Some results of number theory

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Abstract

This directory contains formalized proofs of many results of number theory. The proofs of the Chinese Remainder Theorem and Wilson's Theorem are due to Rasmussen. The proof of Gauss's law of quadratic reciprocity is due to Avigad, Gray and Kramer. Proofs can be found in most introductory number theory textbooks; Goldman's *The Queen of Mathematics: a Historically Motivated Guide to Number Theory* provides some historical context.

Avigad, Gray and Kramer have also provided library theories dealing with finite sets and finite sums, divisibility and congruences, parity and residues. The authors are engaged in redesigning and polishing these theories for more serious use. For the latest information in this respect, please see the web page http://www.andrew.cmu.edu/~avigad/isabelle. Other theories contain proofs of Euler's criteria, Gauss' lemma, and the law of quadratic reciprocity. The formalization follows Eisenstein's proof, which is the one most commonly found in introductory textbooks; in particular, it follows the presentation in Niven and Zuckerman, *The Theory of Numbers*.

To avoid having to count roots of polynomials, however, we relied on a trick previously used by David Russinoff in formalizing quadratic reciprocity for the Boyer-Moore theorem prover; see Russinoff, David, "A mechanical proof of quadratic reciprocity," *Journal of Automated Reasoning* 8:3-21, 1992. We are grateful to Larry Paulson for calling our attention to this reference.

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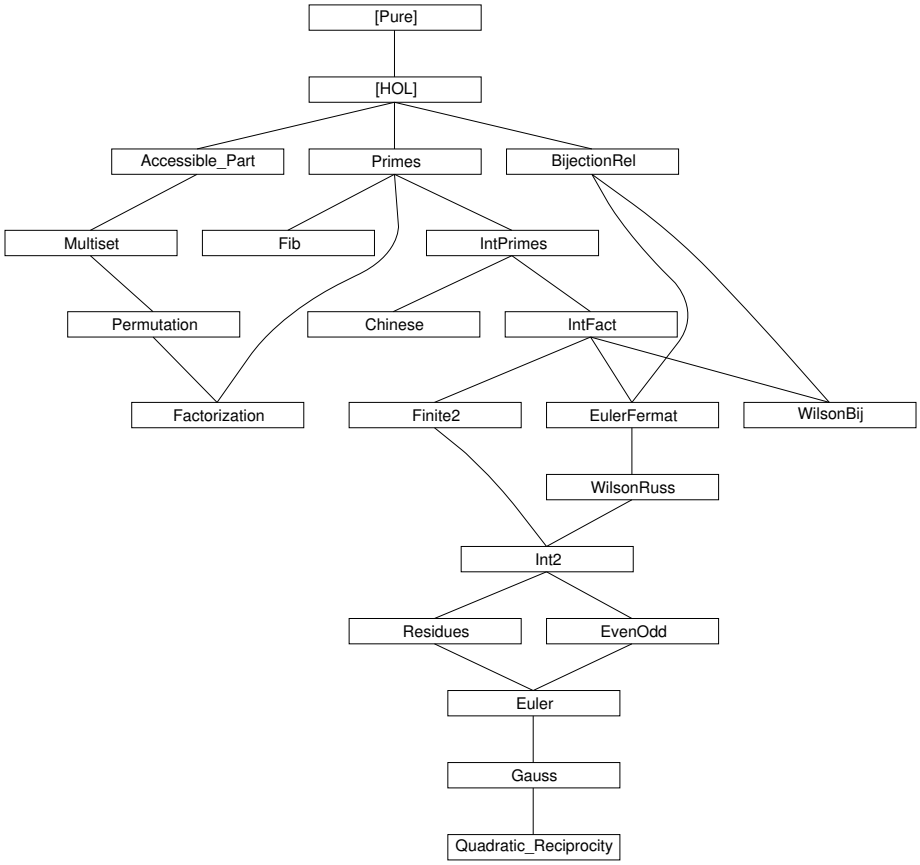

1 The Fibonacci function

theory *Fib* **imports** *Primes* **begin**

Fibonacci numbers: proofs of laws taken from: R. L. Graham, D. E. Knuth, O. Patashnik. Concrete Mathematics. (Addison-Wesley, 1989)

consts *fib* :: *nat* => *nat*
recdef *fib* *measure* (λ*x*. *x*)
  zero: *fib 0* = *0*
  one: *fib* (*Suc 0*) = *Suc 0*
  Suc-Suc: *fib* (*Suc* (*Suc x*)) = *fib x* + *fib* (*Suc x*)

The difficulty in these proofs is to ensure that the induction hypotheses are applied before the definition of *fib*. Towards this end, the *fib* equations are not declared to the Simplifier and are applied very selectively at first.

We disable *fib.Suc-Suc* for simplification ...

declare *fib.Suc-Suc* [*simp del*]

...then prove a version that has a more restrictive pattern.

lemma *fib-Suc3*: *fib* (*Suc* (*Suc* (*Suc n*))) = *fib* (*Suc n*) + *fib* (*Suc* (*Suc n*))
  by (*rule fib.Suc-Suc*)

Concrete Mathematics, page 280

lemma *fib-add*: *fib* (*Suc* (*n* + *k*)) = *fib* (*Suc k*) ∗ *fib* (*Suc n*) + *fib k* ∗ *fib n*
  apply (*induct n rule*: *fib.induct*)
    prefer *3*

simplify the LHS just enough to apply the induction hypotheses

    apply (*simp add*: *fib-Suc3*)
    apply (*simp-all add*: *fib.Suc-Suc add-mult-distrib2*)
    done

lemma *fib-Suc-neq-0*: *fib* (*Suc n*) ≠ *0*
  apply (*induct n rule*: *fib.induct*)
    apply (*simp-all add*: *fib.Suc-Suc*)
  done

lemma *fib-Suc-gr-0*: *0* < *fib* (*Suc n*)
  by (*insert fib-Suc-neq-0* [*of n*], *simp*)

lemma *fib-gr-0*: *0* < *n* ==> *0* < *fib n*
  by (*case-tac n*, *auto simp add*: *fib-Suc-gr-0*)

Concrete Mathematics, page 278: Cassini's identity. The proof is much easier using integers, not natural numbers!

lemma *fib-Cassini-int*:
int (*fib* (*Suc* (*Suc n*)) ∗ *fib n*) =
(*if n mod 2 = 0 then int* (*fib* (*Suc n*) ∗ *fib* (*Suc n*)) − *1*
else int (*fib* (*Suc n*) ∗ *fib* (*Suc n*)) + *1*)
apply (*induct n rule*: *fib.induct*)
apply (*simp add*: *fib.Suc-Suc*)
apply (*simp add*: *fib.Suc-Suc mod-Suc*)
apply (*simp add*: *fib.Suc-Suc add-mult-distrib add-mult-distrib2*
mod-Suc zmult-int [*symmetric*])
apply *presburger*
done

We now obtain a version for the natural numbers via the coercion function *int*.

theorem *fib-Cassini*:
fib (*Suc* (*Suc n*)) ∗ *fib n* =
(*if n mod 2 = 0 then fib* (*Suc n*) ∗ *fib* (*Suc n*) − *1*
else fib (*Suc n*) ∗ *fib* (*Suc n*) + *1*)
apply (*rule int-int-eq* [*THEN iffD1*])
apply (*simp add*: *fib-Cassini-int*)
apply (*subst zdiff-int* [*symmetric*])
apply (*insert fib-Suc-gr-0* [*of n*], *simp-all*)
done

Toward Law 6.111 of Concrete Mathematics

lemma *gcd-fib-Suc-eq-1*: *gcd* (*fib n*, *fib* (*Suc n*)) = *Suc 0*
apply (*induct n rule*: *fib.induct*)
prefer *3*
apply (*simp add*: *gcd-commute fib-Suc3*)
apply (*simp-all add*: *fib.Suc-Suc*)
done

lemma *gcd-fib-add*: *gcd* (*fib m*, *fib* (*n* + *m*)) = *gcd* (*fib m*, *fib n*)
apply (*simp add*: *gcd-commute* [*of fib m*])
apply (*case-tac m*)
apply *simp*
apply (*simp add*: *fib-add*)
apply (*simp add*: *add-commute gcd-non-0* [*OF fib-Suc-gr-0*])
apply (*simp add*: *gcd-non-0* [*OF fib-Suc-gr-0*, *symmetric*])
apply (*simp add*: *gcd-fib-Suc-eq-1 gcd-mult-cancel*)
done

lemma *gcd-fib-diff*: *m* ≤ *n* ==> *gcd* (*fib m*, *fib* (*n* − *m*)) = *gcd* (*fib m*, *fib n*)
by (*simp add*: *gcd-fib-add* [*symmetric*, *of* - *n*−*m*])

lemma *gcd-fib-mod*: *0* < *m* ==> *gcd* (*fib m*, *fib* (*n mod m*)) = *gcd* (*fib m*, *fib n*)
apply (*induct n rule*: *nat-less-induct*)
apply (*simp add*: *mod-if gcd-fib-diff mod-geq*)

done

lemma *fib-gcd*: *fib* (*gcd* (*m*, *n*)) = *gcd* (*fib m*, *fib n*) — Law 6.111
apply (*induct m n rule*: *gcd-induct*)
apply (*simp-all add*: *gcd-non-0 gcd-commute gcd-fib-mod*)
done

theorem *fib-mult-eq-setsum*:
fib (*Suc n*) ∗ *fib n* = (∑ *k* ∈ {..*n*}. *fib k* ∗ *fib k*)
apply (*induct n rule*: *fib.induct*)
apply (*auto simp add*: *atMost-Suc fib.Suc-Suc*)
apply (*simp add*: *add-mult-distrib add-mult-distrib2*)
done

end

2 Fundamental Theorem of Arithmetic (unique factorization into primes)

theory *Factorization* **imports** *Primes Permutation* **begin**

2.1 Definitions

consts
primel :: *nat list* => *bool*
nondec :: *nat list* => *bool*
prod :: *nat list* => *nat*
oinsert :: *nat* => *nat list* => *nat list*
sort :: *nat list* => *nat list*

defs
primel-def: *primel xs* == ∀ *p* ∈ *set xs*. *prime p*

primrec
nondec [] = *True*
nondec (*x* # *xs*) = (*case xs of* [] => *True* | *y* # *ys* => *x* ≤ *y* ∧ *nondec xs*)

primrec
prod [] = *Suc 0*
prod (*x* # *xs*) = *x* ∗ *prod xs*

primrec
oinsert x [] = [*x*]
oinsert x (*y* # *ys*) = (*if x* ≤ *y then x* # *y* # *ys else y* # *oinsert x ys*)

primrec
sort [] = []

sort (*x* # *xs*) = *oinsert x* (*sort xs*)

2.2 Arithmetic

lemma *one-less-m*: (*m*::*nat*) ≠ *m* ∗ *k* ==> *m* ≠ *Suc 0* ==> *Suc 0* < *m*
apply (*case-tac m*)
apply *auto*
done

lemma *one-less-k*: (*m*::*nat*) ≠ *m* ∗ *k* ==> *Suc 0* < *m* ∗ *k* ==> *Suc 0* < *k*
apply (*case-tac k*)
apply *auto*
done

lemma *mult-left-cancel*: (*0*::*nat*) < *k* ==> *k* ∗ *n* = *k* ∗ *m* ==> *n* = *m*
apply *auto*
done

lemma *mn-eq-m-one*: (*0*::*nat*) < *m* ==> *m* ∗ *n* = *m* ==> *n* = *Suc 0*
apply (*case-tac n*)
apply *auto*
done

lemma *prod-mn-less-k*:
(*0*::*nat*) < *n* ==> *0* < *k* ==> *Suc 0* < *m* ==> *m* ∗ *n* = *k* ==> *n* < *k*
apply (*induct m*)
apply *auto*
done

2.3 Prime list and product

lemma *prod-append*: *prod* (*xs* @ *ys*) = *prod xs* ∗ *prod ys*
apply (*induct xs*)
apply (*simp-all add*: *mult-assoc*)
done

lemma *prod-xy-prod*:
prod (*x* # *xs*) = *prod* (*y* # *ys*) ==> *x* ∗ *prod xs* = *y* ∗ *prod ys*
apply *auto*
done

lemma *primel-append*: *primel* (*xs* @ *ys*) = (*primel xs* ∧ *primel ys*)
apply (*unfold primel-def*)
apply *auto*
done

lemma *prime-primel*: *prime n* ==> *primel* [*n*] ∧ *prod* [*n*] = *n*
apply (*unfold primel-def*)
apply *auto*
done

lemma *prime-nd-one*: *prime p ==> ¬ p dvd Suc 0*
apply (*unfold prime-def dvd-def*)
apply *auto*
done

lemma *hd-dvd-prod*: *prod (x # xs) = prod ys ==> x dvd (prod ys)*
apply (*unfold dvd-def*)
apply (*rule exI*)
apply (*rule sym*)
apply *simp*
done

lemma *primel-tl*: *primel (x # xs) ==> primel xs*
apply (*unfold primel-def*)
apply *auto*
done

lemma *primel-hd-tl*: (*primel (x # xs)*) = (*prime x ∧ primel xs*)
apply (*unfold primel-def*)
apply *auto*
done

lemma *primes-eq*: *prime p ==> prime q ==> p dvd q ==> p = q*
apply (*unfold prime-def*)
apply *auto*
done

lemma *primel-one-empty*: *primel xs ==> prod xs = Suc 0 ==> xs = []*
apply (*unfold primel-def prime-def*)
apply (*case-tac xs*)
apply *simp-all*
done

lemma *prime-g-one*: *prime p ==> Suc 0 < p*
apply (*unfold prime-def*)
apply *auto*
done

lemma *prime-g-zero*: *prime p ==> 0 < p*
apply (*unfold prime-def*)
apply *auto*
done

lemma *primel-nempty-g-one* [*rule-format*]:
primel xs −−> xs ≠ [] −−> Suc 0 < prod xs
apply (*unfold primel-def prime-def*)
apply (*induct xs*)
apply (*auto elim*: *one-less-mult*)

done

lemma *primel-prod-gz*: *primel xs ==> 0 < prod xs*
apply (*unfold primel-def prime-def*)
apply (*induct xs*)
apply *auto*
done

2.4 Sorting

lemma *nondec-oinsert* [*rule-format*]: *nondec xs --> nondec* (*oinsert x xs*)
apply (*induct xs*)
apply (*case-tac* [*2*] *xs*)
apply (*simp-all cong del*: *list.weak-case-cong*)
done

lemma *nondec-sort*: *nondec* (*sort xs*)
apply (*induct xs*)
apply *simp-all*
apply (*erule nondec-oinsert*)
done

lemma *x-less-y-oinsert*: $x \leq y$ *==> l = y # ys ==> x # l = oinsert x l*
apply *simp-all*
done

lemma *nondec-sort-eq* [*rule-format*]: *nondec xs --> xs = sort xs*
apply (*induct xs*)
apply *safe*
apply *simp-all*
apply (*case-tac xs*)
apply *simp-all*
apply (*case-tac xs*)
apply *simp*
apply (*rule-tac y = aa* **and** *ys = list* **in** *x-less-y-oinsert*)
apply *simp-all*
done

lemma *oinsert-x-y*: *oinsert x* (*oinsert y l*) = *oinsert y* (*oinsert x l*)
apply (*induct l*)
apply *auto*
done

2.5 Permutation

lemma *perm-primel* [*rule-format*]: *xs <~~> ys ==> primel xs --> primel ys*
apply (*unfold primel-def*)
apply (*erule perm.induct*)
apply *simp*
apply *simp*

apply (*simp* (*no-asm*))
apply *blast*
apply *blast*
done

lemma *perm-prod* [*rule-format*]: *xs* <~~> *ys* ==> *prod xs* = *prod ys*
apply (*erule perm.induct*)
apply (*simp-all add*: *mult-ac*)
done

lemma *perm-subst-oinsert*: *xs* <~~> *ys* ==> *oinsert a xs* <~~> *oinsert a ys*
apply (*erule perm.induct*)
apply *auto*
done

lemma *perm-oinsert*: *x* # *xs* <~~> *oinsert x xs*
apply (*induct xs*)
apply *auto*
done

lemma *perm-sort*: *xs* <~~> *sort xs*
apply (*induct xs*)
apply (*auto intro*: *perm-oinsert elim*: *perm-subst-oinsert*)
done

lemma *perm-sort-eq*: *xs* <~~> *ys* ==> *sort xs* = *sort ys*
apply (*erule perm.induct*)
apply (*simp-all add*: *oinsert-x-y*)
done

2.6 Existence

lemma *ex-nondec-lemma*:
primel xs ==> ∃ *ys*. *primel ys* ∧ *nondec ys* ∧ *prod ys* = *prod xs*
apply (*blast intro*: *nondec-sort perm-prod perm-primel perm-sort perm-sym*)
done

lemma *not-prime-ex-mk*:
Suc 0 < *n* ∧ ¬ *prime n* ==>
∃ *m k*. *Suc 0* < *m* ∧ *Suc 0* < *k* ∧ *m* < *n* ∧ *k* < *n* ∧ *n* = *m* ∗ *k*
apply (*unfold prime-def dvd-def*)
apply (*auto intro*: *n-less-m-mult-n n-less-n-mult-m one-less-m one-less-k*)
done

lemma *split-primel*:
primel xs ==> *primel ys* ==> ∃ *l*. *primel l* ∧ *prod l* = *prod xs* ∗ *prod ys*
apply (*rule exI*)
apply *safe*
apply (*rule-tac* [*2*] *prod-append*)

apply (*simp add*: *primel-append*)
done

lemma *factor-exists* [*rule-format*]: *Suc 0 < n −−> (∃ l. primel l ∧ prod l = n)*
apply (*induct n rule*: *nat-less-induct*)
apply (*rule impI*)
apply (*case-tac prime n*)
apply (*rule exI*)
apply (*erule prime-primel*)
apply (*cut-tac n = n* **in** *not-prime-ex-mk*)
apply (*auto intro*!: *split-primel*)
done

lemma *nondec-factor-exists*: *Suc 0 < n ==> ∃ l. primel l ∧ nondec l ∧ prod l = n*
apply (*erule factor-exists* [*THEN exE*])
apply (*blast intro*!: *ex-nondec-lemma*)
done

2.7 Uniqueness

lemma *prime-dvd-mult-list* [*rule-format*]:
prime p ==> p dvd (prod xs) −−> (∃ m. m:set xs ∧ p dvd m)
apply (*induct xs*)
apply (*force simp add*: *prime-def*)
apply (*force dest*: *prime-dvd-mult*)
done

lemma *hd-xs-dvd-prod*:
primel (x # xs) ==> primel ys ==> prod (x # xs) = prod ys
==> ∃ m. m ∈ set ys ∧ x dvd m
apply (*rule prime-dvd-mult-list*)
apply (*simp add*: *primel-hd-tl*)
apply (*erule hd-dvd-prod*)
done

lemma *prime-dvd-eq*: *primel (x # xs) ==> primel ys ==> m ∈ set ys ==> x dvd m ==> x = m*
apply (*rule primes-eq*)
apply (*auto simp add*: *primel-def primel-hd-tl*)
done

lemma *hd-xs-eq-prod*:
primel (x # xs) ==>
primel ys ==> prod (x # xs) = prod ys ==> x ∈ set ys
apply (*frule hd-xs-dvd-prod*)
apply *auto*
apply (*drule prime-dvd-eq*)
apply *auto*

done

lemma *perm-primel-ex*:
primel (*x* # *xs*) ==>
primel ys ==> *prod* (*x* # *xs*) = *prod ys* ==> ∃ *l*. *ys* <~~> (*x* # *l*)
apply (*rule exI*)
apply (*rule perm-remove*)
apply (*erule hd-xs-eq-prod*)
apply *simp-all*
done

lemma *primel-prod-less*:
primel (*x* # *xs*) ==>
primel ys ==> *prod* (*x* # *xs*) = *prod ys* ==> *prod xs* < *prod ys*
apply (*auto intro*: *prod-mn-less-k prime-g-one primel-prod-gz simp add*: *primel-hd-tl*)
done

lemma *prod-one-empty*:
primel xs ==> *p* ∗ *prod xs* = *p* ==> *prime p* ==> *xs* = []
apply (*auto intro*: *primel-one-empty simp add*: *prime-def*)
done

lemma *uniq-ex-aux*:
∀ *m*. *m* < *prod ys* −−> (∀ *xs ys*. *primel xs* ∧ *primel ys* ∧
prod xs = *prod ys* ∧ *prod xs* = *m* −−> *xs* <~~> *ys*) ==>
primel list ==> *primel x* ==> *prod list* = *prod x* ==> *prod x* < *prod ys*
==> *x* <~~> *list*
apply *simp*
done

lemma *factor-unique* [*rule-format*]:
∀ *xs ys*. *primel xs* ∧ *primel ys* ∧ *prod xs* = *prod ys* ∧ *prod xs* = *n*
−−> *xs* <~~> *ys*
apply (*induct n rule*: *nat-less-induct*)
apply *safe*
apply (*case-tac xs*)
apply (*force intro*: *primel-one-empty*)
apply (*rule perm-primel-ex* [*THEN exE*])
apply *simp-all*
apply (*rule perm.trans* [*THEN perm-sym*])
apply *assumption*
apply (*rule perm.Cons*)
apply (*case-tac x* = [])
apply (*simp add*: *perm-sing-eq primel-hd-tl*)
apply (*rule-tac p* = *a* **in** *prod-one-empty*)
apply *simp-all*
apply (*erule uniq-ex-aux*)
apply (*auto intro*: *primel-tl perm-primel simp add*: *primel-hd-tl*)
apply (*rule-tac k* = *a* **and** *n* = *prod list* **and** *m* = *prod x* **in** *mult-left-cancel*)

```
  apply (rule-tac [3] x = a in primel-prod-less)
   apply (rule-tac [2] prod-xy-prod)
   apply (rule-tac [2] s = prod ys in HOL.trans)
    apply (erule-tac [3] perm-prod)
   apply (erule-tac [5] perm-prod [symmetric])
  apply (auto intro: perm-primel prime-g-zero)
 done

lemma perm-nondec-unique:
   xs <~~> ys ==> nondec xs ==> nondec ys ==> xs = ys
 apply (rule HOL.trans)
  apply (rule HOL.trans)
   apply (erule nondec-sort-eq)
  apply (erule perm-sort-eq)
 apply (erule nondec-sort-eq [symmetric])
 done

lemma unique-prime-factorization [rule-format]:
   ∀ n. Suc 0 < n --> (∃!l. primel l ∧ nondec l ∧ prod l = n)
 apply safe
  apply (erule nondec-factor-exists)
 apply (rule perm-nondec-unique)
   apply (rule factor-unique)
   apply simp-all
 done

end
```

3 Divisibility and prime numbers (on integers)

theory *IntPrimes* **imports** *Primes* **begin**

The *dvd* relation, GCD, Euclid's extended algorithm, primes, congruences (all on the Integers). Comparable to theory *Primes*, but *dvd* is included here as it is not present in main HOL. Also includes extended GCD and congruences not present in *Primes*.

3.1 Definitions

```
consts
  xzgcda :: int * int * int * int * int * int * int * int => int * int * int

recdef xzgcda
  measure ((λ(m, n, r', r, s', s, t', t). nat r)
    :: int * int * int * int *int * int * int * int => nat)
  xzgcda (m, n, r', r, s', s, t', t) =
        (if r ≤ 0 then (r', s', t')
```

```
    else xzgcda (m, n, r, r' mod r,
                 s, s' - (r' div r) * s,
                 t, t' - (r' div r) * t))
```

constdefs

```
  zgcd :: int * int => int
  zgcd == λ(x,y). int (gcd (nat (abs x), nat (abs y)))

  zprime :: int ⇒ bool
  zprime p == 1 < p ∧ (∀ m. 0 <= m & m dvd p --> m = 1 ∨ m = p)

  xzgcd :: int => int => int * int * int
  xzgcd m n == xzgcda (m, n, m, n, 1, 0, 0, 1)

  zcong :: int => int => int => bool    ((1[- = -] '(mod -')))
  [a = b] (mod m) == m dvd (a - b)
```

gcd lemmas

lemma *gcd-add1-eq*: *gcd* (*m* + *k*, *k*) = *gcd* (*m* + *k*, *m*)
  by (*simp add*: *gcd-commute*)

lemma *gcd-diff2*: *m* ≤ *n* ==> *gcd* (*n*, *n* − *m*) = *gcd* (*n*, *m*)
  apply (*subgoal-tac n* = *m* + (*n* − *m*))
   apply (*erule ssubst*, *rule gcd-add1-eq*, *simp*)
  done

3.2 Euclid's Algorithm and GCD

lemma *zgcd-0* [*simp*]: *zgcd* (*m*, *0*) = *abs m*
  by (*simp add*: *zgcd-def abs-if*)

lemma *zgcd-0-left* [*simp*]: *zgcd* (*0*, *m*) = *abs m*
  by (*simp add*: *zgcd-def abs-if*)

lemma *zgcd-zminus* [*simp*]: *zgcd* (−*m*, *n*) = *zgcd* (*m*, *n*)
  by (*simp add*: *zgcd-def*)

lemma *zgcd-zminus2* [*simp*]: *zgcd* (*m*, −*n*) = *zgcd* (*m*, *n*)
  by (*simp add*: *zgcd-def*)

lemma *zgcd-non-0*: *0* < *n* ==> *zgcd* (*m*, *n*) = *zgcd* (*n*, *m mod n*)
  apply (*frule-tac b* = *n* **and** *a* = *m* **in** *pos-mod-sign*)
  apply (*simp del*: *pos-mod-sign add*: *zgcd-def abs-if nat-mod-distrib*)
 apply (*auto simp add*: *gcd-non-0 nat-mod-distrib* [*symmetric*] *zmod-zminus1-eq-if*)
  apply (*frule-tac a* = *m* **in** *pos-mod-bound*)
  apply (*simp del*: *pos-mod-bound add*: *nat-diff-distrib gcd-diff2 nat-le-eq-zle*)
  done

lemma *zgcd-eq*: *zgcd* (*m*, *n*) = *zgcd* (*n*, *m mod n*)

apply (*case-tac* $n = 0$, *simp add*: *DIVISION-BY-ZERO*)
apply (*auto simp add*: *linorder-neq-iff zgcd-non-0*)
apply (*cut-tac* $m = -m$ **and** $n = -n$ **in** *zgcd-non-0*, *auto*)
done

lemma *zgcd-1* [*simp*]: *zgcd* (m, *1*) = *1*
by (*simp add*: *zgcd-def abs-if*)

lemma *zgcd-0-1-iff* [*simp*]: (*zgcd* (*0*, m) = *1*) = (*abs* m = *1*)
by (*simp add*: *zgcd-def abs-if*)

lemma *zgcd-zdvd1* [*iff*]: *zgcd* (m, n) *dvd* m
by (*simp add*: *zgcd-def abs-if int-dvd-iff*)

lemma *zgcd-zdvd2* [*iff*]: *zgcd* (m, n) *dvd* n
by (*simp add*: *zgcd-def abs-if int-dvd-iff*)

lemma *zgcd-greatest-iff*: k *dvd zgcd* (m, n) = (k *dvd* m ∧ k *dvd* n)
by (*simp add*: *zgcd-def abs-if int-dvd-iff dvd-int-iff nat-dvd-iff*)

lemma *zgcd-commute*: *zgcd* (m, n) = *zgcd* (n, m)
by (*simp add*: *zgcd-def gcd-commute*)

lemma *zgcd-1-left* [*simp*]: *zgcd* (*1*, m) = *1*
by (*simp add*: *zgcd-def gcd-1-left*)

lemma *zgcd-assoc*: *zgcd* (*zgcd* (k, m), n) = *zgcd* (k, *zgcd* (m, n))
by (*simp add*: *zgcd-def gcd-assoc*)

lemma *zgcd-left-commute*: *zgcd* (k, *zgcd* (m, n)) = *zgcd* (m, *zgcd* (k, n))
apply (*rule zgcd-commute* [*THEN trans*])
apply (*rule zgcd-assoc* [*THEN trans*])
apply (*rule zgcd-commute* [*THEN arg-cong*])
done

lemmas *zgcd-ac* = *zgcd-assoc zgcd-commute zgcd-left-commute*
— addition is an AC-operator

lemma *zgcd-zmult-distrib2*: *0* ≤ k ==> k * *zgcd* (m, n) = *zgcd* (k * m, k * n)
by (*simp del*: *minus-mult-right* [*symmetric*]
add: *minus-mult-right nat-mult-distrib zgcd-def abs-if*
mult-less-0-iff gcd-mult-distrib2 [*symmetric*] *zmult-int* [*symmetric*])

lemma *zgcd-zmult-distrib2-abs*: *zgcd* (k * m, k * n) = *abs* k * *zgcd* (m, n)
by (*simp add*: *abs-if zgcd-zmult-distrib2*)

lemma *zgcd-self* [*simp*]: *0* ≤ m ==> *zgcd* (m, m) = m
by (*cut-tac* $k = m$ **and** m = *1* **and** n = *1* **in** *zgcd-zmult-distrib2*, *simp-all*)

lemma *zgcd-zmult-eq-self* [*simp*]: $0 \leq k \Longrightarrow zgcd\ (k, k * n) = k$
by (*cut-tac* $k = k$ **and** $m = 1$ **and** $n = n$ **in** *zgcd-zmult-distrib2*, *simp-all*)

lemma *zgcd-zmult-eq-self2* [*simp*]: $0 \leq k \Longrightarrow zgcd\ (k * n, k) = k$
by (*cut-tac* $k = k$ **and** $m = n$ **and** $n = 1$ **in** *zgcd-zmult-distrib2*, *simp-all*)

lemma *zrelprime-zdvd-zmult-aux*:
$zgcd\ (n, k) = 1 \Longrightarrow k\ dvd\ m * n \Longrightarrow 0 \leq m \Longrightarrow k\ dvd\ m$
apply (*subgoal-tac* $m = zgcd\ (m * n, m * k)$)
apply (*erule ssubst*, *rule zgcd-greatest-iff* [*THEN iffD2*])
apply (*simp-all add*: *zgcd-zmult-distrib2* [*symmetric*] *zero-le-mult-iff*)
done

lemma *zrelprime-zdvd-zmult*: $zgcd\ (n, k) = 1 \Longrightarrow k\ dvd\ m * n \Longrightarrow k\ dvd\ m$
apply (*case-tac* $0 \leq m$)
apply (*blast intro*: *zrelprime-zdvd-zmult-aux*)
apply (*subgoal-tac* $k\ dvd\ -m$)
apply (*rule-tac* [2] *zrelprime-zdvd-zmult-aux*, *auto*)
done

lemma *zgcd-geq-zero*: $0 <= zgcd(x,y)$
by (*auto simp add*: *zgcd-def*)

