append*)
apply(*simp add*: *Mul-Mutator-def mul-mutator-defs mul-collector-defs nth-append nth-map-upt*)
— Collector
apply(*rule Mul-Collector*)
— Mutator
apply(*erule Mul-Mutator*)
— Interference freedom
apply(*simp add*:*Mul-interfree-Collector-Mutator*)
apply(*simp add*:*Mul-interfree-Mutator-Collector*)
apply(*simp add*:*Mul-interfree-Mutator-Mutator*)
— Weakening of the postcondition
apply(*case-tac n*)
 apply(*simp add*:*Mul-Collector-def mul-mutator-defs mul-collector-defs nth-append*)
apply(*simp add*:*Mul-Mutator-def mul-mutator-defs mul-collector-defs nth-append*)
done

end

Chapter 3

The Rely-Guarantee Method

3.1 Abstract Syntax

theory *RG-Com* **imports** *Main* **begin**

Semantics of assertions and boolean expressions (bexp) as sets of states. Syntax of commands *com* and parallel commands *par-com*.

types
$'a\ bexp = 'a\ set$

datatype $'a\ com =$
$\quad Basic\ 'a \Rightarrow 'a$
$\mid Seq\ 'a\ com\ 'a\ com$
$\mid Cond\ 'a\ bexp\ 'a\ com\ 'a\ com$
$\mid While\ 'a\ bexp\ 'a\ com$
$\mid Await\ 'a\ bexp\ 'a\ com$

types $'a\ par\text{-}com = (('a\ com)\ option)\ list$

end

3.2 Operational Semantics

theory *RG-Tran*
imports *RG-Com*
begin

3.2.1 Semantics of Component Programs

Environment transitions

types $'a\ conf = (('a\ com)\ option) \times 'a$

consts *etran* $::\ ('a\ conf \times 'a\ conf)\ set$
syntax *-etran* $::\ 'a\ conf \Rightarrow 'a\ conf \Rightarrow bool\ \ (\text{-} -e\rightarrow \text{-}\ [81,81]\ 80)$

translations $P\ -e\rightarrow\ Q\ \rightleftharpoons (P,Q) \in etran$
inductive *etran*
intros
$Env: (P,\ s)\ -e\rightarrow (P,\ t)$

Component transitions

consts *ctran* :: $('a\ conf \times 'a\ conf)\ set$
syntax
-ctran :: $'a\ conf \Rightarrow 'a\ conf \Rightarrow bool$ (- $-c\rightarrow$ - [81,81] 80)
-ctran-∗:: $'a\ conf \Rightarrow 'a\ conf \Rightarrow bool$ (- $-c{*}\rightarrow$ - [81,81] 80)
translations
$P\ -c\rightarrow\ Q\ \rightleftharpoons (P,Q) \in ctran$
$P\ -c{*}\rightarrow\ Q \rightleftharpoons (P,Q) \in ctran\hat{}*$

inductive *ctran*
intros
$Basic: (Some(Basic\ f),\ s)\ -c\rightarrow (None,\ f\ s)$

$Seq1: (Some\ P0,\ s)\ -c\rightarrow (None,\ t) \Longrightarrow (Some(Seq\ P0\ P1),\ s)\ -c\rightarrow (Some\ P1,\ t)$

$Seq2: (Some\ P0,\ s)\ -c\rightarrow (Some\ P2,\ t) \Longrightarrow (Some(Seq\ P0\ P1),\ s)\ -c\rightarrow (Some(Seq\ P2\ P1),\ t)$

$CondT: s{\in}b \Longrightarrow (Some(Cond\ b\ P1\ P2),\ s)\ -c\rightarrow (Some\ P1,\ s)$
$CondF: s{\notin}b \Longrightarrow (Some(Cond\ b\ P1\ P2),\ s)\ -c\rightarrow (Some\ P2,\ s)$

$WhileF: s{\notin}b \Longrightarrow (Some(While\ b\ P),\ s)\ -c\rightarrow (None,\ s)$
$WhileT: s{\in}b \Longrightarrow (Some(While\ b\ P),\ s)\ -c\rightarrow (Some(Seq\ P\ (While\ b\ P)),\ s)$

$Await: [\![s{\in}b;\ (Some\ P,\ s)\ -c{*}\rightarrow (None,\ t)]\!] \Longrightarrow (Some(Await\ b\ P),\ s)\ -c\rightarrow (None,\ t)$

monos *rtrancl-mono*

3.2.2 Semantics of Parallel Programs

types $'a\ par\text{-}conf = ('a\ par\text{-}com) \times 'a$
consts
par-etran :: $('a\ par\text{-}conf \times 'a\ par\text{-}conf)\ set$
par-ctran :: $('a\ par\text{-}conf \times 'a\ par\text{-}conf)\ set$
syntax
-par-etran:: $['a\ par\text{-}conf,'a\ par\text{-}conf] \Rightarrow bool$ (- $-pe\rightarrow$ - [81,81] 80)
-par-ctran:: $['a\ par\text{-}conf,'a\ par\text{-}conf] \Rightarrow bool$ (- $-pc\rightarrow$ - [81,81] 80)
translations
$P\ -pe\rightarrow\ Q\ \rightleftharpoons (P,Q) \in par\text{-}etran$
$P\ -pc\rightarrow\ Q\ \rightleftharpoons (P,Q) \in par\text{-}ctran$

inductive *par-etran*

intros
  ParEnv: $(Ps,\ s)\ -pe\rightarrow\ (Ps,\ t)$

inductive *par-ctran*
intros
  ParComp: $[\![i<length\ Ps;\ (Ps!i,\ s)\ -c\rightarrow\ (r,\ t)]\!] \Longrightarrow (Ps,\ s)\ -pc\rightarrow\ (Ps[i:=r],\ t)$

3.2.3 Computations

Sequential computations

types $'a\ confs = ('a\ conf)\ list$
consts $cptn :: ('a\ confs)\ set$
inductive *cptn*
intros
  CptnOne: $[(P,s)] \in cptn$
  CptnEnv: $(P,\ t)\#xs \in cptn \Longrightarrow (P,s)\#(P,t)\#xs \in cptn$
  CptnComp: $[\![(P,s)\ -c\rightarrow\ (Q,t);\ (Q,\ t)\#xs \in cptn\ ]\!] \Longrightarrow (P,s)\#(Q,t)\#xs \in cptn$

constdefs
  $cp :: ('a\ com)\ option \Rightarrow\ 'a \Rightarrow ('a\ confs)\ set$
  $cp\ P\ s \equiv \{l.\ l!0=(P,s) \wedge l \in cptn\}$

Parallel computations

types $'a\ par\text{-}confs = ('a\ par\text{-}conf)\ list$
consts $par\text{-}cptn :: ('a\ par\text{-}confs)\ set$
inductive *par-cptn*
intros
  ParCptnOne: $[(P,s)] \in par\text{-}cptn$
  ParCptnEnv: $(P,t)\#xs \in par\text{-}cptn \Longrightarrow (P,s)\#(P,t)\#xs \in par\text{-}cptn$
  ParCptnComp: $[\![\ (P,s)\ -pc\rightarrow\ (Q,t);\ (Q,t)\#xs \in par\text{-}cptn\ ]\!] \Longrightarrow (P,s)\#(Q,t)\#xs$
$\in par\text{-}cptn$

constdefs
  $par\text{-}cp ::\ 'a\ par\text{-}com \Rightarrow\ 'a \Rightarrow ('a\ par\text{-}confs)\ set$
  $par\text{-}cp\ P\ s \equiv \{l.\ l!0=(P,s) \wedge l \in par\text{-}cptn\}$

3.2.4 Modular Definition of Computation

constdefs
  $lift ::\ 'a\ com \Rightarrow\ 'a\ conf \Rightarrow\ 'a\ conf$
  $lift\ Q \equiv \lambda(P,\ s).\ (if\ P=None\ then\ (Some\ Q,s)\ else\ (Some(Seq\ (the\ P)\ Q),\ s))$

consts $cptn\text{-}mod :: ('a\ confs)\ set$
inductive *cptn-mod*
intros
  CptnModOne: $[(P,\ s)] \in cptn\text{-}mod$
  CptnModEnv: $(P,\ t)\#xs \in cptn\text{-}mod \Longrightarrow (P,\ s)\#(P,\ t)\#xs \in cptn\text{-}mod$

CptnModNone: $[\![$(*Some P*, *s*) $-c\rightarrow$ (*None*, *t*); (*None*, *t*)#*xs* $\in$ *cptn-mod* $]\!]$ $\Longrightarrow$ (*Some P*,*s*)#(*None*, *t*)#*xs* $\in$*cptn-mod*
CptnModCondT: $[\![$(*Some P0*, *s*)#*ys* $\in$ *cptn-mod*; *s* $\in$ *b* $]\!]$ $\Longrightarrow$ (*Some*(*Cond b P0 P1*), *s*)#(*Some P0*, *s*)#*ys* $\in$ *cptn-mod*
CptnModCondF: $[\![$(*Some P1*, *s*)#*ys* $\in$ *cptn-mod*; *s* $\notin$ *b* $]\!]$ $\Longrightarrow$ (*Some*(*Cond b P0 P1*), *s*)#(*Some P1*, *s*)#*ys* $\in$ *cptn-mod*
CptnModSeq1: $[\![$(*Some P0*, *s*)#*xs* $\in$ *cptn-mod*; *zs*=*map* (*lift P1*) *xs* $]\!]$
$\Longrightarrow$ (*Some*(*Seq P0 P1*), *s*)#*zs* $\in$ *cptn-mod*
CptnModSeq2:
$[\![$(*Some P0*, *s*)#*xs* $\in$ *cptn-mod*; *fst*(*last* ((*Some P0*, *s*)#*xs*)) = *None*;
(*Some P1*, *snd*(*last* ((*Some P0*, *s*)#*xs*)))#*ys* $\in$ *cptn-mod*;
zs=(*map* (*lift P1*) *xs*)@*ys* $]\!]$ $\Longrightarrow$ (*Some*(*Seq P0 P1*), *s*)#*zs* $\in$ *cptn-mod*

CptnModWhile1:
$[\![$ (*Some P*, *s*)#*xs* $\in$ *cptn-mod*; *s* $\in$ *b*; *zs*=*map* (*lift* (*While b P*)) *xs* $]\!]$
$\Longrightarrow$ (*Some*(*While b P*), *s*)#(*Some*(*Seq P* (*While b P*)), *s*)#*zs* $\in$ *cptn-mod*
CptnModWhile2:
$[\![$ (*Some P*, *s*)#*xs* $\in$ *cptn-mod*; *fst*(*last* ((*Some P*, *s*)#*xs*))=*None*; *s* $\in$ *b*;
zs=(*map* (*lift* (*While b P*)) *xs*)@*ys*;
(*Some*(*While b P*), *snd*(*last* ((*Some P*, *s*)#*xs*)))#*ys* $\in$ *cptn-mod*$]\!]$
$\Longrightarrow$ (*Some*(*While b P*), *s*)#(*Some*(*Seq P* (*While b P*)), *s*)#*zs* $\in$ *cptn-mod*

3.2.5 Equivalence of Both Definitions.

lemma *last-length*: ((*a*#*xs*)!(*length xs*))=*last* (*a*#*xs*)
apply *simp*
apply(*induct xs*,*simp+*)
apply(*case-tac xs*)
apply *simp-all*
done

lemma *div-seq* [*rule-format*]: *list* $\in$ *cptn-mod* $\Longrightarrow$
($\forall$ *s P Q zs*. *list*=(*Some* (*Seq P Q*), *s*)#*zs* $\longrightarrow$
($\exists$ *xs*. (*Some P*, *s*)#*xs* $\in$ *cptn-mod* $\wedge$ (*zs*=(*map* (*lift Q*) *xs*) $\vee$
(*fst*(((*Some P*, *s*)#*xs*)!*length xs*)=*None* $\wedge$
($\exists$ *ys*. (*Some Q*, *snd*(((*Some P*, *s*)#*xs*)!*length xs*))#*ys* $\in$ *cptn-mod*
$\wedge$ *zs*=(*map* (*lift* (*Q*)) *xs*)@*ys*)))))
apply(*erule cptn-mod.induct*)
apply *simp-all*
apply *clarify*
apply(*force intro*:*CptnModOne*)
apply *clarify*
apply(*erule-tac x*=*Pa* **in** *allE*)
apply(*erule-tac x*=*Q* **in** *allE*)
apply *simp*
apply *clarify*
apply(*erule disjE*)
apply(*rule-tac x*=(*Some Pa*,*t*)#*xsa* **in** *exI*)
apply(*rule conjI*)

```
    apply clarify
    apply(erule CptnModEnv)
   apply(rule disjI1)
   apply(simp add:lift-def)
  apply clarify
  apply(rule-tac x=(Some Pa,t)#xsa in exI)
  apply(rule conjI)
   apply(erule CptnModEnv)
  apply(rule disjI2)
  apply(rule conjI)
   apply(case-tac xsa,simp,simp)
  apply(rule-tac x=ys in exI)
  apply(rule conjI)
   apply simp
  apply(simp add:lift-def)
 apply clarify
 apply(erule ctran.elims,simp-all)
apply clarify
apply(rule-tac x=xs in exI)
apply simp
apply clarify
apply(rule-tac x=xs in exI)
apply(simp add: last-length)
done

lemma cptn-onlyif-cptn-mod-aux [rule-format]:
  ∀ s Q t xs.((Some a, s), Q, t) ∈ ctran ⟶ (Q, t) # xs ∈ cptn-mod
  ⟶ (Some a, s) # (Q, t) # xs ∈ cptn-mod
apply(induct a)
apply simp-all
— basic
apply clarify
apply(erule ctran.elims,simp-all)
apply(rule CptnModNone,rule Basic,simp)
apply clarify
apply(erule ctran.elims,simp-all)
— Seq1
apply(rule-tac xs=[(None,ta)] in CptnModSeq2)
  apply(erule CptnModNone)
  apply(rule CptnModOne)
 apply simp
apply simp
apply(simp add:lift-def)
— Seq2
apply(erule-tac x=sa in allE)
apply(erule-tac x=Some P2 in allE)
apply(erule allE,erule impE, assumption)
apply(drule div-seq,simp)
apply force
```

apply *clarify*
apply(*erule disjE*)
apply *clarify*
apply(*erule allE*,*erule impE*, *assumption*)
apply(*erule-tac CptnModSeq1*)
apply(*simp add*:*lift-def*)
apply *clarify*
apply(*erule allE*,*erule impE*, *assumption*)
apply(*erule-tac CptnModSeq2*)
apply (*simp add*:*last-length*)
apply (*simp add*:*last-length*)
apply(*simp add*:*lift-def*)
— Cond
apply *clarify*
apply(*erule ctran.elims*,*simp-all*)
apply(*force elim*: *CptnModCondT*)
apply(*force elim*: *CptnModCondF*)
— While
apply *clarify*
apply(*erule ctran.elims*,*simp-all*)
apply(*rule CptnModNone*,*erule WhileF*,*simp*)
apply(*drule div-seq*,*force*)
apply *clarify*
apply (*erule disjE*)
apply(*force elim*:*CptnModWhile1*)
apply *clarify*
apply(*force simp add*:*last-length elim*:*CptnModWhile2*)
— await
apply *clarify*
apply(*erule ctran.elims*,*simp-all*)
apply(*rule CptnModNone*,*erule Await*,*simp+*)
done

lemma *cptn-onlyif-cptn-mod* [*rule-format*]: $c \in cptn \Longrightarrow c \in cptn\text{-}mod$
apply(*erule cptn.induct*)
apply(*rule CptnModOne*)
apply(*erule CptnModEnv*)
apply(*case-tac P*)
apply *simp*
apply(*erule ctran.elims*,*simp-all*)
apply(*force elim*:*cptn-onlyif-cptn-mod-aux*)
done

lemma *lift-is-cptn*: $c \in cptn \Longrightarrow map\ (lift\ P)\ c \in cptn$
apply(*erule cptn.induct*)
apply(*force simp add*:*lift-def CptnOne*)
apply(*force intro*:*CptnEnv simp add*:*lift-def*)
apply(*force simp add*:*lift-def intro*:*CptnComp Seq2 Seq1 elim*:*ctran.elims*)
done

lemma *cptn-append-is-cptn* [*rule-format*]:
∀ b a. b#c1∈cptn ⟶ a#c2∈cptn ⟶ (b#c1)!length c1=a ⟶ b#c1@c2∈cptn
apply(*induct c1*)
apply *simp*
apply *clarify*
apply(*erule cptn.elims*,*simp-all*)
apply(*force intro*:*CptnEnv*)
apply(*force elim*:*CptnComp*)
done

lemma *last-lift*: ⟦xs≠[]; fst(xs!(length xs − (Suc 0)))=None⟧
⟹ fst((map (lift P) xs)!(length (map (lift P) xs)− (Suc 0)))=(Some P)
apply(*case-tac* (*xs* ! (*length xs* − (*Suc 0*))))
apply (*simp add*:*lift-def*)
done

lemma *last-fst* [*rule-format*]: P((a#x)!length x) ⟶ ¬P a ⟶ P (x!(length x − (Suc 0)))
apply(*induct x*,*simp+*)
done

lemma *last-fst-esp*:
fst(((Some a,s)#xs)!(length xs))=None ⟹ fst(xs!(length xs − (Suc 0)))=None
apply(*erule last-fst*)
apply *simp*
done

lemma *last-snd*: xs≠[] ⟹
snd(((map (lift P) xs))!(length (map (lift P) xs) − (Suc 0)))=snd(xs!(length xs − (Suc 0)))
apply(*case-tac* (*xs* ! (*length xs* − (*Suc 0*))),*simp*)
apply (*simp add*:*lift-def*)
done

lemma *Cons-lift*: (Some (Seq P Q), s) # (map (lift Q) xs) = map (lift Q) ((Some P, s) # xs)
by(*simp add*:*lift-def*)

lemma *Cons-lift-append*:
(Some (Seq P Q), s) # (map (lift Q) xs) @ ys = map (lift Q) ((Some P, s) # xs)@ ys
by(*simp add*:*lift-def*)

lemma *lift-nth*: i<length xs ⟹ map (lift Q) xs ! i = lift Q (xs! i)
by (*simp add*:*lift-def*)

lemma *snd-lift*: i< length xs ⟹ snd(lift Q (xs ! i))= snd (xs ! i)
apply(*case-tac xs*!*i*)

apply(*simp add*:*lift-def*)
done

lemma *cptn-if-cptn-mod*: $c \in$ *cptn-mod* $\Longrightarrow c \in$ *cptn*
apply(*erule cptn-mod.induct*)
apply(*rule CptnOne*)
apply(*erule CptnEnv*)
apply(*erule CptnComp*,*simp*)
apply(*rule CptnComp*)
apply(*erule CondT*,*simp*)
apply(*rule CptnComp*)
apply(*erule CondF*,*simp*)
— Seq1
apply(*erule cptn.elims*,*simp-all*)
apply(*rule CptnOne*)
apply *clarify*
apply(*drule-tac P=P1* **in** *lift-is-cptn*)
apply(*simp add*:*lift-def*)
apply(*rule CptnEnv*,*simp*)
apply *clarify*
apply(*simp add*:*lift-def*)
apply(*rule conjI*)
apply *clarify*
apply(*rule CptnComp*)
apply(*rule Seq1*,*simp*)
apply(*drule-tac P=P1* **in** *lift-is-cptn*)
apply(*simp add*:*lift-def*)
apply *clarify*
apply(*rule CptnComp*)
apply(*rule Seq2*,*simp*)
apply(*drule-tac P=P1* **in** *lift-is-cptn*)
apply(*simp add*:*lift-def*)
— Seq2
apply(*rule cptn-append-is-cptn*)
apply(*drule-tac P=P1* **in** *lift-is-cptn*)
apply(*simp add*:*lift-def*)
apply *simp*
apply(*case-tac* $xs \neq []$)
apply(*drule-tac P=P1* **in** *last-lift*)
apply(*rule last-fst-esp*)
apply (*simp add*:*last-length*)
apply(*simp add*:*Cons-lift del*:*map.simps*)
apply(*rule conjI*, *clarify*, *simp*)
apply(*case-tac* (((*Some P0*, *s*) # *xs*) ! *length xs*))
apply *clarify*
apply (*simp add*:*lift-def last-length*)
apply (*simp add*:*last-length*)
— While1
apply(*rule CptnComp*)

apply(*rule WhileT*,*simp*)
apply(*drule-tac P=While b P* **in** *lift-is-cptn*)
apply(*simp add*:*lift-def*)
— While2
apply(*rule CptnComp*)
apply(*rule WhileT*,*simp*)
apply(*rule cptn-append-is-cptn*)
apply(*drule-tac P=While b P* **in** *lift-is-cptn*)
  apply(*simp add*:*lift-def*)
 apply *simp*
apply(*case-tac xs*≠[])
 apply(*drule-tac P=While b P* **in** *last-lift*)
  apply(*rule last-fst-esp*,*simp add*:*last-length*)
 apply(*simp add*:*Cons-lift del*:*map.simps*)
 apply(*rule conjI*, *clarify*, *simp*)
 apply(*case-tac* (((*Some P*, *s*) # *xs*) ! *length xs*))
 apply *clarify*
 apply (*simp add*:*last-length lift-def*)
apply *simp*
done

theorem *cptn-iff-cptn-mod*: (*c* ∈ *cptn*) = (*c* ∈ *cptn-mod*)
apply(*rule iffI*)
 apply(*erule cptn-onlyif-cptn-mod*)
apply(*erule cptn-if-cptn-mod*)
done

