The Hahn-Banach Theorem for Real Vector Spaces

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Abstract

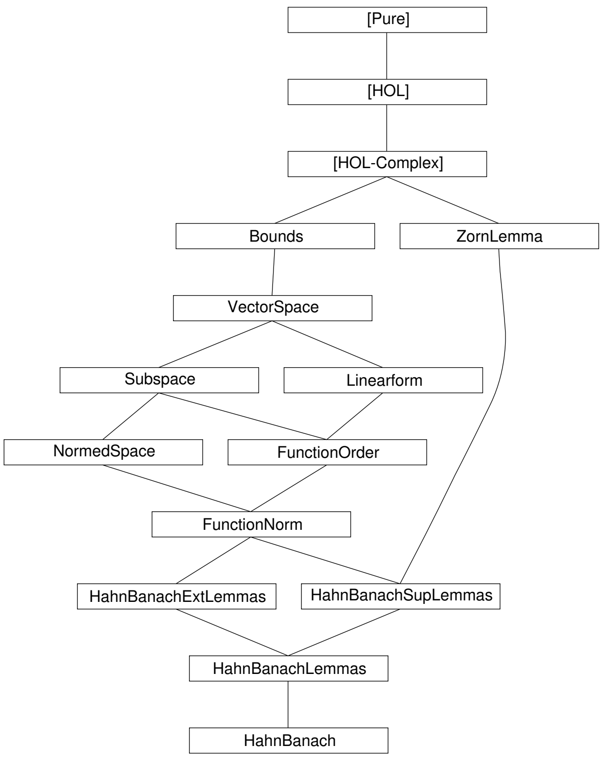
The Hahn-Banach Theorem is one of the most fundamental results in functional analysis. We present a fully formal proof of two versions of the theorem, one for general linear spaces and another for normed spaces. This development is based on simply-typed classical set-theory, as provided by Isabelle/HOL.

Contents

1 Preface

This is a fully formal proof of the Hahn-Banach Theorem. It closely follows the informal presentation given in Heuser's textbook [1, § 36]. Another formal proof of the same theorem has been done in Mizar [3]. A general overview of the relevance and history of the Hahn-Banach Theorem is given by Narici and Beckenstein [2].

The document is structured as follows. The first part contains definitions of basic notions of linear algebra: vector spaces, subspaces, normed spaces, continuous linear-forms, norm of functions and an order on functions by domain extension. The second part contains some lemmas about the supremum (w.r.t. the function order) and extension of non-maximal functions. With these preliminaries, the main proof of the theorem (in its two versions) is conducted in the third part. The dependencies of individual theories are as follows.



Part I
Basic Notions

2 Bounds

theory *Bounds* **imports** *Main Real* **begin**

locale *lub* =
  fixes A **and** x
  assumes *least* [*intro?*]: $(\bigwedge a.\ a \in A \Longrightarrow a \leq b) \Longrightarrow x \leq b$
    and *upper* [*intro?*]: $a \in A \Longrightarrow a \leq x$

lemmas [*elim?*] = *lub.least lub.upper*

constdefs
  the-lub :: $'a$::*order set* $\Rightarrow$ $'a$
  the-lub $A \equiv$ *The* (*lub* A)

syntax (*xsymbols*)
  the-lub :: $'a$::*order set* $\Rightarrow$ $'a$ ($\bigsqcup$ - [*90*] *90*)

lemma *the-lub-equality* [*elim?*]:
  includes *lub*
  shows $\bigsqcup A = (x::'a::order)$
proof (*unfold the-lub-def*)
  from *lub-axioms* **show** *The* (*lub* A) = x
  proof
    fix x' **assume** *lub'*: *lub* A x'
    show $x' = x$
    proof (*rule order-antisym*)
      from *lub'* **show** $x' \leq x$
      proof
        fix a **assume** $a \in A$
        then show $a \leq x$ **..**
      qed
      show $x \leq x'$
      proof
        fix a **assume** $a \in A$
        with *lub'* **show** $a \leq x'$ **..**
      qed
    qed
  qed
qed

lemma *the-lubI-ex*:
  assumes *ex*: $\exists x.$ *lub* A x
  shows *lub* A ($\bigsqcup A$)
proof −
  from *ex* **obtain** x **where** x: *lub* A x **..**
  also from x **have** [*symmetric*]: $\bigsqcup A = x$ **..**
  finally show *?thesis* **.**
qed