This is merely a sanity check on zprime, since the previous version denoted the empty set.

lemma *zprime 2*
apply (*auto simp add*: *zprime-def*)
apply (*frule zdvd-imp-le*, *simp*)
apply (*auto simp add*: *order-le-less dvd-def*)
done

lemma *zprime-imp-zrelprime*:
$zprime\ p \Longrightarrow \neg\ p\ dvd\ n \Longrightarrow zgcd\ (n, p) = 1$
apply (*auto simp add*: *zprime-def*)
apply (*drule-tac* $x = zgcd(n, p)$ **in** *allE*)
apply (*auto simp add*: *zgcd-zdvd2* [*of n p*] *zgcd-geq-zero*)
apply (*insert zgcd-zdvd1* [*of n p*], *auto*)
done

lemma *zless-zprime-imp-zrelprime*:
$zprime\ p \Longrightarrow 0 < n \Longrightarrow n < p \Longrightarrow zgcd\ (n, p) = 1$
apply (*erule zprime-imp-zrelprime*)
apply (*erule zdvd-not-zless*, *assumption*)
done

lemma *zprime-zdvd-zmult*:
$0 \leq (m::int) \Longrightarrow zprime\ p \Longrightarrow p\ dvd\ m * n \Longrightarrow p\ dvd\ m \vee p\ dvd\ n$
apply *safe*
apply (*rule zrelprime-zdvd-zmult*)

```
 apply (rule zprime-imp-zrelprime, auto)
done

lemma zgcd-zadd-zmult [simp]: zgcd (m + n * k, n) = zgcd (m, n)
 apply (rule zgcd-eq [THEN trans])
 apply (simp add: zmod-zadd1-eq)
 apply (rule zgcd-eq [symmetric])
 done

lemma zgcd-zdvd-zgcd-zmult: zgcd (m, n) dvd zgcd (k * m, n)
 apply (simp add: zgcd-greatest-iff)
 apply (blast intro: zdvd-trans)
 done

lemma zgcd-zmult-zdvd-zgcd:
    zgcd (k, n) = 1 ==> zgcd (k * m, n) dvd zgcd (m, n)
 apply (simp add: zgcd-greatest-iff)
 apply (rule-tac n = k in zrelprime-zdvd-zmult)
  prefer 2
  apply (simp add: zmult-commute)
 apply (subgoal-tac zgcd (k, zgcd (k * m, n)) = zgcd (k * m, zgcd (k, n)))
  apply simp
 apply (simp (no-asm) add: zgcd-ac)
 done

lemma zgcd-zmult-cancel: zgcd (k, n) = 1 ==> zgcd (k * m, n) = zgcd (m, n)
 by (simp add: zgcd-def nat-abs-mult-distrib gcd-mult-cancel)

lemma zgcd-zgcd-zmult:
    zgcd (k, m) = 1 ==> zgcd (n, m) = 1 ==> zgcd (k * n, m) = 1
 by (simp add: zgcd-zmult-cancel)

lemma zdvd-iff-zgcd: 0 < m ==> (m dvd n) = (zgcd (n, m) = m)
 apply safe
  apply (rule-tac [2] n = zgcd (n, m) in zdvd-trans)
   apply (rule-tac [3] zgcd-zdvd1, simp-all)
 apply (unfold dvd-def, auto)
 done
```

3.3 Congruences

```
lemma zcong-1 [simp]: [a = b] (mod 1)
 by (unfold zcong-def, auto)

lemma zcong-refl [simp]: [k = k] (mod m)
 by (unfold zcong-def, auto)

lemma zcong-sym: [a = b] (mod m) = [b = a] (mod m)
 apply (unfold zcong-def dvd-def, auto)
```

```
 apply (rule-tac [!] x = -k in exI, auto)
done

lemma zcong-zadd:
   [a = b] (mod m) ==> [c = d] (mod m) ==> [a + c = b + d] (mod m)
apply (unfold zcong-def)
apply (rule-tac s = (a - b) + (c - d) in subst)
 apply (rule-tac [2] zdvd-zadd, auto)
done

lemma zcong-zdiff:
   [a = b] (mod m) ==> [c = d] (mod m) ==> [a - c = b - d] (mod m)
apply (unfold zcong-def)
apply (rule-tac s = (a - b) - (c - d) in subst)
 apply (rule-tac [2] zdvd-zdiff, auto)
done

lemma zcong-trans:
   [a = b] (mod m) ==> [b = c] (mod m) ==> [a = c] (mod m)
apply (unfold zcong-def dvd-def, auto)
apply (rule-tac x = k + ka in exI)
apply (simp add: zadd-ac zadd-zmult-distrib2)
done

lemma zcong-zmult:
   [a = b] (mod m) ==> [c = d] (mod m) ==> [a * c = b * d] (mod m)
apply (rule-tac b = b * c in zcong-trans)
 apply (unfold zcong-def)
 apply (rule-tac s = c * (a - b) in subst)
  apply (rule-tac [3] s = b * (c - d) in subst)
   prefer 4
   apply (blast intro: zdvd-zmult)
  prefer 2
  apply (blast intro: zdvd-zmult)
 apply (simp-all add: zdiff-zmult-distrib2 zmult-commute)
done

lemma zcong-scalar: [a = b] (mod m) ==> [a * k = b * k] (mod m)
by (rule zcong-zmult, simp-all)

lemma zcong-scalar2: [a = b] (mod m) ==> [k * a = k * b] (mod m)
by (rule zcong-zmult, simp-all)

lemma zcong-zmult-self: [a * m = b * m] (mod m)
apply (unfold zcong-def)
apply (rule zdvd-zdiff, simp-all)
done

lemma zcong-square:
```

```
  [| zprime p;  0 < a;  [a * a = 1] (mod p)|]
   ==> [a = 1] (mod p) ∨ [a = p - 1] (mod p)
 apply (unfold zcong-def)
 apply (rule zprime-zdvd-zmult)
  apply (rule-tac [3] s = a * a - 1 + p * (1 - a) in subst)
   prefer 4
   apply (simp add: zdvd-reduce)
  apply (simp-all add: zdiff-zmult-distrib zmult-commute zdiff-zmult-distrib2)
 done

lemma zcong-cancel:
 0 ≤ m ==>
   zgcd (k, m) = 1 ==> [a * k = b * k] (mod m) = [a = b] (mod m)
 apply safe
  prefer 2
  apply (blast intro: zcong-scalar)
 apply (case-tac b < a)
  prefer 2
  apply (subst zcong-sym)
  apply (unfold zcong-def)
  apply (rule-tac [!] zrelprime-zdvd-zmult)
   apply (simp-all add: zdiff-zmult-distrib)
 apply (subgoal-tac m dvd (-(a * k - b * k)))
  apply simp
 apply (subst zdvd-zminus-iff, assumption)
 done

lemma zcong-cancel2:
 0 ≤ m ==>
   zgcd (k, m) = 1 ==> [k * a = k * b] (mod m) = [a = b] (mod m)
 by (simp add: zmult-commute zcong-cancel)

lemma zcong-zgcd-zmult-zmod:
 [a = b] (mod m) ==> [a = b] (mod n) ==> zgcd (m, n) = 1
   ==> [a = b] (mod m * n)
 apply (unfold zcong-def dvd-def, auto)
 apply (subgoal-tac m dvd n * ka)
  apply (subgoal-tac m dvd ka)
   apply (case-tac [2] 0 ≤ ka)
    prefer 3
    apply (subst zdvd-zminus-iff [symmetric])
    apply (rule-tac n = n in zrelprime-zdvd-zmult)
     apply (simp add: zgcd-commute)
    apply (simp add: zmult-commute zdvd-zminus-iff)
   prefer 2
   apply (rule-tac n = n in zrelprime-zdvd-zmult)
    apply (simp add: zgcd-commute)
   apply (simp add: zmult-commute)
  apply (auto simp add: dvd-def)
```

done

lemma *zcong-zless-imp-eq*:
0 ≤ a ==>
a < m ==> 0 ≤ b ==> b < m ==> [a = b] (mod m) ==> a = b
apply (*unfold zcong-def dvd-def*, *auto*)
apply (*drule-tac f = λz. z mod m* **in** *arg-cong*)
apply (*cut-tac x = a* **and** *y = b* **in** *linorder-less-linear*, *auto*)
apply (*subgoal-tac [2] (a − b) mod m = a − b*)
apply (*rule-tac [3] mod-pos-pos-trivial*, *auto*)
apply (*subgoal-tac (m + (a − b)) mod m = m + (a − b)*)
apply (*rule-tac [2] mod-pos-pos-trivial*, *auto*)
done

lemma *zcong-square-zless*:
zprime p ==> 0 < a ==> a < p ==>
*[a * a = 1] (mod p) ==> a = 1 ∨ a = p − 1*
apply (*cut-tac p = p* **and** *a = a* **in** *zcong-square*)
apply (*simp add*: *zprime-def*)
apply (*auto intro*: *zcong-zless-imp-eq*)
done

lemma *zcong-not*:
0 < a ==> a < m ==> 0 < b ==> b < a ==> ¬ [a = b] (mod m)
apply (*unfold zcong-def*)
apply (*rule zdvd-not-zless*, *auto*)
done

lemma *zcong-zless-0*:
0 ≤ a ==> a < m ==> [a = 0] (mod m) ==> a = 0
apply (*unfold zcong-def dvd-def*, *auto*)
apply (*subgoal-tac 0 < m*)
apply (*simp add*: *zero-le-mult-iff*)
apply (*subgoal-tac m * k < m * 1*)
apply (*drule mult-less-cancel-left* [*THEN iffD1*])
apply (*auto simp add*: *linorder-neq-iff*)
done

lemma *zcong-zless-unique*:
0 < m ==> (∃!b. 0 ≤ b ∧ b < m ∧ [a = b] (mod m))
apply *auto*
apply (*subgoal-tac [2] [b = y] (mod m)*)
apply (*case-tac [2] b = 0*)
apply (*case-tac [3] y = 0*)
apply (*auto intro*: *zcong-trans zcong-zless-0 zcong-zless-imp-eq order-less-le*
simp add: *zcong-sym*)
apply (*unfold zcong-def dvd-def*)
apply (*rule-tac x = a mod m* **in** *exI*, *auto*)
apply (*rule-tac x = −(a div m)* **in** *exI*)

apply (*simp add*: *diff-eq-eq eq-diff-eq add-commute*)
done

lemma *zcong-iff-lin*: $([a = b]\ (mod\ m)) = (\exists k.\ b = a + m * k)$
apply (*unfold zcong-def dvd-def*, *auto*)
apply (*rule-tac* [!] $x = -k$ **in** *exI*, *auto*)
done

lemma *zgcd-zcong-zgcd*:
$0 < m$ ==>
zgcd $(a, m) = 1$ ==> $[a = b]\ (mod\ m)$ ==> *zgcd* $(b, m) = 1$
by (*auto simp add*: *zcong-iff-lin*)

lemma *zcong-zmod-aux*:
$a - b = (m::int) * (a\ div\ m - b\ div\ m) + (a\ mod\ m - b\ mod\ m)$
by(*simp add*: *zdiff-zmult-distrib2 add-diff-eq eq-diff-eq add-ac*)

lemma *zcong-zmod*: $[a = b]\ (mod\ m) = [a\ mod\ m = b\ mod\ m]\ (mod\ m)$
apply (*unfold zcong-def*)
apply (*rule-tac* $t = a - b$ **in** *ssubst*)
apply (*rule-tac* $m = m$ **in** *zcong-zmod-aux*)
apply (*rule trans*)
apply (*rule-tac* [*2*] $k = m$ **and** $m = a\ div\ m - b\ div\ m$ **in** *zdvd-reduce*)
apply (*simp add*: *zadd-commute*)
done

lemma *zcong-zmod-eq*: $0 < m$ ==> $[a = b]\ (mod\ m) = (a\ mod\ m = b\ mod\ m)$
apply *auto*
apply (*rule-tac* $m = m$ **in** *zcong-zless-imp-eq*)
prefer *5*
apply (*subst zcong-zmod* [*symmetric*], *simp-all*)
apply (*unfold zcong-def dvd-def*)
apply (*rule-tac* $x = a\ div\ m - b\ div\ m$ **in** *exI*)
apply (*rule-tac* $m1 = m$ **in** *zcong-zmod-aux* [*THEN trans*], *auto*)
done

lemma *zcong-zminus* [*iff*]: $[a = b]\ (mod\ -m) = [a = b]\ (mod\ m)$
by (*auto simp add*: *zcong-def*)

lemma *zcong-zero* [*iff*]: $[a = b]\ (mod\ 0) = (a = b)$
by (*auto simp add*: *zcong-def*)

lemma $[a = b]\ (mod\ m) = (a\ mod\ m = b\ mod\ m)$
apply (*case-tac* $m = 0$, *simp add*: *DIVISION-BY-ZERO*)
apply (*simp add*: *linorder-neq-iff*)
apply (*erule disjE*)
prefer *2* **apply** (*simp add*: *zcong-zmod-eq*)

Remainding case: $m < 0$

apply (*rule-tac* $t = m$ **in** *zminus-zminus* [*THEN subst*])
apply (*subst zcong-zminus*)
apply (*subst zcong-zmod-eq*, *arith*)
apply (*frule neg-mod-bound* [*of* - *a*], *frule neg-mod-bound* [*of* - *b*])
apply (*simp add*: *zmod-zminus2-eq-if del*: *neg-mod-bound*)
done

3.4 Modulo

lemma *zmod-zdvd-zmod*:
$0 < (m::int) \Longrightarrow m\ dvd\ b \Longrightarrow (a\ mod\ b\ mod\ m) = (a\ mod\ m)$
apply (*unfold dvd-def*, *auto*)
apply (*subst zcong-zmod-eq* [*symmetric*])
prefer *2*
apply (*subst zcong-iff-lin*)
apply (*rule-tac* $x = k * (a\ div\ (m * k))$ **in** *exI*)
apply (*simp add*:*zmult-assoc* [*symmetric*], *assumption*)
done

3.5 Extended GCD

declare *xzgcda.simps* [*simp del*]

lemma *xzgcd-correct-aux1*:
zgcd $(r', r) = k \longrightarrow 0 < r \longrightarrow$
$(\exists\, sn\ tn.\ xzgcda\ (m, n, r', r, s', s, t', t) = (k, sn, tn))$
apply (*rule-tac* $u = m$ **and** $v = n$ **and** $w = r'$ **and** $x = r$ **and** $y = s'$ **and**
$z = s$ **and** $aa = t'$ **and** $ab = t$ **in** *xzgcda.induct*)
apply (*subst zgcd-eq*)
apply (*subst xzgcda.simps*, *auto*)
apply (*case-tac* $r'\ mod\ r = 0$)
prefer *2*
apply (*frule-tac* $a = r'$ **in** *pos-mod-sign*, *auto*)
apply (*rule exI*)
apply (*rule exI*)
apply (*subst xzgcda.simps*, *auto*)
done

lemma *xzgcd-correct-aux2*:
$(\exists\, sn\ tn.\ xzgcda\ (m, n, r', r, s', s, t', t) = (k, sn, tn)) \longrightarrow 0 < r \longrightarrow$
zgcd $(r', r) = k$
apply (*rule-tac* $u = m$ **and** $v = n$ **and** $w = r'$ **and** $x = r$ **and** $y = s'$ **and**
$z = s$ **and** $aa = t'$ **and** $ab = t$ **in** *xzgcda.induct*)
apply (*subst zgcd-eq*)
apply (*subst xzgcda.simps*)
apply (*auto simp add*: *linorder-not-le*)
apply (*case-tac* $r'\ mod\ r = 0$)
prefer *2*
apply (*frule-tac* $a = r'$ **in** *pos-mod-sign*, *auto*)
apply (*erule-tac* $P = xzgcda\ ?u = ?v$ **in** *rev-mp*)

apply (*subst xzgcda.simps*, *auto*)
done

lemma *xzgcd-correct*:
$0 < n \Longrightarrow (zgcd\ (m,\ n) = k) = (\exists\, s\ t.\ xzgcd\ m\ n = (k,\ s,\ t))$
apply (*unfold xzgcd-def*)
apply (*rule iffI*)
apply (*rule-tac* [*2*] *xzgcd-correct-aux2* [*THEN mp*, *THEN mp*])
apply (*rule xzgcd-correct-aux1* [*THEN mp*, *THEN mp*], *auto*)
done

xzgcd linear

lemma *xzgcda-linear-aux1*:
$(a - r * b) * m + (c - r * d) * (n::int) =$
$(a * m + c * n) - r * (b * m + d * n)$
by (*simp add*: *zdiff-zmult-distrib zadd-zmult-distrib2 zmult-assoc*)

lemma *xzgcda-linear-aux2*:
$r' = s' * m + t' * n \Longrightarrow r = s * m + t * n$
$\Longrightarrow (r'\ mod\ r) = (s' - (r'\ div\ r) * s) * m + (t' - (r'\ div\ r) * t) * (n::int)$
apply (*rule trans*)
apply (*rule-tac* [*2*] *xzgcda-linear-aux1* [*symmetric*])
apply (*simp add*: *eq-diff-eq mult-commute*)
done

lemma *order-le-neq-implies-less*: $(x::'a::order) \leq y \Longrightarrow x \neq y \Longrightarrow x < y$
by (*rule iffD2* [*OF order-less-le conjI*])

lemma *xzgcda-linear* [*rule-format*]:
$0 < r \longrightarrow xzgcda\ (m,\ n,\ r',\ r,\ s',\ s,\ t',\ t) = (rn,\ sn,\ tn) \longrightarrow$
$r' = s' * m + t' * n \longrightarrow\ r = s * m + t * n \longrightarrow rn = sn * m + tn * n$
apply (*rule-tac* $u = m$ **and** $v = n$ **and** $w = r'$ **and** $x = r$ **and** $y = s'$ **and**
$z = s$ **and** $aa = t'$ **and** $ab = t$ **in** *xzgcda.induct*)
apply (*subst xzgcda.simps*)
apply (*simp* (*no-asm*))
apply (*rule impI*)+
apply (*case-tac* $r'\ mod\ r = 0$)
apply (*simp add*: *xzgcda.simps*, *clarify*)
apply (*subgoal-tac* $0 < r'\ mod\ r$)
apply (*rule-tac* [*2*] *order-le-neq-implies-less*)
apply (*rule-tac* [*2*] *pos-mod-sign*)
apply (*cut-tac* $m = m$ **and** $n = n$ **and** $r' = r'$ **and** $r = r$ **and** $s' = s'$ **and**
$s = s$ **and** $t' = t'$ **and** $t = t$ **in** *xzgcda-linear-aux2*, *auto*)
done

lemma *xzgcd-linear*:
$0 < n \Longrightarrow xzgcd\ m\ n = (r,\ s,\ t) \Longrightarrow r = s * m + t * n$
apply (*unfold xzgcd-def*)
apply (*erule xzgcda-linear*, *assumption*, *auto*)

```
  done

lemma zgcd-ex-linear:
    0 < n ==> zgcd (m, n) = k ==> (∃ s t. k = s * m + t * n)
  apply (simp add: xzgcd-correct, safe)
  apply (rule exI)+
  apply (erule xzgcd-linear, auto)
  done

lemma zcong-lineq-ex:
    0 < n ==> zgcd (a, n) = 1 ==> ∃ x. [a * x = 1] (mod n)
  apply (cut-tac m = a and n = n and k = 1 in zgcd-ex-linear, safe)
  apply (rule-tac x = s in exI)
  apply (rule-tac b = s * a + t * n in zcong-trans)
   prefer 2
   apply simp
  apply (unfold zcong-def)
  apply (simp (no-asm) add: zmult-commute zdvd-zminus-iff)
  done

lemma zcong-lineq-unique:
  0 < n ==>
    zgcd (a, n) = 1 ==> ∃!x. 0 ≤ x ∧ x < n ∧ [a * x = b] (mod n)
  apply auto
   apply (rule-tac [2] zcong-zless-imp-eq)
       apply (tactic ⟪ stac (thm zcong-cancel2 RS sym) 6 ⟫)
         apply (rule-tac [8] zcong-trans)
          apply (simp-all (no-asm-simp))
   prefer 2
   apply (simp add: zcong-sym)
  apply (cut-tac a = a and n = n in zcong-lineq-ex, auto)
  apply (rule-tac x = x * b mod n in exI, safe)
    apply (simp-all (no-asm-simp))
  apply (subst zcong-zmod)
  apply (subst zmod-zmult1-eq [symmetric])
  apply (subst zcong-zmod [symmetric])
  apply (subgoal-tac [a * x * b = 1 * b] (mod n))
   apply (rule-tac [2] zcong-zmult)
    apply (simp-all add: zmult-assoc)
  done

end
```

4 The Chinese Remainder Theorem

```
theory Chinese imports IntPrimes begin
```

The Chinese Remainder Theorem for an arbitrary finite number of equa-

tions. (The one-equation case is included in theory *IntPrimes*. Uses functions for indexing.[1]

4.1 Definitions

consts
funprod :: (*nat* => *int*) => *nat* => *nat* => *int*
funsum :: (*nat* => *int*) => *nat* => *nat* => *int*

primrec
funprod f i 0 = *f i*
funprod f i (*Suc n*) = *f* (*Suc* (*i* + *n*)) ∗ *funprod f i n*

primrec
funsum f i 0 = *f i*
funsum f i (*Suc n*) = *f* (*Suc* (*i* + *n*)) + *funsum f i n*

consts
m-cond :: *nat* => (*nat* => *int*) => *bool*
km-cond :: *nat* => (*nat* => *int*) => (*nat* => *int*) => *bool*
lincong-sol ::
nat => (*nat* => *int*) => (*nat* => *int*) => (*nat* => *int*) => *int* => *bool*

mhf :: (*nat* => *int*) => *nat* => *nat* => *int*
xilin-sol ::
nat => *nat* => (*nat* => *int*) => (*nat* => *int*) => (*nat* => *int*) => *int*
x-sol :: *nat* => (*nat* => *int*) => (*nat* => *int*) => (*nat* => *int*) => *int*

defs
m-cond-def:
m-cond n mf ==
(∀ *i*. *i* ≤ *n* −−> 0 < *mf i*) ∧
(∀ *i j*. *i* ≤ *n* ∧ *j* ≤ *n* ∧ *i* ≠ *j* −−> *zgcd* (*mf i*, *mf j*) = *1*)

km-cond-def:
km-cond n kf mf == ∀ *i*. *i* ≤ *n* −−> *zgcd* (*kf i*, *mf i*) = *1*

lincong-sol-def:
lincong-sol n kf bf mf x == ∀ *i*. *i* ≤ *n* −−> *zcong* (*kf i* ∗ *x*) (*bf i*) (*mf i*)

mhf-def:
mhf mf n i ==
if i = *0 then funprod mf* (*Suc 0*) (*n* − *Suc 0*)
else if i = *n then funprod mf 0* (*n* − *Suc 0*)
else funprod mf 0 (*i* − *Suc 0*) ∗ *funprod mf* (*Suc i*) (*n* − *Suc 0* − *i*)

xilin-sol-def:

[1] Maybe *funprod* and *funsum* should be based on general *fold* on indices?

```
  xilin-sol i n kf bf mf ==
    if 0 < n ∧ i ≤ n ∧ m-cond n mf ∧ km-cond n kf mf then
      (SOME x. 0 ≤ x ∧ x < mf i ∧ zcong (kf i * mhf mf n i * x) (bf i) (mf i))
    else 0

x-sol-def:
  x-sol n kf bf mf == funsum (λi. xilin-sol i n kf bf mf * mhf mf n i) 0 n
```

funprod and *funsum*

```
lemma funprod-pos: (∀ i. i ≤ n --> 0 < mf i) ==> 0 < funprod mf 0 n
 apply (induct n)
  apply auto
 apply (simp add: zero-less-mult-iff)
 done

lemma funprod-zgcd [rule-format (no-asm)]:
 (∀ i. k ≤ i ∧ i ≤ k + l --> zgcd (mf i, mf m) = 1) -->
   zgcd (funprod mf k l, mf m) = 1
 apply (induct l)
  apply simp-all
 apply (rule impI)+
 apply (subst zgcd-zmult-cancel)
 apply auto
 done

lemma funprod-zdvd [rule-format]:
   k ≤ i --> i ≤ k + l --> mf i dvd funprod mf k l
 apply (induct l)
  apply auto
   apply (rule-tac [1] zdvd-zmult2)
   apply (rule-tac [2] zdvd-zmult)
   apply (subgoal-tac i = Suc (k + l))
   apply (simp-all (no-asm-simp))
 done

lemma funsum-mod:
   funsum f k l mod m = funsum (λi. (f i) mod m) k l mod m
 apply (induct l)
  apply auto
 apply (rule trans)
  apply (rule zmod-zadd1-eq)
 apply simp
 apply (rule zmod-zadd-right-eq [symmetric])
 done

lemma funsum-zero [rule-format (no-asm)]:
   (∀ i. k ≤ i ∧ i ≤ k + l --> f i = 0) --> (funsum f k l) = 0
 apply (induct l)
  apply auto
```

done

lemma *funsum-oneelem* [*rule-format* (*no-asm*)]:
k ≤ *j* −−> *j* ≤ *k* + *l* −−>
(∀ *i*. *k* ≤ *i* ∧ *i* ≤ *k* + *l* ∧ *i* ≠ *j* −−> *f i* = *0*) −−>
funsum f k l = *f j*
apply (*induct l*)
prefer *2*
apply *clarify*
defer
apply *clarify*
apply (*subgoal-tac k* = *j*)
apply (*simp-all* (*no-asm-simp*))
apply (*case-tac Suc* (*k* + *l*) = *j*)
apply (*subgoal-tac funsum f k l* = *0*)
apply (*rule-tac* [*2*] *funsum-zero*)
apply (*subgoal-tac* [*3*] *f* (*Suc* (*k* + *l*)) = *0*)
apply (*subgoal-tac* [*3*] *j* ≤ *k* + *l*)
prefer *4*
apply *arith*
apply *auto*
done

4.2 Chinese: uniqueness

lemma *zcong-funprod-aux*:
m-cond n mf ==> *km-cond n kf mf*
==> *lincong-sol n kf bf mf x* ==> *lincong-sol n kf bf mf y*
==> [*x* = *y*] (*mod mf n*)
apply (*unfold m-cond-def km-cond-def lincong-sol-def*)
apply (*rule iffD1*)
apply (*rule-tac k* = *kf n* **in** *zcong-cancel2*)
apply (*rule-tac* [*3*] *b* = *bf n* **in** *zcong-trans*)
prefer *4*
apply (*subst zcong-sym*)
defer
apply (*rule order-less-imp-le*)
apply *simp-all*
done

lemma *zcong-funprod* [*rule-format*]:
m-cond n mf −−> *km-cond n kf mf* −−>
lincong-sol n kf bf mf x −−> *lincong-sol n kf bf mf y* −−>
[*x* = *y*] (*mod funprod mf 0 n*)
apply (*induct n*)
apply (*simp-all* (*no-asm*))
apply (*blast intro*: *zcong-funprod-aux*)
apply (*rule impI*)+
apply (*rule zcong-zgcd-zmult-zmod*)

```
  apply (blast intro: zcong-funprod-aux)
  prefer 2
  apply (subst zgcd-commute)
  apply (rule funprod-zgcd)
 apply (auto simp add: m-cond-def km-cond-def lincong-sol-def)
done
```