3.3 Validity of Correctness Formulas

3.3.1 Validity for Component Programs.

types *'a rgformula* = *'a com* × *'a set* × (*'a* × *'a*) *set* × (*'a* × *'a*) *set* × *'a set*

constdefs
  assum :: (*'a set* × (*'a* × *'a*) *set*) ⇒ (*'a confs*) *set*
  assum ≡ λ(*pre*, *rely*). {*c*. *snd*(*c*!*0*) ∈ *pre* ∧ (∀ *i*. *Suc i*<*length c* ⟶
        c!*i* −*e*→ *c*!(*Suc i*) ⟶ (*snd*(*c*!*i*), *snd*(*c*!*Suc i*)) ∈ *rely*)}

  comm :: ((*'a* × *'a*) *set* × *'a set*) ⇒ (*'a confs*) *set*
  comm ≡ λ(*guar*, *post*). {*c*. (∀ *i*. *Suc i*<*length c* ⟶
        c!*i* −*c*→ *c*!(*Suc i*) ⟶ (*snd*(*c*!*i*), *snd*(*c*!*Suc i*)) ∈ *guar*) ∧
        (*fst* (*last c*) = *None* ⟶ *snd* (*last c*) ∈ *post*)}

  com-validity :: *'a com* ⇒ *'a set* ⇒ (*'a* × *'a*) *set* ⇒ (*'a* × *'a*) *set* ⇒ *'a set* ⇒ *bool*

        (⊨ - *sat* [-, -, -, -] [*60*,*0*,*0*,*0*,*0*] *45*)
  ⊨ *P sat* [*pre*, *rely*, *guar*, *post*] ≡
   ∀ *s*. *cp* (*Some P*) *s* ∩ *assum*(*pre*, *rely*) ⊆ *comm*(*guar*, *post*)

3.3.2 Validity for Parallel Programs.

constdefs
All-None :: ((*'a com*) *option*) *list* ⇒ *bool*
All-None xs ≡ ∀ *c*∈*set xs*. *c*=*None*

par-assum :: (*'a set* × (*'a* × *'a*) *set*) ⇒ (*'a par-confs*) *set*
par-assum ≡ λ(*pre*, *rely*). {*c*. *snd*(*c*!*0*) ∈ *pre* ∧ (∀ *i*. *Suc i*<*length c* ⟶
c!*i* −*pe*→ *c*!*Suc i* ⟶ (*snd*(*c*!*i*), *snd*(*c*!*Suc i*)) ∈ *rely*)}

par-comm :: ((*'a* × *'a*) *set* × *'a set*) ⇒ (*'a par-confs*) *set*
par-comm ≡ λ(*guar*, *post*). {*c*. (∀ *i*. *Suc i*<*length c* ⟶
c!*i* −*pc*→ *c*!*Suc i* ⟶ (*snd*(*c*!*i*), *snd*(*c*!*Suc i*)) ∈ *guar*) ∧
(*All-None* (*fst* (*last c*)) ⟶ *snd*(*last c*) ∈ *post*)}

par-com-validity :: *'a par-com* ⇒ *'a set* ⇒ (*'a* × *'a*) *set* ⇒ (*'a* × *'a*) *set*
⇒ *'a set* ⇒ *bool* (⊨ - *SAT* [-, -, -, -] [*60*,*0*,*0*,*0*,*0*] *45*)
⊨ *Ps SAT* [*pre*, *rely*, *guar*, *post*] ≡
∀ *s*. *par-cp Ps s* ∩ *par-assum*(*pre*, *rely*) ⊆ *par-comm*(*guar*, *post*)

3.3.3 Compositionality of the Semantics

Definition of the conjoin operator

constdefs
same-length :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool*
same-length c clist ≡ (∀ *i*<*length clist*. *length*(*clist*!*i*)=*length c*)

same-state :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool*
same-state c clist ≡ (∀ *i* <*length clist*. ∀ *j*<*length c*. *snd*(*c*!*j*) = *snd*((*clist*!*i*)!*j*))

same-program :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool*
same-program c clist ≡ (∀ *j*<*length c*. *fst*(*c*!*j*) = *map* (λ*x*. *fst*(*nth x j*)) *clist*)

compat-label :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool*
compat-label c clist ≡ (∀ *j*. *Suc j*<*length c* ⟶
(*c*!*j* −*pc*→ *c*!*Suc j* ∧ (∃ *i*<*length clist*. (*clist*!*i*)!*j* −*c*→ (*clist*!*i*)! *Suc j* ∧
(∀ *l*<*length clist*. *l*≠*i* ⟶ (*clist*!*l*)!*j* −*e*→ (*clist*!*l*)! *Suc j*))) ∨
(*c*!*j* −*pe*→ *c*!*Suc j* ∧ (∀ *i*<*length clist*. (*clist*!*i*)!*j* −*e*→ (*clist*!*i*)! *Suc j*)))

conjoin :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool* (- ∝ - [*65*,*65*] *64*)
c ∝ *clist* ≡ (*same-length c clist*) ∧ (*same-state c clist*) ∧ (*same-program c clist*)
∧ (*compat-label c clist*)

Some previous lemmas

lemma *list-eq-if* [*rule-format*]:
∀ *ys*. *xs*=*ys* ⟶ (*length xs* = *length ys*) ⟶ (∀ *i*<*length xs*. *xs*!*i*=*ys*!*i*)
apply (*induct xs*)
apply *simp*

apply *clarify*
done

lemma *list-eq*: $(length\ xs = length\ ys \wedge (\forall\ i{<}length\ xs.\ xs!i{=}ys!i)) = (xs{=}ys)$
apply(*rule iffI*)
 apply *clarify*
 apply(*erule nth-equalityI*)
 apply *simp+*
done

lemma *nth-tl*: $[\![\ ys!0{=}a;\ ys{\neq}[]\ ]\!] \Longrightarrow ys{=}(a\#(tl\ ys))$
apply(*case-tac ys*)
 apply *simp+*
done

lemma *nth-tl-if* [*rule-format*]: $ys{\neq}[] \longrightarrow ys!0{=}a \longrightarrow P\ ys \longrightarrow P\ (a\#(tl\ ys))$
apply(*induct ys*)
 apply *simp+*
done

lemma *nth-tl-onlyif* [*rule-format*]: $ys{\neq}[] \longrightarrow ys!0{=}a \longrightarrow P\ (a\#(tl\ ys)) \longrightarrow P\ ys$
apply(*induct ys*)
 apply *simp+*
done

lemma *seq-not-eq1*: $Seq\ c1\ c2{\neq}c1$
apply(*rule com.induct*)
apply *simp-all*
apply *clarify*
done

lemma *seq-not-eq2*: $Seq\ c1\ c2{\neq}c2$
apply(*rule com.induct*)
apply *simp-all*
apply *clarify*
done

lemma *if-not-eq1*: $Cond\ b\ c1\ c2\ {\neq}c1$
apply(*rule com.induct*)
apply *simp-all*
apply *clarify*
done

lemma *if-not-eq2*: $Cond\ b\ c1\ c2{\neq}c2$
apply(*rule com.induct*)
apply *simp-all*
apply *clarify*
done

lemmas *seq-and-if-not-eq* [*simp*] = *seq-not-eq1 seq-not-eq2*
seq-not-eq1 [*THEN not-sym*] *seq-not-eq2* [*THEN not-sym*]
if-not-eq1 if-not-eq2 if-not-eq1 [*THEN not-sym*] *if-not-eq2* [*THEN not-sym*]

lemma *prog-not-eq-in-ctran-aux* [*rule-format*]: $(P,s)\ -c\rightarrow\ (Q,t) \Longrightarrow (P{\neq}Q)$
apply(*erule ctran.induct*)
apply *simp-all*
done

lemma *prog-not-eq-in-ctran* [*simp*]: $\neg\ (P,s)\ -c\rightarrow\ (P,t)$
apply *clarify*
apply(*drule prog-not-eq-in-ctran-aux*)
apply *simp*
done

lemma *prog-not-eq-in-par-ctran-aux* [*rule-format*]: $(P,s)\ -pc\rightarrow\ (Q,t) \Longrightarrow (P{\neq}Q)$
apply(*erule par-ctran.induct*)
apply(*drule prog-not-eq-in-ctran-aux*)
apply *clarify*
apply(*drule list-eq-if*)
 apply *simp-all*
apply *force*
done

lemma *prog-not-eq-in-par-ctran* [*simp*]: $\neg\ (P,s)\ -pc\rightarrow\ (P,t)$
apply *clarify*
apply(*drule prog-not-eq-in-par-ctran-aux*)
apply *simp*
done

lemma *tl-in-cptn*: $[\![\ a\#xs\ \in cptn;\ xs{\neq}[]\ ]\!] \Longrightarrow xs{\in}cptn$
apply(*force elim*:*cptn.elims*)
done

lemma *tl-zero*[*rule-format*]:
  $P\ (ys!Suc\ j) \longrightarrow Suc\ j{<}length\ ys \longrightarrow ys{\neq}[] \longrightarrow P\ (tl(ys)!j)$
apply(*induct ys*)
 apply *simp-all*
done

3.3.4 The Semantics is Compositional

lemma *aux-if* [*rule-format*]:
  $\forall\, xs\ s\ clist.\ (length\ clist = length\ xs \wedge (\forall\, i{<}length\ xs.\ (xs!i,s)\#clist!i \in cptn)$
  $\wedge\ ((xs,\ s)\#ys \propto map\ (\lambda i.\ (fst\ i,s)\#snd\ i)\ (zip\ xs\ clist))$
   $\longrightarrow (xs,\ s)\#ys \in par\text{-}cptn)$
apply(*induct ys*)
 apply(*clarify*)
 apply(*rule ParCptnOne*)

```
apply(clarify)
apply(simp add:conjoin-def compat-label-def)
apply clarify
apply(erule-tac x=0 and P=λj. ?H j ⟶ (?P j ∨ ?Q j) in all-dupE,simp)
apply(erule disjE)
— first step is a Component step
 apply clarify
 apply simp
 apply(subgoal-tac a=(xs[i:=(fst(clist!i!0))]))
  apply(subgoal-tac b=snd(clist!i!0),simp)
   prefer 2
   apply(simp add: same-state-def)
   apply(erule-tac x=i in allE,erule impE,assumption,
       erule-tac x=1 and P=λj. (?H j) ⟶ (snd (?d j))=(snd (?e j)) in allE,simp)
  prefer 2
  apply(simp add:same-program-def)
  apply(erule-tac x=1 and P=λj. ?H j ⟶ (fst (?s j))=(?t j) in allE,simp)
  apply(rule nth-equalityI,simp)
  apply clarify
  apply(case-tac i=ia,simp,simp)
  apply(erule-tac x=ia and P=λj. ?H j ⟶ ?I j ⟶ ?J j in allE)
  apply(drule-tac t=i in not-sym,simp)
  apply(erule etran.elims,simp)
 apply(rule ParCptnComp)
  apply(erule ParComp,simp)
— applying the induction hypothesis
 apply(erule-tac x=xs[i := fst (clist ! i ! 0)] in allE)
 apply(erule-tac x=snd (clist ! i ! 0) in allE)
 apply(erule mp)
 apply(rule-tac x=map tl clist in exI,simp)
 apply(rule conjI,clarify)
  apply(case-tac i=ia,simp)
   apply(rule nth-tl-if)
     apply(force simp add:same-length-def length-Suc-conv)
    apply simp
   apply(erule allE,erule impE,assumption,erule tl-in-cptn)
   apply(force simp add:same-length-def length-Suc-conv)
  apply(rule nth-tl-if)
    apply(force simp add:same-length-def length-Suc-conv)
   apply(simp add:same-state-def)
   apply(erule-tac x=ia in allE, erule impE, assumption,
     erule-tac x=1 and P=λj. ?H j ⟶ (snd (?d j))=(snd (?e j)) in allE)
   apply(erule-tac x=ia and P=λj. ?H j ⟶ ?I j ⟶ ?J j in allE)
   apply(drule-tac t=i  in not-sym,simp)
   apply(erule etran.elims,simp)
  apply(erule allE,erule impE,assumption,erule tl-in-cptn)
  apply(force simp add:same-length-def length-Suc-conv)
 apply(simp add:same-length-def same-state-def)
 apply(rule conjI)
```

```
  apply clarify
  apply(case-tac j,simp,simp)
  apply(erule-tac x=ia in allE, erule impE, assumption,
       erule-tac x=Suc(Suc nat) and P=λj. ?H j ⟶ (snd (?d j))=(snd (?e j))
in allE,simp)
  apply(force simp add:same-length-def length-Suc-conv)
 apply(rule conjI)
  apply(simp add:same-program-def)
  apply clarify
  apply(case-tac j,simp)
   apply(rule nth-equalityI,simp)
   apply clarify
   apply(case-tac i=ia,simp,simp)
   apply(erule-tac x=Suc(Suc nat) and P=λj. ?H j ⟶ (fst (?s j))=(?t j) in
allE,simp)
  apply(rule nth-equalityI,simp,simp)
  apply(force simp add:length-Suc-conv)
 apply(rule allI,rule impI)
 apply(erule-tac x=Suc j and P=λj. ?H j ⟶ (?I j ∨ ?J j) in allE,simp)
 apply(erule disjE)
  apply clarify
  apply(rule-tac x=ia in exI,simp)
  apply(case-tac i=ia,simp)
   apply(rule conjI)
    apply(force simp add: length-Suc-conv)
   apply clarify
  apply(erule-tac x=l and P=λj. ?H j ⟶ ?I j ⟶ ?J j in allE,erule impE,assumption)
  apply(erule-tac x=l and P=λj. ?H j ⟶ ?I j ⟶ ?J j in allE,erule impE,assumption)
   apply simp
   apply(case-tac j,simp)
    apply(rule tl-zero)
      apply(erule-tac x=l in allE, erule impE, assumption,
            erule-tac x=1 and P=λj. (?H j) ⟶ (snd (?d j))=(snd (?e j)) in
allE,simp)
      apply(force elim:etran.elims intro:Env)
     apply force
    apply force
   apply simp
   apply(rule tl-zero)
     apply(erule tl-zero)
      apply force
     apply force
    apply force
   apply force
  apply(rule conjI,simp)
   apply(rule nth-tl-if)
     apply force
    apply(erule-tac x=ia  in allE, erule impE, assumption,
         erule-tac x=1 and P=λj. ?H j ⟶ (snd (?d j))=(snd (?e j)) in allE)
```