lemma *lub-compat*: *lub A x* = *isLub UNIV A x*
proof −
  have *isUb UNIV A* = (λx. *A* ∗<= *x* ∧ *x* ∈ *UNIV*)
    by (*rule ext*) (*simp only*: *isUb-def*)
  then show *?thesis*
    by (*simp only*: *lub-def isLub-def leastP-def setge-def setle-def*) *blast*
qed

lemma *real-complete*:
  fixes *A* :: *real set*
  assumes *nonempty*: ∃ *a*. *a* ∈ *A*
    and *ex-upper*: ∃ *y*. ∀ *a* ∈ *A*. *a* ≤ *y*
  shows ∃ *x*. *lub A x*
proof −
  from *ex-upper* **have** ∃ *y*. *isUb UNIV A y*
    by (*unfold isUb-def setle-def*) *blast*
  with *nonempty* **have** ∃ *x*. *isLub UNIV A x*
    by (*rule reals-complete*)
  then show *?thesis* **by** (*simp only*: *lub-compat*)
qed

end

3 Vector spaces

theory *VectorSpace* **imports** *Real Bounds Zorn* **begin**

3.1 Signature

For the definition of real vector spaces a type $'a$ of the sort {*plus*, *minus*, *zero*} is considered, on which a real scalar multiplication · is declared.

consts
  prod :: *real* ⇒ $'a$::{*plus*, *minus*, *zero*} ⇒ $'a$ (**infixr** $'(*')$ *70*)

syntax (*xsymbols*)
  prod :: *real* ⇒ $'a$ ⇒ $'a$ (**infixr** · *70*)
syntax (*HTML* **output**)
  prod :: *real* ⇒ $'a$ ⇒ $'a$ (**infixr** · *70*)

3.2 Vector space laws

A *vector space* is a non-empty set V of elements from $'a$ with the following vector space laws: The set V is closed under addition and scalar multiplication, addition is associative and commutative; − x is the inverse of x w. r. t. addition and *0* is the neutral element of addition. Addition and multiplication are distributive; scalar multiplication is associative and the real number *1* is the neutral element of scalar multiplication.

locale *vectorspace* = *var V* +
  assumes *non-empty* [*iff*, *intro?*]: *V* ≠ {}

and *add-closed* [*iff*]: $x \in V \Longrightarrow y \in V \Longrightarrow x + y \in V$
and *mult-closed* [*iff*]: $x \in V \Longrightarrow a \cdot x \in V$
and *add-assoc*: $x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow (x + y) + z = x + (y + z)$
and *add-commute*: $x \in V \Longrightarrow y \in V \Longrightarrow x + y = y + x$
and *diff-self* [*simp*]: $x \in V \Longrightarrow x - x = \mathit{0}$
and *add-zero-left* [*simp*]: $x \in V \Longrightarrow \mathit{0} + x = x$
and *add-mult-distrib1*: $x \in V \Longrightarrow y \in V \Longrightarrow a \cdot (x + y) = a \cdot x + a \cdot y$
and *add-mult-distrib2*: $x \in V \Longrightarrow (a + b) \cdot x = a \cdot x + b \cdot x$
and *mult-assoc*: $x \in V \Longrightarrow (a * b) \cdot x = a \cdot (b \cdot x)$
and *mult-1* [*simp*]: $x \in V \Longrightarrow \mathit{1} \cdot x = x$
and *negate-eq1*: $x \in V \Longrightarrow - x = (- \mathit{1}) \cdot x$
and *diff-eq1*: $x \in V \Longrightarrow y \in V \Longrightarrow x - y = x + - y$

lemma (**in** *vectorspace*) *negate-eq2*: $x \in V \Longrightarrow (- \mathit{1}) \cdot x = - x$
by (*rule negate-eq1* [*symmetric*])

lemma (**in** *vectorspace*) *negate-eq2a*: $x \in V \Longrightarrow -\mathit{1} \cdot x = - x$
by (*simp add*: *negate-eq1*)

lemma (**in** *vectorspace*) *diff-eq2*: $x \in V \Longrightarrow y \in V \Longrightarrow x + - y = x - y$
by (*rule diff-eq1* [*symmetric*])

lemma (**in** *vectorspace*) *diff-closed* [*iff*]: $x \in V \Longrightarrow y \in V \Longrightarrow x - y \in V$
by (*simp add*: *diff-eq1 negate-eq1*)