4.3 Chinese: existence

```
lemma unique-xi-sol:
 0 < n ==> i ≤ n ==> m-cond n mf ==> km-cond n kf mf
  ==> ∃!x. 0 ≤ x ∧ x < mf i ∧ [kf i * mhf mf n i * x = bf i] (mod mf i)
 apply (rule zcong-lineq-unique)
  apply (tactic ⟪ stac (thm zgcd-zmult-cancel) 2 ⟫)
  apply (unfold m-cond-def km-cond-def mhf-def)
  apply (simp-all (no-asm-simp))
 apply safe
  apply (tactic ⟪ stac (thm zgcd-zmult-cancel) 3 ⟫)
   apply (rule-tac [!] funprod-zgcd)
   apply safe
   apply simp-all
 apply (subgoal-tac i<n)
  prefer 2
  apply arith
 apply (case-tac [2] i)
  apply simp-all
done

lemma x-sol-lin-aux:
  0 < n ==> i ≤ n ==> j ≤ n ==> j ≠ i ==> mf j dvd mhf mf n i
 apply (unfold mhf-def)
 apply (case-tac i = 0)
  apply (case-tac [2] i = n)
   apply (simp-all (no-asm-simp))
   apply (case-tac [3] j < i)
    apply (rule-tac [3] zdvd-zmult2)
    apply (rule-tac [4] zdvd-zmult)
    apply (rule-tac [!] funprod-zdvd)
       apply arith+
done

lemma x-sol-lin:
 0 < n ==> i ≤ n
  ==> x-sol n kf bf mf mod mf i =
    xilin-sol i n kf bf mf * mhf mf n i mod mf i
 apply (unfold x-sol-def)
 apply (subst funsum-mod)
 apply (subst funsum-oneelem)
   apply auto
```

```
  apply (subst zdvd-iff-zmod-eq-0 [symmetric])
  apply (rule zdvd-zmult)
  apply (rule x-sol-lin-aux)
  apply auto
  done
```

4.4 Chinese

```
lemma chinese-remainder:
  0 < n ==> m-cond n mf ==> km-cond n kf mf
    ==> ∃!x. 0 ≤ x ∧ x < funprod mf 0 n ∧ lincong-sol n kf bf mf x
  apply safe
   apply (rule-tac [2] m = funprod mf 0 n in zcong-zless-imp-eq)
      apply (rule-tac [6] zcong-funprod)
        apply auto
  apply (rule-tac x = x-sol n kf bf mf mod funprod mf 0 n in exI)
  apply (unfold lincong-sol-def)
  apply safe
    apply (tactic ⟪ stac (thm zcong-zmod) 3 ⟫)
    apply (tactic ⟪ stac (thm zmod-zmult-distrib) 3 ⟫)
    apply (tactic ⟪ stac (thm zmod-zdvd-zmod) 3 ⟫)
      apply (tactic ⟪ stac (thm x-sol-lin) 5 ⟫)
       apply (tactic ⟪ stac (thm zmod-zmult-distrib RS sym) 7 ⟫)
       apply (tactic ⟪ stac (thm zcong-zmod RS sym) 7 ⟫)
       apply (subgoal-tac [7]
         0 ≤ xilin-sol i n kf bf mf ∧ xilin-sol i n kf bf mf < mf i
         ∧ [kf i * mhf mf n i * xilin-sol i n kf bf mf = bf i] (mod mf i))
        prefer 7
        apply (simp add: zmult-ac)
       apply (unfold xilin-sol-def)
       apply (tactic ⟪ Asm-simp-tac 7 ⟫)
       apply (rule-tac [7] ex1-implies-ex [THEN someI-ex])
       apply (rule-tac [7] unique-xi-sol)
          apply (rule-tac [4] funprod-zdvd)
           apply (unfold m-cond-def)
           apply (rule funprod-pos [THEN pos-mod-sign])
           apply (rule-tac [2] funprod-pos [THEN pos-mod-bound])
           apply auto
  done

end
```

5 Bijections between sets

```
theory BijectionRel imports Main begin
```

Inductive definitions of bijections between two different sets and between the same set. Theorem for relating the two definitions.

consts
bijR :: (*'a* => *'b* => *bool*) => (*'a set* * *'b set*) *set*

inductive *bijR P*
intros
empty [*simp*]: ({}, {}) ∈ *bijR P*
insert: *P a b* ==> *a* ∉ *A* ==> *b* ∉ *B* ==> (*A*, *B*) ∈ *bijR P*
==> (*insert a A*, *insert b B*) ∈ *bijR P*

Add extra condition to *insert*: ∀ *b*∈*B*. ¬ *P a b* (and similar for *A*).

constdefs
bijP :: (*'a* => *'a* => *bool*) => *'a set* => *bool*
bijP P F == ∀ *a b*. *a* ∈ *F* ∧ *P a b* --> *b* ∈ *F*

uniqP :: (*'a* => *'a* => *bool*) => *bool*
uniqP P == ∀ *a b c d*. *P a b* ∧ *P c d* --> (*a* = *c*) = (*b* = *d*)

symP :: (*'a* => *'a* => *bool*) => *bool*
symP P == ∀ *a b*. *P a b* = *P b a*

consts
bijER :: (*'a* => *'a* => *bool*) => *'a set set*

inductive *bijER P*
intros
empty [*simp*]: {} ∈ *bijER P*
insert1: *P a a* ==> *a* ∉ *A* ==> *A* ∈ *bijER P* ==> *insert a A* ∈ *bijER P*
insert2: *P a b* ==> *a* ≠ *b* ==> *a* ∉ *A* ==> *b* ∉ *A* ==> *A* ∈ *bijER P*
==> *insert a* (*insert b A*) ∈ *bijER P*

bijR

lemma *fin-bijRl*: (*A*, *B*) ∈ *bijR P* ==> *finite A*
apply (*erule bijR.induct*)
apply *auto*
done

lemma *fin-bijRr*: (*A*, *B*) ∈ *bijR P* ==> *finite B*
apply (*erule bijR.induct*)
apply *auto*
done

lemma *aux-induct*:
finite F ==> *F* ⊆ *A* ==> *P* {} ==>
(!!*F a*. *F* ⊆ *A* ==> *a* ∈ *A* ==> *a* ∉ *F* ==> *P F* ==> *P* (*insert a F*))
==> *P F*
proof −
case *rule-context*
assume *major*: *finite F*
and *subs*: *F* ⊆ *A*

```
  show ?thesis
    apply (rule subs [THEN rev-mp])
    apply (rule major [THEN finite-induct])
     apply (blast intro: rule-context)+
    done
qed

lemma inj-func-bijR-aux1:
    A ⊆ B ==> a ∉ A ==> a ∈ B ==> inj-on f B ==> f a ∉ f ‘ A
  apply (unfold inj-on-def)
  apply auto
  done

lemma inj-func-bijR-aux2:
  ∀ a. a ∈ A −−> P a (f a) ==> inj-on f A ==> finite A ==> F <= A
    ==> (F, f ‘ F) ∈ bijR P
  apply (rule-tac F = F and A = A in aux-induct)
     apply (rule finite-subset)
      apply auto
  apply (rule bijR.insert)
     apply (rule-tac [3] inj-func-bijR-aux1)
       apply auto
  done

lemma inj-func-bijR:
  ∀ a. a ∈ A −−> P a (f a) ==> inj-on f A ==> finite A
    ==> (A, f ‘ A) ∈ bijR P
  apply (rule inj-func-bijR-aux2)
     apply auto
  done
```

bijER

```
lemma fin-bijER: A ∈ bijER P ==> finite A
  apply (erule bijER.induct)
    apply auto
  done

lemma aux1:
  a ∉ A ==> a ∉ B ==> F ⊆ insert a A ==> F ⊆ insert a B ==> a ∈ F
    ==> ∃ C. F = insert a C ∧ a ∉ C ∧ C <= A ∧ C <= B
  apply (rule-tac x = F − {a} in exI)
  apply auto
  done

lemma aux2: a ≠ b ==> a ∉ A ==> b ∉ B ==> a ∈ F ==> b ∈ F
    ==> F ⊆ insert a A ==> F ⊆ insert b B
    ==> ∃ C. F = insert a (insert b C) ∧ a ∉ C ∧ b ∉ C ∧ C ⊆ A ∧ C ⊆ B
  apply (rule-tac x = F − {a, b} in exI)
  apply auto
```

```
  done

lemma aux-uniq: uniqP P ==> P a b ==> P c d ==> (a = c) = (b = d)
  apply (unfold uniqP-def)
  apply auto
  done

lemma aux-sym: symP P ==> P a b = P b a
  apply (unfold symP-def)
  apply auto
  done

lemma aux-in1:
    uniqP P ==> b ∉ C ==> P b b ==> bijP P (insert b C) ==> bijP P C
  apply (unfold bijP-def)
  apply auto
  apply (subgoal-tac b ≠ a)
   prefer 2
   apply clarify
  apply (simp add: aux-uniq)
  apply auto
  done

lemma aux-in2:
  symP P ==> uniqP P ==> a ∉ C ==> b ∉ C ==> a ≠ b ==> P a b
    ==> bijP P (insert a (insert b C)) ==> bijP P C
  apply (unfold bijP-def)
  apply auto
  apply (subgoal-tac aa ≠ a)
   prefer 2
   apply clarify
  apply (subgoal-tac aa ≠ b)
   prefer 2
   apply clarify
  apply (simp add: aux-uniq)
  apply (subgoal-tac ba ≠ a)
   apply auto
  apply (subgoal-tac P a aa)
   prefer 2
   apply (simp add: aux-sym)
  apply (subgoal-tac b = aa)
   apply (rule-tac [2] iffD1)
    apply (rule-tac [2] a = a and c = a and P = P in aux-uniq)
      apply auto
  done

lemma aux-foo: ∀ a b. Q a ∧ P a b --> R b ==> P a b ==> Q a ==> R b
  apply auto
  done
```

lemma *aux-bij*: *bijP P F* ==> *symP P* ==> *P a b* ==> $(a \in F) = (b \in F)$
apply (*unfold bijP-def*)
apply (*rule iffI*)
apply (*erule-tac* [!] *aux-foo*)
apply *simp-all*
apply (*rule iffD2*)
apply (*rule-tac* $P = P$ **in** *aux-sym*)
apply *simp-all*
done

lemma *aux-bijRER*:
$(A, B) \in$ *bijR P* ==> *uniqP P* ==> *symP P*
==> $\forall F.$ *bijP P F* $\wedge$ $F \subseteq A \wedge F \subseteq B$ --> $F \in$ *bijER P*
apply (*erule bijR.induct*)
apply *simp*
apply (*case-tac* $a = b$)
apply *clarify*
apply (*case-tac* $b \in F$)
prefer *2*
apply (*simp add*: *subset-insert*)
apply (*cut-tac* $F = F$ **and** $a = b$ **and** $A = A$ **and** $B = B$ **in** *aux1*)
prefer *6*
apply *clarify*
apply (*rule bijER.insert1*)
apply *simp-all*
apply (*subgoal-tac bijP P C*)
apply *simp*
apply (*rule aux-in1*)
apply *simp-all*
apply *clarify*
apply (*case-tac* $a \in F$)
apply (*case-tac* [!] $b \in F$)
apply (*cut-tac* $F = F$ **and** $a = a$ **and** $b = b$ **and** $A = A$ **and** $B = B$
in *aux2*)
apply (*simp-all add*: *subset-insert*)
apply *clarify*
apply (*rule bijER.insert2*)
apply *simp-all*
apply (*subgoal-tac bijP P C*)
apply *simp*
apply (*rule aux-in2*)
apply *simp-all*
apply (*subgoal-tac* $b \in F$)
apply (*rule-tac* [*2*] *iffD1*)
apply (*rule-tac* [*2*] $a = a$ **and** $F = F$ **and** $P = P$ **in** *aux-bij*)
apply (*simp-all* (*no-asm-simp*))
apply (*subgoal-tac* [*2*] $a \in F$)

```
    apply (rule-tac [3] iffD2)
     apply (rule-tac [3] b = b and F = F and P = P in aux-bij)
      apply auto
  done

lemma bijR-bijER:
  (A, A) ∈ bijR P ==>
    bijP P A ==> uniqP P ==> symP P ==> A ∈ bijER P
  apply (cut-tac A = A and B = A and P = P in aux-bijRER)
    apply auto
  done

end
```

6 Factorial on integers

theory *IntFact* **imports** *IntPrimes* **begin**

Factorial on integers and recursively defined set including all Integers from *2* up to *a*. Plus definition of product of finite set.

```
consts
  zfact :: int => int
  d22set :: int => int set

recdef zfact  measure ((λn. nat n) :: int => nat)
  zfact n = (if n ≤ 0 then 1 else n * zfact (n - 1))

recdef d22set  measure ((λa. nat a) :: int => nat)
  d22set a = (if 1 < a then insert a (d22set (a - 1)) else {})
```

d22set — recursively defined set including all integers from *2* up to *a*

```
declare d22set.simps [simp del]


lemma d22set-induct:
  (!!a. P {} a) ==>
    (!!a. 1 < (a::int) ==> P (d22set (a - 1)) (a - 1)
      ==> P (d22set a) a)
    ==> P (d22set u) u
proof -
  case rule-context
  show ?thesis
    apply (rule d22set.induct)
    apply safe
     apply (case-tac [2] 1 < a)
      apply (rule-tac [2] rule-context)
```

```
    apply (simp-all (no-asm-simp))
   apply (simp-all (no-asm-simp) add: d22set.simps rule-context)
  done
qed

lemma d22set-g-1 [rule-format]: b ∈ d22set a −−> 1 < b
 apply (induct a rule: d22set-induct)
  prefer 2
  apply (subst d22set.simps)
  apply auto
 done

lemma d22set-le [rule-format]: b ∈ d22set a −−> b ≤ a
 apply (induct a rule: d22set-induct)
  prefer 2
  apply (subst d22set.simps)
  apply auto
 done

lemma d22set-le-swap: a < b ==> b ∉ d22set a
 apply (auto dest: d22set-le)
 done

lemma d22set-mem [rule-format]: 1 < b −−> b ≤ a −−> b ∈ d22set a
 apply (induct a rule: d22set.induct)
 apply auto
  apply (simp-all add: d22set.simps)
 done

lemma d22set-fin: finite (d22set a)
 apply (induct a rule: d22set-induct)
  prefer 2
  apply (subst d22set.simps)
  apply auto
 done


declare zfact.simps [simp del]

lemma d22set-prod-zfact: ∏(d22set a) = zfact a
 apply (induct a rule: d22set.induct)
 apply safe
  apply (simp add: d22set.simps zfact.simps)
 apply (subst d22set.simps)
 apply (subst zfact.simps)
 apply (case-tac 1 < a)
  prefer 2
  apply (simp add: d22set.simps zfact.simps)
 apply (simp add: d22set-fin d22set-le-swap)
```

done

end

7 Fermat's Little Theorem extended to Euler's Totient function

theory *EulerFermat* **imports** *BijectionRel IntFact* **begin**

Fermat's Little Theorem extended to Euler's Totient function. More abstract approach than Boyer-Moore (which seems necessary to achieve the extended version).

7.1 Definitions and lemmas

consts
RsetR :: *int => int set set*
BnorRset :: *int * int => int set*
norRRset :: *int => int set*
noXRRset :: *int => int => int set*
phi :: *int => nat*
is-RRset :: *int set => int => bool*
RRset2norRR :: *int set => int => int => int*

inductive *RsetR m*
intros
empty [*simp*]: {} ∈ *RsetR m*
insert: *A* ∈ *RsetR m* ==> *zgcd* (*a*, *m*) = *1* ==>
∀ *a'*. *a'* ∈ *A* --> ¬ *zcong a a' m* ==> *insert a A* ∈ *RsetR m*

recdef *BnorRset*
measure ((λ(*a*, *m*). *nat a*) :: *int * int => nat*)
BnorRset (*a*, *m*) =
(*if 0 < a then*
let na = BnorRset (*a − 1*, *m*)
in (*if zgcd* (*a*, *m*) = *1 then insert a na else na*)
else {})

defs
norRRset-def: *norRRset m == BnorRset* (*m − 1*, *m*)
noXRRset-def: *noXRRset m x* == (λ*a*. *a * x*) ‘ *norRRset m*
phi-def: *phi m == card* (*norRRset m*)
is-RRset-def: *is-RRset A m == A* ∈ *RsetR m* ∧ *card A = phi m*
RRset2norRR-def:
RRset2norRR A m a ==
(*if 1 < m* ∧ *is-RRset A m* ∧ *a* ∈ *A then*
SOME b. *zcong a b m* ∧ *b* ∈ *norRRset m*

else 0)

constdefs
zcongm :: *int* => *int* => *int* => *bool*
zcongm m == λ*a b*. *zcong a b m*

lemma *abs-eq-1-iff* [*iff*]: (*abs z* = (*1*::*int*)) = (*z* = *1* ∨ *z* = −*1*)
— LCP: not sure why this lemma is needed now
by (*auto simp add*: *abs-if*)

norRRset

declare *BnorRset.simps* [*simp del*]

lemma *BnorRset-induct*:
(!!*a m*. *P* {} *a m*) ==>
(!!*a m*. *0* < (*a*::*int*) ==> *P* (*BnorRset* (*a* − *1*, *m*::*int*)) (*a* − *1*) *m*
==> *P* (*BnorRset*(*a*,*m*)) *a m*)
==> *P* (*BnorRset*(*u*,*v*)) *u v*
proof −
case *rule-context*
show *?thesis*
apply (*rule BnorRset.induct*, *safe*)
apply (*case-tac* [*2*] *0* < *a*)
apply (*rule-tac* [*2*] *rule-context*, *simp-all*)
apply (*simp-all add*: *BnorRset.simps rule-context*)
done
qed

lemma *Bnor-mem-zle* [*rule-format*]: *b* ∈ *BnorRset* (*a*, *m*) −−> *b* ≤ *a*
apply (*induct a m rule*: *BnorRset-induct*)
prefer *2*
apply (*subst BnorRset.simps*)
apply (*unfold Let-def*, *auto*)
done

lemma *Bnor-mem-zle-swap*: *a* < *b* ==> *b* ∉ *BnorRset* (*a*, *m*)
by (*auto dest*: *Bnor-mem-zle*)

lemma *Bnor-mem-zg* [*rule-format*]: *b* ∈ *BnorRset* (*a*, *m*) −−> *0* < *b*
apply (*induct a m rule*: *BnorRset-induct*)
prefer *2*
apply (*subst BnorRset.simps*)
apply (*unfold Let-def*, *auto*)
done

lemma *Bnor-mem-if* [*rule-format*]:
zgcd (*b*, *m*) = *1* −−> *0* < *b* −−> *b* ≤ *a* −−> *b* ∈ *BnorRset* (*a*, *m*)
apply (*induct a m rule*: *BnorRset.induct*, *auto*)
apply (*subst BnorRset.simps*)

```
 defer
 apply (subst BnorRset.simps)
 apply (unfold Let-def, auto)
done

lemma Bnor-in-RsetR [rule-format]: a < m --> BnorRset (a, m) ∈ RsetR m
 apply (induct a m rule: BnorRset-induct, simp)
 apply (subst BnorRset.simps)
 apply (unfold Let-def, auto)
 apply (rule RsetR.insert)
  apply (rule-tac [3] allI)
  apply (rule-tac [3] impI)
  apply (rule-tac [3] zcong-not)
    apply (subgoal-tac [6] a' ≤ a - 1)
     apply (rule-tac [7] Bnor-mem-zle)
     apply (rule-tac [5] Bnor-mem-zg, auto)
 done

lemma Bnor-fin: finite (BnorRset (a, m))
 apply (induct a m rule: BnorRset-induct)
  prefer 2
  apply (subst BnorRset.simps)
  apply (unfold Let-def, auto)
 done

lemma norR-mem-unique-aux: a ≤ b - 1 ==> a < (b::int)
 apply auto
 done

lemma norR-mem-unique:
 1 < m ==>
  zgcd (a, m) = 1 ==> ∃!b. [a = b] (mod m) ∧ b ∈ norRRset m
 apply (unfold norRRset-def)
 apply (cut-tac a = a and m = m in zcong-zless-unique, auto)
  apply (rule-tac [2] m = m in zcong-zless-imp-eq)
    apply (auto intro: Bnor-mem-zle Bnor-mem-zg zcong-trans
      order-less-imp-le norR-mem-unique-aux simp add: zcong-sym)
 apply (rule-tac x = b in exI, safe)
 apply (rule Bnor-mem-if)
  apply (case-tac [2] b = 0)
   apply (auto intro: order-less-le [THEN iffD2])
  prefer 2
  apply (simp only: zcong-def)
  apply (subgoal-tac zgcd (a, m) = m)
   prefer 2
   apply (subst zdvd-iff-zgcd [symmetric])
    apply (rule-tac [4] zgcd-zcong-zgcd)
     apply (simp-all add: zdvd-zminus-iff zcong-sym)
 done
```

noXRRset

lemma *RRset-gcd* [*rule-format*]:
is-RRset A m ==> a ∈ A --> zgcd (a, m) = 1
apply (*unfold is-RRset-def*)
apply (*rule RsetR.induct*, *auto*)
done

lemma *RsetR-zmult-mono*:
A ∈ RsetR m ==>
*0 < m ==> zgcd (x, m) = 1 ==> (λa. a * x) ‘ A ∈ RsetR m*
apply (*erule RsetR.induct*, *simp-all*)
apply (*rule RsetR.insert*, *auto*)
apply (*blast intro*: *zgcd-zgcd-zmult*)
apply (*simp add*: *zcong-cancel*)
done

lemma *card-nor-eq-noX*:
0 < m ==>
zgcd (x, m) = 1 ==> card (noXRRset m x) = card (norRRset m)
apply (*unfold norRRset-def noXRRset-def*)
apply (*rule card-image*)
apply (*auto simp add*: *inj-on-def Bnor-fin*)
apply (*simp add*: *BnorRset.simps*)
done

lemma *noX-is-RRset*:
0 < m ==> zgcd (x, m) = 1 ==> is-RRset (noXRRset m x) m
apply (*unfold is-RRset-def phi-def*)
apply (*auto simp add*: *card-nor-eq-noX*)
apply (*unfold noXRRset-def norRRset-def*)
apply (*rule RsetR-zmult-mono*)
apply (*rule Bnor-in-RsetR*, *simp-all*)
done

lemma *aux-some*:
1 < m ==> is-RRset A m ==> a ∈ A
==> zcong a (SOME b. [a = b] (mod m) ∧ b ∈ norRRset m) m ∧
(SOME b. [a = b] (mod m) ∧ b ∈ norRRset m) ∈ norRRset m
apply (*rule norR-mem-unique* [*THEN ex1-implies-ex*, *THEN someI-ex*])
apply (*rule-tac* [*2*] *RRset-gcd*, *simp-all*)
done

lemma *RRset2norRR-correct*:
1 < m ==> is-RRset A m ==> a ∈ A ==>
[a = RRset2norRR A m a] (mod m) ∧ RRset2norRR A m a ∈ norRRset m
apply (*unfold RRset2norRR-def*, *simp*)
apply (*rule aux-some*, *simp-all*)
done

lemmas *RRset2norRR-correct1* =
RRset2norRR-correct [*THEN conjunct1*, *standard*]
lemmas *RRset2norRR-correct2* =
RRset2norRR-correct [*THEN conjunct2*, *standard*]

lemma *RsetR-fin*: $A \in RsetR\ m \Longrightarrow finite\ A$
by (*erule RsetR.induct*, *auto*)

lemma *RRset-zcong-eq* [*rule-format*]:
$1 < m \Longrightarrow$
is-RRset A m $\Longrightarrow [a = b]\ (mod\ m) \Longrightarrow a \in A \longrightarrow b \in A \longrightarrow a = b$
apply (*unfold is-RRset-def*)
apply (*rule RsetR.induct*)
apply (*auto simp add*: *zcong-sym*)
done

lemma *aux*:
$P\ (SOME\ a.\ P\ a) \Longrightarrow Q\ (SOME\ a.\ Q\ a) \Longrightarrow$
$(SOME\ a.\ P\ a) = (SOME\ a.\ Q\ a) \Longrightarrow \exists a.\ P\ a \wedge Q\ a$
apply *auto*
done

lemma *RRset2norRR-inj*:
$1 < m \Longrightarrow$ *is-RRset A m* $\Longrightarrow$ *inj-on* (*RRset2norRR A m*) *A*
apply (*unfold RRset2norRR-def inj-on-def*, *auto*)
apply (*subgoal-tac* $\exists b.\ ([x = b]\ (mod\ m) \wedge b \in norRRset\ m) \wedge$
$([y = b]\ (mod\ m) \wedge b \in norRRset\ m)$)
apply (*rule-tac* [*2*] *aux*)
apply (*rule-tac* [*3*] *aux-some*)
apply (*rule-tac* [*2*] *aux-some*)
apply (*rule RRset-zcong-eq*, *auto*)
apply (*rule-tac* $b = b$ **in** *zcong-trans*)
apply (*simp-all add*: *zcong-sym*)
done

lemma *RRset2norRR-eq-norR*:
$1 < m \Longrightarrow$ *is-RRset A m* $\Longrightarrow$ *RRset2norRR A m* ‘ *A* = *norRRset m*
apply (*rule card-seteq*)
prefer *3*
apply (*subst card-image*)
apply (*rule-tac RRset2norRR-inj*, *auto*)
apply (*rule-tac* [*3*] *RRset2norRR-correct2*, *auto*)
apply (*unfold is-RRset-def phi-def norRRset-def*)
apply (*auto simp add*: *Bnor-fin*)
done

lemma *Bnor-prod-power-aux*: $a \notin A \Longrightarrow inj\ f \Longrightarrow f\ a \notin f$ ‘ *A*
by (*unfold inj-on-def*, *auto*)

lemma *Bnor-prod-power* [*rule-format*]:
x ≠ *0* ==> *a* < *m* --> ∏((λ*a*. *a* * *x*) ' *BnorRset* (*a*, *m*)) =
∏(*BnorRset*(*a*, *m*)) * *x*^*card* (*BnorRset* (*a*, *m*))
apply (*induct a m rule*: *BnorRset-induct*)
prefer *2*
apply (*simplesubst BnorRset.simps*) — multiple redexes
apply (*unfold Let-def*, *auto*)
apply (*simp add*: *Bnor-fin Bnor-mem-zle-swap*)
apply (*subst setprod-insert*)
apply (*rule-tac* [*2*] *Bnor-prod-power-aux*)
apply (*unfold inj-on-def*)
apply (*simp-all add*: *zmult-ac Bnor-fin finite-imageI*
Bnor-mem-zle-swap)
done

7.2 Fermat

lemma *bijzcong-zcong-prod*:
(*A*, *B*) ∈ *bijR* (*zcongm m*) ==> [∏*A* = ∏*B*] (*mod m*)
apply (*unfold zcongm-def*)
apply (*erule bijR.induct*)
apply (*subgoal-tac* [*2*] *a* ∉ *A* ∧ *b* ∉ *B* ∧ *finite A* ∧ *finite B*)
apply (*auto intro*: *fin-bijRl fin-bijRr zcong-zmult*)
done

lemma *Bnor-prod-zgcd* [*rule-format*]:
a < *m* --> *zgcd* (∏(*BnorRset*(*a*, *m*)), *m*) = *1*
apply (*induct a m rule*: *BnorRset-induct*)
prefer *2*
apply (*subst BnorRset.simps*)
apply (*unfold Let-def*, *auto*)
apply (*simp add*: *Bnor-fin Bnor-mem-zle-swap*)
apply (*blast intro*: *zgcd-zgcd-zmult*)
done

theorem *Euler-Fermat*:
0 < *m* ==> *zgcd* (*x*, *m*) = *1* ==> [*x*^(*phi m*) = *1*] (*mod m*)
apply (*unfold norRRset-def phi-def*)
apply (*case-tac x* = *0*)
apply (*case-tac* [*2*] *m* = *1*)
apply (*rule-tac* [*3*] *iffD1*)
apply (*rule-tac* [*3*] *k* = ∏(*BnorRset*(*m* − *1*, *m*))
in *zcong-cancel2*)
prefer *5*
apply (*subst Bnor-prod-power* [*symmetric*])
apply (*rule-tac* [*7*] *Bnor-prod-zgcd*, *simp-all*)
apply (*rule bijzcong-zcong-prod*)
apply (*fold norRRset-def noXRRset-def*)

```
  apply (subst RRset2norRR-eq-norR [symmetric])
   apply (rule-tac [3] inj-func-bijR, auto)
    apply (unfold zcongm-def)
    apply (rule-tac [2] RRset2norRR-correct1)
     apply (rule-tac [5] RRset2norRR-inj)
      apply (auto intro: order-less-le [THEN iffD2]
        simp add: noX-is-RRset)
  apply (unfold noXRRset-def norRRset-def)
  apply (rule finite-imageI)
  apply (rule Bnor-fin)
  done

lemma Bnor-prime:
  ⟦ zprime p; a < p ⟧ ⟹ card (BnorRset (a, p)) = nat a
  apply (induct a p rule: BnorRset.induct)
  apply (subst BnorRset.simps)
  apply (unfold Let-def, auto simp add:zless-zprime-imp-zrelprime)
  apply (subgoal-tac finite (BnorRset (a - 1,m)))
   apply (subgoal-tac a ~: BnorRset (a - 1,m))
    apply (auto simp add: card-insert-disjoint Suc-nat-eq-nat-zadd1)
   apply (frule Bnor-mem-zle, arith)
  apply (frule Bnor-fin)
  done

lemma phi-prime: zprime p ==> phi p = nat (p - 1)
  apply (unfold phi-def norRRset-def)
  apply (rule Bnor-prime, auto)
  done

theorem Little-Fermat:
    zprime p ==> ¬ p dvd x ==> [x^(nat (p - 1)) = 1] (mod p)
  apply (subst phi-prime [symmetric])
   apply (rule-tac [2] Euler-Fermat)
    apply (erule-tac [3] zprime-imp-zrelprime)
    apply (unfold zprime-def, auto)
  done

end
```

8 Wilson's Theorem according to Russinoff

theory *WilsonRuss* **imports** *EulerFermat* **begin**

Wilson's Theorem following quite closely Russinoff's approach using Boyer-Moore (using finite sets instead of lists, though).