apply(*erule-tac x=ia* **and** *P*=λ*j*. *?H j* $\longrightarrow$ *?I j* $\longrightarrow$ *?J j* **in** *allE*)
apply(*drule-tac t=i* **in** *not-sym*,*simp*)
apply(*erule etran.elims*,*simp*)
apply(*erule tl-zero*)
apply *force*
apply *force*
apply *clarify*
apply(*case-tac i=l*,*simp*)
apply(*rule nth-tl-if*)
apply(*erule-tac x=l* **and** *P*=λ*j*. *?H j* $\longrightarrow$ (*length* (*?s j*) = *?t*) **in** *allE*,*force*)
apply *simp*
apply(*erule-tac P*=λ*j*. *?H j* $\longrightarrow$ *?I j* $\longrightarrow$ *?J j* **in** *allE*,*erule impE*,*assumption*,*erule impE*,*assumption*)
apply(*erule tl-zero*,*force*)
apply(*erule-tac x=l* **and** *P*=λ*j*. *?H j* $\longrightarrow$ (*length* (*?s j*) = *?t*) **in** *allE*,*force*)
apply(*rule nth-tl-if*)
apply(*erule-tac x=l* **and** *P*=λ*j*. *?H j* $\longrightarrow$ (*length* (*?s j*) = *?t*) **in** *allE*,*force*)
apply(*erule-tac x=l* **in** *allE*, *erule impE*, *assumption*,
erule-tac x=1 **and** *P*=λ*j*. *?H j* $\longrightarrow$ (*snd* (*?d j*))=(*snd* (*?e j*)) **in** *allE*)
apply(*erule-tac x=l* **and** *P*=λ*j*. *?H j* $\longrightarrow$ *?I j* $\longrightarrow$ *?J j* **in** *allE*,*erule impE*,*assumption*,*simp*)
apply(*erule etran.elims*,*simp*)
apply(*rule tl-zero*)
apply *force*
apply *force*
apply(*erule-tac x=l* **and** *P*=λ*j*. *?H j* $\longrightarrow$ (*length* (*?s j*) = *?t*) **in** *allE*,*force*)
apply(*rule disjI2*)
apply(*case-tac j*,*simp*)
apply *clarify*
apply(*rule tl-zero*)
apply(*erule-tac x=ia* **and** *P*=λ*j*. *?H j* $\longrightarrow$ *?I j*$\in$*etran* **in** *allE*,*erule impE*,*assumption*)
apply(*case-tac i=ia*,*simp*,*simp*)
apply(*erule-tac x=ia* **in** *allE*, *erule impE*, *assumption*,
erule-tac x=1 **and** *P*=λ*j*. *?H j* $\longrightarrow$ (*snd* (*?d j*))=(*snd* (*?e j*)) **in** *allE*)
apply(*erule-tac x=ia* **and** *P*=λ*j*. *?H j* $\longrightarrow$ *?I j* $\longrightarrow$ *?J j* **in** *allE*,*erule impE*,*assumption*,*simp*)
apply(*force elim*:*etran.elims intro*:*Env*)
apply *force*
apply(*erule-tac x=ia* **and** *P*=λ*j*. *?H j* $\longrightarrow$ (*length* (*?s j*) = *?t*) **in** *allE*,*force*)
apply *simp*
apply *clarify*
apply(*rule tl-zero*)
apply(*rule tl-zero*,*force*)
apply *force*
apply(*erule-tac x=ia* **and** *P*=λ*j*. *?H j* $\longrightarrow$ (*length* (*?s j*) = *?t*) **in** *allE*,*force*)
apply *force*
apply(*erule-tac x=ia* **and** *P*=λ*j*. *?H j* $\longrightarrow$ (*length* (*?s j*) = *?t*) **in** *allE*,*force*)
— first step is an environmental step

```
apply clarify
apply(erule par-etran.elims)
apply simp
apply(rule ParCptnEnv)
apply(erule-tac x=Ps in allE)
apply(erule-tac x=t in allE)
apply(erule mp)
apply(rule-tac x=map tl clist in exI,simp)
apply(rule conjI)
 apply clarify
 apply(erule-tac x=i and P=λj. ?H j ⟶ (?I ?s j) ∈ cptn in allE,simp)
 apply(erule cptn.elims)
   apply(simp add:same-length-def)
   apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE,force)
  apply(simp add:same-state-def)
  apply(erule-tac x=i  in allE, erule impE, assumption,
   erule-tac x=1 and P=λj. ?H j ⟶ (snd (?d j))=(snd (?e j)) in allE,simp)
 apply(erule-tac x=i and P=λj. ?H j ⟶ ?J j ∈etran in allE,simp)
 apply(erule etran.elims,simp)
apply(simp add:same-state-def same-length-def)
apply(rule conjI,clarify)
 apply(case-tac j,simp,simp)
 apply(erule-tac x=i  in allE, erule impE, assumption,
        erule-tac x=Suc(Suc nat) and P=λj. ?H j ⟶ (snd (?d j))=(snd (?e j))
in allE,simp)
 apply(rule tl-zero)
   apply(simp)
  apply force
 apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE,force)
apply(rule conjI)
 apply(simp add:same-program-def)
 apply clarify
 apply(case-tac j,simp)
  apply(rule nth-equalityI,simp)
  apply clarify
  apply simp
 apply(erule-tac x=Suc(Suc nat) and P=λj. ?H j ⟶ (fst (?s j))=(?t j) in
allE,simp)
 apply(rule nth-equalityI,simp,simp)
 apply(force simp add:length-Suc-conv)
apply(rule allI,rule impI)
apply(erule-tac x=Suc j and P=λj. ?H j ⟶ (?I j ∨ ?J j) in allE,simp)
apply(erule disjE)
 apply clarify
 apply(rule-tac x=i in exI,simp)
 apply(rule conjI)
  apply(erule-tac x=i and P=λi. ?H i ⟶ ?J i ∈etran in allE, erule impE,
assumption)
  apply(erule etran.elims,simp)
```

```
  apply(erule-tac x=i  in allE, erule impE, assumption,
      erule-tac x=1 and P=λj. (?H j) ⟶ (snd (?d j))=(snd (?e j)) in allE,simp)
  apply(rule nth-tl-if)
   apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE,force)
  apply simp
 apply(erule tl-zero,force)
  apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE,force)
 apply clarify
 apply(erule-tac x=l and P=λi. ?H i ⟶ ?J i ∈etran in allE, erule impE,
assumption)
 apply(erule etran.elims,simp)
 apply(erule-tac x=l  in allE, erule impE, assumption,
     erule-tac x=1 and P=λj. (?H j) ⟶ (snd (?d j))=(snd (?e j)) in allE,simp)
 apply(rule nth-tl-if)
   apply(erule-tac x=l and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE,force)
  apply simp
  apply(rule tl-zero,force)
  apply force
 apply(erule-tac x=l and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE,force)
apply(rule disjI2)
apply simp
apply clarify
apply(case-tac j,simp)
 apply(rule tl-zero)
    apply(erule-tac x=i and P=λi. ?H i ⟶ ?J i ∈etran in allE, erule impE,
assumption)
    apply(erule-tac x=i and P=λi. ?H i ⟶ ?J i ∈etran in allE, erule impE,
assumption)
   apply(force elim:etran.elims intro:Env)
  apply force
 apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE,force)
apply simp
apply(rule tl-zero)
  apply(rule tl-zero,force)
   apply force
  apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE,force)
 apply force
apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE,force)
done

lemma less-Suc-0 [iff]: (n < Suc 0) = (n = 0)
by auto

lemma aux-onlyif [rule-format]: ∀ xs s. (xs, s)#ys ∈ par-cptn ⟶
  (∃ clist. (length clist = length xs) ∧
  (xs, s)#ys ∝ map (λi. (fst i,s)#(snd i)) (zip xs clist) ∧
  (∀ i<length xs. (xs!i,s)#(clist!i) ∈ cptn))
apply(induct ys)
 apply(clarify)
```

```
 apply(rule-tac x=map (λi. []) [0..<length xs] in exI)
 apply(simp add: conjoin-def same-length-def same-state-def same-program-def compat-label-def)
 apply(rule conjI)
  apply(rule nth-equalityI,simp,simp)
 apply(force intro: cptn.intros)
apply(clarify)
apply(erule par-cptn.elims,simp)
 apply simp
 apply(erule-tac x=xs in allE)
 apply(erule-tac x=t in allE,simp)
 apply clarify
 apply(rule-tac x=(map (λj. (P!j, t)#(clist!j)) [0..<length P]) in exI,simp)
 apply(rule conjI)
  prefer 2
  apply clarify
  apply(rule CptnEnv,simp)
 apply(simp add:conjoin-def same-length-def same-state-def)
 apply (rule conjI)
  apply clarify
  apply(case-tac j,simp,simp)
 apply(rule conjI)
  apply(simp add:same-program-def)
  apply clarify
  apply(case-tac j,simp)
   apply(rule nth-equalityI,simp,simp)
  apply simp
  apply(rule nth-equalityI,simp,simp)
 apply(simp add:compat-label-def)
 apply clarify
 apply(case-tac j,simp)
  apply(simp add:ParEnv)
  apply clarify
  apply(simp add:Env)
 apply simp
 apply(erule-tac x=nat in allE,erule impE, assumption)
 apply(erule disjE,simp)
  apply clarify
  apply(rule-tac x=i in exI,simp)
 apply force
apply(erule par-ctran.elims,simp)
apply(erule-tac x=Ps[i:=r] in allE)
apply(erule-tac x=ta in allE,simp)
apply clarify
apply(rule-tac x=(map (λj. (Ps!j, ta)#(clist!j)) [0..<length Ps]) [i:=((r, ta)#(clist!i))]
in exI,simp)
apply(rule conjI)
 prefer 2
 apply clarify
 apply(case-tac i=ia,simp)
```

```
  apply(erule CptnComp)
  apply(erule-tac x=ia and P=λj. ?H j ⟶ (?I j ∈ cptn) in allE,simp)
 apply simp
 apply(erule-tac x=ia in allE)
 apply(rule CptnEnv,simp)
apply(simp add:conjoin-def)
apply (rule conjI)
 apply(simp add:same-length-def)
 apply clarify
 apply(case-tac i=ia,simp,simp)
apply(rule conjI)
 apply(simp add:same-state-def)
 apply clarify
 apply(case-tac j, simp, simp (no-asm-simp))
 apply(case-tac i=ia,simp,simp)
apply(rule conjI)
 apply(simp add:same-program-def)
 apply clarify
 apply(case-tac j,simp)
  apply(rule nth-equalityI,simp,simp)
 apply simp
 apply(rule nth-equalityI,simp,simp)
 apply(erule-tac x=nat and P=λj. ?H j ⟶ (fst (?a j))=((?b j)) in allE)
 apply(case-tac nat)
  apply clarify
  apply(case-tac i=ia,simp,simp)
 apply clarify
 apply(case-tac i=ia,simp,simp)
apply(simp add:compat-label-def)
apply clarify
apply(case-tac j)
 apply(rule conjI,simp)
  apply(erule ParComp,assumption)
  apply clarify
  apply(rule-tac x=i in exI,simp)
 apply clarify
 apply(rule Env)
apply simp
apply(erule-tac x=nat and P=λj. ?H j ⟶ (?P j ∨ ?Q j) in allE,simp)
apply(erule disjE)
 apply clarify
 apply(rule-tac x=ia in exI,simp)
 apply(rule conjI)
  apply(case-tac i=ia,simp,simp)
 apply clarify
 apply(case-tac i=l,simp)
  apply(case-tac l=ia,simp,simp)
 apply(erule-tac x=l in allE,erule impE,assumption,erule impE, assumption,simp)
 apply simp
```

```
 apply(erule-tac x=l in allE,erule impE,assumption,erule impE, assumption,simp)
apply clarify
apply(erule-tac x=ia and P=λj. ?H j ⟶ (?P j)∈etran in allE, erule impE,
assumption)
apply(case-tac i=ia,simp,simp)
done

lemma one-iff-aux: xs≠[] ⟹ (∀ ys. ((xs, s)#ys ∈ par-cptn) =
 (∃ clist. length clist= length xs ∧
 ((xs, s)#ys ∝ map (λi. (fst i,s)#(snd i)) (zip xs clist)) ∧
 (∀ i<length xs. (xs!i,s)#(clist!i) ∈ cptn))) =
 (par-cp (xs) s = {c. ∃ clist. (length clist)=(length xs) ∧
 (∀ i<length clist. (clist!i) ∈ cp(xs!i) s) ∧ c ∝ clist})
apply (rule iffI)
 apply(rule subset-antisym)
  apply(rule subsetI)
  apply(clarify)
  apply(simp add:par-cp-def cp-def)
  apply(case-tac x)
   apply(force elim:par-cptn.elims)
  apply simp
  apply(erule-tac x=list in allE)
  apply clarify
  apply simp
  apply(rule-tac x=map (λi. (fst i, s) # snd i) (zip xs clist) in exI,simp)
 apply(rule subsetI)
 apply(clarify)
 apply(case-tac x)
  apply(erule-tac x=0 in allE)
  apply(simp add:cp-def conjoin-def same-length-def same-program-def same-state-def
compat-label-def)
  apply clarify
  apply(erule cptn.elims,force,force,force)
 apply(simp add:par-cp-def conjoin-def  same-length-def same-program-def same-state-def
compat-label-def)
 apply clarify
 apply(erule-tac x=0 and P=λj. ?H j ⟶ (length (?s j) = ?t) in all-dupE)
 apply(subgoal-tac a = xs)
  apply(subgoal-tac b = s,simp)
   prefer 3
   apply(erule-tac x=0 and P=λj. ?H j ⟶ (fst (?s j))=((?t j)) in allE)
   apply (simp add:cp-def)
   apply(rule nth-equalityI,simp,simp)
  prefer 2
  apply(erule-tac x=0 in allE)
  apply (simp add:cp-def)
  apply(erule-tac x=0 and P=λj. ?H j ⟶ (∀ i. ?T i ⟶ (snd (?d j i))=(snd
(?e j i))) in allE,simp)
 apply(erule-tac x=0 and P=λj. ?H j ⟶ (snd (?d j))=(snd (?e j)) in allE,simp)
```

apply(*erule-tac x=list* **in** *allE*)
apply(*rule-tac x=map tl clist* **in** *exI*,*simp*)
apply(*rule conjI*)
 apply *clarify*
 apply(*case-tac j*,*simp*)
  apply(*erule-tac x=i* **in** *allE*, *erule impE*, *assumption*,
   erule-tac x=0 **and** $P=\lambda j.\ ?H\,j \longrightarrow (snd\ (?d\,j))=(snd\ (?e\ j))$ **in** *allE*,*simp*)
 apply(*erule-tac x=i* **in** *allE*, *erule impE*, *assumption*,
   erule-tac x=Suc nat **and** $P=\lambda j.\ ?H\ j \longrightarrow (snd\ (?d\ j))=(snd\ (?e\ j))$ **in** *allE*)
 apply(*erule-tac x=i* **and** $P=\lambda j.\ ?H\ j \longrightarrow (length\ (?s\ j) = ?t)$ **in** *allE*)
 apply(*case-tac clist!i*,*simp*,*simp*)
apply(*rule conjI*)
 apply *clarify*
 apply(*rule nth-equalityI*,*simp*,*simp*)
 apply(*case-tac j*)
  apply *clarify*
  apply(*erule-tac x=i* **in** *allE*)
  apply(*simp add:cp-def*)
 apply *clarify*
 apply *simp*
 apply(*erule-tac x=i* **and** $P=\lambda j.\ ?H\ j \longrightarrow (length\ (?s\ j) = ?t)$ **in** *allE*)
 apply(*case-tac clist!i*,*simp*,*simp*)
apply(*thin-tac* $?H = (\exists\, i.\ ?J\ i)$)
apply(*rule conjI*)
 apply *clarify*
 apply(*erule-tac x=j* **in** *allE*,*erule impE*, *assumption*,*erule disjE*)
  apply *clarify*
  apply(*rule-tac x=i* **in** *exI*,*simp*)
  apply(*case-tac j*,*simp*)
   apply(*rule conjI*)
    apply(*erule-tac x=i* **in** *allE*)
    apply(*simp add:cp-def*)
    apply(*erule-tac x=i* **and** $P=\lambda j.\ ?H\ j \longrightarrow (length\ (?s\ j) = ?t)$ **in** *allE*)
    apply(*case-tac clist!i*,*simp*,*simp*)
   apply *clarify*
   apply(*erule-tac x=l* **in** *allE*)
   apply(*erule-tac x=l* **and** $P=\lambda j.\ ?H\ j \longrightarrow ?I\ j \longrightarrow ?J\ j$ **in** *allE*)
   apply *clarify*
   apply(*simp add:cp-def*)
   apply(*erule-tac x=l* **and** $P=\lambda j.\ ?H\ j \longrightarrow (length\ (?s\ j) = ?t)$ **in** *allE*)
   apply(*case-tac clist!l*,*simp*,*simp*)
  apply *simp*
  apply(*rule conjI*)
   apply(*erule-tac x=i* **and** $P=\lambda j.\ ?H\ j \longrightarrow (length\ (?s\ j) = ?t)$ **in** *allE*)
   apply(*case-tac clist!i*,*simp*,*simp*)
  apply *clarify*
  apply(*erule-tac x=l* **and** $P=\lambda j.\ ?H\ j \longrightarrow ?I\ j \longrightarrow ?J\ j$ **in** *allE*)
  apply(*erule-tac x=l* **and** $P=\lambda j.\ ?H\ j \longrightarrow (length\ (?s\ j) = ?t)$ **in** *allE*)

```
   apply(case-tac clist!l,simp,simp)
  apply clarify
  apply(erule-tac x=i in allE)
  apply(simp add:cp-def)
  apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE)
  apply(case-tac clist!i,simp)
  apply(rule nth-tl-if,simp,simp)
  apply(erule-tac x=i and P=λj. ?H j ⟶ (?P j)∈etran in allE, erule impE,
assumption,simp)
  apply(simp add:cp-def)
  apply clarify
  apply(rule nth-tl-if)
   apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE)
   apply(case-tac clist!i,simp,simp)
  apply force
 apply force
apply clarify
apply(rule iffI)
 apply(simp add:par-cp-def)
 apply(erule-tac c=(xs, s) # ys in equalityCE)
  apply simp
  apply clarify
  apply(rule-tac x=map tl clist in exI)
  apply simp
  apply (rule conjI)
   apply(simp add:conjoin-def cp-def)
   apply(rule conjI)
    apply clarify
    apply(unfold same-length-def)
    apply clarify
    apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE,simp)
   apply(rule conjI)
    apply(simp add:same-state-def)
    apply clarify
    apply(erule-tac x=i in allE, erule impE, assumption,
       erule-tac x=j and P=λj. ?H j ⟶ (snd (?d j))=(snd (?e j)) in allE)
    apply(case-tac j,simp)
    apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE)
    apply(case-tac clist!i,simp,simp)
   apply(rule conjI)
    apply(simp add:same-program-def)
    apply clarify
    apply(rule nth-equalityI,simp,simp)
    apply(case-tac j,simp)
    apply clarify
    apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE)
    apply(case-tac clist!i,simp,simp)
   apply clarify
   apply(simp add:compat-label-def)
```

```
   apply(rule allI,rule impI)
   apply(erule-tac x=j in allE,erule impE, assumption)
   apply(erule disjE)
    apply clarify
    apply(rule-tac x=i in exI,simp)
    apply(rule conjI)
     apply(erule-tac x=i in allE)
     apply(case-tac j,simp)
      apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE)
      apply(case-tac clist!i,simp,simp)
     apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE)
     apply(case-tac clist!i,simp,simp)
    apply clarify
    apply(erule-tac x=l and P=λj. ?H j ⟶ ?I j ⟶ ?J j in allE)
    apply(erule-tac x=l and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE)
    apply(case-tac clist!l,simp,simp)
    apply(erule-tac x=l in allE,simp)
   apply(rule disjI2)
   apply clarify
   apply(rule tl-zero)
     apply(case-tac j,simp,simp)
     apply(rule tl-zero,force)
      apply force
     apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE,force)
    apply force
   apply(erule-tac x=i and P=λj. ?H j ⟶ (length (?s j) = ?t) in allE,force)
  apply clarify
  apply(erule-tac x=i in allE)
  apply(simp add:cp-def)
  apply(rule nth-tl-if)
    apply(simp add:conjoin-def)
    apply clarify
    apply(simp add:same-length-def)
    apply(erule-tac x=i in allE,simp)
   apply simp
  apply simp
 apply simp
apply clarify
apply(erule-tac c=(xs, s) # ys in equalityCE)
 apply(simp add:par-cp-def)
apply simp
apply(erule-tac x=map (λi. (fst i, s) # snd i) (zip xs clist) in allE)
apply simp
apply clarify
apply(simp add:cp-def)
done

theorem one: xs≠[] ⟹
 par-cp xs s = {c. ∃ clist. (length clist)=(length xs) ∧
```

$(\forall\, i{<}length\ clist.\ (clist!i) \in cp(xs!i)\ s) \wedge c \propto clist\}$

apply(*frule one-iff-aux*)
apply(*drule sym*)
apply(*erule iffD2*)
apply *clarify*
apply(*rule iffI*)
apply(*erule aux-onlyif*)
apply *clarify*
apply(*force intro*: *aux-if*)
done

end

3.4 The Proof System

theory *RG-Hoare* **imports** *RG-Tran* **begin**

3.4.1 Proof System for Component Programs

declare *Un-subset-iff* [*iff del*]

constdefs
$stable :: {'a}\ set \Rightarrow ({'a} \times {'a})\ set \Rightarrow bool$
$stable \equiv \lambda f\ g.\ (\forall\, x\ y.\ x \in f \longrightarrow (x,\ y) \in g \longrightarrow y \in f)$

consts $rghoare :: ({'a}\ rgformula)\ set$
syntax
$\text{-}rghoare :: [{'a}\ com,\ {'a}\ set,\ ({'a} \times {'a})\ set,\ ({'a} \times {'a})\ set,\ {'a}\ set] \Rightarrow bool$
$(\vdash\ \text{-}\ sat\ [\text{-},\ \text{-},\ \text{-},\ \text{-}]\ [60,0,0,0,0]\ 45)$
translations
$\vdash P\ sat\ [pre,\ rely,\ guar,\ post] \rightleftharpoons (P,\ pre,\ rely,\ guar,\ post) \in rghoare$

inductive *rghoare*
intros
$Basic: [\![\ pre \subseteq \{s.\ f\ s \in post\};\ \{(s,t).\ s \in pre \wedge (t{=}f\ s \vee t{=}s)\} \subseteq guar;$
$stable\ pre\ rely;\ stable\ post\ rely\ ]\!]$
$\Longrightarrow \vdash Basic\ f\ sat\ [pre,\ rely,\ guar,\ post]$

$Seq: [\![\ \vdash P\ sat\ [pre,\ rely,\ guar,\ mid];\ \vdash Q\ sat\ [mid,\ rely,\ guar,\ post]\ ]\!]$
$\Longrightarrow \vdash Seq\ P\ Q\ sat\ [pre,\ rely,\ guar,\ post]$

$Cond: [\![\ stable\ pre\ rely;\ \vdash P1\ sat\ [pre \cap b,\ rely,\ guar,\ post];$
$\vdash P2\ sat\ [pre \cap -b,\ rely,\ guar,\ post];\ \forall\, s.\ (s,s){\in}guar\ ]\!]$
$\Longrightarrow \vdash Cond\ b\ P1\ P2\ sat\ [pre,\ rely,\ guar,\ post]$

$While: [\![\ stable\ pre\ rely;\ (pre \cap -b) \subseteq post;\ stable\ post\ rely;$
$\vdash P\ sat\ [pre \cap b,\ rely,\ guar,\ pre];\ \forall\, s.\ (s,s){\in}guar\ ]\!]$
$\Longrightarrow \vdash While\ b\ P\ sat\ [pre,\ rely,\ guar,\ post]$

$Await$: $\llbracket$ $stable\ pre\ rely$; $stable\ post\ rely$;
$\forall V. \vdash P\ sat\ [pre \cap b \cap \{V\}, \{(s, t).\ s = t\},$
$UNIV, \{s.\ (V, s) \in guar\} \cap post]$ $\rrbracket$
$\Longrightarrow \vdash Await\ b\ P\ sat\ [pre, rely, guar, post]$