lemma (**in** *vectorspace*) *neg-closed* [*iff*]: $x \in V \Longrightarrow - x \in V$
by (*simp add*: *negate-eq1*)

lemma (**in** *vectorspace*) *add-left-commute*:
$x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow x + (y + z) = y + (x + z)$
proof –
assume *xyz*: $x \in V \;\; y \in V \;\; z \in V$
hence $x + (y + z) = (x + y) + z$
by (*simp only*: *add-assoc*)
also from *xyz* **have** ... $= (y + x) + z$ **by** (*simp only*: *add-commute*)
also from *xyz* **have** ... $= y + (x + z)$ **by** (*simp only*: *add-assoc*)
finally show *?thesis* **.**
qed

theorems (**in** *vectorspace*) *add-ac* =
add-assoc add-commute add-left-commute

The existence of the zero element of a vector space follows from the non-emptiness of carrier set.

lemma (**in** *vectorspace*) *zero* [*iff*]: $\mathit{0} \in V$
proof –
from *non-empty* **obtain** x **where** x: $x \in V$ **by** *blast*
then have $\mathit{0} = x - x$ **by** (*rule diff-self* [*symmetric*])
also from x **have** ... $\in V$ **by** (*rule diff-closed*)
finally show *?thesis* **.**
qed

lemma (**in** *vectorspace*) *add-zero-right* [*simp*]:
$x \in V \Longrightarrow \;\; x + \mathit{0} = x$

proof −
  assume x: $x \in V$
  from *this* **and** *zero* **have** $x + 0 = 0 + x$ **by** (*rule add-commute*)
  also from x **have** ... $= x$ **by** (*rule add-zero-left*)
  finally show *?thesis* **.**
qed

lemma (**in** *vectorspace*) *mult-assoc2*:
    $x \in V \Longrightarrow a \cdot b \cdot x = (a * b) \cdot x$
  by (*simp only*: *mult-assoc*)

lemma (**in** *vectorspace*) *diff-mult-distrib1*:
    $x \in V \Longrightarrow y \in V \Longrightarrow a \cdot (x - y) = a \cdot x - a \cdot y$
  by (*simp add*: *diff-eq1 negate-eq1 add-mult-distrib1 mult-assoc2*)

lemma (**in** *vectorspace*) *diff-mult-distrib2*:
  $x \in V \Longrightarrow (a - b) \cdot x = a \cdot x - (b \cdot x)$
proof −
  assume x: $x \in V$
  have $(a - b) \cdot x = (a + - b) \cdot x$
    by (*simp add*: *real-diff-def*)
  also have ... $= a \cdot x + (- b) \cdot x$
    by (*rule add-mult-distrib2*)
  also from x **have** ... $= a \cdot x + - (b \cdot x)$
    by (*simp add*: *negate-eq1 mult-assoc2*)
  also from x **have** ... $= a \cdot x - (b \cdot x)$
    by (*simp add*: *diff-eq1*)
  finally show *?thesis* **.**
qed

lemmas (**in** *vectorspace*) *distrib* =
  add-mult-distrib1 add-mult-distrib2
  diff-mult-distrib1 diff-mult-distrib2

Further derived laws:

lemma (**in** *vectorspace*) *mult-zero-left* [*simp*]:
  $x \in V \Longrightarrow 0 \cdot x = 0$
proof −
  assume x: $x \in V$
  have $0 \cdot x = (1 - 1) \cdot x$ **by** *simp*
  also have ... $= (1 + - 1) \cdot x$ **by** *simp*
  also have ... $= 1 \cdot x + (- 1) \cdot x$
    by (*rule add-mult-distrib2*)
  also from x **have** ... $= x + (- 1) \cdot x$ **by** *simp*
  also from x **have** ... $= x + - x$ **by** (*simp add*: *negate-eq2a*)
  also from x **have** ... $= x - x$ **by** (*simp add*: *diff-eq2*)
  also from x **have** ... $= 0$ **by** *simp*
  finally show *?thesis* **.**
qed

lemma (**in** *vectorspace*) *mult-zero-right* [*simp*]:
  $a \cdot 0 = (0::'a)$
proof −
  have $a \cdot 0 = a \cdot (0 - (0::'a))$ **by** *simp*

also have ... = $a \cdot 0 - a \cdot 0$
by (*rule diff-mult-distrib1*) *simp-all*
also have ... = *0* **by** *simp*
finally show *?thesis* **.**
qed