8.1 Definitions and lemmas

consts
inv :: *int* => *int* => *int*
wset :: *int* * *int* => *int set*

defs
inv-def: *inv p a* == (*a*^(*nat* (*p* − *2*))) *mod p*

recdef *wset*
measure ((λ(*a*, *p*). *nat a*) :: *int* * *int* => *nat*)
wset (*a*, *p*) =
(*if* *1* < *a* *then*
let *ws* = *wset* (*a* − *1*, *p*)
in (*if* *a* ∈ *ws* *then* *ws* *else* *insert* *a* (*insert* (*inv p a*) *ws*)) *else* {})

inv

lemma *inv-is-inv-aux*: *1* < *m* ==> *Suc* (*nat* (*m* − *2*)) = *nat* (*m* − *1*)
by (*subst int-int-eq* [*symmetric*], *auto*)

lemma *inv-is-inv*:
zprime p ⟹ *0* < *a* ⟹ *a* < *p* ==> [*a* * *inv p a* = *1*] (*mod p*)
apply (*unfold inv-def*)
apply (*subst zcong-zmod*)
apply (*subst zmod-zmult1-eq* [*symmetric*])
apply (*subst zcong-zmod* [*symmetric*])
apply (*subst power-Suc* [*symmetric*])
apply (*subst inv-is-inv-aux*)
apply (*erule-tac* [*2*] *Little-Fermat*)
apply (*erule-tac* [*2*] *zdvd-not-zless*)
apply (*unfold zprime-def*, *auto*)
done

lemma *inv-distinct*:
zprime p ⟹ *1* < *a* ⟹ *a* < *p* − *1* ==> *a* ≠ *inv p a*
apply *safe*
apply (*cut-tac a* = *a* **and** *p* = *p* **in** *zcong-square*)
apply (*cut-tac* [*3*] *a* = *a* **and** *p* = *p* **in** *inv-is-inv*, *auto*)
apply (*subgoal-tac a* = *1*)
apply (*rule-tac* [*2*] *m* = *p* **in** *zcong-zless-imp-eq*)
apply (*subgoal-tac* [*7*] *a* = *p* − *1*)
apply (*rule-tac* [*8*] *m* = *p* **in** *zcong-zless-imp-eq*, *auto*)
done

lemma *inv-not-0*:
zprime p ⟹ *1* < *a* ⟹ *a* < *p* − *1* ==> *inv p a* ≠ *0*
apply *safe*
apply (*cut-tac a* = *a* **and** *p* = *p* **in** *inv-is-inv*)
apply (*unfold zcong-def*, *auto*)

apply (*subgoal-tac* ¬ *p dvd 1*)
apply (*rule-tac* [*2*] *zdvd-not-zless*)
apply (*subgoal-tac p dvd 1*)
prefer *2*
apply (*subst zdvd-zminus-iff* [*symmetric*], *auto*)
done

lemma *inv-not-1*:
zprime p ⟹ *1 < a* ⟹ *a < p − 1* ==> *inv p a ≠ 1*
apply *safe*
apply (*cut-tac a = a* **and** *p = p* **in** *inv-is-inv*)
prefer *4*
apply *simp*
apply (*subgoal-tac a = 1*)
apply (*rule-tac* [*2*] *zcong-zless-imp-eq*, *auto*)
done

lemma *inv-not-p-minus-1-aux*: [*a ∗* (*p − 1*) *= 1*] (*mod p*) *=* [*a = p − 1*] (*mod p*)
apply (*unfold zcong-def*)
apply (*simp add*: *OrderedGroup.diff-diff-eq diff-diff-eq2 zdiff-zmult-distrib2*)
apply (*rule-tac s = p dvd* −((*a + 1*) + (*p ∗* −*a*)) **in** *trans*)
apply (*simp add*: *mult-commute*)
apply (*subst zdvd-zminus-iff*)
apply (*subst zdvd-reduce*)
apply (*rule-tac s = p dvd* (*a + 1*) + (*p ∗ −1*) **in** *trans*)
apply (*subst zdvd-reduce*, *auto*)
done

lemma *inv-not-p-minus-1*:
zprime p ⟹ *1 < a* ⟹ *a < p − 1* ==> *inv p a ≠ p − 1*
apply *safe*
apply (*cut-tac a = a* **and** *p = p* **in** *inv-is-inv*, *auto*)
apply (*simp add*: *inv-not-p-minus-1-aux*)
apply (*subgoal-tac a = p − 1*)
apply (*rule-tac* [*2*] *zcong-zless-imp-eq*, *auto*)
done

lemma *inv-g-1*:
zprime p ⟹ *1 < a* ⟹ *a < p − 1* ==> *1 < inv p a*
apply (*case-tac 0≤ inv p a*)
apply (*subgoal-tac inv p a ≠ 1*)
apply (*subgoal-tac inv p a ≠ 0*)
apply (*subst order-less-le*)
apply (*subst zle-add1-eq-le* [*symmetric*])
apply (*subst order-less-le*)
apply (*rule-tac* [*2*] *inv-not-0*)
apply (*rule-tac* [*5*] *inv-not-1*, *auto*)
apply (*unfold inv-def zprime-def*, *simp*)

done

lemma *inv-less-p-minus-1*:
zprime p ⟹ 1 < a ⟹ a < p − 1 ==> inv p a < p − 1
apply (*case-tac inv p a < p*)
apply (*subst order-less-le*)
apply (*simp add*: *inv-not-p-minus-1*, *auto*)
apply (*unfold inv-def zprime-def*, *simp*)
done

lemma *inv-inv-aux*: *5 ≤ p ==>*
*nat (p − 2) * nat (p − 2) = Suc (nat (p − 1) * nat (p − 3))*
apply (*subst int-int-eq* [*symmetric*])
apply (*simp add*: *zmult-int* [*symmetric*])
apply (*simp add*: *zdiff-zmult-distrib zdiff-zmult-distrib2*)
done

lemma *zcong-zpower-zmult*:
*[x^y = 1] (mod p) ⟹ [x^(y * z) = 1] (mod p)*
apply (*induct z*)
apply (*auto simp add*: *zpower-zadd-distrib*)
apply (*subgoal-tac zcong (x^y * x^(y * z)) (1 * 1) p*)
apply (*rule-tac* [*2*] *zcong-zmult*, *simp-all*)
done

lemma *inv-inv*: *zprime p ⟹*
5 ≤ p ⟹ 0 < a ⟹ a < p ==> inv p (inv p a) = a
apply (*unfold inv-def*)
apply (*subst zpower-zmod*)
apply (*subst zpower-zpower*)
apply (*rule zcong-zless-imp-eq*)
prefer *5*
apply (*subst zcong-zmod*)
apply (*subst mod-mod-trivial*)
apply (*subst zcong-zmod* [*symmetric*])
apply (*subst inv-inv-aux*)
apply (*subgoal-tac* [*2*]
*zcong (a * a^(nat (p − 1) * nat (p − 3))) (a * 1) p*)
apply (*rule-tac* [*3*] *zcong-zmult*)
apply (*rule-tac* [*4*] *zcong-zpower-zmult*)
apply (*erule-tac* [*4*] *Little-Fermat*)
apply (*rule-tac* [*4*] *zdvd-not-zless*, *simp-all*)
done

wset

declare *wset.simps* [*simp del*]

lemma *wset-induct*:
(!!a p. P {} a p) ⟹

```
    (!!a p. 1 < (a::int) ⟹ P (wset (a - 1, p)) (a - 1) p
      ==> P (wset (a, p)) a p)
    ==> P (wset (u, v)) u v
proof -
  case rule-context
  show ?thesis
    apply (rule wset.induct, safe)
     apply (case-tac [2] 1 < a)
      apply (rule-tac [2] rule-context, simp-all)
      apply (simp-all add: wset.simps rule-context)
    done
qed

lemma wset-mem-imp-or [rule-format]:
  1 < a ⟹ b ∉ wset (a - 1, p)
    ==> b ∈ wset (a, p) --> b = a ∨ b = inv p a
  apply (subst wset.simps)
  apply (unfold Let-def, simp)
  done

lemma wset-mem-mem [simp]: 1 < a ==> a ∈ wset (a, p)
  apply (subst wset.simps)
  apply (unfold Let-def, simp)
  done

lemma wset-subset: 1 < a ⟹ b ∈ wset (a - 1, p) ==> b ∈ wset (a, p)
  apply (subst wset.simps)
  apply (unfold Let-def, auto)
  done

lemma wset-g-1 [rule-format]:
    zprime p --> a < p - 1 --> b ∈ wset (a, p) --> 1 < b
  apply (induct a p rule: wset-induct, auto)
  apply (case-tac b = a)
   apply (case-tac [2] b = inv p a)
    apply (subgoal-tac [3] b = a ∨ b = inv p a)
     apply (rule-tac [4] wset-mem-imp-or)
       prefer 2
       apply simp
       apply (rule inv-g-1, auto)
  done

lemma wset-less [rule-format]:
    zprime p --> a < p - 1 --> b ∈ wset (a, p) --> b < p - 1
  apply (induct a p rule: wset-induct, auto)
  apply (case-tac b = a)
   apply (case-tac [2] b = inv p a)
    apply (subgoal-tac [3] b = a ∨ b = inv p a)
     apply (rule-tac [4] wset-mem-imp-or)
```

```
    prefer 2
    apply simp
    apply (rule inv-less-p-minus-1, auto)
  done

lemma wset-mem [rule-format]:
  zprime p −−>
    a < p − 1 −−> 1 < b −−> b ≤ a −−> b ∈ wset (a, p)
  apply (induct a p rule: wset.induct, auto)
  apply (rule-tac wset-subset)
  apply (simp (no-asm-simp))
  apply auto
  done

lemma wset-mem-inv-mem [rule-format]:
  zprime p −−> 5 ≤ p −−> a < p − 1 −−> b ∈ wset (a, p)
    −−> inv p b ∈ wset (a, p)
  apply (induct a p rule: wset-induct, auto)
   apply (case-tac b = a)
    apply (subst wset.simps)
    apply (unfold Let-def)
    apply (rule-tac [3] wset-subset, auto)
  apply (case-tac b = inv p a)
   apply (simp (no-asm-simp))
   apply (subst inv-inv)
      apply (subgoal-tac [6] b = a ∨ b = inv p a)
       apply (rule-tac [7] wset-mem-imp-or, auto)
  done

lemma wset-inv-mem-mem:
  zprime p ⟹ 5 ≤ p ⟹ a < p − 1 ⟹ 1 < b ⟹ b < p − 1
    ⟹ inv p b ∈ wset (a, p) ⟹ b ∈ wset (a, p)
  apply (rule-tac s = inv p (inv p b) and t = b in subst)
   apply (rule-tac [2] wset-mem-inv-mem)
     apply (rule inv-inv, simp-all)
  done

lemma wset-fin: finite (wset (a, p))
  apply (induct a p rule: wset-induct)
   prefer 2
   apply (subst wset.simps)
   apply (unfold Let-def, auto)
  done

lemma wset-zcong-prod-1 [rule-format]:
  zprime p −−>
    5 ≤ p −−> a < p − 1 −−> [(∏ x∈wset(a, p). x) = 1] (mod p)
  apply (induct a p rule: wset-induct)
   prefer 2
```

```
 apply (subst wset.simps)
 apply (unfold Let-def, auto)
apply (subst setprod-insert)
 apply (tactic ⟪ stac (thm setprod-insert) 3 ⟫)
  apply (subgoal-tac [5]
    zcong (a * inv p a * (∏x∈ wset(a - 1, p). x)) (1 * 1) p)
   prefer 5
   apply (simp add: zmult-assoc)
  apply (rule-tac [5] zcong-zmult)
   apply (rule-tac [5] inv-is-inv)
    apply (tactic Clarify-tac 4)
    apply (subgoal-tac [4] a ∈ wset (a - 1, p))
     apply (rule-tac [5] wset-inv-mem-mem)
         apply (simp-all add: wset-fin)
apply (rule inv-distinct, auto)
done

lemma d22set-eq-wset: zprime p ==> d22set (p - 2) = wset (p - 2, p)
 apply safe
  apply (erule wset-mem)
   apply (rule-tac [2] d22set-g-1)
   apply (rule-tac [3] d22set-le)
   apply (rule-tac [4] d22set-mem)
    apply (erule-tac [4] wset-g-1)
     prefer 6
     apply (subst zle-add1-eq-le [symmetric])
     apply (subgoal-tac p - 2 + 1 = p - 1)
      apply (simp (no-asm-simp))
      apply (erule wset-less, auto)
done
```

8.2 Wilson

```
lemma prime-g-5: zprime p ⟹ p ≠ 2 ⟹ p ≠ 3 ==> 5 ≤ p
 apply (unfold zprime-def dvd-def)
 apply (case-tac p = 4, auto)
  apply (rule notE)
   prefer 2
   apply assumption
  apply (simp (no-asm))
  apply (rule-tac x = 2 in exI)
  apply (safe, arith)
   apply (rule-tac x = 2 in exI, auto)
done

theorem Wilson-Russ:
   zprime p ==> [zfact (p - 1) = -1] (mod p)
 apply (subgoal-tac [(p - 1) * zfact (p - 2) = -1 * 1] (mod p))
  apply (rule-tac [2] zcong-zmult)
```

```
  apply (simp only: zprime-def)
  apply (subst zfact.simps)
  apply (rule-tac t = p - 1 - 1 and s = p - 2 in subst, auto)
 apply (simp only: zcong-def)
 apply (simp (no-asm-simp))
apply (case-tac p = 2)
 apply (simp add: zfact.simps)
apply (case-tac p = 3)
 apply (simp add: zfact.simps)
apply (subgoal-tac 5 ≤ p)
 apply (erule-tac [2] prime-g-5)
  apply (subst d22set-prod-zfact [symmetric])
  apply (subst d22set-eq-wset)
   apply (rule-tac [2] wset-zcong-prod-1, auto)
done

end
```

9 Wilson's Theorem using a more abstract approach

theory *WilsonBij* **imports** *BijectionRel IntFact* **begin**

Wilson's Theorem using a more "abstract" approach based on bijections between sets. Does not use Fermat's Little Theorem (unlike Russinoff).

9.1 Definitions and lemmas

constdefs

```
  reciR :: int => int => int => bool
  reciR p ==
    λa b. zcong (a * b) 1 p ∧ 1 < a ∧ a < p - 1 ∧ 1 < b ∧ b < p - 1
  inv :: int => int => int
  inv p a ==
    if zprime p ∧ 0 < a ∧ a < p then
      (SOME x. 0 ≤ x ∧ x < p ∧ zcong (a * x) 1 p)
    else 0
```

Inverse

lemma *inv-correct*:

```
  zprime p ==> 0 < a ==> a < p
    ==> 0 ≤ inv p a ∧ inv p a < p ∧ [a * inv p a = 1] (mod p)
  apply (unfold inv-def)
  apply (simp (no-asm-simp))
  apply (rule zcong-lineq-unique [THEN ex1-implies-ex, THEN someI-ex])
   apply (erule-tac [2] zless-zprime-imp-zrelprime)
```

```
  apply (unfold zprime-def)
  apply auto
 done

lemmas inv-ge = inv-correct [THEN conjunct1, standard]
lemmas inv-less = inv-correct [THEN conjunct2, THEN conjunct1, standard]
lemmas inv-is-inv = inv-correct [THEN conjunct2, THEN conjunct2, standard]

lemma inv-not-0:
 zprime p ==> 1 < a ==> a < p - 1 ==> inv p a ≠ 0
 — same as WilsonRuss
 apply safe
 apply (cut-tac a = a and p = p in inv-is-inv)
  apply (unfold zcong-def)
  apply auto
 apply (subgoal-tac ¬ p dvd 1)
  apply (rule-tac [2] zdvd-not-zless)
  apply (subgoal-tac p dvd 1)
  prefer 2
  apply (subst zdvd-zminus-iff [symmetric])
  apply auto
 done

lemma inv-not-1:
 zprime p ==> 1 < a ==> a < p - 1 ==> inv p a ≠ 1
 — same as WilsonRuss
 apply safe
 apply (cut-tac a = a and p = p in inv-is-inv)
  prefer 4
  apply simp
  apply (subgoal-tac a = 1)
   apply (rule-tac [2] zcong-zless-imp-eq)
     apply auto
 done

lemma aux: [a * (p - 1) = 1] (mod p) = [a = p - 1] (mod p)
 — same as WilsonRuss
 apply (unfold zcong-def)
 apply (simp add: OrderedGroup.diff-diff-eq diff-diff-eq2 zdiff-zmult-distrib2)
 apply (rule-tac s = p dvd -((a + 1) + (p * -a)) in trans)
  apply (simp add: mult-commute)
 apply (subst zdvd-zminus-iff)
 apply (subst zdvd-reduce)
 apply (rule-tac s = p dvd (a + 1) + (p * -1) in trans)
  apply (subst zdvd-reduce)
  apply auto
 done

lemma inv-not-p-minus-1:
```

```
  zprime p ==> 1 < a ==> a < p - 1 ==> inv p a ≠ p - 1
  — same as WilsonRuss
  apply safe
  apply (cut-tac a = a and p = p in inv-is-inv)
    apply auto
  apply (simp add: aux)
  apply (subgoal-tac a = p - 1)
   apply (rule-tac [2] zcong-zless-imp-eq)
     apply auto
  done
```

Below is slightly different as we don't expand *inv* but use "*correct*" theorems.

```
lemma inv-g-1: zprime p ==> 1 < a ==> a < p - 1 ==> 1 < inv p a
  apply (subgoal-tac inv p a ≠ 1)
   apply (subgoal-tac inv p a ≠ 0)
    apply (subst order-less-le)
    apply (subst zle-add1-eq-le [symmetric])
    apply (subst order-less-le)
    apply (rule-tac [2] inv-not-0)
     apply (rule-tac [5] inv-not-1)
      apply auto
  apply (rule inv-ge)
   apply auto
  done

lemma inv-less-p-minus-1:
  zprime p ==> 1 < a ==> a < p - 1 ==> inv p a < p - 1
  — ditto
  apply (subst order-less-le)
  apply (simp add: inv-not-p-minus-1 inv-less)
  done
```

Bijection

```
lemma aux1: 1 < x ==> 0 ≤ (x::int)
  apply auto
  done

lemma aux2: 1 < x ==> 0 < (x::int)
  apply auto
  done

lemma aux3: x ≤ p - 2 ==> x < (p::int)
  apply auto
  done

lemma aux4: x ≤ p - 2 ==> x < (p::int) - 1
  apply auto
  done
```

```
lemma inv-inj: zprime p ==> inj-on (inv p) (d22set (p - 2))
  apply (unfold inj-on-def)
  apply auto
  apply (rule zcong-zless-imp-eq)
     apply (tactic ⟪ stac (thm zcong-cancel RS sym) 5 ⟫)
      apply (rule-tac [7] zcong-trans)
       apply (tactic ⟪ stac (thm zcong-sym) 8 ⟫)
       apply (erule-tac [7] inv-is-inv)
        apply (tactic Asm-simp-tac 9)
        apply (erule-tac [9] inv-is-inv)
         apply (rule-tac [6] zless-zprime-imp-zrelprime)
           apply (rule-tac [8] inv-less)
             apply (rule-tac [7] inv-g-1 [THEN aux2])
               apply (unfold zprime-def)
               apply (auto intro: d22set-g-1 d22set-le
                 aux1 aux2 aux3 aux4)
  done

lemma inv-d22set-d22set:
    zprime p ==> inv p ` d22set (p - 2) = d22set (p - 2)
  apply (rule endo-inj-surj)
    apply (rule d22set-fin)
   apply (erule-tac [2] inv-inj)
  apply auto
  apply (rule d22set-mem)
   apply (erule inv-g-1)
    apply (subgoal-tac [3] inv p xa < p - 1)
     apply (erule-tac [4] inv-less-p-minus-1)
      apply (auto intro: d22set-g-1 d22set-le aux4)
  done

lemma d22set-d22set-bij:
    zprime p ==> (d22set (p - 2), d22set (p - 2)) ∈ bijR (reciR p)
  apply (unfold reciR-def)
  apply (rule-tac s = (d22set (p - 2), inv p ` d22set (p - 2)) in subst)
   apply (simp add: inv-d22set-d22set)
  apply (rule inj-func-bijR)
    apply (rule-tac [3] d22set-fin)
   apply (erule-tac [2] inv-inj)
  apply auto
     apply (erule inv-is-inv)
      apply (erule-tac [5] inv-g-1)
       apply (erule-tac [7] inv-less-p-minus-1)
        apply (auto intro: d22set-g-1 d22set-le aux2 aux3 aux4)
  done

lemma reciP-bijP: zprime p ==> bijP (reciR p) (d22set (p - 2))
  apply (unfold reciR-def bijP-def)
  apply auto
```

```
 apply (rule d22set-mem)
  apply auto
 done

lemma reciP-uniq: zprime p ==> uniqP (reciR p)
 apply (unfold reciR-def uniqP-def)
 apply auto
  apply (rule zcong-zless-imp-eq)
     apply (tactic ⟪ stac (thm zcong-cancel2 RS sym) 5 ⟫)
       apply (rule-tac [7] zcong-trans)
        apply (tactic ⟪ stac (thm zcong-sym) 8 ⟫)
        apply (rule-tac [6] zless-zprime-imp-zrelprime)
          apply auto
 apply (rule zcong-zless-imp-eq)
    apply (tactic ⟪ stac (thm zcong-cancel RS sym) 5 ⟫)
      apply (rule-tac [7] zcong-trans)
       apply (tactic ⟪ stac (thm zcong-sym) 8 ⟫)
       apply (rule-tac [6] zless-zprime-imp-zrelprime)
         apply auto
 done

lemma reciP-sym: zprime p ==> symP (reciR p)
 apply (unfold reciR-def symP-def)
 apply (simp add: zmult-commute)
 apply auto
 done

lemma bijER-d22set: zprime p ==> d22set (p - 2) ∈ bijER (reciR p)
 apply (rule bijR-bijER)
   apply (erule d22set-d22set-bij)
  apply (erule reciP-bijP)
 apply (erule reciP-uniq)
 apply (erule reciP-sym)
 done
```

9.2 Wilson

```
lemma bijER-zcong-prod-1:
   zprime p ==> A ∈ bijER (reciR p) ==> [∏ A = 1] (mod p)
 apply (unfold reciR-def)
 apply (erule bijER.induct)
  apply (subgoal-tac [2] a = 1 ∨ a = p - 1)
   apply (rule-tac [3] zcong-square-zless)
     apply auto
 apply (subst setprod-insert)
   prefer 3
   apply (subst setprod-insert)
    apply (auto simp add: fin-bijER)
 apply (subgoal-tac zcong ((a * b) * ∏ A) (1 * 1) p)
```

```
 apply (simp add: zmult-assoc)
 apply (rule zcong-zmult)
 apply auto
 done

theorem Wilson-Bij: zprime p ==> [zfact (p - 1) = -1] (mod p)
 apply (subgoal-tac zcong ((p - 1) * zfact (p - 2)) (-1 * 1) p)
 apply (rule-tac [2] zcong-zmult)
 apply (simp add: zprime-def)
 apply (subst zfact.simps)
 apply (rule-tac t = p - 1 - 1 and s = p - 2 in subst)
 apply auto
 apply (simp add: zcong-def)
 apply (subst d22set-prod-zfact [symmetric])
 apply (rule bijER-zcong-prod-1)
 apply (rule-tac [2] bijER-d22set)
 apply auto
 done

end
```

10 Finite Sets and Finite Sums

```
theory Finite2
imports IntFact
begin
```

These are useful for combinatorial and number-theoretic counting arguments.

Note. This theory is being revised. See the web page http://www.andrew.cmu.edu/~avigad/isabelle.