$Conseq$: $\llbracket$ $pre \subseteq pre'$; $rely \subseteq rely'$; $guar' \subseteq guar$; $post' \subseteq post$;
$\vdash P\ sat\ [pre', rely', guar', post']$ $\rrbracket$
$\Longrightarrow \vdash P\ sat\ [pre, rely, guar, post]$

constdefs
$Pre :: 'a\ rgformula \Rightarrow 'a\ set$
$Pre\ x \equiv fst(snd\ x)$
$Post :: 'a\ rgformula \Rightarrow 'a\ set$
$Post\ x \equiv snd(snd(snd(snd\ x)))$
$Rely :: 'a\ rgformula \Rightarrow ('a \times 'a)\ set$
$Rely\ x \equiv fst(snd(snd\ x))$
$Guar :: 'a\ rgformula \Rightarrow ('a \times 'a)\ set$
$Guar\ x \equiv fst(snd(snd(snd\ x)))$
$Com :: 'a\ rgformula \Rightarrow 'a\ com$
$Com\ x \equiv fst\ x$

3.4.2 Proof System for Parallel Programs

types $'a\ par\text{-}rgformula = ('a\ rgformula)\ list \times 'a\ set \times ('a \times 'a)\ set \times ('a \times 'a)\ set \times 'a\ set$

consts $par\text{-}rghoare :: ('a\ par\text{-}rgformula)\ set$
syntax
$\text{-}par\text{-}rghoare :: ('a\ rgformula)\ list \Rightarrow 'a\ set \Rightarrow ('a \times 'a)\ set \Rightarrow ('a \times 'a)\ set \Rightarrow 'a\ set \Rightarrow bool$ $(\vdash\ \text{-}\ SAT\ [\text{-}, \text{-}, \text{-}, \text{-}]\ [60,0,0,0,0]\ 45)$
translations
$\vdash Ps\ SAT\ [pre, rely, guar, post] \rightleftharpoons (Ps, pre, rely, guar, post) \in par\text{-}rghoare$

inductive $par\text{-}rghoare$
intros
$Parallel$:
$\llbracket$ $\forall i{<}length\ xs.\ rely \cup (\bigcup j{\in}\{j.\ j{<}length\ xs \wedge j{\neq}i\}.\ Guar(xs!j)) \subseteq Rely(xs!i)$;
$(\bigcup j{\in}\{j.\ j{<}length\ xs\}.\ Guar(xs!j)) \subseteq guar$;
$pre \subseteq (\bigcap i{\in}\{i.\ i{<}length\ xs\}.\ Pre(xs!i))$;
$(\bigcap i{\in}\{i.\ i{<}length\ xs\}.\ Post(xs!i)) \subseteq post$;
$\forall i{<}length\ xs. \vdash Com(xs!i)\ sat\ [Pre(xs!i),Rely(xs!i),Guar(xs!i),Post(xs!i)]$ $\rrbracket$
$\Longrightarrow \vdash xs\ SAT\ [pre, rely, guar, post]$

3.5 Soundness

Some previous lemmas

lemma $tl\text{-}of\text{-}assum\text{-}in\text{-}assum$:

```
  (P, s) # (P, t) # xs ∈ assum (pre, rely) ⟹ stable pre rely
  ⟹ (P, t) # xs ∈ assum (pre, rely)
apply(simp add:assum-def)
apply clarify
apply(rule conjI)
 apply(erule-tac x=0 in allE)
 apply(simp (no-asm-use)only:stable-def)
 apply(erule allE,erule allE,erule impE,assumption,erule mp)
 apply(simp add:Env)
apply clarify
apply(erule-tac x=Suc i in allE)
apply simp
done

lemma etran-in-comm:
  (P, t) # xs ∈ comm(guar, post) ⟹ (P, s) # (P, t) # xs ∈ comm(guar, post)
apply(simp add:comm-def)
apply clarify
apply(case-tac i,simp+)
done

lemma ctran-in-comm:
  ⟦(s, s) ∈ guar; (Q, s) # xs ∈ comm(guar, post)⟧
  ⟹ (P, s) # (Q, s) # xs ∈ comm(guar, post)
apply(simp add:comm-def)
apply clarify
apply(case-tac i,simp+)
done

lemma takecptn-is-cptn [rule-format, elim!]:
  ∀j. c ∈ cptn ⟶ take (Suc j) c ∈ cptn
apply(induct c)
 apply(force elim: cptn.elims)
apply clarify
apply(case-tac j)
 apply simp
 apply(rule CptnOne)
apply simp
apply(force intro:cptn.intros elim:cptn.elims)
done

lemma dropcptn-is-cptn [rule-format,elim!]:
  ∀j<length c. c ∈ cptn ⟶ drop j c ∈ cptn
apply(induct c)
 apply(force elim: cptn.elims)
apply clarify
apply(case-tac j,simp+)
apply(erule cptn.elims)
  apply simp
```

apply *force*
apply *force*
done

lemma *takepar-cptn-is-par-cptn* [*rule-format*,*elim*]:
$\forall j.\ c \in par\text{-}cptn \longrightarrow take\ (Suc\ j)\ c \in par\text{-}cptn$
apply(*induct c*)
apply(*force elim*: *cptn.elims*)
apply *clarify*
apply(*case-tac j*,*simp*)
apply(*rule ParCptnOne*)
apply(*force intro*:*par-cptn.intros elim*:*par-cptn.elims*)
done

lemma *droppar-cptn-is-par-cptn* [*rule-format*]:
$\forall j{<}length\ c.\ c \in par\text{-}cptn \longrightarrow drop\ j\ c \in par\text{-}cptn$
apply(*induct c*)
apply(*force elim*: *par-cptn.elims*)
apply *clarify*
apply(*case-tac j*,*simp+*)
apply(*erule par-cptn.elims*)
apply *simp*
apply *force*
apply *force*
done

lemma *tl-of-cptn-is-cptn*: $[\![x\ \#\ xs \in cptn;\ xs \neq []]\!] \Longrightarrow xs\ \in cptn$
apply(*subgoal-tac 1 < length (x # xs)*)
apply(*drule dropcptn-is-cptn*,*simp+*)
done

lemma *not-ctran-None* [*rule-format*]:
$\forall s.\ (None,\ s)\#xs \in cptn \longrightarrow (\forall i{<}length\ xs.\ ((None,\ s)\#xs)!i\ -e\rightarrow\ xs!i)$
apply(*induct xs*,*simp+*)
apply *clarify*
apply(*erule cptn.elims*,*simp*)
apply *simp*
apply(*case-tac i*,*simp*)
apply(*rule Env*)
apply *simp*
apply(*force elim*:*ctran.elims*)
done

lemma *cptn-not-empty* [*simp*]:$[] \notin cptn$
apply(*force elim*:*cptn.elims*)
done

lemma *etran-or-ctran* [*rule-format*]:
$\forall m\ i.\ x{\in}cptn \longrightarrow m \leq length\ x$

$\longrightarrow (\forall i.\ Suc\ i < m \longrightarrow \neg\ x!i\ -c\rightarrow\ x!Suc\ i) \longrightarrow Suc\ i < m$
$\longrightarrow x!i\ -e\rightarrow\ x!Suc\ i$
apply(*induct x*,*simp*)
apply *clarify*
apply(*erule cptn.elims*,*simp*)
 apply(*case-tac i*,*simp*)
  apply(*rule Env*)
 apply *simp*
 apply(*erule-tac x=m − 1* **in** *allE*)
 apply(*case-tac m*,*simp*,*simp*)
 apply(*subgoal-tac* $(\forall i.\ Suc\ i < nata \longrightarrow (((P,\ t)\ \#\ xs)\ !\ i,\ xs\ !\ i) \notin ctran)$)
  apply *force*
 apply *clarify*
 apply(*erule-tac x=Suc ia* **in** *allE*,*simp*)
apply(*erule-tac x=0* **and** $P{=}\lambda j.\ ?H\ j \longrightarrow (?J\ j) \notin ctran$ **in** *allE*,*simp*)
done

lemma *etran-or-ctran2* [*rule-format*]:
$\forall i.\ Suc\ i{<}length\ x \longrightarrow x{\in}cptn \longrightarrow (x!i\ -c\rightarrow\ x!Suc\ i \longrightarrow \neg\ x!i\ -e\rightarrow\ x!Suc\ i)$
$\vee\ (x!i\ -e\rightarrow\ x!Suc\ i \longrightarrow \neg\ x!i\ -c\rightarrow\ x!Suc\ i)$
apply(*induct x*)
 apply *simp*
apply *clarify*
apply(*erule cptn.elims*,*simp*)
 apply(*case-tac i*,*simp+*)
apply(*case-tac i*,*simp*)
 apply(*force elim*:*etran.elims*)
apply *simp*
done

lemma *etran-or-ctran2-disjI1*:
$[\![\ x{\in}cptn;\ Suc\ i{<}length\ x;\ x!i\ -c\rightarrow\ x!Suc\ i]\!] \Longrightarrow \neg\ x!i\ -e\rightarrow\ x!Suc\ i$
by(*drule etran-or-ctran2*,*simp-all*)

lemma *etran-or-ctran2-disjI2*:
$[\![\ x{\in}cptn;\ Suc\ i{<}length\ x;\ x!i\ -e\rightarrow\ x!Suc\ i]\!] \Longrightarrow \neg\ x!i\ -c\rightarrow\ x!Suc\ i$
by(*drule etran-or-ctran2*,*simp-all*)

lemma *not-ctran-None2* [*rule-format*]:
$[\![\ (None,\ s)\ \#\ xs\ {\in}cptn;\ i{<}length\ xs]\!] \Longrightarrow \neg\ ((None,\ s)\ \#\ xs)\ !\ i\ -c\rightarrow\ xs\ !\ i$
apply(*frule not-ctran-None*,*simp*)
apply(*case-tac i*,*simp*)
 apply(*force elim*:*etran.elims*)
apply *simp*
apply(*rule etran-or-ctran2-disjI2*,*simp-all*)
apply(*force intro*:*tl-of-cptn-is-cptn*)
done

lemma *Ex-first-occurrence* [*rule-format*]: $P\ (n{::}nat) \longrightarrow (\exists m.\ P\ m \wedge (\forall i{<}m.\ \neg$

P i))
apply(*rule nat-less-induct*)
apply *clarify*
apply(*case-tac* $\forall m.\ m<n \longrightarrow \neg\ P\ m$)
apply *auto*
done

lemma *stability* [*rule-format*]:
$\forall j\ k.\ x \in cptn \longrightarrow stable\ p\ rely \longrightarrow j\leq k \longrightarrow k<length\ x \longrightarrow snd(x!j)\in p \longrightarrow$
$(\forall i.\ (Suc\ i)<length\ x \longrightarrow$
$(x!i\ -e\rightarrow\ x!(Suc\ i)) \longrightarrow (snd(x!i),\ snd(x!(Suc\ i))) \in rely) \longrightarrow$
$(\forall i.\ j\leq i \wedge i<k \longrightarrow x!i\ -e\rightarrow\ x!Suc\ i) \longrightarrow snd(x!k)\in p \wedge fst(x!j)=fst(x!k)$
apply(*induct x*)
apply *clarify*
apply(*force elim*:*cptn.elims*)
apply *clarify*
apply(*erule cptn.elims*,*simp*)
apply *simp*
apply(*case-tac k*,*simp*,*simp*)
apply(*case-tac j*,*simp*)
apply(*erule-tac* $x=0$ **in** *allE*)
apply(*erule-tac* $x=nat$ **and** $P=\lambda j.\ (0\leq j) \longrightarrow (?J\ j)$ **in** *allE*,*simp*)
apply(*subgoal-tac* $t\in p$)
apply(*subgoal-tac* $(\forall i.\ i < length\ xs \longrightarrow ((P,\ t)\ \#\ xs)\ !\ i\ -e\rightarrow\ xs\ !\ i \longrightarrow (snd\ (((P,\ t)\ \#\ xs)\ !\ i),\ snd\ (xs\ !\ i)) \in rely))$
apply *clarify*
apply(*erule-tac* $x=Suc\ i$ **and** $P=\lambda j.\ (?H\ j) \longrightarrow (?J\ j)\in etran$ **in** *allE*,*simp*)
apply *clarify*
apply(*erule-tac* $x=Suc\ i$ **and** $P=\lambda j.\ (?H\ j) \longrightarrow (?J\ j) \longrightarrow (?T\ j)\in rely$ **in** *allE*,*simp*)
apply(*erule-tac* $x=0$ **and** $P=\lambda j.\ (?H\ j) \longrightarrow (?J\ j)\in etran \longrightarrow ?T\ j$ **in** *allE*,*simp*)
apply(*simp*(*no-asm-use*) *only*:*stable-def*)
apply(*erule-tac* $x=s$ **in** *allE*)
apply(*erule-tac* $x=t$ **in** *allE*)
apply *simp*
apply(*erule mp*)
apply(*erule mp*)
apply(*rule Env*)
apply *simp*
apply(*erule-tac* $x=nata$ **in** *allE*)
apply(*erule-tac* $x=nat$ **and** $P=\lambda j.\ (?s\leq j) \longrightarrow (?J\ j)$ **in** *allE*,*simp*)
apply(*subgoal-tac* $(\forall i.\ i < length\ xs \longrightarrow ((P,\ t)\ \#\ xs)\ !\ i\ -e\rightarrow\ xs\ !\ i \longrightarrow (snd\ (((P,\ t)\ \#\ xs)\ !\ i),\ snd\ (xs\ !\ i)) \in rely))$
apply *clarify*
apply(*erule-tac* $x=Suc\ i$ **and** $P=\lambda j.\ (?H\ j) \longrightarrow (?J\ j)\in etran$ **in** *allE*,*simp*)
apply *clarify*
apply(*erule-tac* $x=Suc\ i$ **and** $P=\lambda j.\ (?H\ j) \longrightarrow (?J\ j) \longrightarrow (?T\ j)\in rely$ **in** *allE*,*simp*)
apply(*case-tac k*,*simp*,*simp*)

apply(*case-tac j*)
 apply(*erule-tac x=0* **and** *P*=λj. (*?H j*) $\longrightarrow$ (*?J j*)$\in$*etran* **in** *allE*,*simp*)
 apply(*erule etran.elims*,*simp*)
apply(*erule-tac x=nata* **in** *allE*)
apply(*erule-tac x=nat* **and** *P*=λj. (*?s*$\leq$*j*) $\longrightarrow$ (*?J j*) **in** *allE*,*simp*)
apply(*subgoal-tac* ($\forall$ *i*. *i* < *length xs* $\longrightarrow$ ((*Q*, *t*) # *xs*) ! *i* −*e*→ *xs* ! *i* $\longrightarrow$ (*snd* (((*Q*, *t*) # *xs*) ! *i*), *snd* (*xs* ! *i*)) $\in$ *rely*))
 apply *clarify*
 apply(*erule-tac x=Suc i* **and** *P*=λj. (*?H j*) $\longrightarrow$ (*?J j*)$\in$*etran* **in** *allE*,*simp*)
apply *clarify*
apply(*erule-tac x=Suc i* **and** *P*=λj. (*?H j*) $\longrightarrow$ (*?J j*) $\longrightarrow$ (*?T j*)$\in$*rely* **in** *allE*,*simp*)
done

3.5.1 Soundness of the System for Component Programs

Soundness of the Basic rule

lemma *unique-ctran-Basic* [*rule-format*]:
 $\forall$ *s i*. *x* $\in$ *cptn* $\longrightarrow$ *x* ! *0* = (*Some* (*Basic f*), *s*) $\longrightarrow$
 Suc i<*length x* $\longrightarrow$ *x*!*i* −*c*→ *x*!*Suc i* $\longrightarrow$
 ($\forall$ *j*. *Suc j*<*length x* $\longrightarrow$ *i*$\neq$*j* $\longrightarrow$ *x*!*j* −*e*→ *x*!*Suc j*)
apply(*induct x*,*simp*)
apply *simp*
apply *clarify*
apply(*erule cptn.elims*,*simp*)
 apply(*case-tac i*,*simp+*)
 apply *clarify*
 apply(*case-tac j*,*simp*)
  apply(*rule Env*)
 apply *simp*
apply *clarify*
apply *simp*
apply(*case-tac i*)
 apply(*case-tac j*,*simp*,*simp*)
 apply(*erule ctran.elims*,*simp-all*)
 apply(*force elim*: *not-ctran-None*)
apply(*ind-cases* ((*Some* (*Basic f*), *sa*), *Q*, *t*) $\in$ *ctran*)
apply *simp*
apply(*drule-tac i=nat* **in** *not-ctran-None*,*simp*)
apply(*erule etran.elims*,*simp*)
done

lemma *exists-ctran-Basic-None* [*rule-format*]:
 $\forall$ *s i*. *x* $\in$ *cptn* $\longrightarrow$ *x* ! *0* = (*Some* (*Basic f*), *s*)
 $\longrightarrow$ *i*<*length x* $\longrightarrow$ *fst*(*x*!*i*)=*None* $\longrightarrow$ ($\exists$ *j*<*i*. *x*!*j* −*c*→ *x*!*Suc j*)
apply(*induct x*,*simp*)
apply *simp*
apply *clarify*
apply(*erule cptn.elims*,*simp*)
 apply(*case-tac i*,*simp*,*simp*)

```
 apply(erule-tac x=nat in allE,simp)
 apply clarify
 apply(rule-tac x=Suc j in exI,simp,simp)
apply clarify
apply(case-tac i,simp,simp)
apply(rule-tac x=0 in exI,simp)
done

lemma Basic-sound:
  ⟦pre ⊆ {s. f s ∈ post}; {(s, t). s ∈ pre ∧ t = f s} ⊆ guar;
  stable pre rely; stable post rely⟧
  ⟹ ⊨ Basic f sat [pre, rely, guar, post]
apply(unfold com-validity-def)
apply clarify
apply(simp add:comm-def)
apply(rule conjI)
 apply clarify
 apply(simp add:cp-def assum-def)
 apply clarify
 apply(frule-tac j=0 and k=i and p=pre in stability)
      apply simp-all
   apply(erule-tac x=ia in allE,simp)
  apply(erule-tac i=i and f=f in unique-ctran-Basic,simp-all)
 apply(erule subsetD,simp)
 apply(case-tac x!i)
 apply clarify
 apply(drule-tac s=Some (Basic f) in sym,simp)
 apply(thin-tac ∀ j. ?H j)
 apply(force elim:ctran.elims)
apply clarify
apply(simp add:cp-def)
apply clarify
apply(frule-tac i=length x − 1 and f=f in exists-ctran-Basic-None,simp+)
  apply(case-tac x,simp+)
  apply(rule last-fst-esp,simp add:last-length)
 apply (case-tac x,simp+)
apply(simp add:assum-def)
apply clarify
apply(frule-tac j=0 and k=j and p=pre in stability)
     apply simp-all
    apply arith
  apply(erule-tac x=i in allE,simp)
 apply(erule-tac i=j and f=f in unique-ctran-Basic,simp-all)
  apply arith
 apply arith
apply(case-tac x!j)
apply clarify
apply simp
apply(drule-tac s=Some (Basic f) in sym,simp)
```

apply(*case-tac x*!*Suc j*,*simp*)
apply(*rule ctran.elims*,*simp*)
apply(*simp-all*)
apply(*drule-tac c*=*sa* **in** *subsetD*,*simp*)
apply *clarify*
apply(*frule-tac j*=*Suc j* **and** *k*=*length x* − *1* **and** *p*=*post* **in** *stability*,*simp-all*)
 apply(*case-tac x*,*simp*+)
 apply(*erule-tac x*=*i* **in** *allE*)
apply(*erule-tac i*=*j* **and** *f*=*f* **in** *unique-ctran-Basic*,*simp-all*)
  apply *arith*+
apply(*case-tac x*)
apply(*simp add*:*last-length*)+
done