lemma (**in** *vectorspace*) *minus-mult-cancel* [*simp*]:
$x \in V \Longrightarrow (- a) \cdot - x = a \cdot x$
by (*simp add*: *negate-eq1 mult-assoc2*)

lemma (**in** *vectorspace*) *add-minus-left-eq-diff*:
$x \in V \Longrightarrow y \in V \Longrightarrow - x + y = y - x$
proof −
assume *xy*: $x \in V$ $y \in V$
hence $- x + y = y + - x$ **by** (*simp add*: *add-commute*)
also from *xy* **have** ... = $y - x$ **by** (*simp add*: *diff-eq1*)
finally show *?thesis* **.**
qed

lemma (**in** *vectorspace*) *add-minus* [*simp*]:
$x \in V \Longrightarrow x + - x = 0$
by (*simp add*: *diff-eq2*)

lemma (**in** *vectorspace*) *add-minus-left* [*simp*]:
$x \in V \Longrightarrow - x + x = 0$
by (*simp add*: *diff-eq2 add-commute*)

lemma (**in** *vectorspace*) *minus-minus* [*simp*]:
$x \in V \Longrightarrow - (- x) = x$
by (*simp add*: *negate-eq1 mult-assoc2*)

lemma (**in** *vectorspace*) *minus-zero* [*simp*]:
$- (0::'a) = 0$
by (*simp add*: *negate-eq1*)

lemma (**in** *vectorspace*) *minus-zero-iff* [*simp*]:
$x \in V \Longrightarrow (- x = 0) = (x = 0)$
proof
assume *x*: $x \in V$
{
from *x* **have** $x = - (- x)$ **by** (*simp add*: *minus-minus*)
also assume $- x = 0$
also have − ... = *0* **by** (*rule minus-zero*)
finally show $x = 0$ **.**
next
assume $x = 0$
then show $- x = 0$ **by** *simp*
}
qed

lemma (**in** *vectorspace*) *add-minus-cancel* [*simp*]:
$x \in V \Longrightarrow y \in V \Longrightarrow x + (- x + y) = y$
by (*simp add*: *add-assoc* [*symmetric*] *del*: *add-commute*)

lemma (**in** *vectorspace*) *minus-add-cancel* [*simp*]:
  $x \in V \Longrightarrow y \in V \Longrightarrow - x + (x + y) = y$
 by (*simp add*: *add-assoc* [*symmetric*] *del*: *add-commute*)

lemma (**in** *vectorspace*) *minus-add-distrib* [*simp*]:
  $x \in V \Longrightarrow y \in V \Longrightarrow - (x + y) = - x + - y$
 by (*simp add*: *negate-eq1 add-mult-distrib1*)

lemma (**in** *vectorspace*) *diff-zero* [*simp*]:
  $x \in V \Longrightarrow x - 0 = x$
 by (*simp add*: *diff-eq1*)

lemma (**in** *vectorspace*) *diff-zero-right* [*simp*]:
  $x \in V \Longrightarrow 0 - x = - x$
 by (*simp add*: *diff-eq1*)

lemma (**in** *vectorspace*) *add-left-cancel*:
 $x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow (x + y = x + z) = (y = z)$
proof
 assume *x*: $x \in V$ **and** *y*: $y \in V$ **and** *z*: $z \in V$
 {
  from *y* **have** $y = 0 + y$ **by** *simp*
  also from *x y* **have** ... $= (- x + x) + y$ **by** *simp*
  also from *x y* **have** ... $= - x + (x + y)$
   by (*simp add*: *add-assoc neg-closed*)
  also assume $x + y = x + z$
  also from *x z* **have** $- x + (x + z) = - x + x + z$
   by (*simp add*: *add-assoc* [*symmetric*] *neg-closed*)
  also from *x z* **have** ... $= z$ **by** *simp*
  finally show $y = z$ **.**
 next
  assume $y = z$
  then show $x + y = x + z$ **by** (*simp only*:)
 }
qed

lemma (**in** *vectorspace*) *add-right-cancel*:
  $x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow (y + x = z + x) = (y = z)$
 by (*simp only*: *add-commute add-left-cancel*)

lemma (**in** *vectorspace*) *add-assoc-cong*:
 $x \in V \Longrightarrow y \in V \Longrightarrow x' \in V \Longrightarrow y' \in V \Longrightarrow z \in V$
  $\Longrightarrow x + y = x' + y' \Longrightarrow x + (y + z) = x' + (y' + z)$
 by (*simp only*: *add-assoc* [*symmetric*])