10.1 Useful properties of sums and products

```
lemma setsum-same-function-zcong:
assumes a: ∀ x ∈ S. [f x = g x](mod m)
shows [setsum f S = setsum g S] (mod m)
proof cases
 assume finite S
 thus ?thesis using a by induct (simp-all add: zcong-zadd)
next
 assume infinite S thus ?thesis by(simp add:setsum-def)
qed

lemma setprod-same-function-zcong:
assumes a: ∀ x ∈ S. [f x = g x](mod m)
shows [setprod f S = setprod g S] (mod m)
```

proof *cases*
assume *finite S*
thus *?thesis* **using** *a* **by** *induct* (*simp-all add*: *zcong-zmult*)
next
assume *infinite S* **thus** *?thesis* **by**(*simp add*:*setprod-def*)
qed

lemma *setsum-const*: *finite X ==> setsum* (%*x*. (*c* :: *int*)) *X* = *c* ∗ *int*(*card X*)
apply (*induct set*: *Finites*)
apply (*auto simp add*: *left-distrib right-distrib int-eq-of-nat*)
done

lemma *setsum-const2*: *finite X ==> int* (*setsum* (%*x*. (*c* :: *nat*)) *X*) =
int(*c*) ∗ *int*(*card X*)
apply (*induct set*: *Finites*)
apply (*auto simp add*: *zadd-zmult-distrib2*)
done

lemma *setsum-const-mult*: *finite A ==> setsum* (%*x*. *c* ∗ ((*f x*)::*int*)) *A* =
c ∗ *setsum f A*
apply (*induct set*: *Finites*, *auto*)
by (*auto simp only*: *zadd-zmult-distrib2*)

10.2 Cardinality of explicit finite sets

lemma *finite-surjI*: [| *B* ⊆ *f* ‘ *A*; *finite A* |] ==> *finite B*
by (*simp add*: *finite-subset finite-imageI*)

lemma *bdd-nat-set-l-finite*: *finite* { *y*::*nat* . *y* < *x*}
apply (*rule-tac N* = {*y*. *y* < *x*} **and** *n* = *x* **in** *bounded-nat-set-is-finite*)
by *auto*

lemma *bdd-nat-set-le-finite*: *finite* { *y*::*nat* . *y* ≤ *x* }
apply (*subgoal-tac* { *y*::*nat* . *y* ≤ *x* } = { *y*::*nat* . *y* < *Suc x*})
by (*auto simp add*: *bdd-nat-set-l-finite*)

lemma *bdd-int-set-l-finite*: *finite* { *x*::*int* . 0 ≤ *x* & *x* < *n*}
apply (*subgoal-tac* {(*x* :: *int*). 0 ≤ *x* & *x* < *n*} ⊆
int ‘ {(*x* :: *nat*). *x* < *nat n*})
apply (*erule finite-surjI*)
apply (*auto simp add*: *bdd-nat-set-l-finite image-def*)
apply (*rule-tac x* = *nat x* **in** *exI*, *simp*)
done

lemma *bdd-int-set-le-finite*: *finite* {*x*::*int*. 0 ≤ *x* & *x* ≤ *n*}
apply (*subgoal-tac* {*x*. 0 ≤ *x* & *x* ≤ *n*} = {*x*. 0 ≤ *x* & *x* < *n* + *1*})
apply (*erule ssubst*)
apply (*rule bdd-int-set-l-finite*)
by *auto*

lemma *bdd-int-set-l-l-finite*: *finite* {*x*::*int*. *0* < *x* & *x* < *n*}
apply (*subgoal-tac* {*x*::*int*. *0* < *x* & *x* < *n*} ⊆ {*x*::*int*. *0* ≤ *x* & *x* < *n*})
by (*auto simp add*: *bdd-int-set-l-finite finite-subset*)

lemma *bdd-int-set-l-le-finite*: *finite* {*x*::*int*. *0* < *x* & *x* ≤ *n*}
apply (*subgoal-tac* {*x*::*int*. *0* < *x* & *x* ≤ *n*} ⊆ {*x*::*int*. *0* ≤ *x* & *x* ≤ *n*})
by (*auto simp add*: *bdd-int-set-le-finite finite-subset*)

lemma *card-bdd-nat-set-l*: *card* {*y*::*nat* . *y* < *x*} = *x*
apply (*induct-tac x*, *force*)
proof −
  fix *n*::*nat*
  assume *card* {*y*. *y* < *n*} = *n*
  have {*y*. *y* < *Suc n*} = *insert n* {*y*. *y* < *n*}
    by *auto*
  then have *card* {*y*. *y* < *Suc n*} = *card* (*insert n* {*y*. *y* < *n*})
    by *auto*
  also have ... = *Suc* (*card* {*y*. *y* < *n*})
    apply (*rule card-insert-disjoint*)
    by (*auto simp add*: *bdd-nat-set-l-finite*)
  finally show *card* {*y*. *y* < *Suc n*} = *Suc n*
    by (*simp add*: *prems*)
qed

lemma *card-bdd-nat-set-le*: *card* { *y*::*nat*. *y* ≤ *x*} = *Suc x*
apply (*subgoal-tac* { *y*::*nat*. *y* ≤ *x*} = { *y*::*nat*. *y* < *Suc x*})
by (*auto simp add*: *card-bdd-nat-set-l*)

lemma *card-bdd-int-set-l*: *0* ≤ (*n*::*int*) ==> *card* {*y*. *0* ≤ *y* & *y* < *n*} = *nat n*
proof −
  fix *n*::*int*
  assume *0* ≤ *n*
  have *inj-on* (%*y*. *int y*) {*y*. *y* < *nat n*}
    by (*auto simp add*: *inj-on-def*)
  hence *card* (*int* ‘ {*y*. *y* < *nat n*}) = *card* {*y*. *y* < *nat n*}
    by (*rule card-image*)
  also from *prems* **have** *int* ‘ {*y*. *y* < *nat n*} = {*y*. *0* ≤ *y* & *y* < *n*}
    apply (*auto simp add*: *zless-nat-eq-int-zless image-def*)
    apply (*rule-tac x* = *nat x* **in** *exI*)
    by (*auto simp add*: *nat-0-le*)
  also have *card* {*y*. *y* < *nat n*} = *nat n*
    by (*rule card-bdd-nat-set-l*)
  finally show *card* {*y*. *0* ≤ *y* & *y* < *n*} = *nat n* **.**
qed

lemma *card-bdd-int-set-le*: *0* ≤ (*n*::*int*) ==> *card* {*y*. *0* ≤ *y* & *y* ≤ *n*} =
  nat n + *1*
apply (*subgoal-tac* {*y*. *0* ≤ *y* & *y* ≤ *n*} = {*y*. *0* ≤ *y* & *y* < *n*+*1*})

apply (*insert card-bdd-int-set-l* [*of n+1*])
by (*auto simp add*: *nat-add-distrib*)

lemma *card-bdd-int-set-l-le*: *0* ≤ (*n*::*int*) ==>
  card {*x*. *0* < *x* & *x* ≤ *n*} = *nat n*
proof −
 fix *n*::*int*
 assume *0* ≤ *n*
 have *inj-on* (%*x*. *x+1*) {*x*. *0* ≤ *x* & *x* < *n*}
  by (*auto simp add*: *inj-on-def*)
 hence *card* ((%*x*. *x+1*) ' {*x*. *0* ≤ *x* & *x* < *n*}) =
   card {*x*. *0* ≤ *x* & *x* < *n*}
  by (*rule card-image*)
 also from *prems* **have** ... = *nat n*
  by (*rule card-bdd-int-set-l*)
 also have (%*x*. *x* + *1*) ' {*x*. *0* ≤ *x* & *x* < *n*} = {*x*. *0* < *x* & *x*<= *n*}
  apply (*auto simp add*: *image-def*)
  apply (*rule-tac x* = *x* − *1* **in** *exI*)
  by *arith*
 finally show *card* {*x*. *0* < *x* & *x* ≤ *n*} = *nat n*.
qed

lemma *card-bdd-int-set-l-l*: *0* < (*n*::*int*) ==>
  card {*x*. *0* < *x* & *x* < *n*} = *nat n* − *1*
 apply (*subgoal-tac* {*x*. *0* < *x* & *x* < *n*} = {*x*. *0* < *x* & *x* ≤ *n* − *1*})
 apply (*insert card-bdd-int-set-l-le* [*of n* − *1*])
 by (*auto simp add*: *nat-diff-distrib*)

lemma *int-card-bdd-int-set-l-l*: *0* < *n* ==>
  int(*card* {*x*. *0* < *x* & *x* < *n*}) = *n* − *1*
 apply (*auto simp add*: *card-bdd-int-set-l-l*)
 apply (*subgoal-tac Suc 0* ≤ *nat n*)
 apply (*auto simp add*: *zdiff-int* [*THEN sym*])
 apply (*subgoal-tac 0* < *nat n*, *arith*)
 by (*simp add*: *zero-less-nat-eq*)

lemma *int-card-bdd-int-set-l-le*: *0* ≤ *n* ==>
  int(*card* {*x*. *0* < *x* & *x* ≤ *n*}) = *n*
 by (*auto simp add*: *card-bdd-int-set-l-le*)

10.3 Cardinality of finite cartesian products

10.4 Lemmas for counting arguments

lemma *setsum-bij-eq*: [| *finite A*; *finite B*; *f* ' *A* ⊆ *B*; *inj-on f A*;
  g ' *B* ⊆ *A*; *inj-on g B* |] ==> *setsum g B* = *setsum* (*g* ∘ *f*) *A*
apply (*frule-tac h* = *g* **and** *f* = *f* **in** *setsum-reindex*)
apply (*subgoal-tac setsum g B* = *setsum g* (*f* ' *A*))
apply (*simp add*: *inj-on-def*)
apply (*subgoal-tac card A* = *card B*)

apply (*drule-tac* $A = f$ ` A **and** $B = B$ **in** *card-seteq*)
apply (*auto simp add*: *card-image*)
apply (*frule-tac* $A = A$ **and** $B = B$ **and** $f = f$ **in** *card-inj-on-le*, *auto*)
apply (*frule-tac* $A = B$ **and** $B = A$ **and** $f = g$ **in** *card-inj-on-le*)
by *auto*

lemma *setprod-bij-eq*: [| *finite* A; *finite* B; f ` $A \subseteq B$; *inj-on* f A;
g ` $B \subseteq A$; *inj-on* g B |] ==> *setprod* g B = *setprod* $(g \circ f)$ A
apply (*frule-tac* $h = g$ **and** $f = f$ **in** *setprod-reindex*)
apply (*subgoal-tac setprod* g B = *setprod* g $(f$ ` $A)$)
apply (*simp add*: *inj-on-def*)
apply (*subgoal-tac card* A = *card* B)
apply (*drule-tac* $A = f$ ` A **and** $B = B$ **in** *card-seteq*)
apply (*auto simp add*: *card-image*)
apply (*frule-tac* $A = A$ **and** $B = B$ **and** $f = f$ **in** *card-inj-on-le*, *auto*)
by (*frule-tac* $A = B$ **and** $B = A$ **and** $f = g$ **in** *card-inj-on-le*, *auto*)

end

11 Integers: Divisibility and Congruences

theory *Int2* **imports** *Finite2 WilsonRuss* **begin**

Note. This theory is being revised. See the web page http://www.andrew.cmu.edu/~avigad/isabelle.

constdefs
MultInv :: *int* => *int* => *int*
MultInv p x == x ^ *nat* $(p - 2)$

lemma *zpower-zdvd-prop1* [*rule-format*]: $((0 < n)$ & $(p$ *dvd* $y))$ −−>
p *dvd* $((y$::*int*$)$ ^ $n)$
by (*induct-tac* n, *auto simp add*: *zdvd-zmult zdvd-zmult2* [*of* p y])

lemma *zdvd-bounds*: n *dvd* m ==> $(m \leq (0$::*int*$) \mid n \leq m)$
proof −
assume n *dvd* m
then have ~$(0 < m$ & $m < n)$
apply (*insert zdvd-not-zless* [*of* m n])
by (*rule contrapos-pn*, *auto*)
then have (~$0 < m \mid$ ~$m < n)$ **by** *auto*
then show *?thesis* **by** *auto*
qed

lemma *aux4*: $-(m * n) = (-m) * (n::int)$
by *auto*

lemma *zprime-zdvd-zmult-better*: [| *zprime p*; *p dvd* $(m * n)$ |] ==>
$(p\ dvd\ m) \mid (p\ dvd\ n)$
apply (*case-tac* $0 \leq m$)
apply (*simp add*: *zprime-zdvd-zmult*)
by (*insert zprime-zdvd-zmult* [*of* $-m\ p\ n$], *auto*)

lemma *zpower-zdvd-prop2* [*rule-format*]: *zprime p* −−> *p dvd* $((y::int)\ \hat{}\ n)$
−−> $0 < n$ −−> *p dvd y*
apply (*induct-tac n*, *auto*)
apply (*frule zprime-zdvd-zmult-better*, *auto*)
done

lemma *stupid*: $(0 :: int) \leq y$ ==> $x \leq x + y$
by *arith*

lemma *div-prop1*: [| $0 < z$; $(x::int) < y * z$ |] ==> $x\ div\ z < y$
proof −
assume $0 < z$
then have $(x\ div\ z) * z \leq (x\ div\ z) * z + x\ mod\ z$
apply (*rule-tac* $x = x\ div\ z * z$ **in** *stupid*)
by (*simp add*: *pos-mod-sign*)
also have ... $= x$
by (*auto simp add*: *zmod-zdiv-equality* [*THEN sym*] *zmult-ac*)
also assume $x < y * z$
finally show *?thesis*
by (*auto simp add*: *prems mult-less-cancel-right*, *insert prems*, *arith*)
qed

lemma *div-prop2*: [| $0 < z$; $(x::int) < (y * z) + z$ |] ==> $x\ div\ z \leq y$
proof −
assume $0 < z$ **and** $x < (y * z) + z$
then have $x < (y + 1) * z$ **by** (*auto simp add*: *int-distrib*)
then have $x\ div\ z < y + 1$
by (*rule-tac* $y = y + 1$ **in** *div-prop1*, *auto simp add*: *prems*)
then show *?thesis* **by** *auto*
qed

lemma *zdiv-leq-prop*: [| $0 < y$ |] ==> $y * (x\ div\ y) \leq (x::int)$
proof−
assume $0 < y$
from *zmod-zdiv-equality* **have** $x = y * (x\ div\ y) + x\ mod\ y$ **by** *auto*
moreover have $0 \leq x\ mod\ y$
by (*auto simp add*: *prems pos-mod-sign*)
ultimately show *?thesis*
by *arith*
qed

lemma *zcong-eq-zdvd-prop*: $[x = 0](mod\ p) = (p\ dvd\ x)$
by (*auto simp add*: *zcong-def*)

lemma *zcong-id*: $[m = 0]\ (mod\ m)$
by (*auto simp add*: *zcong-def zdvd-0-right*)

lemma *zcong-shift*: $[a = b]\ (mod\ m) ==> [a + c = b + c]\ (mod\ m)$
by (*auto simp add*: *zcong-refl zcong-zadd*)

lemma *zcong-zpower*: $[x = y](mod\ m) ==> [x\hat{}z = y\hat{}z](mod\ m)$
by (*induct-tac z*, *auto simp add*: *zcong-zmult*)

lemma *zcong-eq-trans*: $[|\ [a = b](mod\ m);\ b = c;\ [c = d](mod\ m)\ |] ==>$
$[a = d](mod\ m)$
by (*auto*, *rule-tac b = c* **in** *zcong-trans*)

lemma *aux1*: $a - b = (c::int) ==> a = c + b$
by *auto*

lemma *zcong-zmult-prop1*: $[a = b](mod\ m) ==> ([c = a * d](mod\ m) =$
$[c = b * d]\ (mod\ m))$
apply (*auto simp add*: *zcong-def dvd-def*)
apply (*rule-tac x = ka + k * d* **in** *exI*)
apply (*drule aux1*)+
apply (*auto simp add*: *int-distrib*)
apply (*rule-tac x = ka - k * d* **in** *exI*)
apply (*drule aux1*)+
apply (*auto simp add*: *int-distrib*)
done

lemma *zcong-zmult-prop2*: $[a = b](mod\ m) ==>$
$([c = d * a](mod\ m) = [c = d * b]\ (mod\ m))$
by (*auto simp add*: *zmult-ac zcong-zmult-prop1*)

lemma *zcong-zmult-prop3*: $[|\ zprime\ p;\ \sim[x = 0]\ (mod\ p);$
$\sim[y = 0]\ (mod\ p)\ |] ==> \sim[x * y = 0]\ (mod\ p)$
apply (*auto simp add*: *zcong-def*)
apply (*drule zprime-zdvd-zmult-better*, *auto*)
done

lemma *zcong-less-eq*: $[|\ 0 < x;\ 0 < y;\ 0 < m;\ [x = y]\ (mod\ m);$
$x < m;\ y < m\ |] ==> x = y$

apply (*simp add*: *zcong-zmod-eq*)
apply (*subgoal-tac* (*x mod m*) = *x*)
apply (*subgoal-tac* (*y mod m*) = *y*)
apply *simp*
apply (*rule-tac* [*1−2*] *mod-pos-pos-trivial*)
by *auto*

lemma *zcong-neg-1-impl-ne-1*: [| *2* < *p*; [*x* = *−1*] (*mod p*) |] ==>
~([*x* = *1*] (*mod p*))
proof
assume *2* < *p* **and** [*x* = *1*] (*mod p*) **and** [*x* = *−1*] (*mod p*)
then have [*1* = *−1*] (*mod p*)
apply (*auto simp add*: *zcong-sym*)
apply (*drule zcong-trans*, *auto*)
done
then have [*1* + *1* = *−1* + *1*] (*mod p*)
by (*simp only*: *zcong-shift*)
then have [*2* = *0*] (*mod p*)
by *auto*
then have *p dvd 2*
by (*auto simp add*: *dvd-def zcong-def*)
with *prems* **show** *False*
by (*auto simp add*: *zdvd-not-zless*)
qed

lemma *zcong-zero-equiv-div*: [*a* = *0*] (*mod m*) = (*m dvd a*)
by (*auto simp add*: *zcong-def*)

lemma *zcong-zprime-prod-zero*: [| *zprime p*; *0* < *a* |] ==>
[*a* * *b* = *0*] (*mod p*) ==> [*a* = *0*] (*mod p*) | [*b* = *0*] (*mod p*)
by (*auto simp add*: *zcong-zero-equiv-div zprime-zdvd-zmult*)

lemma *zcong-zprime-prod-zero-contra*: [| *zprime p*; *0* < *a* |] ==>
~[*a* = *0*](*mod p*) & ~[*b* = *0*](*mod p*) ==> ~[*a* * *b* = *0*] (*mod p*)
apply *auto*
apply (*frule-tac a* = *a* **and** *b* = *b* **and** *p* = *p* **in** *zcong-zprime-prod-zero*)
by *auto*

lemma *zcong-not-zero*: [| *0* < *x*; *x* < *m* |] ==> ~[*x* = *0*] (*mod m*)
by (*auto simp add*: *zcong-zero-equiv-div zdvd-not-zless*)

lemma *zcong-zero*: [| *0* ≤ *x*; *x* < *m*; [*x* = *0*](*mod m*) |] ==> *x* = *0*
apply (*drule order-le-imp-less-or-eq*, *auto*)
by (*frule-tac m* = *m* **in** *zcong-not-zero*, *auto*)

lemma *all-relprime-prod-relprime*: [| *finite A*; ∀ *x* ∈ *A*. (*zgcd*(*x*,*y*) = *1*) |]
==> *zgcd* (*setprod id A*,*y*) = *1*
by (*induct set*: *Finites*, *auto simp add*: *zgcd-zgcd-zmult*)

lemma *MultInv-prop1*: [| *2* < *p*; [*x* = *y*] (*mod p*) |] ==>
  [(*MultInv p x*) = (*MultInv p y*)] (*mod p*)
 by (*auto simp add*: *MultInv-def zcong-zpower*)

lemma *MultInv-prop2*: [| *2* < *p*; *zprime p*; ~([*x* = *0*](*mod p*)) |] ==>
 [(*x* * (*MultInv p x*)) = *1*] (*mod p*)
proof (*simp add*: *MultInv-def zcong-eq-zdvd-prop*)
 assume *2* < *p* **and** *zprime p* **and** ~ *p dvd x*
 have *x* * *x* ^ *nat* (*p* − *2*) = *x* ^ (*nat* (*p* − *2*) + *1*)
  by *auto*
 also from *prems* **have** *nat* (*p* − *2*) + *1* = *nat* (*p* − *2* + *1*)
  by (*simp only*: *nat-add-distrib*, *auto*)
 also have *p* − *2* + *1* = *p* − *1* **by** *arith*
 finally have [*x* * *x* ^ *nat* (*p* − *2*) = *x* ^ *nat* (*p* − *1*)] (*mod p*)
  by (*rule ssubst*, *auto*)
 also from *prems* **have** [*x* ^ *nat* (*p* − *1*) = *1*] (*mod p*)
  by (*auto simp add*: *Little-Fermat*)
 finally (*zcong-trans*) **show** [*x* * *x* ^ *nat* (*p* − *2*) = *1*] (*mod p*)**.**
qed

lemma *MultInv-prop2a*: [| *2* < *p*; *zprime p*; ~([*x* = *0*](*mod p*)) |] ==>
  [(*MultInv p x*) * *x* = *1*] (*mod p*)
 by (*auto simp add*: *MultInv-prop2 zmult-ac*)

lemma *aux-1*: *2* < *p* ==> ((*nat p*) − *2*) = (*nat* (*p* − *2*))
 by (*simp add*: *nat-diff-distrib*)

lemma *aux-2*: *2* < *p* ==> *0* < *nat* (*p* − *2*)
 by *auto*

lemma *MultInv-prop3*: [| *2* < *p*; *zprime p*; ~([*x* = *0*](*mod p*)) |] ==>
  ~([*MultInv p x* = *0*](*mod p*))
 apply (*auto simp add*: *MultInv-def zcong-eq-zdvd-prop aux-1*)
 apply (*drule aux-2*)
 apply (*drule zpower-zdvd-prop2*, *auto*)
done

lemma *aux--1*: [| *2* < *p*; *zprime p*; ~([*x* = *0*](*mod p*))|] ==>
  [(*MultInv p* (*MultInv p x*)) = (*x* * (*MultInv p x*) *
   (*MultInv p* (*MultInv p x*)))] (*mod p*)
 apply (*drule MultInv-prop2*, *auto*)
 apply (*drule-tac k* = *MultInv p* (*MultInv p x*) **in** *zcong-scalar*, *auto*)
 apply (*auto simp add*: *zcong-sym*)

done

lemma *aux--2*: [| *2 < p*; *zprime p*; ~([*x = 0*](*mod p*))|] ==>
[(*x* * (*MultInv p x*) * (*MultInv p* (*MultInv p x*))) = *x*] (*mod p*)
apply (*frule MultInv-prop3*, *auto*)
apply (*insert MultInv-prop2* [*of p MultInv p x*], *auto*)
apply (*drule MultInv-prop2*, *auto*)
apply (*drule-tac k = x* **in** *zcong-scalar2*, *auto*)
apply (*auto simp add*: *zmult-ac*)
done

lemma *MultInv-prop4*: [| *2 < p*; *zprime p*; ~([*x = 0*](*mod p*)) |] ==>
[(*MultInv p* (*MultInv p x*)) = *x*] (*mod p*)
apply (*frule aux--1*, *auto*)
apply (*drule aux--2*, *auto*)
apply (*drule zcong-trans*, *auto*)
done

lemma *MultInv-prop5*: [| *2 < p*; *zprime p*; ~([*x = 0*](*mod p*));
~([*y = 0*](*mod p*)); [(*MultInv p x*) = (*MultInv p y*)] (*mod p*) |] ==>
[*x = y*] (*mod p*)
apply (*drule-tac a = MultInv p x* **and** *b = MultInv p y* **and**
m = p **and** *k = x* **in** *zcong-scalar*)
apply (*insert MultInv-prop2* [*of p x*], *simp*)
apply (*auto simp only*: *zcong-sym* [*of MultInv p x* * *x*])
apply (*auto simp add*: *zmult-ac*)
apply (*drule zcong-trans*, *auto*)
apply (*drule-tac a = x* * *MultInv p y* **and** *k = y* **in** *zcong-scalar*, *auto*)
apply (*insert MultInv-prop2a* [*of p y*], *auto simp add*: *zmult-ac*)
apply (*insert zcong-zmult-prop2* [*of y* * *MultInv p y 1 p y x*])
apply (*auto simp add*: *zcong-sym*)
done

lemma *MultInv-zcong-prop1*: [| *2 < p*; [*j = k*] (*mod p*) |] ==>
[*a* * *MultInv p j = a* * *MultInv p k*] (*mod p*)
by (*drule MultInv-prop1*, *auto simp add*: *zcong-scalar2*)

lemma *aux---1*: [*j = a* * *MultInv p k*] (*mod p*) ==>
[*j* * *k = a* * *MultInv p k* * *k*] (*mod p*)
by (*auto simp add*: *zcong-scalar*)

lemma *aux---2*: [|*2 < p*; *zprime p*; ~([*k = 0*](*mod p*));
[*j* * *k = a* * *MultInv p k* * *k*] (*mod p*) |] ==> [*j* * *k = a*] (*mod p*)
apply (*insert MultInv-prop2a* [*of p k*] *zcong-zmult-prop2*
[*of MultInv p k* * *k 1 p j* * *k a*])
apply (*auto simp add*: *zmult-ac*)
done

lemma *aux---3*: [*j* * *k = a*] (*mod p*) ==> [(*MultInv p j*) * *j* * *k =*

```
    (MultInv p j) * a] (mod p)
  by (auto simp add: zmult-assoc zcong-scalar2)

lemma aux---4: [|2 < p; zprime p; ~([j = 0](mod p));
    [(MultInv p j) * j * k = (MultInv p j) * a] (mod p) |]
      ==> [k = a * (MultInv p j)] (mod p)
  apply (insert MultInv-prop2a [of p j] zcong-zmult-prop1
    [of MultInv p j * j 1 p MultInv p j * a k])
  apply (auto simp add: zmult-ac zcong-sym)
done

lemma MultInv-zcong-prop2: [| 2 < p; zprime p; ~([k = 0](mod p));
    ~([j = 0](mod p)); [j = a * MultInv p k] (mod p) |] ==>
    [k = a * MultInv p j] (mod p)
  apply (drule aux---1)
  apply (frule aux---2, auto)
  by (drule aux---3, drule aux---4, auto)

lemma MultInv-zcong-prop3: [| 2 < p; zprime p; ~([a = 0](mod p));
    ~([k = 0](mod p)); ~([j = 0](mod p));
    [a * MultInv p j = a * MultInv p k] (mod p) |] ==>
      [j = k] (mod p)
  apply (auto simp add: zcong-eq-zdvd-prop [of a p])
  apply (frule zprime-imp-zrelprime, auto)
  apply (insert zcong-cancel2 [of p a MultInv p j MultInv p k], auto)
  apply (drule MultInv-prop5, auto)
done

end
```

12 Residue Sets

theory *Residues* **imports** *Int2* **begin**

Note. This theory is being revised. See the web page http://www.andrew.cmu.edu/~avigad/isabelle.

```
constdefs
  ResSet      :: int => int set => bool
  ResSet m X == ∀ y1 y2. (((y1 ∈ X) & (y2 ∈ X) & [y1 = y2] (mod m)) -->
    y1 = y2)

  StandardRes :: int => int => int
  StandardRes m x == x mod m

  QuadRes     :: int => int => bool
  QuadRes m x == ∃ y. ([(y ^ 2) = x] (mod m))

  Legendre    :: int => int => int
```

Legendre a p == (if ([a = 0] (mod p)) then 0
else if (QuadRes p a) then 1
else −1)

SR :: int => int set
SR p == {x. (0 ≤ x) & (x < p)}

SRStar :: int => int set
SRStar p == {x. (0 < x) & (x < p)}

12.1 Properties of StandardRes

lemma *StandardRes-prop1*: *[x = StandardRes m x] (mod m)*
by (*auto simp add*: *StandardRes-def zcong-zmod*)

lemma *StandardRes-prop2*: *0 < m ==> (StandardRes m x1 = StandardRes m x2)*
= ([x1 = x2] (mod m))
by (*auto simp add*: *StandardRes-def zcong-zmod-eq*)

lemma *StandardRes-prop3*: *(~[x = 0] (mod p)) = (~(StandardRes p x = 0))*
by (*auto simp add*: *StandardRes-def zcong-def zdvd-iff-zmod-eq-0*)

lemma *StandardRes-prop4*: *2 < m*
*==> [StandardRes m x * StandardRes m y = (x * y)] (mod m)*
by (*auto simp add*: *StandardRes-def zcong-zmod-eq*
zmod-zmult-distrib [of x y m])

lemma *StandardRes-lbound*: *0 < p ==> 0 ≤ StandardRes p x*
by (*auto simp add*: *StandardRes-def pos-mod-sign*)

lemma *StandardRes-ubound*: *0 < p ==> StandardRes p x < p*
by (*auto simp add*: *StandardRes-def pos-mod-bound*)

lemma *StandardRes-eq-zcong*:
(StandardRes m x = 0) = ([x = 0](mod m))
by (*auto simp add*: *StandardRes-def zcong-eq-zdvd-prop dvd-def*)

12.2 Relations between StandardRes, SRStar, and SR

lemma *SRStar-SR-prop*: *x ∈ SRStar p ==> x ∈ SR p*
by (*auto simp add*: *SRStar-def SR-def*)

lemma *StandardRes-SR-prop*: *x ∈ SR p ==> StandardRes p x = x*
by (*auto simp add*: *SR-def StandardRes-def mod-pos-pos-trivial*)

lemma *StandardRes-SRStar-prop1*: *2 < p ==> (StandardRes p x ∈ SRStar p)*
= (~[x = 0] (mod p))
apply (*auto simp add*: *StandardRes-prop3 StandardRes-def*
SRStar-def pos-mod-bound)

apply (*subgoal-tac* $0 < p$)
by (*drule-tac* $a = x$ **in** *pos-mod-sign*, *arith*, *simp*)

lemma *StandardRes-SRStar-prop1a*: $x \in SRStar\ p \Longrightarrow \sim([x = 0]\ (mod\ p))$
by (*auto simp add*: *SRStar-def zcong-def zdvd-not-zless*)

lemma *StandardRes-SRStar-prop2*: [| $2 < p$; *zprime p*; $x \in SRStar\ p$ |]
$\Longrightarrow StandardRes\ p\ (MultInv\ p\ x) \in SRStar\ p$
apply (*frule-tac* $x = (MultInv\ p\ x)$ **in** *StandardRes-SRStar-prop1*, *simp*)
apply (*rule MultInv-prop3*)
apply (*auto simp add*: *SRStar-def zcong-def zdvd-not-zless*)
done

lemma *StandardRes-SRStar-prop3*: $x \in SRStar\ p \Longrightarrow StandardRes\ p\ x = x$
by (*auto simp add*: *SRStar-SR-prop StandardRes-SR-prop*)

lemma *StandardRes-SRStar-prop4*: [| *zprime p*; $2 < p$; $x \in SRStar\ p$ |]
$\Longrightarrow StandardRes\ p\ x \in SRStar\ p$
by (*frule StandardRes-SRStar-prop3*, *auto*)

lemma *SRStar-mult-prop1*: [| *zprime p*; $2 < p$; $x \in SRStar\ p$; $y \in SRStar\ p$|]
$\Longrightarrow (StandardRes\ p\ (x * y)) :: SRStar\ p$
apply (*frule-tac* $x = x$ **in** *StandardRes-SRStar-prop4*, *auto*)
apply (*frule-tac* $x = y$ **in** *StandardRes-SRStar-prop4*, *auto*)
apply (*auto simp add*: *StandardRes-SRStar-prop1 zcong-zmult-prop3*)
done

lemma *SRStar-mult-prop2*: [| *zprime p*; $2 < p$; $\sim([a = 0](mod\ p))$;
$x \in SRStar\ p$ |]
$\Longrightarrow StandardRes\ p\ (a * MultInv\ p\ x) \in SRStar\ p$
apply (*frule-tac* $x = x$ **in** *StandardRes-SRStar-prop2*, *auto*)
apply (*frule-tac* $x = MultInv\ p\ x$ **in** *StandardRes-SRStar-prop1*)
apply (*auto simp add*: *StandardRes-SRStar-prop1 zcong-zmult-prop3*)
done

lemma *SRStar-card*: $2 < p \Longrightarrow int(card(SRStar\ p)) = p - 1$
by (*auto simp add*: *SRStar-def int-card-bdd-int-set-l-l*)

lemma *SRStar-finite*: $2 < p \Longrightarrow finite(\ SRStar\ p)$
by (*auto simp add*: *SRStar-def bdd-int-set-l-l-finite*)