Soundness of the Await rule

lemma *unique-ctran-Await* [*rule-format*]:
  $\forall s\ i.\ x \in cptn \longrightarrow x\ !\ 0 = (Some\ (Await\ b\ c),\ s) \longrightarrow$
  $Suc\ i{<}length\ x \longrightarrow x!i\ {-c\rightarrow}\ x!Suc\ i \longrightarrow$
  $(\forall j.\ Suc\ j{<}length\ x \longrightarrow i{\neq}j \longrightarrow x!j\ {-e\rightarrow}\ x!Suc\ j)$
apply(*induct x*,*simp*+)
apply *clarify*
apply(*erule cptn.elims*,*simp*)
 apply(*case-tac i*,*simp*+)
 apply *clarify*
 apply(*case-tac j*,*simp*)
  apply(*rule Env*)
 apply *simp*
apply *clarify*
apply *simp*
apply(*case-tac i*)
 apply(*case-tac j*,*simp*,*simp*)
 apply(*erule ctran.elims*,*simp-all*)
 apply(*force elim*: *not-ctran-None*)
apply(*ind-cases* ((*Some* (*Await b c*), *sa*), *Q*, *t*) ∈ *ctran*,*simp*)
apply(*drule-tac i*=*nat* **in** *not-ctran-None*,*simp*)
apply(*erule etran.elims*,*simp*)
done

lemma *exists-ctran-Await-None* [*rule-format*]:
  $\forall s\ i.\ \ x \in cptn \longrightarrow x\ !\ 0 = (Some\ (Await\ b\ c),\ s)$
  $\longrightarrow i{<}length\ x \longrightarrow fst(x!i){=}None \longrightarrow (\exists j{<}i.\ x!j\ {-c\rightarrow}\ x!Suc\ j)$
apply(*induct x*,*simp*+)
apply *clarify*
apply(*erule cptn.elims*,*simp*)
 apply(*case-tac i*,*simp*+)
 apply(*erule-tac x*=*nat* **in** *allE*,*simp*)
 apply *clarify*
 apply(*rule-tac x*=*Suc j* **in** *exI*,*simp*,*simp*)

```
apply clarify
apply(case-tac i,simp,simp)
apply(rule-tac x=0 in exI,simp)
done

lemma Star-imp-cptn:
  (P, s) −c∗→ (R, t) ⟹ ∃ l ∈ cp P s. (last l)=(R, t)
  ∧ (∀ i. Suc i<length l ⟶ l!i −c→ l!Suc i)
apply (erule converse-rtrancl-induct2)
 apply(rule-tac x=[(R,t)] in bexI)
  apply simp
 apply(simp add:cp-def)
 apply(rule CptnOne)
apply clarify
apply(rule-tac x=(a, b)#l in bexI)
 apply (rule conjI)
  apply(case-tac l,simp add:cp-def)
  apply(simp add:last-length)
 apply clarify
apply(case-tac i,simp)
apply(simp add:cp-def)
apply force
apply(simp add:cp-def)
 apply(case-tac l)
 apply(force elim:cptn.elims)
apply simp
apply(erule CptnComp)
apply clarify
done

lemma Await-sound:
  ⟦stable pre rely; stable post rely;
  ∀ V. ⊢ P sat [pre ∩ b ∩ {s. s = V}, {(s, t). s = t},
                UNIV, {s. (V, s) ∈ guar} ∩ post] ∧
  ⊨ P sat [pre ∩ b ∩ {s. s = V}, {(s, t). s = t},
                UNIV, {s. (V, s) ∈ guar} ∩ post] ⟧
  ⟹ ⊨ Await b P sat [pre, rely, guar, post]
apply(unfold com-validity-def)
apply clarify
apply(simp add:comm-def)
apply(rule conjI)
 apply clarify
 apply(simp add:cp-def assum-def)
 apply clarify
 apply(frule-tac j=0 and k=i and p=pre in stability,simp-all)
   apply(erule-tac x=ia in allE,simp)
  apply(subgoal-tac x∈ cp (Some(Await b P)) s)
  apply(erule-tac i=i in unique-ctran-Await,force,simp-all)
  apply(simp add:cp-def)
```

— here starts the different part.
apply(*erule ctran.elims*,*simp-all*)
apply(*drule Star-imp-cptn*)
apply *clarify*
apply(*erule-tac x=sa* **in** *allE*)
apply *clarify*
apply(*erule-tac x=sa* **in** *allE*)
apply(*drule-tac c=l* **in** *subsetD*)
apply (*simp add*:*cp-def*)
apply *clarify*
apply(*erule-tac x=ia* **and** $P=\lambda i.$ *?H i* $\longrightarrow$ (*?J i*,*?I i*)$\in$*ctran* **in** *allE*,*simp*)
apply(*erule etran.elims*,*simp*)
apply *simp*
apply *clarify*
apply(*simp add*:*cp-def*)
apply *clarify*
apply(*frule-tac i=length x* $-$ *1* **in** *exists-ctran-Await-None*,*force*)
apply (*case-tac x*,*simp*+)
apply(*rule last-fst-esp*,*simp add*:*last-length*)
apply(*case-tac x*, (*simp add*:*cptn-not-empty*)+)
apply *clarify*
apply(*simp add*:*assum-def*)
apply *clarify*
apply(*frule-tac j=0* **and** *k=j* **and** *p=pre* **in** *stability*,*simp-all*)
apply *arith*
apply(*erule-tac x=i* **in** *allE*,*simp*)
apply(*erule-tac i=j* **in** *unique-ctran-Await*,*force*,*simp-all*)
apply *arith*
apply *arith*
apply(*case-tac x*!*j*)
apply *clarify*
apply *simp*
apply(*drule-tac s=Some* (*Await b P*) **in** *sym*,*simp*)
apply(*case-tac x*!*Suc j*,*simp*)
apply(*rule ctran.elims*,*simp*)
apply(*simp-all*)
apply(*drule Star-imp-cptn*)
apply *clarify*
apply(*erule-tac x=sa* **in** *allE*)
apply *clarify*
apply(*erule-tac x=sa* **in** *allE*)
apply(*drule-tac c=l* **in** *subsetD*)
apply (*simp add*:*cp-def*)
apply *clarify*
apply(*erule-tac x=i* **and** $P=\lambda i.$ *?H i* $\longrightarrow$ (*?J i*,*?I i*)$\in$*ctran* **in** *allE*,*simp*)
apply(*erule etran.elims*,*simp*)
apply *simp*
apply *clarify*
apply(*frule-tac j=Suc j* **and** *k=length x* $-$ *1* **and** *p=post* **in** *stability*,*simp-all*)

apply(*case-tac x*,*simp*+)
apply(*erule-tac x*=*i* **in** *allE*)
apply(*erule-tac i*=*j* **in** *unique-ctran-Await*,*force*,*simp-all*)
apply *arith*+
apply(*case-tac x*)
apply(*simp add*:*last-length*)+
done

Soundness of the Conditional rule

lemma *Cond-sound*:
$\llbracket$ *stable pre rely*; $\models$ *P1 sat* [*pre* $\cap$ *b*, *rely*, *guar*, *post*];
$\models$ *P2 sat* [*pre* $\cap$ $-$ *b*, *rely*, *guar*, *post*]; $\forall$ *s*. (*s*,*s*)$\in$*guar*$\rrbracket$
$\Longrightarrow$ $\models$ (*Cond b P1 P2*) *sat* [*pre*, *rely*, *guar*, *post*]
apply(*unfold com-validity-def*)
apply *clarify*
apply(*simp add*:*cp-def comm-def*)
apply(*case-tac* $\exists$ *i*. *Suc i*<*length x* $\wedge$ *x*!*i* $-c\rightarrow$ *x*!*Suc i*)
prefer *2*
apply *simp*
apply *clarify*
apply(*frule-tac j*=*0* **and** *k*=*length x* $-$ *1* **and** *p*=*pre* **in** *stability*,*simp*+)
apply(*case-tac x*,*simp*+)
apply(*simp add*:*assum-def*)
apply(*simp add*:*assum-def*)
apply(*erule-tac m*=*length x* **in** *etran-or-ctran*,*simp*+)
apply(*case-tac x*,*simp*+)
apply(*case-tac x*, (*simp add*:*last-length*)+)
apply(*erule exE*)
apply(*drule-tac n*=*i* **and** *P*=λ*i*. *?H i* $\wedge$ (*?J i*,*?I i*)$\in$ *ctran* **in** *Ex-first-occurrence*)
apply *clarify*
apply (*simp add*:*assum-def*)
apply(*frule-tac j*=*0* **and** *k*=*m* **and** *p*=*pre* **in** *stability*,*simp*+)
apply(*erule-tac m*=*Suc m* **in** *etran-or-ctran*,*simp*+)
apply(*erule ctran.elims*,*simp-all*)
apply(*erule-tac x*=*sa* **in** *allE*)
apply(*drule-tac c*=*drop* (*Suc m*) *x* **in** *subsetD*)
apply *simp*
apply(*rule conjI*)
apply *force*
apply *clarify*
apply(*subgoal-tac* (*Suc m*) + *i* $\leq$ *length x*)
apply(*subgoal-tac* (*Suc m*) + (*Suc i*) $\leq$ *length x*)
apply(*rotate-tac* $-$*2*)
apply *simp*
apply(*erule-tac x*=*Suc* (*m* + *i*) **and** *P*=λ*j*. *?H j* $\longrightarrow$ *?J j* $\longrightarrow$ *?I j* **in** *allE*)
apply(*subgoal-tac Suc* (*Suc* (*m* + *i*)) < *length x*,*simp*)
apply *arith*
apply *arith*

```
  apply arith
 apply simp
 apply clarify
 apply(case-tac i≤m)
  apply(drule le-imp-less-or-eq)
  apply(erule disjE)
   apply(erule-tac x=i in allE, erule impE, assumption)
   apply simp+
 apply(erule-tac x=i − (Suc m) and P=λj. ?H j ⟶ ?J j ⟶ (?I j)∈guar in allE)
  apply(subgoal-tac (Suc m)+(i − Suc m) ≤ length x)
   apply(subgoal-tac (Suc m)+Suc (i − Suc m) ≤ length x)
    apply(rotate-tac −2)
    apply simp
    apply(erule mp)
    apply arith
   apply arith
  apply arith
  apply(case-tac length (drop (Suc m) x),simp)
  apply(erule-tac x=sa in allE)
  back
  apply(drule-tac c=drop (Suc m) x in subsetD,simp)
   apply(rule conjI)
    apply force
   apply clarify
   apply(subgoal-tac (Suc m) + i ≤ length x)
    apply(subgoal-tac (Suc m) + (Suc i) ≤ length x)
     apply(rotate-tac −2)
     apply simp
     apply(erule-tac x=Suc (m + i) and P=λj. ?H j ⟶ ?J j ⟶ ?I j in allE)
     apply(subgoal-tac Suc (Suc (m + i)) < length x)
      apply simp
     apply arith
    apply arith
   apply arith
  apply simp
  apply clarify
  apply(case-tac i≤m)
   apply(drule le-imp-less-or-eq)
   apply(erule disjE)
    apply(erule-tac x=i in allE, erule impE, assumption)
    apply simp
   apply simp
  apply(erule-tac x=i − (Suc m) and P=λj. ?H j ⟶ ?J j ⟶ (?I j)∈guar in allE)
  apply(subgoal-tac (Suc m)+(i − Suc m) ≤ length x)
   apply(subgoal-tac (Suc m)+Suc (i − Suc m) ≤ length x)
    apply(rotate-tac −2)
    apply simp
```

```
    apply(erule mp)
   apply arith
  apply arith
 apply arith
 done
```

Soundness of the Sequential rule

```
inductive-cases Seq-cases [elim!]: (Some (Seq P Q), s) −c→ t

lemma last-lift-not-None: fst ((lift Q) ((x#xs)!(length xs))) ≠ None
apply(subgoal-tac length xs<length (x # xs))
 apply(drule-tac Q=Q in  lift-nth)
 apply(erule ssubst)
 apply (simp add:lift-def)
 apply(case-tac (x # xs) ! length xs,simp)
apply simp
done

declare map-eq-Cons-conv [simp del] Cons-eq-map-conv [simp del]
lemma Seq-sound1 [rule-format]:
  x∈ cptn-mod ⟹ ∀ s P. x !0=(Some (Seq P Q), s) ⟶
  (∀ i<length x. fst(x!i)≠Some Q) ⟶
  (∃ xs∈ cp (Some P) s. x=map (lift Q) xs)
apply(erule cptn-mod.induct)
apply(unfold cp-def)
apply safe
apply simp-all
    apply(simp add:lift-def)
    apply(rule-tac x=[(Some Pa, sa)] in exI,simp add:CptnOne)
   apply(subgoal-tac (∀ i < Suc (length xs). fst (((Some (Seq Pa Q), t) # xs) ! i)
≠ Some Q))
    apply clarify
    apply(rule-tac x=(Some Pa, sa) #(Some Pa, t) # zs in exI,simp)
    apply(rule conjI,erule CptnEnv)
    apply(simp (no-asm-use) add:lift-def)
   apply clarify
   apply(erule-tac x=Suc i in allE, simp)
  apply(ind-cases ((Some (Seq Pa Q), sa), None, t) ∈ ctran)
 apply(rule-tac x=(Some P, sa) # xs in exI, simp add:cptn-iff-cptn-mod lift-def)
apply(erule-tac x=length xs in allE, simp)
apply(simp only:Cons-lift-append)
apply(subgoal-tac length xs < length ((Some P, sa) # xs))
 apply(simp only :nth-append length-map last-length nth-map)
 apply(case-tac last((Some P, sa) # xs))
 apply(simp add:lift-def)
apply simp
done
declare map-eq-Cons-conv [simp del] Cons-eq-map-conv [simp del]
```

lemma *Seq-sound2* [*rule-format*]:
x ∈ *cptn* ⟹ ∀ *s P i*. *x*!*0*=(*Some* (*Seq P Q*), *s*) ⟶ *i*<*length x*
⟶ *fst*(*x*!*i*)=*Some Q* ⟶
(∀ *j*<*i*. *fst*(*x*!*j*)≠(*Some Q*)) ⟶
(∃ *xs ys*. *xs* ∈ *cp* (*Some P*) *s* ∧ *length xs*=*Suc i*
∧ *ys* ∈ *cp* (*Some Q*) (*snd*(*xs* !*i*)) ∧ *x*=(*map* (*lift Q*) *xs*)@*tl ys*)
apply(*erule cptn.induct*)
apply(*unfold cp-def*)
apply *safe*
apply *simp-all*
apply(*case-tac i*,*simp*+)
apply(*erule allE*,*erule impE*,*assumption*,*simp*)
apply *clarify*
apply(*subgoal-tac* (∀ *j* < *nat*. *fst* (((*Some* (*Seq Pa Q*), *t*) # *xs*) ! *j*) ≠ *Some Q*),*clarify*)
prefer *2*
apply *force*
apply(*case-tac xsa*,*simp*,*simp*)
apply(*rule-tac x*=(*Some Pa*, *sa*) #(*Some Pa*, *t*) # *list* **in** *exI*,*simp*)
apply(*rule conjI*,*erule CptnEnv*)
apply(*simp* (*no-asm-use*) *add*:*lift-def*)
apply(*rule-tac x*=*ys* **in** *exI*,*simp*)
apply(*ind-cases* ((*Some* (*Seq Pa Q*), *sa*), *t*) ∈ *ctran*)
apply *simp*
apply(*rule-tac x*=(*Some Pa*, *sa*)#[(*None*, *ta*)] **in** *exI*,*simp*)
apply(*rule conjI*)
apply(*drule-tac xs*=[] **in** *CptnComp*,*force simp add*:*CptnOne*,*simp*)
apply(*case-tac i*, *simp*+)
apply(*case-tac nat*,*simp*+)
apply(*rule-tac x*=(*Some Q*,*ta*)#*xs* **in** *exI*,*simp add*:*lift-def*)
apply(*case-tac nat*,*simp*+)
apply(*force*)
apply(*case-tac i*, *simp*+)
apply(*case-tac nat*,*simp*+)
apply(*erule-tac x*=*Suc nata* **in** *allE*,*simp*)
apply *clarify*
apply(*subgoal-tac* (∀ *j*<*Suc nata*. *fst* (((*Some* (*Seq P2 Q*), *ta*) # *xs*) ! *j*) ≠ *Some Q*),*clarify*)
prefer *2*
apply *clarify*
apply *force*
apply(*rule-tac x*=(*Some Pa*, *sa*)#(*Some P2*, *ta*)#(*tl xsa*) **in** *exI*,*simp*)
apply(*rule conjI*,*erule CptnComp*)
apply(*rule nth-tl-if*,*force*,*simp*+)
apply(*rule-tac x*=*ys* **in** *exI*,*simp*)
apply(*rule conjI*)
apply(*rule nth-tl-if*,*force*,*simp*+)
apply(*rule tl-zero*,*simp*+)

```
 apply force
apply(rule conjI,simp add:lift-def)
apply(subgoal-tac lift Q (Some P2, ta) =(Some (Seq P2 Q), ta))
 apply(simp add:Cons-lift del:map.simps)
 apply(rule nth-tl-if)
   apply force
  apply simp+
apply(simp add:lift-def)
done


lemma last-lift-not-None2: fst ((lift Q) (last (x#xs))) ≠ None
apply(simp only:last-length [THEN sym])
apply(subgoal-tac length xs<length (x # xs))
 apply(drule-tac Q=Q in  lift-nth)
 apply(erule ssubst)
 apply (simp add:lift-def)
 apply(case-tac (x # xs) ! length xs,simp)
apply simp
done

lemma Seq-sound:
  ⟦⊨ P sat [pre, rely, guar, mid]; ⊨ Q sat [mid, rely, guar, post]⟧
  ⟹ ⊨ Seq P Q sat [pre, rely, guar, post]
apply(unfold com-validity-def)
apply clarify
apply(case-tac ∃ i<length x. fst(x!i)=Some Q)
 prefer 2
 apply (simp add:cp-def cptn-iff-cptn-mod)
 apply clarify
 apply(frule-tac Seq-sound1,force)
  apply force
 apply clarify
 apply(erule-tac x=s in allE,simp)
 apply(drule-tac c=xs in subsetD,simp add:cp-def cptn-iff-cptn-mod)
  apply(simp add:assum-def)
  apply clarify
  apply(erule-tac P=λj. ?H j ⟶ ?J j ⟶ ?I j in allE,erule impE, assumption)
  apply(simp add:snd-lift)
  apply(erule mp)
  apply(force elim:etran.elims intro:Env simp add:lift-def)
 apply(simp add:comm-def)
 apply(rule conjI)
  apply clarify
  apply(erule-tac P=λj. ?H j ⟶ ?J j ⟶ ?I j in allE,erule impE, assumption)
  apply(simp add:snd-lift)
  apply(erule mp)
  apply(case-tac (xs!i))
  apply(case-tac (xs! Suc i))
```

```
  apply(case-tac fst(xs!i))
   apply(erule-tac x=i in allE, simp add:lift-def)
  apply(case-tac fst(xs!Suc i))
   apply(force simp add:lift-def)
  apply(force simp add:lift-def)
 apply clarify
 apply(case-tac xs,simp add:cp-def)
 apply clarify
 apply (simp del:map.simps)
 apply(subgoal-tac (map (lift Q) ((a, b) # list))≠[])
  apply(drule last-conv-nth)
  apply (simp del:map.simps)
  apply(simp only:last-lift-not-None)
 apply simp
— ∃ i<length x. fst (x ! i) = Some Q
apply(erule exE)
apply(drule-tac n=i and P=λi. i < length x ∧ fst (x ! i) = Some Q in Ex-first-occurrence)
apply clarify
apply (simp add:cp-def)
 apply clarify
 apply(frule-tac i=m in Seq-sound2,force)
  apply simp+
apply clarify
apply(simp add:comm-def)
apply(erule-tac x=s in allE)
apply(drule-tac c=xs in subsetD,simp)
 apply(case-tac xs=[],simp)
 apply(simp add:cp-def assum-def nth-append)
 apply clarify
 apply(erule-tac x=i in allE)
  back
 apply(simp add:snd-lift)
 apply(erule mp)
 apply(force elim:etran.elims intro:Env simp add:lift-def)
apply simp
apply clarify
apply(erule-tac x=snd(xs!m) in allE)
apply(drule-tac c=ys in subsetD,simp add:cp-def assum-def)
 apply(case-tac xs≠[])
 apply(drule last-conv-nth,simp)
 apply(rule conjI)
  apply(erule mp)
  apply(case-tac xs!m)
  apply(case-tac fst(xs!m),simp)
  apply(simp add:lift-def nth-append)
 apply clarify
 apply(erule-tac x=m+i in allE)
 back
 back
```

```
 apply(case-tac ys,(simp add:nth-append)+)
 apply (case-tac i, (simp add:snd-lift)+)
  apply(erule mp)
  apply(case-tac xs!m)
  apply(force elim:etran.elims intro:Env simp add:lift-def)
 apply simp
apply simp
apply clarify
apply(rule conjI,clarify)
 apply(case-tac i<m,simp add:nth-append)
  apply(simp add:snd-lift)
  apply(erule allE, erule impE, assumption, erule mp)
  apply(case-tac (xs ! i))
  apply(case-tac (xs ! Suc i))
  apply(case-tac fst(xs ! i),force simp add:lift-def)
  apply(case-tac fst(xs ! Suc i))
   apply (force simp add:lift-def)
  apply (force simp add:lift-def)
 apply(erule-tac x=i−m in allE)
 back
 back
 apply(subgoal-tac Suc (i − m) < length ys,simp)
  prefer 2
  apply arith
 apply(simp add:nth-append snd-lift)
 apply(rule conjI,clarify)
  apply(subgoal-tac i=m)
   prefer 2
   apply arith
  apply clarify
  apply(simp add:cp-def)
  apply(rule tl-zero)
    apply(erule mp)
    apply(case-tac lift Q (xs!m),simp add:snd-lift)
    apply(case-tac xs!m,case-tac fst(xs!m),simp add:lift-def snd-lift)
     apply(case-tac ys,simp+)
    apply(simp add:lift-def)
   apply simp
  apply force
 apply clarify
 apply(rule tl-zero)
   apply(rule tl-zero)
     apply (subgoal-tac i−m=Suc(i−Suc m))
      apply simp
      apply(erule mp)
      apply(case-tac ys,simp+)
     apply arith
    apply arith
   apply force
```

```
  apply arith
 apply force
apply clarify
apply(case-tac (map (lift Q) xs @ tl ys)≠[])
 apply(drule last-conv-nth)
 apply(simp add: snd-lift nth-append)
 apply(rule conjI,clarify)
  apply(case-tac ys,simp+)
 apply clarify
 apply(case-tac ys,simp+)
done
```