lemma (**in** *vectorspace*) *mult-left-commute*:
  $x \in V \Longrightarrow a \cdot b \cdot x = b \cdot a \cdot x$
 by (*simp add*: *real-mult-commute mult-assoc2*)

lemma (**in** *vectorspace*) *mult-zero-uniq*:
 $x \in V \Longrightarrow x \neq 0 \Longrightarrow a \cdot x = 0 \Longrightarrow a = 0$
proof (*rule classical*)
 assume *a*: $a \neq 0$
 assume *x*: $x \in V$ $x \neq 0$ **and** *ax*: $a \cdot x = 0$

from x a **have** $x = (inverse\ a * a) \cdot x$ **by** *simp*
also have ... $= inverse\ a \cdot (a \cdot x)$ **by** (*rule mult-assoc*)
also from *ax* **have** ... $= inverse\ a \cdot 0$ **by** *simp*
also have ... $= 0$ **by** *simp*
finally have $x = 0$ **.**
thus $a = 0$ **by** *contradiction*
qed

lemma (**in** *vectorspace*) *mult-left-cancel*:
$x \in V \Longrightarrow y \in V \Longrightarrow a \neq 0 \Longrightarrow (a \cdot x = a \cdot y) = (x = y)$
proof
assume x: $x \in V$ **and** y: $y \in V$ **and** a: $a \neq 0$
from x **have** $x = 1 \cdot x$ **by** *simp*
also from a **have** ... $= (inverse\ a * a) \cdot x$ **by** *simp*
also from x **have** ... $= inverse\ a \cdot (a \cdot x)$
by (*simp only*: *mult-assoc*)
also assume $a \cdot x = a \cdot y$
also from a y **have** $inverse\ a \cdot ... = y$
by (*simp add*: *mult-assoc2*)
finally show $x = y$ **.**
next
assume $x = y$
then show $a \cdot x = a \cdot y$ **by** (*simp only*:)
qed

lemma (**in** *vectorspace*) *mult-right-cancel*:
$x \in V \Longrightarrow x \neq 0 \Longrightarrow (a \cdot x = b \cdot x) = (a = b)$
proof
assume x: $x \in V$ **and** *neq*: $x \neq 0$
{
from x **have** $(a - b) \cdot x = a \cdot x - b \cdot x$
by (*simp add*: *diff-mult-distrib2*)
also assume $a \cdot x = b \cdot x$
with x **have** $a \cdot x - b \cdot x = 0$ **by** *simp*
finally have $(a - b) \cdot x = 0$ **.**
with x *neq* **have** $a - b = 0$ **by** (*rule mult-zero-uniq*)
thus $a = b$ **by** *simp*
next
assume $a = b$
then show $a \cdot x = b \cdot x$ **by** (*simp only*:)
}
qed

lemma (**in** *vectorspace*) *eq-diff-eq*:
$x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow (x = z - y) = (x + y = z)$
proof
assume x: $x \in V$ **and** y: $y \in V$ **and** z: $z \in V$
{
assume $x = z - y$
hence $x + y = z - y + y$ **by** *simp*
also from y z **have** ... $= z + - y + y$
by (*simp add*: *diff-eq1*)
also have ... $= z + (- y + y)$
by (*rule add-assoc*) (*simp-all add*: y z)

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    also from y z have ... = z + 0
      by (simp only: add-minus-left)
    also from z have ... = z
      by (simp only: add-zero-right)
    finally show x + y = z .
  next
    assume x + y = z
    hence z − y = (x + y) − y by simp
    also from x y have ... = x + y + − y
      by (simp add: diff-eq1)
    also have ... = x + (y + − y)
      by (rule add-assoc) (simp-all add: x y)
    also from x y have ... = x by simp
    finally show x = z − y ..
  }
qed

lemma (in vectorspace) add-minus-eq-minus:
  x ∈ V ⟹ y ∈ V ⟹ x + y = 0 ⟹ x = − y
proof −
  assume x: x ∈ V and y: y ∈ V
  from x y have x = (− y + y) + x by simp
  also from x y have ... = − y + (x + y) by (simp add: add-ac)
  also assume x + y = 0
  also from y have − y + 0 = − y by simp
  finally show x = − y .
qed