12.3 Properties relating ResSets with StandardRes

lemma *aux*: $x\ mod\ m = y\ mod\ m \Longrightarrow [x = y]\ (mod\ m)$
apply (*subgoal-tac* $x = y \Longrightarrow [x = y](mod\ m)$)
apply (*subgoal-tac* $[x\ mod\ m = y\ mod\ m]\ (mod\ m) \Longrightarrow [x = y]\ (mod\ m)$)
apply (*auto simp add*: *zcong-zmod* [*of x y m*])
done

lemma *StandardRes-inj-on-ResSet*: *ResSet m X ==> (inj-on (StandardRes m) X)*
apply (*auto simp add*: *ResSet-def StandardRes-def inj-on-def*)
apply (*drule-tac m = m* **in** *aux*, *auto*)
done

lemma *StandardRes-Sum*: [| *finite X*; *0 < m* |]
==> [*setsum f X = setsum (StandardRes m o f) X*](*mod m*)
apply (*rule-tac F = X* **in** *finite-induct*)
apply (*auto intro*!: *zcong-zadd simp add*: *StandardRes-prop1*)
done

lemma *SR-pos*: *0 < m ==> (StandardRes m ' X) ⊆ {x. 0 ≤ x & x < m}*
by (*auto simp add*: *StandardRes-ubound StandardRes-lbound*)

lemma *ResSet-finite*: *0 < m ==> ResSet m X ==> finite X*
apply (*rule-tac f = StandardRes m* **in** *finite-imageD*)
apply (*rule-tac B = {x. (0 :: int) ≤ x & x < m}* **in** *finite-subset*)
by (*auto simp add*: *StandardRes-inj-on-ResSet bdd-int-set-l-finite SR-pos*)

lemma *mod-mod-is-mod*: [*x = x mod m*](*mod m*)
by (*auto simp add*: *zcong-zmod*)

lemma *StandardRes-prod*: [| *finite X*; *0 < m* |]
==> [*setprod f X = setprod (StandardRes m o f) X*] (*mod m*)
apply (*rule-tac F = X* **in** *finite-induct*)
by (*auto intro*!: *zcong-zmult simp add*: *StandardRes-prop1*)

lemma *ResSet-image*: [| *0 < m*; *ResSet m A*; ∀ *x* ∈ *A*. ∀ *y* ∈ *A*. ([*f x = f y*](*mod m*) −−> *x = y*) |] ==> *ResSet m (f ' A)*
by (*auto simp add*: *ResSet-def*)

lemma *ResSet-SRStar-prop*: *ResSet p (SRStar p)*
by (*auto simp add*: *SRStar-def ResSet-def zcong-zless-imp-eq*)

end

13 Parity: Even and Odd Integers

theory *EvenOdd* **imports** *Int2* **begin**

Note. This theory is being revised. See the web page http://www.andrew.cmu.edu/~avigad/isabelle.

constdefs
zOdd :: *int set*
zOdd == $\{x.\ \exists k.\ x = 2*k + 1\}$
zEven :: *int set*
zEven == $\{x.\ \exists k.\ x = 2 * k\}$

lemma *one-not-even*: ~(*1* ∈ *zEven*)
apply (*simp add*: *zEven-def*)
apply (*rule allI*, *case-tac* $k \leq 0$, *auto*)
done

lemma *even-odd-conj*: ~(*x* ∈ *zOdd* & *x* ∈ *zEven*)
apply (*auto simp add*: *zOdd-def zEven-def*)
proof −
fix *a b*
assume *2* ∗ (*a*::*int*) = *2* ∗ (*b*::*int*) + *1*
then have *2* ∗ (*a*::*int*) − *2* ∗ (*b* :: *int*) = *1*
by *arith*
then have *2* ∗ (*a* − *b*) = *1*
by (*auto simp add*: *zdiff-zmult-distrib*)
moreover have (*2* ∗ (*a* − *b*)):*zEven*
by (*auto simp only*: *zEven-def*)
ultimately show *False*
by (*auto simp add*: *one-not-even*)
qed

lemma *even-odd-disj*: (*x* ∈ *zOdd* | *x* ∈ *zEven*)
by (*simp add*: *zOdd-def zEven-def*, *presburger*)

lemma *not-odd-impl-even*: ~(*x* ∈ *zOdd*) ==> *x* ∈ *zEven*
by (*insert even-odd-disj*, *auto*)

lemma *odd-mult-odd-prop*: (*x*∗*y*):*zOdd* ==> *x* ∈ *zOdd*
apply (*case-tac x* ∈ *zOdd*, *auto*)
apply (*drule not-odd-impl-even*)
apply (*auto simp add*: *zEven-def zOdd-def*)
proof −
fix *a b*
assume *2* ∗ *a* ∗ *y* = *2* ∗ *b* + *1*
then have *2* ∗ *a* ∗ *y* − *2* ∗ *b* = *1*
by *arith*
then have *2* ∗ (*a* ∗ *y* − *b*) = *1*
by (*auto simp add*: *zdiff-zmult-distrib*)

```
    moreover have (2 * (a * y - b)):zEven
      by (auto simp only: zEven-def)
    ultimately show False
      by (auto simp add: one-not-even)
  qed

lemma odd-minus-one-even: x ∈ zOdd ==> (x - 1):zEven
  by (auto simp add: zOdd-def zEven-def)

lemma even-div-2-prop1: x ∈ zEven ==> (x mod 2) = 0
  by (auto simp add: zEven-def)

lemma even-div-2-prop2: x ∈ zEven ==> (2 * (x div 2)) = x
  by (auto simp add: zEven-def)

lemma even-plus-even: [| x ∈ zEven; y ∈ zEven |] ==> x + y ∈ zEven
  apply (auto simp add: zEven-def)
  by (auto simp only: zadd-zmult-distrib2 [THEN sym])

lemma even-times-either: x ∈ zEven ==> x * y ∈ zEven
  by (auto simp add: zEven-def)

lemma even-minus-even: [| x ∈ zEven; y ∈ zEven |] ==> x - y ∈ zEven
  apply (auto simp add: zEven-def)
  by (auto simp only: zdiff-zmult-distrib2 [THEN sym])

lemma odd-minus-odd: [| x ∈ zOdd; y ∈ zOdd |] ==> x - y ∈ zEven
  apply (auto simp add: zOdd-def zEven-def)
  by (auto simp only: zdiff-zmult-distrib2 [THEN sym])

lemma even-minus-odd: [| x ∈ zEven; y ∈ zOdd |] ==> x - y ∈ zOdd
  apply (auto simp add: zOdd-def zEven-def)
  apply (rule-tac x = k - ka - 1 in exI)
  by auto

lemma odd-minus-even: [| x ∈ zOdd; y ∈ zEven |] ==> x - y ∈ zOdd
  apply (auto simp add: zOdd-def zEven-def)
  by (auto simp only: zdiff-zmult-distrib2 [THEN sym])

lemma odd-times-odd: [| x ∈ zOdd;  y ∈ zOdd |] ==> x * y ∈ zOdd
  apply (auto simp add: zOdd-def zadd-zmult-distrib zadd-zmult-distrib2)
  apply (rule-tac x = 2 * ka * k + ka + k in exI)
  by (auto simp add: zadd-zmult-distrib)

lemma odd-iff-not-even: (x ∈ zOdd) = (~ (x ∈ zEven))
  by (insert even-odd-conj even-odd-disj, auto)

lemma even-product: x * y ∈ zEven ==> x ∈ zEven | y ∈ zEven
  by (insert odd-iff-not-even odd-times-odd, auto)
```

lemma *even-diff*: $x - y \in zEven = ((x \in zEven) = (y \in zEven))$
apply (*auto simp add*: *odd-iff-not-even even-minus-even odd-minus-odd even-minus-odd odd-minus-even*)
proof −
assume $x - y \in zEven$ **and** $x \in zEven$
show $y \in zEven$
proof (*rule classical*)
assume $\sim(y \in zEven)$
then have $y \in zOdd$
by (*auto simp add*: *odd-iff-not-even*)
with *prems* **have** $x - y \in zOdd$
by (*simp add*: *even-minus-odd*)
with *prems* **have** *False*
by (*auto simp add*: *odd-iff-not-even*)
thus *?thesis*
by *auto*
qed
next assume $x - y \in zEven$ **and** $y \in zEven$
show $x \in zEven$
proof (*rule classical*)
assume $\sim(x \in zEven)$
then have $x \in zOdd$
by (*auto simp add*: *odd-iff-not-even*)
with *prems* **have** $x - y \in zOdd$
by (*simp add*: *odd-minus-even*)
with *prems* **have** *False*
by (*auto simp add*: *odd-iff-not-even*)
thus *?thesis*
by *auto*
qed
qed

lemma *neg-one-even-power*: $[|\ x \in zEven;\ 0 \leq x\ |] ==> (-1::int)\hat{}(nat\ x) = 1$
proof −
assume $x \in zEven$ **and** $0 \leq x$
then have $\exists k.\ x = 2 * k$
by (*auto simp only*: *zEven-def*)
then show *?thesis*
proof
fix *a*
assume $x = 2 * a$
from *prems* **have** *a*: $0 \leq a$
by *arith*
from *prems* **have** $nat\ x = nat(2 * a)$
by *auto*
also from *a* **have** $nat\ (2 * a) = 2 * nat\ a$
by (*auto simp add*: *nat-mult-distrib*)
finally have $(-1::int)\hat{}nat\ x = (-1)\hat{}(2 * nat\ a)$

```
      by auto
    also have ... = ((-1::int) ^2) ^ (nat a)
      by (auto simp add: zpower-zpower [THEN sym])
    also have (-1::int) ^2 = 1
      by auto
    finally show ?thesis
      by auto
  qed
qed

lemma neg-one-odd-power: [| x ∈ zOdd; 0 ≤ x |] ==> (-1::int) ^(nat x) = -1
proof -
  assume x ∈ zOdd and 0 ≤ x
  then have ∃ k. x = 2 * k + 1
    by (auto simp only: zOdd-def)
  then show ?thesis
    proof
      fix a
      assume x = 2 * a + 1
      from prems have a: 0 ≤ a
        by arith
      from prems have nat x = nat(2 * a + 1)
        by auto
      also from a have nat (2 * a + 1) = 2 * nat a + 1
        by (auto simp add: nat-mult-distrib nat-add-distrib)
      finally have (-1::int) ^nat x = (-1) ^(2 * nat a + 1)
        by auto
      also have ... = ((-1::int) ^2) ^ (nat a) * (-1) ^1
        by (auto simp add: zpower-zpower [THEN sym] zpower-zadd-distrib)
      also have (-1::int) ^2 = 1
        by auto
      finally show ?thesis
        by auto
    qed
qed

lemma neg-one-power-parity: [| 0 ≤ x; 0 ≤ y; (x ∈ zEven) = (y ∈ zEven) |] ==>

  (-1::int) ^(nat x) = (-1::int) ^(nat y)
  apply (insert even-odd-disj [of x])
  apply (insert even-odd-disj [of y])
  by (auto simp add: neg-one-even-power neg-one-odd-power)

lemma one-not-neg-one-mod-m: 2 < m ==> ~([1 = -1] (mod m))
  by (auto simp add: zcong-def zdvd-not-zless)

lemma even-div-2-l: [| y ∈ zEven; x < y |] ==> x div 2 < y div 2
  apply (auto simp only: zEven-def)
  proof -
```

fix k **assume** $x < 2 * k$
then have $x\ div\ 2 < k$ **by** (*auto simp add*: *div-prop1*)
also have $k = (2 * k)\ div\ 2$ **by** *auto*
finally show $x\ div\ 2 < 2 * k\ div\ 2$ **by** *auto*
qed

lemma *even-sum-div-2*: [| $x \in zEven$; $y \in zEven$ |] ==> $(x + y)\ div\ 2 = x\ div\ 2 + y\ div\ 2$
by (*auto simp add*: *zEven-def*, *auto simp add*: *zdiv-zadd1-eq*)

lemma *even-prod-div-2*: [| $x \in zEven$ |] ==> $(x * y)\ div\ 2 = (x\ div\ 2) * y$
by (*auto simp add*: *zEven-def*)

lemma *zprime-zOdd-eq-grt-2*: *zprime* p ==> $(p \in zOdd) = (2 < p)$
apply (*auto simp add*: *zOdd-def zprime-def*)
apply (*drule-tac* $x = 2$ **in** *allE*)
apply (*insert odd-iff-not-even* [*of* p])
by (*auto simp add*: *zOdd-def zEven-def*)

lemma *neg-one-special*: *finite* A ==>
$((-1 :: int)\ \hat{}\ card\ A) * (-1\ \hat{}\ card\ A) = 1$
by (*induct set*: *Finites*, *auto*)

lemma *neg-one-power*: $(-1::int)\hat{}n = 1 \mid (-1::int)\hat{}n = -1$
apply (*induct-tac* n)
by *auto*

lemma *neg-one-power-eq-mod-m*: [| $2 < m$; $[(-1::int)\hat{}j = (-1::int)\hat{}k]\ (mod\ m)$ |]
==> $((-1::int)\hat{}j = (-1::int)\hat{}k)$
apply (*insert neg-one-power* [*of* j])
apply (*insert neg-one-power* [*of* k])
by (*auto simp add*: *one-not-neg-one-mod-m zcong-sym*)

end

14 Euler's criterion

theory *Euler* **imports** *Residues EvenOdd* **begin**

constdefs
MultInvPair :: *int* => *int* => *int* => *int set*
MultInvPair a p j == {*StandardRes p j*, *StandardRes p* $(a * (MultInv\ p\ j))$}
SetS :: *int* => *int* => *int set set*

SetS a p == ((MultInvPair a p) ‘ (SRStar p))

lemma *MultInvPair-prop1a*: [| *zprime p*; *2 < p*; ~([*a = 0*](*mod p*));
X ∈ (SetS a p); *Y ∈ (SetS a p)*;
~((*X ∩ Y*) = {}) |] ==>
X = Y
apply (*auto simp add*: *SetS-def*)
apply (*drule StandardRes-SRStar-prop1a*)+ **defer** *1*
apply (*drule StandardRes-SRStar-prop1a*)+
apply (*auto simp add*: *MultInvPair-def StandardRes-prop2 zcong-sym*)
apply (*drule notE*, *rule MultInv-zcong-prop1*, *auto*)
apply (*drule notE*, *rule MultInv-zcong-prop2*, *auto simp add*: *zcong-sym*)
apply (*drule MultInv-zcong-prop2*, *auto simp add*: *zcong-sym*)
apply (*drule MultInv-zcong-prop3*, *auto simp add*: *zcong-sym*)
apply (*drule MultInv-zcong-prop1*, *auto*)
apply (*drule MultInv-zcong-prop2*, *auto simp add*: *zcong-sym*)
apply (*drule MultInv-zcong-prop2*, *auto simp add*: *zcong-sym*)
apply (*drule MultInv-zcong-prop3*, *auto simp add*: *zcong-sym*)
done

lemma *MultInvPair-prop1b*: [| *zprime p*; *2 < p*; ~([*a = 0*](*mod p*));
X ∈ (SetS a p); *Y ∈ (SetS a p)*;
X ≠ Y |] ==>
X ∩ Y = {}
apply (*rule notnotD*)
apply (*rule notI*)
apply (*drule MultInvPair-prop1a*, *auto*)
done

lemma *MultInvPair-prop1c*: [| *zprime p*; *2 < p*; ~([*a = 0*](*mod p*)) |] ==>
∀ *X ∈ SetS a p*. ∀ *Y ∈ SetS a p*. *X ≠ Y* --> *X∩Y* = {}
by (*auto simp add*: *MultInvPair-prop1b*)

lemma *MultInvPair-prop2*: [| *zprime p*; *2 < p*; ~([*a = 0*](*mod p*)) |] ==>
Union (*SetS a p*) = *SRStar p*
apply (*auto simp add*: *SetS-def MultInvPair-def StandardRes-SRStar-prop4*
SRStar-mult-prop2)
apply (*frule StandardRes-SRStar-prop3*)
apply (*rule bexI*, *auto*)
done

lemma *MultInvPair-distinct*: [| *zprime p*; *2 < p*; ~([*a = 0*] (*mod p*));
~([*j = 0*] (*mod p*));

```
                                ~(QuadRes p a) |]  ==>
                              ~([j = a * MultInv p j] (mod p))
  apply auto
proof -
  assume zprime p and 2 < p and ~([a = 0] (mod p)) and
    ~([j = 0] (mod p)) and ~(QuadRes p a)
  assume [j = a * MultInv p j] (mod p)
  then have [j * j = (a * MultInv p j) * j] (mod p)
    by (auto simp add: zcong-scalar)
  then have a:[j * j = a * (MultInv p j * j)] (mod p)
    by (auto simp add: zmult-ac)
  have [j * j = a] (mod p)
    proof -
      from prems have b: [MultInv p j * j = 1] (mod p)
        by (simp add: MultInv-prop2a)
      from b a show ?thesis
        by (auto simp add: zcong-zmult-prop2)
    qed
  then have [j^2 = a] (mod p)
    apply(subgoal-tac 2 = Suc(Suc(0)))
    apply (erule ssubst)
    apply (auto simp only: power-Suc power-0)
    by auto
  with prems show False
    by (simp add: QuadRes-def)
qed

lemma MultInvPair-card-two: [| zprime p; 2 < p; ~([a = 0] (mod p));
                               ~(QuadRes p a); ~([j = 0] (mod p)) |]  ==>
                            card (MultInvPair a p j) = 2
  apply (auto simp add: MultInvPair-def)
  apply (subgoal-tac ~ (StandardRes p j = StandardRes p (a * MultInv p j)))
  apply auto
  apply (simp only: StandardRes-prop2)
  apply (drule MultInvPair-distinct)
by auto




lemma SetS-finite: 2 < p ==> finite (SetS a p)
  by (auto simp add: SetS-def SRStar-finite [of p] finite-imageI)

lemma SetS-elems-finite: ∀ X ∈ SetS a p. finite X
  by (auto simp add: SetS-def MultInvPair-def)
```

lemma *SetS-elems-card*: [| *zprime p*; *2 < p*; ~([*a = 0*] (*mod p*));
~(*QuadRes p a*) |] ==>
∀ *X* ∈ *SetS a p*. *card X = 2*
apply (*auto simp add*: *SetS-def*)
apply (*frule StandardRes-SRStar-prop1a*)
apply (*rule MultInvPair-card-two*, *auto*)
done

lemma *Union-SetS-finite*: *2 < p* ==> *finite* (*Union* (*SetS a p*))
by (*auto simp add*: *SetS-finite SetS-elems-finite finite-Union*)

lemma *card-setsum-aux*: [| *finite S*; ∀ *X* ∈ *S*. *finite* (*X*::*int set*);
∀ *X* ∈ *S*. *card X = n* |] ==> *setsum card S = setsum* (%*x*. *n*) *S*
by (*induct set*: *Finites*, *auto*)

lemma *SetS-card*: [| *zprime p*; *2 < p*; ~([*a = 0*] (*mod p*)); ~(*QuadRes p a*) |]
==>
int(*card*(*SetS a p*)) = (*p − 1*) *div 2*
proof −
assume *zprime p* **and** *2 < p* **and** ~([*a = 0*] (*mod p*)) **and** ~(*QuadRes p a*)
then have (*p − 1*) = *2 ∗ int*(*card*(*SetS a p*))
proof −
have *p − 1* = *int*(*card*(*Union* (*SetS a p*)))
by (*auto simp add*: *prems MultInvPair-prop2 SRStar-card*)
also have ... = *int* (*setsum card* (*SetS a p*))
by (*auto simp add*: *prems SetS-finite SetS-elems-finite*
MultInvPair-prop1c [*of p a*] *card-Union-disjoint*)
also have ... = *int*(*setsum* (%*x.2*) (*SetS a p*))
apply (*insert prems*)
apply (*auto simp add*: *SetS-elems-card SetS-finite SetS-elems-finite*
card-setsum-aux simp del: *setsum-constant*)
done
also have ... = *2 ∗ int*(*card*(*SetS a p*))
by (*auto simp add*: *prems SetS-finite setsum-const2*)
finally show *?thesis* **.**
qed
from *this* **show** *?thesis*
by *auto*
qed

lemma *SetS-setprod-prop*: [| *zprime p*; *2 < p*; ~([*a = 0*] (*mod p*));
~(*QuadRes p a*); *x* ∈ (*SetS a p*) |] ==>
[∏ *x = a*] (*mod p*)
apply (*auto simp add*: *SetS-def MultInvPair-def*)
apply (*frule StandardRes-SRStar-prop1a*)
apply (*subgoal-tac StandardRes p x* ≠ *StandardRes p* (*a ∗ MultInv p x*))
apply (*auto simp add*: *StandardRes-prop2 MultInvPair-distinct*)
apply (*frule-tac m = p* **and** *x = x* **and** *y* = (*a ∗ MultInv p x*) **in**
StandardRes-prop4)

```
  apply (subgoal-tac [x * (a * MultInv p x) = a * (x * MultInv p x)] (mod p))
  apply (drule-tac a = StandardRes p x * StandardRes p (a * MultInv p x) and
                   b = x * (a * MultInv p x) and
                   c = a * (x * MultInv p x) in  zcong-trans, force)
  apply (frule-tac p = p and x = x in MultInv-prop2, auto)
  apply (drule-tac a = x * MultInv p x and b = 1 in zcong-zmult-prop2)
  apply (auto simp add: zmult-ac)
done

lemma aux1: [| 0 < x; (x::int) < a; x ≠ (a - 1) |] ==> x < a - 1
  by arith

lemma aux2: [| (a::int) < c; b < c |] ==> (a ≤ b | b ≤ a)
  by auto

lemma SRStar-d22set-prop [rule-format]: 2 < p --> (SRStar p) = {1} ∪
    (d22set (p - 1))
  apply (induct p rule: d22set.induct, auto)
  apply (simp add: SRStar-def d22set.simps)
  apply (simp add: SRStar-def d22set.simps, clarify)
  apply (frule aux1)
  apply (frule aux2, auto)
  apply (simp-all add: SRStar-def)
  apply (simp add: d22set.simps)
  apply (frule d22set-le)
  apply (frule d22set-g-1, auto)
done

lemma Union-SetS-setprod-prop1: [| zprime p; 2 < p; ~([a = 0] (mod p)); ~(QuadRes
p a) |] ==>
                                [∏(Union (SetS a p)) = a ^ nat ((p - 1) div 2)] (mod
p)
proof -
  assume zprime p and 2 < p and  ~([a = 0] (mod p)) and ~(QuadRes p a)
  then have [∏(Union (SetS a p)) =
      setprod (setprod (%x. x)) (SetS a p)] (mod p)
    by (auto simp add: SetS-finite SetS-elems-finite
                       MultInvPair-prop1c setprod-Union-disjoint)
  also have [setprod (setprod (%x. x)) (SetS a p) =
      setprod (%x. a) (SetS a p)] (mod p)
    apply (rule setprod-same-function-zcong)
    by (auto simp add: prems SetS-setprod-prop SetS-finite)
  also (zcong-trans) have [setprod (%x. a) (SetS a p) =
      a^(card (SetS a p))] (mod p)
    by (auto simp add: prems SetS-finite setprod-constant)
  finally (zcong-trans) show ?thesis
    apply (rule zcong-trans)
    apply (subgoal-tac card(SetS a p) = nat((p - 1) div 2), auto)
    apply (subgoal-tac nat(int(card(SetS a p))) = nat((p - 1) div 2), force)
```

```
    apply (auto simp add: prems SetS-card)
  done
qed

lemma Union-SetS-setprod-prop2: [| zprime p; 2 < p; ~([a = 0](mod p)) |] ==>

                                  ∏(Union (SetS a p)) = zfact (p - 1)
proof -
  assume zprime p and 2 < p and ~([a = 0](mod p))
  then have ∏(Union (SetS a p)) = ∏(SRStar p)
    by (auto simp add: MultInvPair-prop2)
  also have ... = ∏({1} ∪ (d22set (p - 1)))
    by (auto simp add: prems SRStar-d22set-prop)
  also have ... = zfact(p - 1)
  proof -
    have ~(1 ∈ d22set (p - 1)) & finite( d22set (p - 1))
     apply (insert prems, auto)
     apply (drule d22set-g-1)
     apply (auto simp add: d22set-fin)
    done
    then have ∏({1} ∪ (d22set (p - 1))) = ∏(d22set (p - 1))
      by auto
    then show ?thesis
      by (auto simp add: d22set-prod-zfact)
  qed
  finally show ?thesis .
qed

lemma zfact-prop: [| zprime p; 2 < p; ~([a = 0] (mod p)); ~(QuadRes p a) |]
==>
                [zfact (p - 1) = a ^ nat ((p - 1) div 2)] (mod p)
  apply (frule Union-SetS-setprod-prop1)
  apply (auto simp add: Union-SetS-setprod-prop2)
done

lemma Euler-part1: [| 2 < p; zprime p; ~([x = 0](mod p));
    ~(QuadRes p x) |] ==>
      [x^(nat (((p) - 1) div 2)) = -1](mod p)
  apply (frule zfact-prop, auto)
  apply (frule Wilson-Russ)
  apply (auto simp add: zcong-sym)
```

```
  apply (rule zcong-trans, auto)
done
```

```
lemma aux-1: 0 < p ==> (a::int) ^ nat (p) = a * a ^ (nat (p) - 1)
proof -
  assume 0 < p
  then have a ^ (nat p) =  a ^ (1 + (nat p - 1))
    by (auto simp add: diff-add-assoc)
  also have ... = (a ^ 1) * a ^ (nat(p) - 1)
    by (simp only: zpower-zadd-distrib)
  also have ... = a * a ^ (nat(p) - 1)
    by auto
  finally show ?thesis .
qed

lemma aux-2: [| (2::int) < p; p ∈ zOdd |] ==> 0 < ((p - 1) div 2)
proof -
  assume 2 < p and p ∈ zOdd
  then have (p - 1):zEven
    by (auto simp add: zEven-def zOdd-def)
  then have aux-1: 2 * ((p - 1) div 2) = (p - 1)
    by (auto simp add: even-div-2-prop2)
  then have 1 < (p - 1)
    by auto
  then have  1 < (2 * ((p - 1) div 2))
    by (auto simp add: aux-1)
  then have 0 < (2 * ((p - 1) div 2)) div 2
    by auto
  then show ?thesis by auto
qed

lemma Euler-part2: [| 2 < p; zprime p; [a = 0] (mod p) |] ==> [0 = a ^ nat ((p
- 1) div 2)] (mod p)
  apply (frule zprime-zOdd-eq-grt-2)
  apply (frule aux-2, auto)
  apply (frule-tac a = a in aux-1, auto)
  apply (frule zcong-zmult-prop1, auto)
done
```

lemma *aux--1*: [| ~([x = 0] (mod p)); [y ^ 2 = x] (mod p)|] ==> ~(p dvd y)
apply (*subgoal-tac* [| ~([x = 0] (mod p)); [y ^ 2 = x] (mod p)|] ==>
~([y ^ 2 = 0] (mod p)))
apply (*auto simp add*: *zcong-sym* [*of* y^2 x p] *intro*: *zcong-trans*)
apply (*auto simp add*: *zcong-eq-zdvd-prop intro*: *zpower-zdvd-prop1*)
done

lemma *aux--2*: 2 * nat((p - 1) div 2) = nat (2 * ((p - 1) div 2))
by (*auto simp add*: *nat-mult-distrib*)

lemma *Euler-part3*: [| 2 < p; zprime p; ~([x = 0](mod p)); QuadRes p x |] ==>

[x^(nat (((p) - 1) div 2)) = 1](mod p)
apply (*subgoal-tac* p ∈ *zOdd*)
apply (*auto simp add*: *QuadRes-def*)
apply (*frule aux--1*, *auto*)
apply (*drule-tac* z = nat ((p - 1) div 2) **in** *zcong-zpower*)
apply (*auto simp add*: *zpower-zpower*)
apply (*rule zcong-trans*)
apply (*auto simp add*: *zcong-sym* [*of* x ^ nat ((p - 1) div 2)])
apply (*simp add*: *aux--2*)
apply (*frule odd-minus-one-even*)
apply (*frule even-div-2-prop2*)
apply (*auto intro*: *Little-Fermat simp add*: *zprime-zOdd-eq-grt-2*)
done

theorem *Euler-Criterion*: [| 2 < p; zprime p |] ==> [(*Legendre* a p) =
a^(nat (((p) - 1) div 2))] (mod p)
apply (*auto simp add*: *Legendre-def Euler-part2*)
apply (*frule Euler-part3*, *auto simp add*: *zcong-sym*)
apply (*frule Euler-part1*, *auto simp add*: *zcong-sym*)
done

end

15 Gauss' Lemma

theory *Gauss* **imports** *Euler* **begin**

locale *GAUSS* =
  fixes *p* :: *int*
  fixes *a* :: *int*
  fixes *A* :: *int set*
  fixes *B* :: *int set*
  fixes *C* :: *int set*
  fixes *D* :: *int set*
  fixes *E* :: *int set*
  fixes *F* :: *int set*

  assumes *p-prime*: *zprime p*
  assumes *p-g-2*: *2 < p*
  assumes *p-a-relprime*: ~[*a = 0*](*mod p*)
  assumes *a-nonzero*: *0 < a*

  defines *A-def*: *A == {*(*x*::*int*)*. 0 < x & x ≤ ((p − 1) div 2)}*
  defines *B-def*: *B == (%x. x * a) ‘ A*
  defines *C-def*: *C == (StandardRes p) ‘ B*
  defines *D-def*: *D == C ∩ {x. x ≤ ((p − 1) div 2)}*
  defines *E-def*: *E == C ∩ {x. ((p − 1) div 2) < x}*
  defines *F-def*: *F == (%x. (p − x)) ‘ E*