Soundness of the While rule

```
lemma last-append[rule-format]:
  ∀ xs. ys≠[] ⟶ ((xs@ys)!(length (xs@ys) − (Suc 0)))=(ys!(length ys − (Suc 0)))
apply(induct ys)
 apply simp
apply clarify
apply (simp add:nth-append length-append)
done

lemma assum-after-body:
  ⟦ ⊨ P sat [pre ∩ b, rely, guar, pre];
  (Some P, s) # xs ∈ cptn-mod; fst (last ((Some P, s) # xs)) = None; s ∈ b;
  (Some (While b P), s) # (Some (Seq P (While b P)), s) #
   map (lift (While b P)) xs @ ys ∈ assum (pre, rely)⟧
  ⟹ (Some (While b P), snd (last ((Some P, s) # xs))) # ys ∈ assum (pre, rely)
apply(simp add:assum-def com-validity-def cp-def cptn-iff-cptn-mod)
apply clarify
apply(erule-tac x=s in allE)
apply(drule-tac c=(Some P, s) # xs in subsetD,simp)
 apply clarify
 apply(erule-tac x=Suc i in allE)
 apply simp
 apply(simp add:Cons-lift-append nth-append snd-lift del:map.simps)
 apply(erule mp)
 apply(erule etran.elims,simp)
 apply(case-tac fst(((Some P, s) # xs) ! i))
  apply(force intro:Env simp add:lift-def)
 apply(force intro:Env simp add:lift-def)
apply(rule conjI)
 apply clarify
 apply(simp add:comm-def last-length)
apply clarify
apply(rule conjI)
 apply(simp add:comm-def)
apply clarify
apply(erule-tac x=Suc(length xs + i) in allE,simp)
```

apply(*case-tac i*, *simp add*:*nth-append Cons-lift-append snd-lift del*:*map.simps*)
apply(*simp add*:*last-length*)
apply(*erule mp*)
apply(*case-tac last xs*)
apply(*simp add*:*lift-def*)
apply(*simp add*:*Cons-lift-append nth-append snd-lift del*:*map.simps*)
done

lemma *While-sound-aux* [*rule-format*]:
$\llbracket$ *pre* $\cap -$ *b* $\subseteq$ *post*; $\models$ *P sat* [*pre* $\cap$ *b*, *rely*, *guar*, *pre*]; $\forall$ *s*. (*s*, *s*) $\in$ *guar*;
stable pre rely; *stable post rely*; *x* $\in$ *cptn-mod* $\rrbracket$
$\Longrightarrow$ $\forall$ *s xs*. *x*=(*Some*(*While b P*),*s*)#*xs* $\longrightarrow$ *x*$\in$*assum*(*pre*, *rely*) $\longrightarrow$ *x* $\in$ *comm* (*guar*, *post*)
apply(*erule cptn-mod.induct*)
apply *safe*
apply (*simp-all del*:*last.simps*)
— 5 subgoals left
apply(*simp add*:*comm-def*)
— 4 subgoals left
apply(*rule etran-in-comm*)
apply(*erule mp*)
apply(*erule tl-of-assum-in-assum*,*simp*)
— While-None
apply(*ind-cases* ((*Some* (*While b P*), *s*), *None*, *t*) $\in$ *ctran*)
apply(*simp add*:*comm-def*)
apply(*simp add*:*cptn-iff-cptn-mod* [*THEN sym*])
apply(*rule conjI*,*clarify*)
apply(*force simp add*:*assum-def*)
apply *clarify*
apply(*rule conjI*, *clarify*)
apply(*case-tac i*,*simp*,*simp*)
apply(*force simp add*:*not-ctran-None2*)
apply(*subgoal-tac* $\forall$ *i*. *Suc i* < *length* ((*None*, *sa*) # *xs*) $\longrightarrow$ (((*None*, *sa*) # *xs*) ! *i*, ((*None*, *sa*) # *xs*) ! *Suc i*)$\in$ *etran*)
prefer *2*
apply *clarify*
apply(*rule-tac m*=*length* ((*None*, *s*) # *xs*) **in** *etran-or-ctran*,*simp*+)
apply(*erule not-ctran-None2*,*simp*)
apply *simp*+
apply(*frule-tac j*=*0* **and** *k*=*length* ((*None*, *s*) # *xs*) − *1* **and** *p*=*post* **in** *stability*,*simp*+)
apply(*force simp add*:*assum-def subsetD*)
apply(*simp add*:*assum-def*)
apply *clarify*
apply(*erule-tac x*=*i* **in** *allE*,*simp*)
apply(*erule-tac x*=*Suc i* **in** *allE*,*simp*)
apply *simp*
apply *clarify*
apply (*simp add*:*last-length*)

— WhileOne
apply(*thin-tac P = While b P ⟶ ?Q*)
apply(*rule ctran-in-comm*,*simp*)
apply(*simp add*:*Cons-lift del*:*map.simps*)
apply(*simp add*:*comm-def del*:*map.simps*)
apply(*rule conjI*)
 apply *clarify*
 apply(*case-tac fst*(((*Some P*, *sa*) # *xs*) ! *i*))
  apply(*case-tac* ((*Some P*, *sa*) # *xs*) ! *i*)
  apply (*simp add*:*lift-def*)
  apply(*ind-cases* (*Some* (*While b P*), *ba*) −*c*→ *t*)
   apply *simp*
  apply *simp*
 apply(*simp add*:*snd-lift del*:*map.simps*)
 apply(*simp only*:*com-validity-def cp-def cptn-iff-cptn-mod*)
 apply(*erule-tac x*=*sa* **in** *allE*)
 apply(*drule-tac c*=(*Some P*, *sa*) # *xs* **in** *subsetD*)
  apply (*simp add*:*assum-def del*:*map.simps*)
  apply *clarify*
  apply(*erule-tac x*=*Suc ia* **in** *allE*,*simp add*:*snd-lift del*:*map.simps*)
  apply(*erule mp*)
  apply(*case-tac fst*(((*Some P*, *sa*) # *xs*) ! *ia*))
   apply(*erule etran.elims*,*simp add*:*lift-def*)
   apply(*rule Env*)
  apply(*erule etran.elims*,*simp add*:*lift-def*)
  apply(*rule Env*)
 apply (*simp add*:*comm-def del*:*map.simps*)
 apply *clarify*
 apply(*erule allE*,*erule impE*,*assumption*)
 apply(*erule mp*)
 apply(*case-tac* ((*Some P*, *sa*) # *xs*) ! *i*)
 apply(*case-tac xs*!*i*)
 apply(*simp add*:*lift-def*)
 apply(*case-tac fst*(*xs*!*i*))
  apply *force*
 apply *force*
— last=None
apply *clarify*
apply(*subgoal-tac* (*map* (*lift* (*While b P*)) ((*Some P*, *sa*) # *xs*))≠[])
 apply(*drule last-conv-nth*)
 apply (*simp del*:*map.simps*)
 apply(*simp only*:*last-lift-not-None*)
apply *simp*
— WhileMore
apply(*thin-tac P = While b P ⟶ ?Q*)
apply(*rule ctran-in-comm*,*simp del*:*last.simps*)
— metiendo la hipotesis antes de dividir la conclusion.
apply(*subgoal-tac* (*Some* (*While b P*), *snd* (*last* ((*Some P*, *sa*) # *xs*))) # *ys* ∈ *assum* (*pre*, *rely*))

apply (*simp del*:*last.simps*)
prefer *2*
apply(*erule assum-after-body*)
apply (*simp del*:*last.simps*)+
— lo de antes.
apply(*simp add*:*comm-def del*:*map.simps last.simps*)
apply(*rule conjI*)
apply *clarify*
apply(*simp only*:*Cons-lift-append*)
apply(*case-tac i*<*length xs*)
apply(*simp add*:*nth-append del*:*map.simps last.simps*)
apply(*case-tac fst*(((*Some P*, *sa*) # *xs*) ! *i*))
apply(*case-tac* ((*Some P*, *sa*) # *xs*) ! *i*)
apply (*simp add*:*lift-def del*:*last.simps*)
apply(*ind-cases* (*Some* (*While b P*), *ba*) $-c\rightarrow$ *t*)
apply *simp*
apply *simp*
apply(*simp add*:*snd-lift del*:*map.simps last.simps*)
apply(*thin-tac* $\forall i.\ i <$ *length ys* $\longrightarrow$ *?P i*)
apply(*simp only*:*com-validity-def cp-def cptn-iff-cptn-mod*)
apply(*erule-tac x*=*sa* **in** *allE*)
apply(*drule-tac c*=(*Some P*, *sa*) # *xs* **in** *subsetD*)
apply (*simp add*:*assum-def del*:*map.simps last.simps*)
apply *clarify*
apply(*erule-tac x*=*Suc ia* **in** *allE*,*simp add*:*nth-append snd-lift del*:*map.simps last.simps*, *erule mp*)
apply(*case-tac fst*(((*Some P*, *sa*) # *xs*) ! *ia*))
apply(*erule etran.elims*,*simp add*:*lift-def*)
apply(*rule Env*)
apply(*erule etran.elims*,*simp add*:*lift-def*)
apply(*rule Env*)
apply (*simp add*:*comm-def del*:*map.simps*)
apply *clarify*
apply(*erule allE*,*erule impE*,*assumption*)
apply(*erule mp*)
apply(*case-tac* ((*Some P*, *sa*) # *xs*) ! *i*)
apply(*case-tac xs*!*i*)
apply(*simp add*:*lift-def*)
apply(*case-tac fst*(*xs*!*i*))
apply *force*
apply *force*
— $i \geq$ *length xs*
apply(*subgoal-tac i*−*length xs* <*length ys*)
prefer *2*
apply *arith*
apply(*erule-tac x*=*i*−*length xs* **in** *allE*,*clarify*)
apply(*case-tac i*=*length xs*)
apply (*simp add*:*nth-append snd-lift del*:*map.simps last.simps*)
apply(*simp add*:*last-length del*:*last.simps*)

apply(*erule mp*)
apply(*case-tac last*((*Some P*, *sa*) # *xs*))
apply(*simp add*:*lift-def del*:*last.simps*)
— *i*>*length xs*
apply(*case-tac i*−*length xs*)
apply *arith*
apply(*simp add*:*nth-append del*:*map.simps last.simps*)
apply(*rotate-tac* −*3*)
apply(*subgoal-tac i*− *Suc* (*length xs*)=*nat*)
prefer *2*
apply *arith*
apply *simp*
— last=None
apply *clarify*
apply(*case-tac ys*)
apply(*simp add*:*Cons-lift del*:*map.simps last.simps*)
apply(*subgoal-tac* (*map* (*lift* (*While b P*)) ((*Some P*, *sa*) # *xs*))≠[])
apply(*drule last-conv-nth*)
apply (*simp del*:*map.simps*)
apply(*simp only*:*last-lift-not-None*)
apply *simp*
apply(*subgoal-tac* ((*Some* (*Seq P* (*While b P*)), *sa*) # *map* (*lift* (*While b P*)) *xs* @ *ys*)≠[])
apply(*drule last-conv-nth*)
apply (*simp del*:*map.simps last.simps*)
apply(*simp add*:*nth-append del*:*last.simps*)
apply(*subgoal-tac* ((*Some* (*While b P*), *snd* (*last* ((*Some P*, *sa*) # *xs*))) # *a* # *list*)≠[])
apply(*drule last-conv-nth*)
apply (*simp del*:*map.simps last.simps*)
apply *simp*
apply *simp*
done

lemma *While-sound*:
$[\![$*stable pre rely*; *pre* ∩ − *b* ⊆ *post*; *stable post rely*;
⊨ *P sat* [*pre* ∩ *b*, *rely*, *guar*, *pre*]; ∀ *s*. (*s*,*s*)∈*guar*$]\!]$
⟹ ⊨ *While b P sat* [*pre*, *rely*, *guar*, *post*]
apply(*unfold com-validity-def*)
apply *clarify*
apply(*erule-tac xs*=*tl x* **in** *While-sound-aux*)
apply(*simp add*:*com-validity-def*)
apply *force*
apply *simp-all*
apply(*simp add*:*cptn-iff-cptn-mod cp-def*)
apply(*simp add*:*cp-def*)
apply *clarify*
apply(*rule nth-equalityI*)
apply *simp-all*

apply(*case-tac x*,*simp*+)
apply *clarify*
apply(*case-tac i*,*simp*+)
apply(*case-tac x*,*simp*+)
done

Soundness of the Rule of Consequence

lemma *Conseq-sound*:
$[\![pre \subseteq pre'; \; rely \subseteq rely'; \; guar' \subseteq guar; \; post' \subseteq post;$
$\models P \; sat \; [pre', \; rely', \; guar', \; post']]\!]$
$\Longrightarrow \models P \; sat \; [pre, \; rely, \; guar, \; post]$
apply(*simp add*:*com-validity-def assum-def comm-def*)
apply *clarify*
apply(*erule-tac x*=*s* **in** *allE*)
apply(*drule-tac c*=*x* **in** *subsetD*)
apply *force*
apply *force*
done

Soundness of the system for sequential component programs

theorem *rgsound*:
$\vdash P \; sat \; [pre, \; rely, \; guar, \; post] \Longrightarrow \models P \; sat \; [pre, \; rely, \; guar, \; post]$
apply(*erule rghoare.induct*)
apply(*force elim*:*Basic-sound*)
apply(*force elim*:*Seq-sound*)
apply(*force elim*:*Cond-sound*)
apply(*force elim*:*While-sound*)
apply(*force elim*:*Await-sound*)
apply(*erule Conseq-sound*,*simp*+)
done

3.5.2 Soundness of the System for Parallel Programs

constdefs
$ParallelCom :: ('a \; rgformula) \; list \Rightarrow 'a \; par\text{-}com$
$ParallelCom \; Ps \equiv map \; (Some \circ fst) \; Ps$

lemma *two*:
$[\![\; \forall i{<}length \; xs. \; rely \cup (\bigcup j{\in}\{j. \; j < length \; xs \wedge j \neq i\}. \; Guar \; (xs \; ! \; j))$
$\subseteq Rely \; (xs \; ! \; i);$
$pre \subseteq (\bigcap i{\in}\{i. \; i < length \; xs\}. \; Pre \; (xs \; ! \; i));$
$\forall i{<}length \; xs.$
$\models Com \; (xs \; ! \; i) \; sat \; [Pre \; (xs \; ! \; i), \; Rely \; (xs \; ! \; i), \; Guar \; (xs \; ! \; i), \; Post \; (xs \; ! \; i)];$
$length \; xs{=}length \; clist; \; x \in par\text{-}cp \; (ParallelCom \; xs) \; s; \; x{\in}par\text{-}assum(pre, \; rely);$
$\forall i{<}length \; clist. \; clist!i{\in}cp \; (Some(Com(xs!i))) \; s; \; x \propto clist \;]\!]$
$\Longrightarrow \forall j \; i. \; i{<}length \; clist \wedge Suc \; j{<}length \; x \longrightarrow (clist!i!j) \; {-c\rightarrow} \; (clist!i!Suc \; j)$
$\longrightarrow (snd(clist!i!j), \; snd(clist!i!Suc \; j)) \in Guar(xs!i)$
apply(*unfold par-cp-def*)

apply (*rule ccontr*)
— By contradiction:
apply (*simp del*: *Un-subset-iff*)
apply(*erule exE*)
— the first c-tran that does not satisfy the guarantee-condition is from σ-*i* at step *m*.
apply(*drule-tac n=j* **and** *P*=$\lambda j.\ \exists\, i.$ *?H i j* **in** *Ex-first-occurrence*)
apply(*erule exE*)
apply *clarify*
— σ-*i* $\in$ *A*(*pre*, *rely-1*)
apply(*subgoal-tac take* (*Suc* (*Suc m*)) (*clist*!*i*) $\in$ *assum*(*Pre*(*xs*!*i*), *Rely*(*xs*!*i*)))
— but this contradicts $\models$ σ-*i sat* [*pre-i*,*rely-i*,*guar-i*,*post-i*]
apply(*erule-tac x=i* **and** *P*=$\lambda i.$ *?H i* $\longrightarrow$ $\models$ (*?J i*) *sat* [*?I i*,*?K i*,*?M i*,*?N i*] **in** *allE*,*erule impE*,*assumption*)
apply(*simp add*:*com-validity-def*)
apply(*erule-tac x=s* **in** *allE*)
apply(*simp add*:*cp-def comm-def*)
apply(*drule-tac c=take* (*Suc* (*Suc m*)) (*clist* ! *i*) **in** *subsetD*)
apply *simp*
apply (*blast intro*: *takecptn-is-cptn*)
apply *simp*
apply *clarify*
apply(*erule-tac x=m* **and** *P*=$\lambda j.$ *?I j* $\wedge$ *?J j* $\longrightarrow$ *?H j* **in** *allE*)
apply (*simp add*:*conjoin-def same-length-def*)
apply(*simp add*:*assum-def del*: *Un-subset-iff*)
apply(*rule conjI*)
apply(*erule-tac x=i* **and** *P*=$\lambda j.$ *?H j* $\longrightarrow$ *?I j* $\in$*cp* (*?K j*) (*?J j*) **in** *allE*)
apply(*simp add*:*cp-def par-assum-def*)
apply(*drule-tac c=s* **in** *subsetD*,*simp*)
apply *simp*
apply *clarify*
apply(*erule-tac x=i* **and** *P*=$\lambda j.$ *?H j* $\longrightarrow$ *?M* $\cup$ *UNION* (*?S j*) (*?T j*) $\subseteq$ (*?L j*) **in** *allE*)
apply(*simp del*: *Un-subset-iff*)
apply(*erule subsetD*)
apply *simp*
apply(*simp add*:*conjoin-def compat-label-def*)
apply *clarify*
apply(*erule-tac x=ia* **and** *P*=$\lambda j.$ *?H j* $\longrightarrow$ (*?P j*) $\vee$ *?Q j* **in** *allE*,*simp*)
— each etran in σ-*1*[*0*...*m*] corresponds to
apply(*erule disjE*)
— a c-tran in some σ-{*ib*}
apply *clarify*
apply(*case-tac i=ib*,*simp*)
apply(*erule etran.elims*,*simp*)
apply(*erule-tac x=ib* **and** *P*=$\lambda i.$ *?H i* $\longrightarrow$ (*?I i*) $\vee$ (*?J i*) **in** *allE*)
apply (*erule etran.elims*)
apply(*case-tac ia=m*,*simp*)
apply *simp*