lemma (in vectorspace) add-minus-eq:
  x ∈ V ⟹ y ∈ V ⟹ x − y = 0 ⟹ x = y
proof −
  assume x: x ∈ V and y: y ∈ V
  assume x − y = 0
  with x y have eq: x + − y = 0 by (simp add: diff-eq1)
  with - - have x = − (− y)
    by (rule add-minus-eq-minus) (simp-all add: x y)
  with x y show x = y by simp
qed

lemma (in vectorspace) add-diff-swap:
  a ∈ V ⟹ b ∈ V ⟹ c ∈ V ⟹ d ∈ V ⟹ a + b = c + d
    ⟹ a − c = d − b
proof −
  assume vs: a ∈ V  b ∈ V  c ∈ V  d ∈ V
    and eq: a + b = c + d
  then have − c + (a + b) = − c + (c + d)
    by (simp add: add-left-cancel)
  also have ... = d by (rule minus-add-cancel)
  finally have eq: − c + (a + b) = d .
  from vs have a − c = (− c + (a + b)) + − b
    by (simp add: add-ac diff-eq1)
  also from vs eq have ...  = d + − b
    by (simp add: add-right-cancel)
  also from vs have ... = d − b by (simp add: diff-eq2)
```

```
  finally show a − c = d − b .
qed

lemma (in vectorspace) vs-add-cancel-21:
  x ∈ V ⟹ y ∈ V ⟹ z ∈ V ⟹ u ∈ V
    ⟹ (x + (y + z) = y + u) = (x + z = u)
proof
  assume vs: x ∈ V  y ∈ V  z ∈ V  u ∈ V
  {
    from vs have x + z = − y + y + (x + z) by simp
    also have ... = − y + (y + (x + z))
      by (rule add-assoc) (simp-all add: vs)
    also from vs have y + (x + z) = x + (y + z)
      by (simp add: add-ac)
    also assume x + (y + z) = y + u
    also from vs have − y + (y + u) = u by simp
    finally show x + z = u .
  next
    assume x + z = u
    with vs show x + (y + z) = y + u
      by (simp only: add-left-commute [of x])
  }
qed

lemma (in vectorspace) add-cancel-end:
  x ∈ V ⟹ y ∈ V ⟹ z ∈ V ⟹ (x + (y + z) = y) = (x = − z)
proof
  assume vs: x ∈ V  y ∈ V  z ∈ V
  {
    assume x + (y + z) = y
    with vs have (x + z) + y = 0 + y
      by (simp add: add-ac)
    with vs have x + z = 0
      by (simp only: add-right-cancel add-closed zero)
    with vs show x = − z by (simp add: add-minus-eq-minus)
  next
    assume eq: x = − z
    hence x + (y + z) = − z + (y + z) by simp
    also have ... = y + (− z + z)
      by (rule add-left-commute) (simp-all add: vs)
    also from vs have ... = y  by simp
    finally show x + (y + z) = y .
  }
qed