15.1 Basic properties of p

lemma (**in** *GAUSS*) *p-odd*: *p ∈ zOdd*
  by (*auto simp add*: *p-prime p-g-2 zprime-zOdd-eq-grt-2*)

lemma (**in** *GAUSS*) *p-g-0*: *0 < p*
  by (*insert p-g-2*, *auto*)

lemma (**in** *GAUSS*) *int-nat*: *int* (*nat* ((*p − 1*) *div 2*)) = (*p − 1*) *div 2*
  by (*insert p-g-2*, *auto simp add*: *pos-imp-zdiv-nonneg-iff*)

lemma (**in** *GAUSS*) *p-minus-one-l*: (*p − 1*) *div 2 < p*
  proof −
    have *p − 1 = (p − 1) div 1* **by** *auto*
    then have (*p − 1*) *div 2 ≤ p − 1*
      apply (*rule ssubst*) **back**
      apply (*rule zdiv-mono2*)
      by (*auto simp add*: *p-g-0*)
    then have (*p − 1*) *div 2 ≤ p − 1*
      by *auto*
    then show *?thesis* **by** *simp*
qed

lemma (**in** *GAUSS*) *p-eq*: *p = (2 * (p − 1) div 2) + 1*
  apply (*insert zdiv-zmult-self2* [*of 2 p − 1*])
by *auto*

lemma *zodd-imp-zdiv-eq*: $x \in zOdd \Longrightarrow 2 * (x - 1)\ div\ 2 = 2 * ((x - 1)\ div\ 2)$
  apply (*frule odd-minus-one-even*)
  apply (*simp add*: *zEven-def*)
  apply (*subgoal-tac* $2 \neq 0$)
  apply (*frule-tac* $b = 2$ :: *int* **and** $a = x - 1$ **in** *zdiv-zmult-self2*)
by (*auto simp add*: *even-div-2-prop2*)

lemma (**in** *GAUSS*) *p-eq2*: $p = (2 * ((p - 1)\ div\ 2)) + 1$
  apply (*insert p-eq p-prime p-g-2 zprime-zOdd-eq-grt-2* [*of p*], *auto*)
by (*frule zodd-imp-zdiv-eq*, *auto*)

15.2 Basic Properties of the Gauss Sets

lemma (**in** *GAUSS*) *finite-A*: *finite* (*A*)
  apply (*auto simp add*: *A-def*)
thm *bdd-int-set-l-finite*
  apply (*subgoal-tac* $\{x.\ 0 < x\ \&\ x \leq (p - 1)\ div\ 2\} \subseteq \{x.\ 0 \leq x\ \&\ x < 1 + (p - 1)\ div\ 2\}$)
by (*auto simp add*: *bdd-int-set-l-finite finite-subset*)

lemma (**in** *GAUSS*) *finite-B*: *finite* (*B*)
  by (*auto simp add*: *B-def finite-A finite-imageI*)

lemma (**in** *GAUSS*) *finite-C*: *finite* (*C*)
  by (*auto simp add*: *C-def finite-B finite-imageI*)

lemma (**in** *GAUSS*) *finite-D*: *finite* (*D*)
  by (*auto simp add*: *D-def finite-Int finite-C*)

lemma (**in** *GAUSS*) *finite-E*: *finite* (*E*)
  by (*auto simp add*: *E-def finite-Int finite-C*)

lemma (**in** *GAUSS*) *finite-F*: *finite* (*F*)
  by (*auto simp add*: *F-def finite-E finite-imageI*)

lemma (**in** *GAUSS*) *C-eq*: $C = D \cup E$
  by (*auto simp add*: *C-def D-def E-def*)

lemma (**in** *GAUSS*) *A-card-eq*: *card A* = *nat* $((p - 1)\ div\ 2)$
  apply (*auto simp add*: *A-def*)
  apply (*insert int-nat*)
  apply (*erule subst*)
  by (*auto simp add*: *card-bdd-int-set-l-le*)

lemma (**in** *GAUSS*) *inj-on-xa-A*: *inj-on* $(\%x.\ x * a)\ A$
  apply (*insert a-nonzero*)
by (*simp add*: *A-def inj-on-def*)

lemma (**in** *GAUSS*) *A-res*: *ResSet p A*
apply (*auto simp add*: *A-def ResSet-def*)
apply (*rule-tac m* = *p* **in** *zcong-less-eq*)
apply (*insert p-g-2*, *auto*)
apply (*subgoal-tac* [*1−2*] (*p* − *1*) *div 2* < *p*)
by (*auto*, *auto simp add*: *p-minus-one-l*)

lemma (**in** *GAUSS*) *B-res*: *ResSet p B*
apply (*insert p-g-2 p-a-relprime p-minus-one-l*)
apply (*auto simp add*: *B-def*)
apply (*rule ResSet-image*)
apply (*auto simp add*: *A-res*)
apply (*auto simp add*: *A-def*)
proof −
fix *x* **fix** *y*
assume *a*: [*x* ∗ *a* = *y* ∗ *a*] (*mod p*)
assume *b*: *0* < *x*
assume *c*: *x* ≤ (*p* − *1*) *div 2*
assume *d*: *0* < *y*
assume *e*: *y* ≤ (*p* − *1*) *div 2*
from *a p-a-relprime p-prime a-nonzero zcong-cancel* [*of p a x y*]
have [*x* = *y*](*mod p*)
by (*simp add*: *zprime-imp-zrelprime zcong-def p-g-0 order-le-less*)
with *zcong-less-eq* [*of x y p*] *p-minus-one-l*
order-le-less-trans [*of x* (*p* − *1*) *div 2 p*]
order-le-less-trans [*of y* (*p* − *1*) *div 2 p*] **show** *x* = *y*
by (*simp add*: *prems p-minus-one-l p-g-0*)
qed

lemma (**in** *GAUSS*) *SR-B-inj*: *inj-on* (*StandardRes p*) *B*
apply (*auto simp add*: *B-def StandardRes-def inj-on-def A-def prems*)
proof −
fix *x* **fix** *y*
assume *a*: *x* ∗ *a mod p* = *y* ∗ *a mod p*
assume *b*: *0* < *x*
assume *c*: *x* ≤ (*p* − *1*) *div 2*
assume *d*: *0* < *y*
assume *e*: *y* ≤ (*p* − *1*) *div 2*
assume *f*: *x* ≠ *y*
from *a* **have** [*x* ∗ *a* = *y* ∗ *a*](*mod p*)
by (*simp add*: *zcong-zmod-eq p-g-0*)
with *p-a-relprime p-prime a-nonzero zcong-cancel* [*of p a x y*]
have [*x* = *y*](*mod p*)
by (*simp add*: *zprime-imp-zrelprime zcong-def p-g-0 order-le-less*)
with *zcong-less-eq* [*of x y p*] *p-minus-one-l*
order-le-less-trans [*of x* (*p* − *1*) *div 2 p*]
order-le-less-trans [*of y* (*p* − *1*) *div 2 p*] **have** *x* = *y*
by (*simp add*: *prems p-minus-one-l p-g-0*)

```
    then have False
      by (simp add: f)
    then show a = 0
      by simp
qed

lemma (in GAUSS) inj-on-pminusx-E: inj-on (%x. p - x) E
  apply (auto simp add: E-def C-def B-def A-def)
  apply (rule-tac g = %x. -1 * (x - p) in inj-on-inverseI)
by auto

lemma (in GAUSS) A-ncong-p: x ∈ A ==> ~[x = 0](mod p)
  apply (auto simp add: A-def)
  apply (frule-tac m = p in zcong-not-zero)
  apply (insert p-minus-one-l)
by auto

lemma (in GAUSS) A-greater-zero: x ∈ A ==> 0 < x
  by (auto simp add: A-def)

lemma (in GAUSS) B-ncong-p: x ∈ B ==> ~[x = 0](mod p)
  apply (auto simp add: B-def)
  apply (frule A-ncong-p)
  apply (insert p-a-relprime p-prime a-nonzero)
  apply (frule-tac a = x and b = a in zcong-zprime-prod-zero-contra)
by (auto simp add: A-greater-zero)

lemma (in GAUSS) B-greater-zero: x ∈ B ==> 0 < x
  apply (insert a-nonzero)
by (auto simp add: B-def mult-pos-pos A-greater-zero)

lemma (in GAUSS) C-ncong-p: x ∈ C ==>  ~[x = 0](mod p)
  apply (auto simp add: C-def)
  apply (frule B-ncong-p)
  apply (subgoal-tac [x = StandardRes p x](mod p))
  defer apply (simp add: StandardRes-prop1)
  apply (frule-tac a = x and b = StandardRes p x and c = 0 in zcong-trans)
by auto

lemma (in GAUSS) C-greater-zero: y ∈ C ==> 0 < y
  apply (auto simp add: C-def)
  proof -
    fix x
    assume a: x ∈ B
    from p-g-0 have 0 ≤ StandardRes p x
      by (simp add: StandardRes-lbound)
    moreover have ~[x = 0] (mod p)
      by (simp add: a B-ncong-p)
    then have StandardRes p x ≠ 0
```

```
    by (simp add: StandardRes-prop3)
  ultimately show 0 < StandardRes p x
    by (simp add: order-le-less)
qed

lemma (in GAUSS) D-ncong-p: x ∈ D ==> ~[x = 0](mod p)
  by (auto simp add: D-def C-ncong-p)

lemma (in GAUSS) E-ncong-p: x ∈ E ==> ~[x = 0](mod p)
  by (auto simp add: E-def C-ncong-p)

lemma (in GAUSS) F-ncong-p: x ∈ F ==> ~[x = 0](mod p)
  apply (auto simp add: F-def)
  proof -
    fix x assume a: x ∈ E assume b: [p - x = 0] (mod p)
    from E-ncong-p have ~[x = 0] (mod p)
      by (simp add: a)
    moreover from a have 0 < x
      by (simp add: a E-def C-greater-zero)
    moreover from a have x < p
      by (auto simp add: E-def C-def p-g-0 StandardRes-ubound)
    ultimately have ~[p - x = 0] (mod p)
      by (simp add: zcong-not-zero)
    from this show False by (simp add: b)
qed

lemma (in GAUSS) F-subset: F ⊆ {x. 0 < x & x ≤ ((p - 1) div 2)}
  apply (auto simp add: F-def E-def)
  apply (insert p-g-0)
  apply (frule-tac x = xa in StandardRes-ubound)
  apply (frule-tac x = x in StandardRes-ubound)
  apply (subgoal-tac xa = StandardRes p xa)
  apply (auto simp add: C-def StandardRes-prop2 StandardRes-prop1)
  proof -
    from zodd-imp-zdiv-eq p-prime p-g-2 zprime-zOdd-eq-grt-2 have
        2 * (p - 1) div 2 = 2 * ((p - 1) div 2)
      by simp
    with p-eq2 show  !!x. [| (p - 1) div 2 < StandardRes p x; x ∈ B |]
        ==> p - StandardRes p x ≤ (p - 1) div 2
      by simp
qed

lemma (in GAUSS) D-subset: D ⊆ {x. 0 < x & x ≤ ((p - 1) div 2)}
  by (auto simp add: D-def C-greater-zero)

lemma (in GAUSS) F-eq: F = {x. ∃ y ∈ A. ( x = p - (StandardRes p (y*a)) &
(p - 1) div 2 < StandardRes p (y*a))}
  by (auto simp add: F-def E-def D-def C-def B-def A-def)
```

lemma (**in** *GAUSS*) *D-eq*: $D = \{x.\ \exists y \in A.\ (\ x = StandardRes\ p\ (y*a)\ \&\ StandardRes\ p\ (y*a) \leq (p - 1)\ div\ 2)\}$
by (*auto simp add*: *D-def C-def B-def A-def*)

lemma (**in** *GAUSS*) *D-leq*: $x \in D \Longrightarrow x \leq (p - 1)\ div\ 2$
by (*auto simp add*: *D-eq*)

lemma (**in** *GAUSS*) *F-ge*: $x \in F \Longrightarrow x \leq (p - 1)\ div\ 2$
apply (*auto simp add*: *F-eq A-def*)
proof −
fix *y*
assume $(p - 1)\ div\ 2 < StandardRes\ p\ (y * a)$
then have $p - StandardRes\ p\ (y * a) < p - ((p - 1)\ div\ 2)$
by *arith*
also from *p-eq2* **have** $... = 2 * ((p - 1)\ div\ 2) + 1 - ((p - 1)\ div\ 2)$
by (*rule subst*, *auto*)
also have $2 * ((p - 1)\ div\ 2) + 1 - (p - 1)\ div\ 2 = (p - 1)\ div\ 2 + 1$
by *arith*
finally show $p - StandardRes\ p\ (y * a) \leq (p - 1)\ div\ 2$
by (*insert zless-add1-eq* [*of* $p - StandardRes\ p\ (y * a)$ $(p - 1)\ div\ 2$],*auto*)
qed

lemma (**in** *GAUSS*) *all-A-relprime*: $\forall x \in A.\ zgcd(x,p) = 1$
apply (*insert p-prime p-minus-one-l*)
by (*auto simp add*: *A-def zless-zprime-imp-zrelprime*)

lemma (**in** *GAUSS*) *A-prod-relprime*: $zgcd((setprod\ id\ A),p) = 1$
by (*insert all-A-relprime finite-A*, *simp add*: *all-relprime-prod-relprime*)

15.3 Relationships Between Gauss Sets

lemma (**in** *GAUSS*) *B-card-eq-A*: $card\ B = card\ A$
apply (*insert finite-A*)
by (*simp add*: *finite-A B-def inj-on-xa-A card-image*)

lemma (**in** *GAUSS*) *B-card-eq*: $card\ B = nat\ ((p - 1)\ div\ 2)$
by (*auto simp add*: *B-card-eq-A A-card-eq*)

lemma (**in** *GAUSS*) *F-card-eq-E*: $card\ F = card\ E$
apply (*insert finite-E*)
by (*simp add*: *F-def inj-on-pminusx-E card-image*)

lemma (**in** *GAUSS*) *C-card-eq-B*: $card\ C = card\ B$
apply (*insert finite-B*)
apply (*subgoal-tac inj-on* (*StandardRes p*) *B*)
apply (*simp add*: *B-def C-def card-image*)
apply (*rule StandardRes-inj-on-ResSet*)
by (*simp add*: *B-res*)

lemma (**in** *GAUSS*) *D-E-disj*: $D \cap E = \{\}$
by (*auto simp add*: *D-def E-def*)

lemma (**in** *GAUSS*) *C-card-eq-D-plus-E*: *card C* = *card D* + *card E*
by (*auto simp add*: *C-eq card-Un-disjoint D-E-disj finite-D finite-E*)

lemma (**in** *GAUSS*) *C-prod-eq-D-times-E*: *setprod id E* ∗ *setprod id D* = *setprod id C*
apply (*insert D-E-disj finite-D finite-E C-eq*)
apply (*frule setprod-Un-disjoint* [*of D E id*])
by *auto*

lemma (**in** *GAUSS*) *C-B-zcong-prod*: [*setprod id C* = *setprod id B*] (*mod p*)
apply (*auto simp add*: *C-def*)
apply (*insert finite-B SR-B-inj*)
apply (*frule-tac f1* = *StandardRes p* **in** *setprod-reindex-id*[*THEN sym*], *auto*)
apply (*rule setprod-same-function-zcong*)
by (*auto simp add*: *StandardRes-prop1 zcong-sym p-g-0*)

lemma (**in** *GAUSS*) *F-Un-D-subset*: $(F \cup D) \subseteq A$
apply (*rule Un-least*)
by (*auto simp add*: *A-def F-subset D-subset*)

lemma *two-eq*: *2* ∗ (*x*::*int*) = *x* + *x*
by *arith*

lemma (**in** *GAUSS*) *F-D-disj*: $(F \cap D) = \{\}$
apply (*simp add*: *F-eq D-eq*)
apply (*auto simp add*: *F-eq D-eq*)
proof −
fix *y* **fix** *ya*
assume *p* − *StandardRes p* (*y* ∗ *a*) = *StandardRes p* (*ya* ∗ *a*)
then have *p* = *StandardRes p* (*y* ∗ *a*) + *StandardRes p* (*ya* ∗ *a*)
by *arith*
moreover have *p dvd p*
by *auto*
ultimately have *p dvd* (*StandardRes p* (*y* ∗ *a*) + *StandardRes p* (*ya* ∗ *a*))
by *auto*
then have *a*: [*StandardRes p* (*y* ∗ *a*) + *StandardRes p* (*ya* ∗ *a*) = *0*] (*mod p*)
by (*auto simp add*: *zcong-def*)
have [*y* ∗ *a* = *StandardRes p* (*y* ∗ *a*)] (*mod p*)
by (*simp only*: *zcong-sym StandardRes-prop1*)
moreover have [*ya* ∗ *a* = *StandardRes p* (*ya* ∗ *a*)] (*mod p*)
by (*simp only*: *zcong-sym StandardRes-prop1*)
ultimately have [*y* ∗ *a* + *ya* ∗ *a* =
StandardRes p (*y* ∗ *a*) + *StandardRes p* (*ya* ∗ *a*)] (*mod p*)
by (*rule zcong-zadd*)
with *a* **have** [*y* ∗ *a* + *ya* ∗ *a* = *0*] (*mod p*)

```
    apply (elim zcong-trans)
    by (simp only: zcong-refl)
  also have y * a + ya * a = a * (y + ya)
    by (simp add: zadd-zmult-distrib2 zmult-commute)
  finally have [a * (y + ya) = 0] (mod p).
  with p-prime a-nonzero zcong-zprime-prod-zero [of p a y + ya]
      p-a-relprime
      have a: [y + ya = 0] (mod p)
    by auto
  assume b: y ∈ A and c: ya: A
  with A-def have 0 < y + ya
    by auto
  moreover from b c A-def have y + ya ≤ (p - 1) div 2 + (p - 1) div 2
    by auto
  moreover from b c p-eq2 A-def have y + ya < p
    by auto
  ultimately show False
    apply simp
    apply (frule-tac m = p in zcong-not-zero)
    by (auto simp add: a)
qed

lemma (in GAUSS) F-Un-D-card: card (F ∪ D) = nat ((p - 1) div 2)
  apply (insert F-D-disj finite-F finite-D)
  proof -
    have card (F ∪ D) = card E + card D
      by (auto simp add: finite-F finite-D F-D-disj
                         card-Un-disjoint F-card-eq-E)
    then have card (F ∪ D) = card C
      by (simp add: C-card-eq-D-plus-E)
    from this show card (F ∪ D) = nat ((p - 1) div 2)
      by (simp add: C-card-eq-B B-card-eq)
qed

lemma (in GAUSS) F-Un-D-eq-A: F ∪ D = A
  apply (insert finite-A F-Un-D-subset A-card-eq F-Un-D-card)
by (auto simp add: card-seteq)

lemma (in GAUSS) prod-D-F-eq-prod-A:
    (setprod id D) * (setprod id F) = setprod id A
  apply (insert F-D-disj finite-D finite-F)
  apply (frule setprod-Un-disjoint [of F D id])
by (auto simp add: F-Un-D-eq-A)

lemma (in GAUSS) prod-F-zcong:
    [setprod id F = ((-1) ^ (card E)) * (setprod id E)] (mod p)
  proof -
    have setprod id F = setprod id (op - p ` E)
      by (auto simp add: F-def)
```

```
  then have setprod id F = setprod (op - p) E
    apply simp
    apply (insert finite-E inj-on-pminusx-E)
    by (frule-tac f = op - p in setprod-reindex-id, auto)
  then have one:
    [setprod id F = setprod (StandardRes p o (op - p)) E] (mod p)
    apply simp
    apply (insert p-g-0 finite-E)
    by (auto simp add: StandardRes-prod)
  moreover have a: ∀ x ∈ E. [p - x = 0 - x] (mod p)
    apply clarify
    apply (insert zcong-id [of p])
    by (rule-tac a = p and m = p and c = x and d = x in zcong-zdiff, auto)
  moreover have b: ∀ x ∈ E. [StandardRes p (p - x) = p - x](mod p)
    apply clarify
    by (simp add: StandardRes-prop1 zcong-sym)
  moreover have ∀ x ∈ E. [StandardRes p (p - x) = - x](mod p)
    apply clarify
    apply (insert a b)
    by (rule-tac b = p - x in zcong-trans, auto)
  ultimately have c:
    [setprod (StandardRes p o (op - p)) E = setprod (uminus) E](mod p)
    apply simp
    apply (insert finite-E p-g-0)
    by (rule setprod-same-function-zcong [of E StandardRes p o (op - p)
                                          uminus p], auto)
  then have two: [setprod id F = setprod (uminus) E](mod p)
    apply (insert one c)
    by (rule zcong-trans [of setprod id F
                          setprod (StandardRes p o op - p) E p
                          setprod uminus E], auto)
  also have setprod uminus E = (setprod id E) * (-1) ^(card E)
    apply (insert finite-E)
    by (induct set: Finites, auto)
  then have setprod uminus E = (-1) ^ (card E) * (setprod id E)
    by (simp add: zmult-commute)
  with two show ?thesis
    by simp
qed
```

15.4 Gauss' Lemma

```
lemma (in GAUSS) aux: setprod id A * -1 ^ card E * a ^ card A * -1 ^ card E
= setprod id A * a ^ card A
  by (auto simp add: finite-E neg-one-special)

theorem (in GAUSS) pre-gauss-lemma:
    [a ^ nat((p - 1) div 2) = (-1) ^ (card E)] (mod p)
  proof -
```

have [*setprod id A = setprod id F * setprod id D*](*mod p*)
by (*auto simp add*: *prod-D-F-eq-prod-A zmult-commute*)
then have [*setprod id A = ((−1) ^(card E) * setprod id E) *
setprod id D] (*mod p*)
apply (*rule zcong-trans*)
by (*auto simp add*: *prod-F-zcong zcong-scalar*)
then have [*setprod id A = ((−1) ^(card E) * setprod id C)*] (*mod p*)
apply (*rule zcong-trans*)
apply (*insert C-prod-eq-D-times-E*, *erule subst*)
by (*subst zmult-assoc*, *auto*)
then have [*setprod id A = ((−1) ^(card E) * setprod id B)*] (*mod p*)
apply (*rule zcong-trans*)
by (*simp add*: *C-B-zcong-prod zcong-scalar2*)
then have [*setprod id A = ((−1) ^(card E) *
(setprod id ((%x. x * a) ‘ A)))*] (*mod p*)
by (*simp add*: *B-def*)
then have [*setprod id A = ((−1) ^(card E) * (setprod (%x. x * a) A))*]
(*mod p*)
by(*simp add*:*finite-A inj-on-xa-A setprod-reindex-id*[*symmetric*])
moreover have *setprod (%x. x * a) A =
setprod (%x. a) A * setprod id A*
by (*insert finite-A*, *induct set*: *Finites*, *auto*)
ultimately have [*setprod id A = ((−1) ^(card E) * (setprod (%x. a) A *
setprod id A))*] (*mod p*)
by *simp*
then have [*setprod id A = ((−1) ^(card E) * a^(card A) *
setprod id A)*](*mod p*)
apply (*rule zcong-trans*)
by (*simp add*: *zcong-scalar2 zcong-scalar finite-A setprod-constant
zmult-assoc*)
then have *a*: [*setprod id A * (−1) ^(card E) =
((−1) ^(card E) * a^(card A) * setprod id A * (−1) ^(card E))*](*mod p*)
by (*rule zcong-scalar*)
then have [*setprod id A * (−1) ^(card E) = setprod id A *
(−1) ^(card E) * a^(card A) * (−1) ^(card E)*](*mod p*)
apply (*rule zcong-trans*)
by (*simp add*: *a mult-commute mult-left-commute*)
then have [*setprod id A * (−1) ^(card E) = setprod id A *
a^(card A)*](*mod p*)
apply (*rule zcong-trans*)
by (*simp add*: *aux*)
with *this zcong-cancel2* [*of p setprod id A −1 ^ card E a ^ card A*]
p-g-0 A-prod-relprime **have** [*−1 ^ card E = a ^ card A*](*mod p*)
by (*simp add*: *order-less-imp-le*)
from *this* **show** *?thesis*
by (*simp add*: *A-card-eq zcong-sym*)
qed

theorem (**in** *GAUSS*) *gauss-lemma*: (*Legendre a p*) = (*−1*) ^ (*card E*)

proof −
from *Euler-Criterion p-prime p-g-2* **have**
[(*Legendre a p*) = *a*^(*nat* (((*p*) − *1*) *div 2*))] (*mod p*)
by *auto*
moreover note *pre-gauss-lemma*
ultimately have [(*Legendre a p*) = (−*1*) ^ (*card E*)] (*mod p*)
by (*rule zcong-trans*)
moreover from *p-a-relprime* **have** (*Legendre a p*) = *1* | (*Legendre a p*) = (−*1*)
by (*auto simp add*: *Legendre-def*)
moreover have (−*1*::*int*) ^ (*card E*) = *1* | (−*1*::*int*) ^ (*card E*) = −*1*
by (*rule neg-one-power*)
ultimately show *?thesis*
by (*auto simp add*: *p-g-2 one-not-neg-one-mod-m zcong-sym*)
qed