apply(*erule-tac x=ia* **and** *P=λj. ?H j ⟶ (∀ i. ?P i j)* **in** *allE*)
apply(*subgoal-tac ia<m,simp*)
prefer *2*
apply *arith*
apply(*erule-tac x=ib* **and** *P=λj. (?I j, ?H j)∈ ctran ⟶ (?P i j)* **in** *allE,simp*)
apply(*simp add:same-state-def*)
apply(*erule-tac x=i* **and** *P=λj. (?T j) ⟶ (∀ i. (?H j i) ⟶ (snd (?d j i))=(snd (?e j i)))* **in** *all-dupE*)
apply(*erule-tac x=ib* **and** *P=λj. (?T j) ⟶ (∀ i. (?H j i) ⟶ (snd (?d j i))=(snd (?e j i)))* **in** *allE,simp*)
— or an e-tran in σ, therefore it satisfies *rely* ∨ *guar*-{*ib*}
apply (*force simp add:par-assum-def same-state-def*)
done

lemma *three* [*rule-format*]:
⟦ *xs≠[]*; ∀ *i<length xs. rely* ∪ (⋃*j∈{j. j < length xs ∧ j ≠ i}. Guar (xs ! j)*)
⊆ *Rely (xs ! i)*;
pre ⊆ (⋂*i∈{i. i < length xs}. Pre (xs ! i)*);
∀ *i<length xs.*
⊨ *Com (xs ! i) sat [Pre (xs ! i), Rely (xs ! i), Guar (xs ! i), Post (xs ! i)]*;
length xs=length clist; x ∈ par-cp (ParallelCom xs) s; x ∈ par-assum(pre, rely);
∀ *i<length clist. clist!i∈cp (Some(Com(xs!i))) s; x ∝ clist* ⟧
⟹ ∀ *j i. i<length clist ∧ Suc j<length x ⟶ (clist!i!j) −e→ (clist!i!Suc j)*
⟶ *(snd(clist!i!j), snd(clist!i!Suc j)) ∈ rely* ∪ (⋃*j∈{j. j < length xs ∧ j ≠ i}. Guar (xs ! j)*)
apply(*drule two*)
apply *simp-all*
apply *clarify*
apply(*simp add:conjoin-def compat-label-def*)
apply *clarify*
apply(*erule-tac x=j* **and** *P=λj. ?H j ⟶ (?J j ∧ (∃ i. ?P i j)) ∨ ?I j* **in** *allE,simp*)
apply(*erule disjE*)
prefer *2*
apply(*force simp add:same-state-def par-assum-def*)
apply *clarify*
apply(*case-tac i=ia,simp*)
apply(*erule etran.elims,simp*)
apply(*erule-tac x=ia* **and** *P=λi. ?H i ⟶ (?I i) ∨ (?J i)* **in** *allE,simp*)
apply(*erule-tac x=j* **and** *P=λj. ∀ i. ?S j i ⟶ (?I j i, ?H j i)∈ ctran ⟶ (?P i j)* **in** *allE*)
apply(*erule-tac x=ia* **and** *P=λj. ?S j ⟶ (?I j, ?H j)∈ ctran ⟶ (?P j)* **in** *allE*)
apply(*simp add:same-state-def*)
apply(*erule-tac x=i* **and** *P=λj. (?T j) ⟶ (∀ i. (?H j i) ⟶ (snd (?d j i))=(snd (?e j i)))* **in** *all-dupE*)
apply(*erule-tac x=ia* **and** *P=λj. (?T j) ⟶ (∀ i. (?H j i) ⟶ (snd (?d j i))=(snd (?e j i)))* **in** *allE,simp*)

done

lemma *four*:
$[\![xs\neq[\,];\ \forall i < \mathit{length}\ xs.\ \mathit{rely} \cup (\bigcup j\in\{j.\ j < \mathit{length}\ xs \wedge j \neq i\}.\ \mathit{Guar}\ (xs\ !\ j))$
$\subseteq \mathit{Rely}\ (xs\ !\ i);$
$(\bigcup j\in\{j.\ j < \mathit{length}\ xs\}.\ \mathit{Guar}\ (xs\ !\ j)) \subseteq \mathit{guar};$
$\mathit{pre} \subseteq (\bigcap i\in\{i.\ i < \mathit{length}\ xs\}.\ \mathit{Pre}\ (xs\ !\ i));$
$\forall i < \mathit{length}\ xs.$
$\models \mathit{Com}\ (xs\ !\ i)\ \mathit{sat}\ [\mathit{Pre}\ (xs\ !\ i),\ \mathit{Rely}\ (xs\ !\ i),\ \mathit{Guar}\ (xs\ !\ i),\ \mathit{Post}\ (xs\ !\ i)];$
$x \in \mathit{par\text{-}cp}\ (\mathit{ParallelCom}\ xs)\ s;\ x \in \mathit{par\text{-}assum}\ (\mathit{pre},\ \mathit{rely});\ \mathit{Suc}\ i < \mathit{length}\ x;$
$x\ !\ i\ {-pc\rightarrow}\ x\ !\ \mathit{Suc}\ i]\!]$
$\Longrightarrow (\mathit{snd}\ (x\ !\ i),\ \mathit{snd}\ (x\ !\ \mathit{Suc}\ i)) \in \mathit{guar}$
apply(*simp add*: *ParallelCom-def del*: *Un-subset-iff*)
apply(*subgoal-tac* (*map* (*Some* $\circ$ *fst*) *xs*)$\neq$[])
prefer *2*
apply *simp*
apply(*frule rev-subsetD*)
apply(*erule one* [*THEN equalityD1*])
apply(*erule subsetD*)
apply (*simp del*: *Un-subset-iff*)
apply *clarify*
apply(*drule-tac pre*=*pre* **and** *rely*=*rely* **and** *x*=*x* **and** *s*=*s* **and** *xs*=*xs* **and** *clist*=*clist* **in** *two*)
apply(*assumption*+)
apply(*erule sym*)
apply(*simp add*:*ParallelCom-def*)
apply *assumption*
apply(*simp add*:*Com-def*)
apply *assumption*
apply(*simp add*:*conjoin-def same-program-def*)
apply *clarify*
apply(*erule-tac* x=i **and** $P=\lambda j.\ ?H\ j \longrightarrow \mathit{fst}(?I\ j)=(?J\ j)$ **in** *all-dupE*)
apply(*erule-tac* x=*Suc i* **and** $P=\lambda j.\ ?H\ j \longrightarrow \mathit{fst}(?I\ j)=(?J\ j)$ **in** *allE*)
apply(*erule par-ctran.elims*,*simp*)
apply(*erule-tac* x=i **and** $P=\lambda j.\ \forall i.\ ?S\ j\ i \longrightarrow (?I\ j\ i,\ ?H\ j\ i)\in \mathit{ctran} \longrightarrow (?P\ i\ j)$ **in** *allE*)
apply(*erule-tac* x=*ia* **and** $P=\lambda j.\ ?S\ j \longrightarrow (?I\ j,\ ?H\ j)\in \mathit{ctran} \longrightarrow (?P\ j)$ **in** *allE*)
apply(*rule-tac* x=*ia* **in** *exI*)
apply(*simp add*:*same-state-def*)
apply(*erule-tac* x=*ia* **and** $P=\lambda j.\ (?T\ j) \longrightarrow (\forall i.\ (?H\ j\ i) \longrightarrow (\mathit{snd}\ (?d\ j\ i))=(\mathit{snd}\ (?e\ j\ i)))$ **in** *all-dupE*,*simp*)
apply(*erule-tac* x=*ia* **and** $P=\lambda j.\ (?T\ j) \longrightarrow (\forall i.\ (?H\ j\ i) \longrightarrow (\mathit{snd}\ (?d\ j\ i))=(\mathit{snd}\ (?e\ j\ i)))$ **in** *allE*,*simp*)
apply(*erule-tac* x=i **and** $P=\lambda j.\ ?H\ j \longrightarrow (\mathit{snd}\ (?d\ j))=(\mathit{snd}\ (?e\ j))$ **in** *all-dupE*)
apply(*erule-tac* x=i **and** $P=\lambda j.\ ?H\ j \longrightarrow (\mathit{snd}\ (?d\ j))=(\mathit{snd}\ (?e\ j))$ **in** *all-dupE*,*simp*)
apply(*erule-tac* x=*Suc i* **and** $P=\lambda j.\ ?H\ j \longrightarrow (\mathit{snd}\ (?d\ j))=(\mathit{snd}\ (?e\ j))$ **in** *allE*,*simp*)
apply(*erule mp*)

```
apply(subgoal-tac r=fst(clist ! ia ! Suc i),simp)
apply(drule-tac i=ia in list-eq-if)
back
apply simp-all
done

lemma parcptn-not-empty [simp]:[] ∉ par-cptn
apply(force elim:par-cptn.elims)
done

lemma five:
  ⟦xs≠[]; ∀ i<length xs. rely ∪ (⋃ j∈{j. j < length xs ∧ j ≠ i}. Guar (xs ! j))
   ⊆ Rely (xs ! i);
   pre ⊆ (⋂ i∈{i. i < length xs}. Pre (xs ! i));
   (⋂ i∈{i. i < length xs}. Post (xs ! i)) ⊆ post;
   ∀ i < length xs.
    ⊨ Com (xs ! i) sat [Pre (xs ! i), Rely (xs ! i), Guar (xs ! i), Post (xs ! i)];
    x ∈ par-cp (ParallelCom xs) s; x ∈ par-assum (pre, rely);
   All-None (fst (last x)) ⟧ ⟹ snd (last x) ∈ post
apply(simp add: ParallelCom-def del: Un-subset-iff)
apply(subgoal-tac (map (Some ∘ fst) xs)≠[])
 prefer 2
 apply simp
apply(frule rev-subsetD)
 apply(erule one [THEN equalityD1])
apply(erule subsetD)
apply(simp del: Un-subset-iff)
apply clarify
apply(subgoal-tac ∀ i<length clist. clist!i∈assum(Pre(xs!i), Rely(xs!i)))
 apply(erule-tac x=i and P=λi. ?H i ⟶ ⊨ (?J i) sat [?I i,?K i,?M i,?N i] in
allE,erule impE,assumption)
 apply(simp add:com-validity-def)
 apply(erule-tac x=s in allE)
 apply(erule-tac x=i and P=λj. ?H j ⟶ (?I j) ∈ cp (?J j) s in allE,simp)
 apply(drule-tac c=clist!i in subsetD)
  apply (force simp add:Com-def)
 apply(simp add:comm-def conjoin-def same-program-def del:last.simps)
 apply clarify
 apply(erule-tac x=length x - 1 and P=λj. ?H j ⟶ fst(?I j)=(?J j) in allE)
 apply (simp add:All-None-def same-length-def)
 apply(erule-tac x=i and P=λj. ?H j ⟶ length(?J j)=(?K j) in allE)
 apply(subgoal-tac length x - 1 < length x,simp)
  apply(case-tac x≠[])
   apply(simp add: last-conv-nth)
   apply(erule-tac x=clist!i in ballE)
    apply(simp add:same-state-def)
    apply(subgoal-tac clist!i≠[])
     apply(simp add: last-conv-nth)
    apply(case-tac x)
```

apply (*force simp add*:*par-cp-def*)
apply (*force simp add*:*par-cp-def*)
apply *force*
apply (*force simp add*:*par-cp-def*)
apply(*case-tac x*)
apply (*force simp add*:*par-cp-def*)
apply (*force simp add*:*par-cp-def*)
apply *clarify*
apply(*simp add*:*assum-def*)
apply(*rule conjI*)
apply(*simp add*:*conjoin-def same-state-def par-cp-def*)
apply *clarify*
apply(*erule-tac* x=*ia* **and** P=λj. (*?T j*) $\longrightarrow$ ($\forall$ *i*. (*?H j i*) $\longrightarrow$ (*snd* (*?d j i*))=(*snd* (*?e j i*))) **in** *allE*,*simp*)
apply(*erule-tac* x=*0* **and** P=λj. *?H j* $\longrightarrow$ (*snd* (*?d j*))=(*snd* (*?e j*)) **in** *allE*)
apply(*case-tac x*,*simp*+)
apply (*simp add*:*par-assum-def*)
apply *clarify*
apply(*drule-tac c*=*snd* (*clist* ! *ia* ! *0*) **in** *subsetD*)
apply *assumption*
apply *simp*
apply *clarify*
apply(*erule-tac x*=*ia* **in** *all-dupE*)
apply(*rule subsetD*, *erule mp*, *assumption*)
apply(*erule-tac pre*=*pre* **and** *rely*=*rely* **and** *x*=*x* **and** *s*=*s* **in** *three*)
apply(*erule-tac x*=*ic* **in** *allE*,*erule mp*)
apply *simp-all*
apply(*simp add*:*ParallelCom-def*)
apply(*force simp add*:*Com-def*)
apply(*simp add*:*conjoin-def same-length-def*)
done

lemma *ParallelEmpty* [*rule-format*]:
$\forall$ *i s*. *x* $\in$ *par-cp* (*ParallelCom* []) *s* $\longrightarrow$
Suc i < *length x* $\longrightarrow$ (*x* ! *i*, *x* ! *Suc i*) $\notin$ *par-ctran*
apply(*induct-tac x*)
apply(*simp add*:*par-cp-def ParallelCom-def*)
apply *clarify*
apply(*case-tac list*,*simp*,*simp*)
apply(*case-tac i*)
apply(*simp add*:*par-cp-def ParallelCom-def*)
apply(*erule par-ctran.elims*,*simp*)
apply(*simp add*:*par-cp-def ParallelCom-def*)
apply *clarify*
apply(*erule par-cptn.elims*,*simp*)
apply *simp*
apply(*erule par-ctran.elims*)
back
apply *simp*

done

theorem *par-rgsound*:
$\vdash$ *c SAT* [*pre*, *rely*, *guar*, *post*] $\Longrightarrow$
$\models$ (*ParallelCom c*) *SAT* [*pre*, *rely*, *guar*, *post*]
apply(*erule par-rghoare.induct*)
apply(*case-tac xs*,*simp*)
apply(*simp add*:*par-com-validity-def par-comm-def*)
apply *clarify*
apply(*case-tac post*=*UNIV*,*simp*)
apply *clarify*
apply(*drule ParallelEmpty*)
apply *assumption*
apply *simp*
apply *clarify*
apply *simp*
apply(*subgoal-tac xs*$\neq$[])
prefer *2*
apply *simp*
apply(*thin-tac xs* = *a* # *list*)
apply(*simp add*:*par-com-validity-def par-comm-def*)
apply *clarify*
apply(*rule conjI*)
apply *clarify*
apply(*erule-tac pre*=*pre* **and** *rely*=*rely* **and** *guar*=*guar* **and** *x*=*x* **and** *s*=*s* **and** *xs*=*xs* **in** *four*)
apply(*assumption*+)
apply *clarify*
apply (*erule allE*, *erule impE*, *assumption*,*erule rgsound*)
apply(*assumption*+)
apply *clarify*
apply(*erule-tac pre*=*pre* **and** *rely*=*rely* **and** *post*=*post* **and** *x*=*x* **and** *s*=*s* **and** *xs*=*xs* **in** *five*)
apply(*assumption*+)
apply *clarify*
apply (*erule allE*, *erule impE*, *assumption*,*erule rgsound*)
apply(*assumption*+)
done

end

3.6 Concrete Syntax

theory *RG-Syntax*
imports *RG-Hoare Quote-Antiquote*
begin

syntax

-Assign :: *idt* ⇒ *'b* ⇒ *'a com* ((´- :=/ -) [70, 65] 61)
-skip :: *'a com* (SKIP)
-Seq :: *'a com* ⇒ *'a com* ⇒ *'a com* ((-;;/ -) [60,61] 60)
-Cond :: *'a bexp* ⇒ *'a com* ⇒ *'a com* ⇒ *'a com* ((0IF -/ THEN -/ ELSE -/FI) [0, 0, 0] 61)
-Cond2 :: *'a bexp* ⇒ *'a com* ⇒ *'a com* ((0IF - THEN - FI) [0,0] 56)
-While :: *'a bexp* ⇒ *'a com* ⇒ *'a com* ((0WHILE - /DO - /OD) [0, 0] 61)
-Await :: *'a bexp* ⇒ *'a com* ⇒ *'a com* ((0AWAIT - /THEN /- /END) [0,0] 61)
-Atom :: *'a com* ⇒ *'a com* ((⟨-⟩) 61)
-Wait :: *'a bexp* ⇒ *'a com* ((0WAIT - END) 61)

translations
´ x := a ⇀ *Basic* ≪´ (*-update-name x a*)≫
SKIP ⇌ *Basic id*
c1;; *c2* ⇌ *Seq c1 c2*
IF b THEN c1 ELSE c2 FI ⇀ *Cond* .{*b*}. *c1 c2*
IF b THEN c FI ⇌ *IF b THEN c ELSE SKIP FI*
WHILE b DO c OD ⇀ *While* .{*b*}. *c*
AWAIT b THEN c END ⇌ *Await* .{*b*}. *c*
⟨*c*⟩ ⇌ *AWAIT True THEN c END*
WAIT b END ⇌ *AWAIT b THEN SKIP END*

nonterminals
prgs

syntax
-PAR :: *prgs* ⇒ *'a* (COBEGIN//-//COEND 60)
-prg :: *'a* ⇒ *prgs* (- 57)
-prgs :: [*'a*, *prgs*] ⇒ *prgs* (-//‖//- [60,57] 57)

translations
-prg a ⇀ [*a*]
-prgs a ps ⇀ *a* # *ps*
-PAR ps ⇀ *ps*

syntax
-prg-scheme :: [*'a*, *'a*, *'a*, *'a*] ⇒ *prgs* (SCHEME [- ≤ - < -] - [0,0,0,60] 57)

translations
-prg-scheme j i k c ⇌ (*map* (λ*i*. *c*) [*j*..<*k*])

Translations for variables before and after a transition:

syntax
-before :: *id* ⇒ *'a* (º-)
-after :: *id* ⇒ *'a* (ª-)

translations

$^{o}x \rightleftharpoons x$ ´*fst*
$^{a}x \rightleftharpoons x$ ´*snd*

print-translation ⟪
let
fun quote-tr′ f (t :: ts) =
Term.list-comb (f $ Syntax.quote-tr′ -antiquote t, ts)
| quote-tr′ - - = raise Match;

val assert-tr′ = quote-tr′ (Syntax.const -Assert);

fun bexp-tr′ name ((Const (Collect, -) $ t) :: ts) =
quote-tr′ (Syntax.const name) (t :: ts)
| bexp-tr′ - - = raise Match;

fun upd-tr′ (x-upd, T) =
(case try (unsuffix RecordPackage.updateN) x-upd of
SOME x => (x, if T = dummyT then T else Term.domain-type T)
| NONE => raise Match);

fun update-name-tr′ (Free x) = Free (upd-tr′ x)
| update-name-tr′ ((c as Const (-free, -)) $ Free x) =
c $ Free (upd-tr′ x)
| update-name-tr′ (Const x) = Const (upd-tr′ x)
| update-name-tr′ - = raise Match;

fun assign-tr′ (Abs (x, -, f $ t $ Bound 0) :: ts) =
quote-tr′ (Syntax.const -Assign $ update-name-tr′ f)
(Abs (x, dummyT, t) :: ts)
| assign-tr′ - = raise Match;
in
[(Collect, assert-tr′), (Basic, assign-tr′),
(Cond, bexp-tr′ -Cond), (While, bexp-tr′ -While-inv)]
end
⟫

end

3.7 Examples

theory *RG-Examples* **imports** *RG-Syntax* **begin**

lemmas *definitions* [*simp*]= *stable-def Pre-def Rely-def Guar-def Post-def Com-def*