end
```

4 Subspaces

theory *Subspace* **imports** *VectorSpace* **begin**

4.1 Definition

A non-empty subset U of a vector space V is a *subspace* of V, iff U is closed under addition and scalar multiplication.

locale *subspace* = *var* U + *var* V +
  assumes *non-empty* [*iff*, *intro*]: $U \neq \{\}$
    and *subset* [*iff*]: $U \subseteq V$
    and *add-closed* [*iff*]: $x \in U \Longrightarrow y \in U \Longrightarrow x + y \in U$
    and *mult-closed* [*iff*]: $x \in U \Longrightarrow a \cdot x \in U$

declare *vectorspace.intro* [*intro?*] *subspace.intro* [*intro?*]

syntax (*symbols*)
  subspace :: $'a$ *set* $\Rightarrow$ $'a$ *set* $\Rightarrow$ *bool*   (**infix** $\trianglelefteq$ *50*)

lemma *subspace-subset* [*elim*]: $U \trianglelefteq V \Longrightarrow U \subseteq V$
  by (*rule subspace.subset*)

lemma (**in** *subspace*) *subsetD* [*iff*]: $x \in U \Longrightarrow x \in V$
  using *subset* **by** *blast*

lemma *subspaceD* [*elim*]: $U \trianglelefteq V \Longrightarrow x \in U \Longrightarrow x \in V$
  by (*rule subspace.subsetD*)

lemma *rev-subspaceD* [*elim?*]: $x \in U \Longrightarrow U \trianglelefteq V \Longrightarrow x \in V$
  by (*rule subspace.subsetD*)

lemma (**in** *subspace*) *diff-closed* [*iff*]:
  includes *vectorspace*
  shows $x \in U \Longrightarrow y \in U \Longrightarrow x - y \in U$
  by (*simp add*: *diff-eq1 negate-eq1*)

Similar as for linear spaces, the existence of the zero element in every subspace follows from the non-emptiness of the carrier set and by vector space laws.

lemma (**in** *subspace*) *zero* [*intro*]:
  includes *vectorspace*
  shows $0 \in U$
proof −
  have $U \neq \{\}$ **by** (*rule U-V.non-empty*)
  then obtain x **where** x: $x \in U$ **by** *blast*
  hence $x \in V$ **..** **hence** $0 = x - x$ **by** *simp*
  also have ... $\in U$ **by** (*rule U-V.diff-closed*)
  finally show *?thesis* **.**
qed

lemma (**in** *subspace*) *neg-closed* [*iff*]:
  includes *vectorspace*
  shows $x \in U \Longrightarrow - x \in U$
  by (*simp add*: *negate-eq1*)

Further derived laws: every subspace is a vector space.

lemma (**in** *subspace*) *vectorspace* [*iff*]:

includes *vectorspace*
shows *vectorspace U*
proof
show $U \neq \{\}$ **..**
fix $x\ y\ z$ **assume** x: $x \in U$ **and** y: $y \in U$ **and** z: $z \in U$
fix $a\ b$ :: *real*
from $x\ y$ **show** $x + y \in U$ **by** *simp*
from x **show** $a \cdot x \in U$ **by** *simp*
from $x\ y\ z$ **show** $(x + y) + z = x + (y + z)$ **by** (*simp add*: *add-ac*)
from $x\ y$ **show** $x + y = y + x$ **by** (*simp add*: *add-ac*)
from x **show** $x - x = 0$ **by** *simp*
from x **show** $0 + x = x$ **by** *simp*
from $x\ y$ **show** $a \cdot (x + y) = a \cdot x + a \cdot y$ **by** (*simp add*: *distrib*)
from x **show** $(a + b) \cdot x = a \cdot x + b \cdot x$ **by** (*simp add*: *distrib*)
from x **show** $(a * b) \cdot x = a \cdot b \cdot x$ **by** (*simp add*: *mult-assoc*)
from x **show** $1 \cdot x = x$ **by** *simp*
from x **show** $- x = - 1 \cdot x$ **by** (*simp add*: *negate-eq1*)
from $x\ y$ **show** $x - y = x + - y$ **by** (*simp add*: *diff-eq1*)
qed

The subspace relation is reflexive.

lemma (**in** *vectorspace*) *subspace-refl* [*intro*]: $V \trianglelefteq V$
proof
show $V \neq \{\}$ **..**
show $V \subseteq V$ **..**
fix $x\ y$ **assume** x: $x \in V$ **and** y: $y \in V$
fix a :: *real*
from $x\ y$ **show** $x + y \in V$ **by** *simp*
from x **show** $a \cdot x \in V$ **by** *simp*
qed

The subspace relation is transitive.

lemma (**in** *vectorspace*) *subspace-trans* [*trans*]:
$U \trianglelefteq V \Longrightarrow V \trianglelefteq W \Longrightarrow U \trianglelefteq W$
proof
assume *uv*: $U \trianglelefteq V$ **and** *vw*: $V \trianglelefteq W$
from *uv* **show** $U \neq \{\}$ **by** (*rule subspace.non-empty*)
show $U \subseteq W$
proof $-$