end

16 The law of Quadratic reciprocity

theory *Quadratic-Reciprocity*
imports *Gauss*
begin

lemma (**in** *GAUSS*) *QRLemma1*: *a* ∗ *setsum id A* =
p ∗ *setsum* (%*x*. ((*x* ∗ *a*) *div p*)) *A* + *setsum id D* + *setsum id E*
proof −
from *finite-A* **have** *a* ∗ *setsum id A* = *setsum* (%*x*. *a* ∗ *x*) *A*
by (*auto simp add*: *setsum-const-mult id-def*)
also have *setsum* (%*x*. *a* ∗ *x*) = *setsum* (%*x*. *x* ∗ *a*)
by (*auto simp add*: *zmult-commute*)
also have *setsum* (%*x*. *x* ∗ *a*) *A* = *setsum id B*
by (*simp add*: *B-def setsum-reindex-id*[*OF inj-on-xa-A*])
also have ... = *setsum* (%*x*. *p* ∗ (*x div p*) + *StandardRes p x*) *B*
by (*auto simp add*: *StandardRes-def zmod-zdiv-equality*)
also have ... = *setsum* (%*x*. *p* ∗ (*x div p*)) *B* + *setsum* (*StandardRes p*) *B*
by (*rule setsum-addf*)
also have *setsum* (*StandardRes p*) *B* = *setsum id C*
by (*auto simp add*: *C-def setsum-reindex-id*[*OF SR-B-inj*])
also from *C-eq* **have** ... = *setsum id* (*D* ∪ *E*)
by *auto*
also from *finite-D finite-E* **have** ... = *setsum id D* + *setsum id E*

```
    apply (rule setsum-Un-disjoint)
    by (auto simp add: D-def E-def)
  also have setsum (%x. p * (x div p)) B =
      setsum ((%x. p * (x div p)) o (%x. (x * a))) A
    by (auto simp add: B-def setsum-reindex inj-on-xa-A)
  also have ... = setsum (%x. p * ((x * a) div p)) A
    by (auto simp add: o-def)
  also from finite-A have setsum (%x. p * ((x * a) div p)) A =
    p * setsum (%x. ((x * a) div p)) A
    by (auto simp add: setsum-const-mult)
  finally show ?thesis by arith
qed

lemma (in GAUSS) QRLemma2: setsum id A = p * int (card E) - setsum id E
+
  setsum id D
proof -
  from F-Un-D-eq-A have setsum id A = setsum id (D ∪ F)
    by (simp add: Un-commute)
  also from F-D-disj finite-D finite-F have
      ... = setsum id D + setsum id F
    apply (simp add: Int-commute)
    by (intro setsum-Un-disjoint)
  also from F-def have F = (%x. (p - x)) ` E
    by auto
  also from finite-E inj-on-pminusx-E have setsum id ((%x. (p - x)) ` E) =
      setsum (%x. (p - x)) E
    by (auto simp add: setsum-reindex)
  also from finite-E have setsum (op - p) E = setsum (%x. p) E - setsum id E
    by (auto simp add: setsum-subtractf id-def)
  also from finite-E have setsum (%x. p) E = p * int(card E)
    by (intro setsum-const)
  finally show ?thesis
    by arith
qed

lemma (in GAUSS) QRLemma3: (a - 1) * setsum id A =
    p * (setsum (%x. ((x * a) div p)) A - int(card E)) + 2 * setsum id E
proof -
  have (a - 1) * setsum id A = a * setsum id A - setsum id A
    by (auto simp add: zdiff-zmult-distrib)
  also note QRLemma1
  also from QRLemma2 have p * (∑ x ∈ A. x * a div p) + setsum id D +
     setsum id E - setsum id A =
      p * (∑ x ∈ A. x * a div p) + setsum id D +
      setsum id E - (p * int (card E) - setsum id E + setsum id D)
    by auto
  also have ... = p * (∑ x ∈ A. x * a div p) -
      p * int (card E) + 2 * setsum id E
```

```
    by arith
  finally show ?thesis
    by (auto simp only: zdiff-zmult-distrib2)
qed

lemma (in GAUSS) QRLemma4: a ∈ zOdd ==>
    (setsum (%x. ((x * a) div p)) A ∈ zEven) = (int(card E): zEven)
proof −
  assume a-odd: a ∈ zOdd
  from QRLemma3 have a: p * (setsum (%x. ((x * a) div p)) A − int(card E))
=
      (a − 1) * setsum id A − 2 * setsum id E
    by arith
  from a-odd have a − 1 ∈ zEven
    by (rule odd-minus-one-even)
  hence (a − 1) * setsum id A ∈ zEven
    by (rule even-times-either)
  moreover have 2 * setsum id E ∈ zEven
    by (auto simp add: zEven-def)
  ultimately have (a − 1) * setsum id A − 2 * setsum id E ∈ zEven
    by (rule even-minus-even)
  with a have p * (setsum (%x. ((x * a) div p)) A − int(card E)): zEven
    by simp
  hence p ∈ zEven | (setsum (%x. ((x * a) div p)) A − int(card E)): zEven
    by (rule EvenOdd.even-product)
  with p-odd have (setsum (%x. ((x * a) div p)) A − int(card E)): zEven
    by (auto simp add: odd-iff-not-even)
  thus ?thesis
    by (auto simp only: even-diff [THEN sym])
qed

lemma (in GAUSS) QRLemma5: a ∈ zOdd ==>
   (−1::int) ^(card E) = (−1::int) ^(nat(setsum (%x. ((x * a) div p)) A))
proof −
  assume a ∈ zOdd
  from QRLemma4 have
    (int(card E): zEven) = (setsum (%x. ((x * a) div p)) A ∈ zEven)..
  moreover have 0 ≤ int(card E)
    by auto
  moreover have 0 ≤ setsum (%x. ((x * a) div p)) A
    proof (intro setsum-nonneg)
      show ∀ x ∈ A. 0 ≤ x * a div p
      proof
        fix x
        assume x ∈ A
        then have 0 ≤ x
          by (auto simp add: A-def)
        with a-nonzero have 0 ≤ x * a
          by (auto simp add: zero-le-mult-iff)
```

```
      with p-g-2 show 0 ≤ x * a div p
        by (auto simp add: pos-imp-zdiv-nonneg-iff)
    qed
  qed
  ultimately have (−1::int) ^nat((int (card E))) =
    (−1) ^nat(((∑ x ∈ A. x * a div p)))
    by (intro neg-one-power-parity, auto)
  also have nat (int(card E)) = card E
    by auto
  finally show ?thesis .
qed

lemma MainQRLemma: [| a ∈ zOdd; 0 < a; ~([a = 0] (mod p)); zprime p; 2 <
p;
  A = {x. 0 < x & x ≤ (p − 1) div 2} |] ==>
  (Legendre a p) = (−1::int) ^(nat(setsum (%x. ((x * a) div p)) A))
  apply (subst GAUSS.gauss-lemma)
  apply (auto simp add: GAUSS-def)
  apply (subst GAUSS.QRLemma5)
by (auto simp add: GAUSS-def)




locale QRTEMP =
  fixes p     :: int
  fixes q     :: int
  fixes P-set :: int set
  fixes Q-set :: int set
  fixes S     :: (int * int) set
  fixes S1    :: (int * int) set
  fixes S2    :: (int * int) set
  fixes f1    :: int => (int * int) set
  fixes f2    :: int => (int * int) set

  assumes p-prime: zprime p
  assumes p-g-2: 2 < p
  assumes q-prime: zprime q
  assumes q-g-2: 2 < q
  assumes p-neq-q:      p ≠ q

  defines P-set-def: P-set == {x. 0 < x & x ≤ ((p − 1) div 2) }
  defines Q-set-def: Q-set == {x. 0 < x & x ≤ ((q − 1) div 2) }
  defines S-def:     S     == P-set <*> Q-set
  defines S1-def:    S1    == { (x, y). (x, y):S & ((p * y) < (q * x)) }
  defines S2-def:    S2    == { (x, y). (x, y):S & ((q * x) < (p * y)) }
```

defines *f1-def*: $f1\ j\ ==\ \{\ (j1,\ y).\ (j1,\ y){:}S\ \&\ j1 = j\ \&\ (y \leq (q * j)\ div\ p)\ \}$
defines *f2-def*: $f2\ j\ ==\ \{\ (x,\ j1).\ (x,\ j1){:}S\ \&\ j1 = j\ \&\ (x \leq (p * j)\ div\ q)\ \}$

lemma (**in** *QRTEMP*) *p-fact*: $0 < (p - 1)\ div\ 2$
proof −
from *prems* **have** $2 < p$ **by** (*simp add*: *QRTEMP-def*)
then have $2 \leq p - 1$ **by** *arith*
then have $2\ div\ 2 \leq (p - 1)\ div\ 2$ **by** (*rule zdiv-mono1*, *auto*)
then show *?thesis* **by** *auto*
qed

lemma (**in** *QRTEMP*) *q-fact*: $0 < (q - 1)\ div\ 2$
proof −
from *prems* **have** $2 < q$ **by** (*simp add*: *QRTEMP-def*)
then have $2 \leq q - 1$ **by** *arith*
then have $2\ div\ 2 \leq (q - 1)\ div\ 2$ **by** (*rule zdiv-mono1*, *auto*)
then show *?thesis* **by** *auto*
qed

lemma (**in** *QRTEMP*) *pb-neq-qa*: $[|1 \leq b;\ b \leq (q - 1)\ div\ 2\ |] ==> (p * b \neq q * a)$
proof
assume $p * b = q * a$ **and** $1 \leq b$ **and** $b \leq (q - 1)\ div\ 2$
then have $q\ dvd\ (p * b)$ **by** (*auto simp add*: *dvd-def*)
with *q-prime p-g-2* **have** $q\ dvd\ p \mid q\ dvd\ b$
by (*auto simp add*: *zprime-zdvd-zmult*)
moreover have $\sim (q\ dvd\ p)$
proof
assume $q\ dvd\ p$
with *p-prime* **have** $q = 1 \mid q = p$
apply (*auto simp add*: *zprime-def QRTEMP-def*)
apply (*drule-tac* $x = q$ **and** $R = False$ **in** *allE*)
apply (*simp add*: *QRTEMP-def*)
apply (*subgoal-tac* $0 \leq q$, *simp add*: *QRTEMP-def*)
apply (*insert prems*)
by (*auto simp add*: *QRTEMP-def*)
with *q-g-2 p-neq-q* **show** *False* **by** *auto*
qed
ultimately have $q\ dvd\ b$ **by** *auto*
then have $q \leq b$
proof −
assume $q\ dvd\ b$
moreover from *prems* **have** $0 < b$ **by** *auto*
ultimately show *?thesis* **by** (*insert zdvd-bounds* [*of q b*], *auto*)
qed
with *prems* **have** $q \leq (q - 1)\ div\ 2$ **by** *auto*
then have $2 * q \leq 2 * ((q - 1)\ div\ 2)$ **by** *arith*

```
  then have 2 * q ≤ q - 1
  proof -
    assume 2 * q ≤ 2 * ((q - 1) div 2)
   with prems have q ∈ zOdd by (auto simp add: QRTEMP-def zprime-zOdd-eq-grt-2)
    with odd-minus-one-even have (q - 1):zEven by auto
    with even-div-2-prop2 have (q - 1) = 2 * ((q - 1) div 2) by auto
    with prems show ?thesis by auto
  qed
  then have p1: q ≤ -1 by arith
  with q-g-2 show False by auto
qed

lemma (in QRTEMP) P-set-finite: finite (P-set)
  by (insert p-fact, auto simp add: P-set-def bdd-int-set-l-le-finite)

lemma (in QRTEMP) Q-set-finite: finite (Q-set)
  by (insert q-fact, auto simp add: Q-set-def bdd-int-set-l-le-finite)

lemma (in QRTEMP) S-finite: finite S
  by (auto simp add: S-def  P-set-finite Q-set-finite finite-cartesian-product)

lemma (in QRTEMP) S1-finite: finite S1
proof -
  have finite S by (auto simp add: S-finite)
  moreover have S1 ⊆ S by (auto simp add: S1-def S-def)
  ultimately show ?thesis by (auto simp add: finite-subset)
qed

lemma (in QRTEMP) S2-finite: finite S2
proof -
  have finite S by (auto simp add: S-finite)
  moreover have S2 ⊆ S by (auto simp add: S2-def S-def)
  ultimately show ?thesis by (auto simp add: finite-subset)
qed

lemma (in QRTEMP) P-set-card: (p - 1) div 2 = int (card (P-set))
  by (insert p-fact, auto simp add: P-set-def card-bdd-int-set-l-le)

lemma (in QRTEMP) Q-set-card: (q - 1) div 2 = int (card (Q-set))
  by (insert q-fact, auto simp add: Q-set-def card-bdd-int-set-l-le)

lemma (in QRTEMP) S-card: ((p - 1) div 2) * ((q - 1) div 2) = int (card(S))
  apply (insert P-set-card Q-set-card P-set-finite Q-set-finite)
  apply (auto simp add: S-def zmult-int setsum-constant)
done

lemma (in QRTEMP) S1-Int-S2-prop: S1 ∩ S2 = {}
  by (auto simp add: S1-def S2-def)
```

lemma (**in** *QRTEMP*) *S1-Union-S2-prop*: $S = S1 \cup S2$
apply (*auto simp add*: *S-def P-set-def Q-set-def S1-def S2-def*)
proof −
fix a **and** b
assume ~ $q * a < p * b$ **and** *b1*: $0 < b$ **and** *b2*: $b \leq (q - 1)$ *div 2*
with *zless-linear* **have** $(p * b < q * a) \mid (p * b = q * a)$ **by** *auto*
moreover from *pb-neq-qa b1 b2* **have** $(p * b \neq q * a)$ **by** *auto*
ultimately show $p * b < q * a$ **by** *auto*
qed

lemma (**in** *QRTEMP*) *card-sum-S1-S2*: $((p - 1)$ *div* $2) * ((q - 1)$ *div* $2) =$
$int(card(S1)) + int(card(S2))$
proof−
have $((p - 1)$ *div* $2) * ((q - 1)$ *div* $2) = int\ (card(S))$
by (*auto simp add*: *S-card*)
also have ... $= int(\ card(S1) + card(S2))$
apply (*insert S1-finite S2-finite S1-Int-S2-prop S1-Union-S2-prop*)
apply (*drule card-Un-disjoint*, *auto*)
done
also have ... $= int(card(S1)) + int(card(S2))$ **by** *auto*
finally show *?thesis* **.**
qed

lemma (**in** *QRTEMP*) *aux1a*: $[|\ 0 < a;\ a \leq (p - 1)$ *div* $2;$
$0 < b;\ b \leq (q - 1)$ *div* $2\ |] ==>$
$(p * b < q * a) = (b \leq q * a\ div\ p)$
proof −
assume $0 < a$ **and** $a \leq (p - 1)$ *div* 2 **and** $0 < b$ **and** $b \leq (q - 1)$ *div* 2
have $p * b < q * a ==> b \leq q * a\ div\ p$
proof −
assume $p * b < q * a$
then have $p * b \leq q * a$ **by** *auto*
then have $(p * b)$ *div* $p \leq (q * a)$ *div* p
by (*rule zdiv-mono1*, *insert p-g-2*, *auto*)
then show $b \leq (q * a)$ *div* p
apply (*subgoal-tac* $p \neq 0$)
apply (*frule zdiv-zmult-self2*, *force*)
by (*insert p-g-2*, *auto*)
qed
moreover have $b \leq q * a\ div\ p ==> p * b < q * a$
proof −
assume $b \leq q * a\ div\ p$
then have $p * b \leq p * ((q * a)$ *div* $p)$
by (*insert p-g-2*, *auto simp add*: *mult-le-cancel-left*)
also have ... $\leq q * a$
by (*rule zdiv-leq-prop*, *insert p-g-2*, *auto*)
finally have $p * b \leq q * a$ **.**
then have $p * b < q * a \mid p * b = q * a$
by (*simp only*: *order-le-imp-less-or-eq*)

```
    moreover have p * b ≠ q * a
      by (rule  pb-neq-qa, insert prems, auto)
    ultimately show ?thesis by auto
  qed
  ultimately show ?thesis ..
qed

lemma (in QRTEMP) aux1b: [| 0 < a; a ≤ (p - 1) div 2;
                            0 < b; b ≤ (q - 1) div 2 |] ==>
                         (q * a < p * b) = (a ≤ p * b div q)
proof -
  assume 0 < a and a ≤ (p - 1) div 2 and 0 < b and b ≤ (q - 1) div 2
  have q * a < p * b ==> a ≤ p * b div q
  proof -
    assume q * a < p * b
    then have q * a ≤ p * b by auto
    then have (q * a) div q ≤ (p * b) div q
      by (rule zdiv-mono1, insert q-g-2, auto)
    then show a ≤ (p * b) div q
      apply (subgoal-tac q ≠ 0)
      apply (frule zdiv-zmult-self2, force)
      by (insert q-g-2, auto)
  qed
  moreover have a ≤ p * b div q ==> q * a < p * b
  proof -
    assume a ≤ p * b div q
    then have q * a ≤ q * ((p * b) div q)
      by (insert q-g-2, auto simp add: mult-le-cancel-left)
    also have ... ≤ p * b
      by (rule zdiv-leq-prop, insert q-g-2, auto)
    finally have q * a ≤ p * b .
    then have q * a < p * b | q * a = p * b
      by (simp only: order-le-imp-less-or-eq)
    moreover have p * b ≠ q * a
      by (rule  pb-neq-qa, insert prems, auto)
    ultimately show ?thesis by auto
  qed
  ultimately show ?thesis ..
qed

lemma aux2: [| zprime p; zprime q; 2 < p; 2 < q |] ==>
          (q * ((p - 1) div 2)) div p ≤ (q - 1) div 2
proof-
  assume zprime p and zprime q and 2 < p and 2 < q

  then have p ∈ zOdd & q ∈ zOdd
    by (auto simp add:  zprime-zOdd-eq-grt-2)
  then have even1: (p - 1):zEven & (q - 1):zEven
    by (auto simp add: odd-minus-one-even)
```

```
  then have even2: (2 * p):zEven & ((q - 1) * p):zEven
    by (auto simp add: zEven-def)
  then have even3: (((q - 1) * p) + (2 * p)):zEven
    by (auto simp: EvenOdd.even-plus-even)

  from prems have q * (p - 1) < ((q - 1) * p) + (2 * p)
    by (auto simp add: int-distrib)
  then have ((p - 1) * q) div 2 < (((q - 1) * p) + (2 * p)) div 2
    apply (rule-tac x = ((p - 1) * q) in even-div-2-l)
    by (auto simp add: even3, auto simp add: zmult-ac)
  also have ((p - 1) * q) div 2 = q * ((p - 1) div 2)
    by (auto simp add: even1 even-prod-div-2)
  also have (((q - 1) * p) + (2 * p)) div 2 = (((q - 1) div 2) * p) + p
    by (auto simp add: even1 even2 even-prod-div-2 even-sum-div-2)
  finally show ?thesis
    apply (rule-tac x =  q * ((p - 1) div 2) and
                    y = (q - 1) div 2 in div-prop2)
    by (insert prems, auto)
qed

lemma (in QRTEMP) aux3a: ∀ j ∈ P-set. int (card (f1 j)) = (q * j) div p
proof
  fix j
  assume j-fact: j ∈ P-set
  have int (card (f1 j)) = int (card {y. y ∈ Q-set & y ≤ (q * j) div p})
  proof -
    have finite (f1 j)
    proof -
      have (f1 j) ⊆ S by (auto simp add: f1-def)
      with S-finite show ?thesis by (auto simp add: finite-subset)
    qed
    moreover have inj-on (%(x,y). y) (f1 j)
      by (auto simp add: f1-def inj-on-def)
    ultimately have card ((%(x,y). y) ` (f1 j)) = card  (f1 j)
      by (auto simp add: f1-def card-image)
    moreover have ((%(x,y). y) ` (f1 j)) = {y. y ∈ Q-set & y ≤ (q * j) div p}
      by (insert prems, auto simp add: f1-def S-def Q-set-def P-set-def
        image-def)
    ultimately show ?thesis by (auto simp add: f1-def)
  qed
  also have ... = int (card {y. 0 < y & y ≤ (q * j) div p})
  proof -
    have {y. y ∈ Q-set & y ≤ (q * j) div p} =
       {y. 0 < y & y ≤ (q * j) div p}
      apply (auto simp add: Q-set-def)
      proof -
        fix x
        assume 0 < x and x ≤ q * j div p
        with j-fact P-set-def  have j ≤ (p - 1) div 2 by auto
```

```
      with q-g-2 have q * j ≤ q * ((p - 1) div 2)
        by (auto simp add: mult-le-cancel-left)
      with p-g-2 have q * j div p ≤ q * ((p - 1) div 2) div p
        by (auto simp add: zdiv-mono1)
      also from prems have ... ≤ (q - 1) div 2
        apply simp apply (insert aux2) by (simp add: QRTEMP-def)
      finally show x ≤ (q - 1) div 2 by (insert prems, auto)
    qed
  then show ?thesis by auto
 qed
 also have ... = (q * j) div p
 proof -
  from j-fact P-set-def have 0 ≤ j by auto
  with q-g-2 have q * 0 ≤ q * j by (auto simp only: mult-left-mono)
  then have 0 ≤ q * j by auto
  then have 0 div p ≤ (q * j) div p
    apply (rule-tac a = 0 in zdiv-mono1)
    by (insert p-g-2, auto)
  also have 0 div p = 0 by auto
  finally show ?thesis by (auto simp add: card-bdd-int-set-l-le)
 qed
 finally show int (card (f1 j)) = q * j div p .
qed

lemma (in QRTEMP) aux3b: ∀ j ∈ Q-set. int (card (f2 j)) = (p * j) div q
proof
 fix j
 assume j-fact: j ∈ Q-set
 have int (card (f2 j)) = int (card {y. y ∈ P-set & y ≤ (p * j) div q})
 proof -
  have finite (f2 j)
  proof -
   have (f2 j) ⊆ S by (auto simp add: f2-def)
   with S-finite show ?thesis by (auto simp add: finite-subset)
  qed
  moreover have inj-on (%(x,y). x) (f2 j)
   by (auto simp add: f2-def inj-on-def)
  ultimately have card ((%(x,y). x) ` (f2 j)) = card (f2 j)
   by (auto simp add: f2-def card-image)
  moreover have ((%(x,y). x) ` (f2 j)) = {y. y ∈ P-set & y ≤ (p * j) div q}
   by (insert prems, auto simp add: f2-def S-def Q-set-def
     P-set-def image-def)
  ultimately show ?thesis by (auto simp add: f2-def)
 qed
 also have ... = int (card {y. 0 < y & y ≤ (p * j) div q})
 proof -
  have {y. y ∈ P-set & y ≤ (p * j) div q} =
     {y. 0 < y & y ≤ (p * j) div q}
   apply (auto simp add: P-set-def)
```

```
      proof −
        fix x
        assume 0 < x and x ≤ p ∗ j div q
        with j-fact Q-set-def have j ≤ (q − 1) div 2 by auto
        with p-g-2 have p ∗ j ≤ p ∗ ((q − 1) div 2)
          by (auto simp add: mult-le-cancel-left)
        with q-g-2 have p ∗ j div q ≤ p ∗ ((q − 1) div 2) div q
          by (auto simp add: zdiv-mono1)
        also from prems have ... ≤ (p − 1) div 2
          by (auto simp add: aux2 QRTEMP-def)
        finally show x ≤ (p − 1) div 2 by (insert prems, auto)
      qed
    then show ?thesis by auto
  qed
  also have ... = (p ∗ j) div q
  proof −
    from j-fact Q-set-def have 0 ≤ j by auto
    with p-g-2 have p ∗ 0 ≤ p ∗ j by (auto simp only: mult-left-mono)
    then have 0 ≤ p ∗ j by auto
    then have 0 div q ≤ (p ∗ j) div q
      apply (rule-tac a = 0 in zdiv-mono1)
      by (insert q-g-2, auto)
    also have 0 div q = 0 by auto
    finally show ?thesis by (auto simp add: card-bdd-int-set-l-le)
  qed
  finally show int (card (f2 j)) = p ∗ j div q .
qed

lemma (in QRTEMP) S1-card: int (card(S1)) = setsum (%j. (q ∗ j) div p) P-set
proof −
  have ∀ x ∈ P-set. finite (f1 x)
  proof
    fix x
    have f1 x ⊆ S by (auto simp add: f1-def)
    with S-finite show finite (f1 x) by (auto simp add: finite-subset)
  qed
  moreover have (∀ x ∈ P-set. ∀ y ∈ P-set. x ≠ y −−> (f1 x) ∩ (f1 y) = {})
    by (auto simp add: f1-def)
  moreover note P-set-finite
  ultimately have int(card (UNION P-set f1)) =
      setsum (%x. int(card (f1 x))) P-set
    by(simp add:card-UN-disjoint int-setsum o-def)
  moreover have S1 = UNION P-set f1
    by (auto simp add: f1-def S-def S1-def S2-def P-set-def Q-set-def aux1a)
  ultimately have int(card (S1)) = setsum (%j. int(card (f1 j))) P-set
    by auto
  also have ... = setsum (%j. q ∗ j div p) P-set
    using aux3a by(fastsimp intro: setsum-cong)
  finally show ?thesis .
```

qed

lemma (**in** *QRTEMP*) *S2-card*: *int* (*card*(*S2*)) = *setsum* (%*j*. (*p* ∗ *j*) *div q*) *Q-set*
proof −
have ∀ *x* ∈ *Q-set*. *finite* (*f2 x*)
proof
fix *x*
have *f2 x* ⊆ *S* **by** (*auto simp add*: *f2-def*)
with *S-finite* **show** *finite* (*f2 x*) **by** (*auto simp add*: *finite-subset*)
qed
moreover have (∀ *x* ∈ *Q-set*. ∀ *y* ∈ *Q-set*. *x* ≠ *y* −−>
(*f2 x*) ∩ (*f2 y*) = {})
by (*auto simp add*: *f2-def*)
moreover note *Q-set-finite*
ultimately have *int*(*card* (*UNION Q-set f2*)) =
setsum (%*x*. *int*(*card* (*f2 x*))) *Q-set*
by(*simp add*:*card-UN-disjoint int-setsum o-def*)
moreover have *S2* = *UNION Q-set f2*
by (*auto simp add*: *f2-def S-def S1-def S2-def P-set-def Q-set-def aux1b*)
ultimately have *int*(*card* (*S2*)) = *setsum* (%*j*. *int*(*card* (*f2 j*))) *Q-set*
by *auto*
also have ... = *setsum* (%*j*. *p* ∗ *j div q*) *Q-set*
using *aux3b* **by**(*fastsimp intro*: *setsum-cong*)
finally show *?thesis* **.**
qed

lemma (**in** *QRTEMP*) *S1-carda*: *int* (*card*(*S1*)) =
setsum (%*j*. (*j* ∗ *q*) *div p*) *P-set*
by (*auto simp add*: *S1-card zmult-ac*)

lemma (**in** *QRTEMP*) *S2-carda*: *int* (*card*(*S2*)) =
setsum (%*j*. (*j* ∗ *p*) *div q*) *Q-set*
by (*auto simp add*: *S2-card zmult-ac*)

lemma (**in** *QRTEMP*) *pq-sum-prop*: (*setsum* (%*j*. (*j* ∗ *p*) *div q*) *Q-set*) +
(*setsum* (%*j*. (*j* ∗ *q*) *div p*) *P-set*) = ((*p* − *1*) *div 2*) ∗ ((*q* − *1*) *div 2*)
proof −
have (*setsum* (%*j*. (*j* ∗ *p*) *div q*) *Q-set*) +
(*setsum* (%*j*. (*j* ∗ *q*) *div p*) *P-set*) = *int* (*card S2*) + *int* (*card S1*)
by (*auto simp add*: *S1-carda S2-carda*)
also have ... = *int* (*card S1*) + *int* (*card S2*)
by *auto*
also have ... = ((*p* − *1*) *div 2*) ∗ ((*q* − *1*) *div 2*)
by (*auto simp add*: *card-sum-S1-S2*)
finally show *?thesis* **.**
qed

lemma *pq-prime-neq*: [| *zprime p*; *zprime q*; *p* ≠ *q* |] ==> (~[*p* = *0*] (*mod q*))
apply (*auto simp add*: *zcong-eq-zdvd-prop zprime-def*)

```
  apply (drule-tac x = q in allE)
  apply (drule-tac x = p in allE)
by auto

lemma (in QRTEMP) QR-short: (Legendre p q) * (Legendre q p) =
    (-1::int) ^nat(((p - 1) div 2)*((q - 1) div 2))
proof -
  from prems have ~([p = 0] (mod q))
    by (auto simp add: pq-prime-neq QRTEMP-def)
  with prems have a1: (Legendre p q) = (-1::int) ^
      nat(setsum (%x. ((x * p) div q)) Q-set)
    apply (rule-tac p = q in  MainQRLemma)
    by (auto simp add: zprime-zOdd-eq-grt-2 QRTEMP-def)
  from prems have ~([q = 0] (mod p))
    apply (rule-tac p = q and q = p in pq-prime-neq)
    apply (simp add: QRTEMP-def)+
    done
  with prems have a2: (Legendre q p) =
      (-1::int) ^ nat(setsum (%x. ((x * q) div p)) P-set)
    apply (rule-tac p = p in  MainQRLemma)
    by (auto simp add: zprime-zOdd-eq-grt-2 QRTEMP-def)
  from a1 a2 have (Legendre p q) * (Legendre q p) =
      (-1::int) ^ nat(setsum (%x. ((x * p) div q)) Q-set) *
        (-1::int) ^ nat(setsum (%x. ((x * q) div p)) P-set)
    by auto
  also have ... = (-1::int) ^ (nat(setsum (%x. ((x * p) div q)) Q-set) +
                  nat(setsum (%x. ((x * q) div p)) P-set))
    by (auto simp add: zpower-zadd-distrib)
  also have nat(setsum (%x. ((x * p) div q)) Q-set) +
      nat(setsum (%x. ((x * q) div p)) P-set) =
        nat((setsum (%x. ((x * p) div q)) Q-set) +
          (setsum (%x. ((x * q) div p)) P-set))
    apply (rule-tac z1 = setsum (%x. ((x * p) div q)) Q-set in
      nat-add-distrib [THEN sym])
    by (auto simp add: S1-carda [THEN sym] S2-carda [THEN sym])
  also have ... = nat(((p - 1) div 2) * ((q - 1) div 2))
    by (auto simp add: pq-sum-prop)
  finally show ?thesis .
qed

theorem Quadratic-Reciprocity:
    [| p ∈ zOdd; zprime p; q ∈ zOdd; zprime q;
       p ≠ q |]
    ==> (Legendre p q) * (Legendre q p) =
        (-1::int) ^nat(((p - 1) div 2)*((q - 1) div 2))
  by (auto simp add: QRTEMP.QR-short zprime-zOdd-eq-grt-2 [THEN sym]
                  QRTEMP-def)

end
```