3.7.1 Set Elements of an Array to Zero

lemma *le-less-trans2*: $[\![(j::nat)<k; i\leq j]\!] \Longrightarrow i<k$
by *simp*

lemma *add-le-less-mono*: $\llbracket$ $(a::nat) < c$; $b \le d$ $\rrbracket \Longrightarrow a + b < c + d$
by *simp*

record *Example1* =
  A :: *nat list*

lemma *Example1*:
$\vdash$ *COBEGIN*
    SCHEME $[0 \le i < n]$
   $(\acute{}A := \acute{}A\ [i := 0]$,
   $\{\!|\ n < length\ \acute{}A\ |\!\}$,
   $\{\!|\ length\ {}^{o}A = length\ {}^{a}A \wedge {}^{o}A\ !\ i = {}^{a}A\ !\ i\ |\!\}$,
   $\{\!|\ length\ {}^{o}A = length\ {}^{a}A \wedge (\forall j<n.\ i \neq j \longrightarrow {}^{o}A\ !\ j = {}^{a}A\ !\ j)\ |\!\}$,
   $\{\!|\ \acute{}A\ !\ i = 0\ |\!\})$
   COEND
 SAT $[\{\!|\ n < length\ \acute{}A\ |\!\}, \{\!|\ {}^{o}A = {}^{a}A\ |\!\}, \{\!|\ True\ |\!\}, \{\!|\ \forall i < n.\ \acute{}A\ !\ i = 0\ |\!\}]$
apply(*rule Parallel*)
apply (*auto intro*!: *Basic*)
done

lemma *Example1-parameterized*:
$k < t \Longrightarrow$
 $\vdash$ *COBEGIN*
   SCHEME $[k*n \le i < (Suc\ k)*n]$ $(\acute{}A := \acute{}A[i := 0]$,
  $\{\!|t*n < length\ \acute{}A|\!\}$,
  $\{\!|t*n < length\ {}^{o}A \wedge length\ {}^{o}A = length\ {}^{a}A \wedge {}^{o}A!i = {}^{a}A!i|\!\}$,
  $\{\!|t*n < length\ {}^{o}A \wedge length\ {}^{o}A = length\ {}^{a}A \wedge (\forall j < length\ {}^{o}A\ .\ i \neq j \longrightarrow {}^{o}A!j =$
${}^{a}A!j)|\!\}$,
  $\{\!|\acute{}A!i = 0|\!\})$
  COEND
 SAT $[\{\!|t*n < length\ \acute{}A|\!\}$,
   $\{\!|t*n < length\ {}^{o}A \wedge length\ {}^{o}A = length\ {}^{a}A \wedge (\forall i<n.\ {}^{o}A!(k*n+i) = {}^{a}A!(k*n+i))|\!\}$,

    $\{\!|t*n < length\ {}^{o}A \wedge length\ {}^{o}A = length\ {}^{a}A \wedge$
     $(\forall i < length\ {}^{o}A\ .\ (i < k*n \longrightarrow {}^{o}A!i = {}^{a}A!i) \wedge ((Suc\ k)*n \le i \longrightarrow {}^{o}A!i =$
${}^{a}A!i))|\!\}$,
    $\{\!|\forall i<n.\ \acute{}A!(k*n+i) = 0|\!\}]$
apply(*rule Parallel*)
  apply *auto*
 apply(*erule-tac* $x=k*n + i$ **in** *allE*)
 apply(*subgoal-tac* $k*n+i < length\ (A\ b)$)
  apply *force*
 apply(*erule le-less-trans2*)
 apply(*case-tac* t,*simp*+)
 apply (*simp add*:*add-commute*)
 apply(*simp add*: *add-le-mono*)
apply(*rule Basic*)
  apply *simp*
  apply *clarify*

```
  apply (subgoal-tac k∗n+i< length (A x))
   apply simp
  apply(erule le-less-trans2)
  apply(case-tac t,simp+)
  apply (simp add:add-commute)
  apply(rule add-le-mono, auto)
done
```

3.7.2 Increment a Variable in Parallel

Two components

```
record Example2 =
 x  :: nat
 c-0 :: nat
 c-1 :: nat

lemma Example2:
 ⊢  COBEGIN
    (⟨ ´x:=´x+1;; ´c-0:=´c-0 + 1 ⟩,
     ⦃´x=´c-0 + ´c-1  ∧ ´c-0=0⦄,
     ⦃ºc-0 = ªc-0 ∧
        (ºx=ºc-0 + ºc-1
        ⟶ ªx = ªc-0 + ªc-1)⦄,
     ⦃ºc-1 = ªc-1 ∧
         (ºx=ºc-0 + ºc-1
         ⟶ ªx =ªc-0 + ªc-1)⦄,
     ⦃´x=´c-0 + ´c-1 ∧ ´c-0=1 ⦄)
  ∥
      (⟨ ´x:=´x+1;; ´c-1:=´c-1+1 ⟩,
     ⦃´x=´c-0 + ´c-1 ∧ ´c-1=0 ⦄,
     ⦃ºc-1 = ªc-1 ∧
        (ºx=ºc-0 + ºc-1
        ⟶ ªx = ªc-0 + ªc-1)⦄,
     ⦃ºc-0 = ªc-0 ∧
         (ºx=ºc-0 + ºc-1
        ⟶ ªx =ªc-0 + ªc-1)⦄,
     ⦃´x=´c-0 + ´c-1 ∧ ´c-1=1⦄)
 COEND
 SAT [⦃´x=0 ∧ ´c-0=0 ∧ ´c-1=0⦄,
      ⦃ºx=ªx ∧  ºc-0= ªc-0 ∧ ºc-1=ªc-1⦄,
      ⦃True⦄,
      ⦃´x=2⦄]
apply(rule Parallel)
  apply simp-all
  apply clarify
  apply(case-tac i)
   apply simp
   apply(rule conjI)
    apply clarify
```

```
      apply simp
     apply clarify
     apply simp
     apply(case-tac j,simp)
     apply simp
    apply simp
    apply(rule conjI)
     apply clarify
     apply simp
    apply clarify
    apply simp
    apply(subgoal-tac j=0)
     apply (rotate-tac -1)
     apply (simp (asm-lr))
    apply arith
   apply clarify
   apply(case-tac i,simp,simp)
  apply clarify
  apply simp
  apply(erule-tac x=0 in all-dupE)
  apply(erule-tac x=1 in allE,simp)
 apply clarify
 apply(case-tac i,simp)
  apply(rule Await)
   apply simp-all
  apply(clarify)
  apply(rule Seq)
   prefer 2
   apply(rule Basic)
    apply simp-all
   apply(rule subset-refl)
  apply(rule Basic)
  apply simp-all
  apply clarify
  apply simp
 apply(rule Await)
  apply simp-all
 apply(clarify)
 apply(rule Seq)
  prefer 2
  apply(rule Basic)
   apply simp-all
  apply(rule subset-refl)
 apply(auto intro!: Basic)
done
```

Parameterized

lemma *Example2-lemma2-aux*: $j<n \Longrightarrow$

$(\sum i{=}0..{<}n.\ (b\ i{::}nat)) =$
$(\sum i{=}0..{<}j.\ b\ i) + b\ j + (\sum i{=}0..{<}n-(Suc\ j)\ .\ b\ (Suc\ j + i))$
apply(*induct n*)
 apply *simp-all*
apply(*simp add*:*less-Suc-eq*)
 apply(*auto*)
apply(*subgoal-tac* $n - j = Suc(n - Suc\ j)$)
  apply *simp*
apply *arith*
done

lemma *Example2-lemma2-aux2*:
  $j \leq s \Longrightarrow (\sum i{::}nat{=}0..{<}j.\ (b\ (s{:=}t))\ i) = (\sum i{=}0..{<}j.\ b\ i)$
apply(*induct j*)
 apply (*simp-all cong*:*setsum-cong*)
done

lemma *Example2-lemma2*:
 $[\![j{<}n;\ b\ j{=}0]\!] \Longrightarrow Suc\ (\sum i{::}nat{=}0..{<}n.\ b\ i){=}(\sum i{=}0..{<}n.\ (b\ (j := Suc\ 0))\ i)$
apply(*frule-tac* $b{=}(b\ (j{:=}(Suc\ 0)))$ **in** *Example2-lemma2-aux*)
apply(*erule-tac* $t{=}setsum\ (b(j := (Suc\ 0)))\ \{0..{<}n\}$ **in** *ssubst*)
apply(*frule-tac* $b{=}b$ **in** *Example2-lemma2-aux*)
apply(*erule-tac* $t{=}setsum\ b\ \{0..{<}n\}$ **in** *ssubst*)
apply(*subgoal-tac* $Suc\ (setsum\ b\ \{0..{<}j\} + b\ j + (\sum i{=}0..{<}n - Suc\ j.\ b\ (Suc\ j$
$+\ i))){=}(setsum\ b\ \{0..{<}j\} + Suc\ (b\ j) + (\sum i{=}0..{<}n - Suc\ j.\ b\ (Suc\ j + i)))$)
apply(*rotate-tac* -1)
apply(*erule ssubst*)
apply(*subgoal-tac* $j{\leq}j$)
 apply(*drule-tac* $b{=}b$ **and** $t{=}(Suc\ 0)$ **in** *Example2-lemma2-aux2*)
apply(*rotate-tac* -1)
apply(*erule ssubst*)
apply *simp-all*
done

lemma *Example2-lemma2-Suc0*: $[\![j{<}n;\ b\ j{=}0]\!] \Longrightarrow$
 $Suc\ (\sum i{::}nat{=}0..{<}\ n.\ b\ i){=}(\sum i{=}0..{<}\ n.\ (b\ (j{:=}Suc\ 0))\ i)$
by(*simp add*:*Example2-lemma2*)

record *Example2-parameterized* =
  $C :: nat \Rightarrow nat$
  $y\ :: nat$

lemma *Example2-parameterized*: $0{<}n \Longrightarrow$
 $\vdash COBEGIN\ SCHEME\ \ [0{\leq}i{<}n]$
  $(\langle\ ´y{:=}´y{+}1;;\ ´C{:=}´C\ (i{:=}1)\ \rangle,$
  $\{\!|´y{=}(\sum i{=}0..{<}n.\ ´C\ i) \wedge ´C\ i{=}0|\!\},$
  $\{\!|{}^{o}C\ i = {}^{a}C\ i\ \wedge$
   $({}^{o}y{=}(\sum i{=}0..{<}n.\ {}^{o}C\ i) \longrightarrow {}^{a}y = (\sum i{=}0..{<}n.\ {}^{a}C\ i))|\!\},$
  $\{\!|(\forall j{<}n.\ i{\neq}j \longrightarrow {}^{o}C\ j = {}^{a}C\ j)\ \wedge$

$({}^{o}y{=}(\sum i{=}0..{<}n.\ {}^{o}C\ i) \longrightarrow {}^{a}y\ {=}(\sum i{=}0..{<}n.\ {}^{a}C\ i))\}$,
$\{\acute{}y{=}(\sum i{=}0..{<}n.\ \acute{}C\ i) \wedge \acute{}C\ i{=}1\})$
COEND
SAT $[\{\acute{}y{=}0 \wedge (\sum i{=}0..{<}n.\ \acute{}C\ i){=}0\ \}, \{{}^{o}C{=}{}^{a}C \wedge {}^{o}y{=}{}^{a}y\}, \{True\}, \{\acute{}y{=}n\}]$
apply(*rule Parallel*)
apply *force*
apply *force*
apply(*force*)
apply *clarify*
apply *simp*
apply(*simp cong*:*setsum-ivl-cong*)
apply *clarify*
apply *simp*
apply(*rule Await*)
apply *simp-all*
apply *clarify*
apply(*rule Seq*)
prefer *2*
apply(*rule Basic*)
apply(*rule subset-refl*)
apply *simp+*
apply(*rule Basic*)
apply *simp*
apply *clarify*
apply *simp*
apply(*simp add*:*Example2-lemma2-Suc0 cong*:*if-cong*)
apply *simp+*
done

3.7.3 Find Least Element

A previous lemma:

lemma *mod-aux* :$[\![i < (n::nat);\ a\ mod\ n = i;\ \ j < a + n;\ j\ mod\ n = i;\ a < j]\!]$
$\Longrightarrow$ *False*
apply(*subgoal-tac a=a div n∗n + a mod n*)
prefer *2* **apply** (*simp* (*no-asm-use*))
apply(*subgoal-tac j=j div n∗n + j mod n*)
prefer *2* **apply** (*simp* (*no-asm-use*))
apply *simp*
apply(*subgoal-tac a div n∗n < j div n∗n*)
prefer *2* **apply** *arith*
apply(*subgoal-tac j div n∗n < (a div n + 1)∗n*)
prefer *2* **apply** *simp*
apply (*simp only*:*mult-less-cancel2*)
apply *arith*
done

record *Example3* =
X :: *nat* ⇒ *nat*

Y :: $nat \Rightarrow nat$

lemma *Example3*: $m\ mod\ n{=}0 \Longrightarrow$
$\vdash$ *COBEGIN*
SCHEME $[0{\le}i{<}n]$
$(WHILE\ (\forall j{<}n.\ {}^\acute{}X\ i < {}^\acute{}Y\ j)\ \ DO$
$IF\ P(B!({}^\acute{}X\ i))\ THEN\ {}^\acute{}Y{:=}{}^\acute{}Y\ (i{:=}{}^\acute{}X\ i)$
$ELSE\ {}^\acute{}X{:=}\ {}^\acute{}X\ (i{:=}({}^\acute{}X\ i){+}\ n)\ FI$
OD,
$\{\!|({}^\acute{}X\ i)\ mod\ n{=}i \wedge (\forall j{<}{}^\acute{}X\ i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge ({}^\acute{}Y\ i{<}m \longrightarrow P(B!({}^\acute{}Y\ i)) \wedge {}^\acute{}Y\ i{\le}\ m{+}i)|\!\}$,
$\{\!|(\forall j{<}n.\ i{\ne}j \longrightarrow {}^aY\ j \le {}^oY\ j) \wedge {}^oX\ i = {}^aX\ i \wedge {}^oY\ i = {}^aY\ i|\!\}$,
$\{\!|(\forall j{<}n.\ i{\ne}j \longrightarrow {}^oX\ j = {}^aX\ j \wedge {}^oY\ j = {}^aY\ j) \wedge {}^aY\ i \le {}^oY\ i|\!\}$,
$\{\!|({}^\acute{}X\ i)\ mod\ n{=}i \wedge (\forall j{<}{}^\acute{}X\ i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge ({}^\acute{}Y\ i{<}m \longrightarrow P(B!({}^\acute{}Y\ i)) \wedge {}^\acute{}Y\ i{\le}\ m{+}i) \wedge (\exists j{<}n.\ {}^\acute{}Y\ j \le {}^\acute{}X\ i)\ |\!\})$
COEND
$SAT\ [\{\!|\ \forall i{<}n.\ {}^\acute{}X\ i{=}i \wedge {}^\acute{}Y\ i{=}m{+}i\ |\!\},\{\!|{}^oX{=}{}^aX \wedge {}^oY{=}{}^aY|\!\},\{\!|True|\!\}$,
$\{\!|\forall i{<}n.\ ({}^\acute{}X\ i)\ mod\ n{=}i \wedge (\forall j{<}{}^\acute{}X\ i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge ({}^\acute{}Y\ i{<}m \longrightarrow P(B!({}^\acute{}Y\ i)) \wedge {}^\acute{}Y\ i{\le}\ m{+}i) \wedge (\exists j{<}n.\ {}^\acute{}Y\ j \le {}^\acute{}X\ i)|\!\}]$
apply(*rule Parallel*)
— 5 subgoals left
apply *force+*
apply *clarify*
apply *simp*
apply(*rule While*)
apply *force*
apply *force*
apply *force*
apply(*rule-tac pre'*$=\{\!|\ {}^\acute{}X\ i\ mod\ n = i \wedge (\forall j.\ j{<}{}^\acute{}X\ i \longrightarrow j\ mod\ n = i \longrightarrow \neg P(B!j)) \wedge ({}^\acute{}Y\ i < n * q \longrightarrow P\ (B!({}^\acute{}Y\ i))) \wedge {}^\acute{}X\ i{<}{}^\acute{}Y\ i|\!\}$ **in** *Conseq*)
apply *force*
apply(*rule subset-refl*)+
apply(*rule Cond*)
apply *force*
apply(*rule Basic*)
apply *force*
apply *fastsimp*
apply *force*
apply *force*
apply(*rule Basic*)
apply *simp*
apply *clarify*
apply *simp*
apply(*case-tac* $X\ x\ (j\ mod\ n){\le}\ j$)
apply(*drule le-imp-less-or-eq*)
apply(*erule disjE*)
apply(*drule-tac* $j{=}j$ **and** $n{=}n$ **and** $i{=}j\ mod\ n$ **and** $a{=}X\ x\ (j\ mod\ n)$ **in**

mod-aux)
apply *assumption+*
apply *simp+*
apply *clarsimp*
apply *fastsimp*
apply *force+*
done

Same but with a list as auxiliary variable:

record *Example3-list* =
X :: *nat list*
Y :: *nat list*

lemma *Example3-list*: m *mod* $n=0 \Longrightarrow \vdash$ (*COBEGIN SCHEME* $[0\leq i<n]$
(*WHILE* $(\forall j<n.\ ´X!i\ <\ ´Y!j)$ *DO*
IF $P(B!(´X!i))$ *THEN* $´Y:=´Y[i:=´X!i]$ *ELSE* $´X:=\ ´X[i:=(´X!i)+\ n]$ *FI*
OD,
$\{\!|n<length\ ´X \wedge n<length\ ´Y \wedge (´X!i)\ mod\ n=i \wedge (\forall j<´X!i.\ j\ mod\ n=i \longrightarrow \neg P(B!j)) \wedge (´Y!i<m \longrightarrow P(B!(´Y!i)) \wedge ´Y!i\leq m+i)|\!\}$,
$\{\!|(\forall j<n.\ i\neq j \longrightarrow {}^{a}Y!j \leq {}^{o}Y!j) \wedge {}^{o}X!i = {}^{a}X!i \wedge$
${}^{o}Y!i = {}^{a}Y!i \wedge length\ {}^{o}X = length\ {}^{a}X \wedge length\ {}^{o}Y = length\ {}^{a}Y|\!\}$,
$\{\!|(\forall j<n.\ i\neq j \longrightarrow {}^{o}X!j = {}^{a}X!j \wedge {}^{o}Y!j = {}^{a}Y!j) \wedge$
${}^{a}Y!i \leq {}^{o}Y!i \wedge length\ {}^{o}X = length\ {}^{a}X \wedge length\ {}^{o}Y = length\ {}^{a}Y|\!\}$,
$\{\!|(´X!i)\ mod\ n=i \wedge (\forall j<´X!i.\ j\ mod\ n=i \longrightarrow \neg P(B!j)) \wedge (´Y!i<m \longrightarrow P(B!(´Y!i))$
$\wedge\ ´Y!i\leq m+i) \wedge (\exists j<n.\ ´Y!j \leq ´X!i)\ |\!\})$ *COEND*)
SAT $[\{\!|n<length\ ´X \wedge n<length\ ´Y \wedge (\forall i<n.\ ´X!i=i \wedge ´Y!i=m+i)\ |\!\}$,
$\{\!|{}^{o}X={}^{a}X \wedge {}^{o}Y={}^{a}Y|\!\}$,
$\{\!|True|\!\}$,
$\{\!|\forall i<n.\ (´X!i)\ mod\ n=i \wedge (\forall j<´X!i.\ j\ mod\ n=i \longrightarrow \neg P(B!j)) \wedge$
$(´Y!i<m \longrightarrow P(B!(´Y!i)) \wedge ´Y!i\leq m+i) \wedge (\exists j<n.\ ´Y!j \leq ´X!i)|\!\}]$
apply(*rule Parallel*)
— 5 subgoals left
apply *force+*
apply *clarify*
apply *simp*
apply(*rule While*)
apply *force*
apply *force*
apply *force*
apply(*rule-tac pre'*=$\{\!|n<length\ ´X \wedge n<length\ ´Y \wedge ´X\ !\ i\ mod\ n = i \wedge (\forall j.\ j < ´X\ !\ i \longrightarrow j\ mod\ n = i \longrightarrow \neg\ P\ (B\ !\ j)) \wedge (´Y\ !\ i < n * q \longrightarrow P\ (B\ !\ (´Y\ !\ i))) \wedge ´X!i<´Y!i|\!\}$ **in** *Conseq*)
apply *force*
apply(*rule subset-refl*)+
apply(*rule Cond*)
apply *force*
apply(*rule Basic*)
apply *force*
apply *force*

```
    apply force
   apply force
  apply(rule Basic)
     apply simp
     apply clarify
     apply simp
     apply(rule allI)
     apply(rule impI)+
     apply(case-tac X x ! i≤ j)
      apply(drule le-imp-less-or-eq)
      apply(erule disjE)
       apply(drule-tac j=j and n=n and i=i and a=X x ! i in mod-aux)
        apply assumption+
       apply simp
apply force+
done

end
```