from *uv* **have** $U \subseteq V$ **by** (*rule subspace.subset*)
also from *vw* **have** $V \subseteq W$ **by** (*rule subspace.subset*)
finally show *?thesis* **.**
qed
fix $x\ y$ **assume** x: $x \in U$ **and** y: $y \in U$
from *uv* **and** $x\ y$ **show** $x + y \in U$ **by** (*rule subspace.add-closed*)
from *uv* **and** x **show** $\bigwedge a.\ a \cdot x \in U$ **by** (*rule subspace.mult-closed*)
qed

4.2 Linear closure

The *linear closure* of a vector x is the set of all scalar multiples of x.

constdefs
lin :: ($'a$::{*minus*, *plus*, *zero*}) $\Rightarrow$ $'a$ *set*

$lin\ x \equiv \{a \cdot x \mid a.\ True\}$

lemma *linI* [*intro*]: $y = a \cdot x \Longrightarrow y \in lin\ x$
  by (*unfold lin-def*) *blast*

lemma *linI'* [*iff*]: $a \cdot x \in lin\ x$
  by (*unfold lin-def*) *blast*

lemma *linE* [*elim*]:
    $x \in lin\ v \Longrightarrow (\bigwedge a{::}real.\ x = a \cdot v \Longrightarrow C) \Longrightarrow C$
  by (*unfold lin-def*) *blast*

Every vector is contained in its linear closure.

lemma (**in** *vectorspace*) *x-lin-x* [*iff*]: $x \in V \Longrightarrow x \in lin\ x$
proof −
  assume $x \in V$
  hence $x = 1 \cdot x$ **by** *simp*
  also have $\ldots \in lin\ x$ **..**
  finally show *?thesis* **.**
qed

lemma (**in** *vectorspace*) *0-lin-x* [*iff*]: $x \in V \Longrightarrow 0 \in lin\ x$
proof
  assume $x \in V$
  thus $0 = 0 \cdot x$ **by** *simp*
qed

Any linear closure is a subspace.

lemma (**in** *vectorspace*) *lin-subspace* [*intro*]:
  $x \in V \Longrightarrow lin\ x \trianglelefteq V$
proof
  assume x: $x \in V$
  thus $lin\ x \neq \{\}$ **by** (*auto simp add*: *x-lin-x*)
  show $lin\ x \subseteq V$
  proof
    fix x' **assume** $x' \in lin\ x$
    then obtain a **where** $x' = a \cdot x$ **..**
    with x **show** $x' \in V$ **by** *simp*
  qed
  fix $x'\ x''$ **assume** x': $x' \in lin\ x$ **and** x'': $x'' \in lin\ x$
  show $x' + x'' \in lin\ x$
  proof −
    from x' **obtain** a' **where** $x' = a' \cdot x$ **..**
    moreover from x'' **obtain** a'' **where** $x'' = a'' \cdot x$ **..**
    ultimately have $x' + x'' = (a' + a'') \cdot x$
      using x **by** (*simp add*: *distrib*)
    also have $\ldots \in lin\ x$ **..**
    finally show *?thesis* **.**
  qed
  fix $a :: real$
  show $a \cdot x' \in lin\ x$
  proof −
    from x' **obtain** a' **where** $x' = a' \cdot x$ **..**
    with x **have** $a \cdot x' = (a * a') \cdot x$ **by** (*simp add*: *mult-assoc*)

also have $\ldots \in$ *lin* x **..**
finally show *?thesis* **.**
qed
qed

Any linear closure is a vector space.

lemma (**in** *vectorspace*) *lin-vectorspace* [*intro*]:
$x \in V \Longrightarrow$ *vectorspace* (*lin* x)
by (*rule subspace.vectorspace*) (*rule lin-subspace*)

4.3 Sum of two vectorspaces

The *sum* of two vectorspaces U and V is the set of all sums of elements from U and V.

instance *set* :: (*plus*) *plus* **..**

defs (**overloaded**)
sum-def: $U + V \equiv \{u + v \mid u\ v.\ u \in U \wedge v \in V\}$

lemma *sumE* [*elim*]:
$x \in U + V \Longrightarrow (\bigwedge u\ v.\ x = u + v \Longrightarrow u \in U \Longrightarrow v \in V \Longrightarrow C) \Longrightarrow C$
by (*unfold sum-def*) *blast*

lemma *sumI* [*intro*]:
$u \in U \Longrightarrow v \in V \Longrightarrow x = u + v \Longrightarrow x \in U + V$
by (*unfold sum-def*) *blast*

lemma *sumI′* [*intro*]:
$u \in U \Longrightarrow v \in V \Longrightarrow u + v \in U + V$
by (*unfold sum-def*) *blast*

U is a subspace of $U + V$.

lemma *subspace-sum1* [*iff*]:
includes *vectorspace* U + *vectorspace* V
shows $U \trianglelefteq U + V$
proof
show $U \neq \{\}$ **..**
show $U \subseteq U + V$
proof
fix x **assume** x: $x \in U$
moreover have $0 \in V$ **..**
ultimately have $x + 0 \in U + V$ **..**
with x **show** $x \in U + V$ **by** *simp*
qed
fix $x\ y$ **assume** x: $x \in U$ **and** $y \in U$
thus $x + y \in U$ **by** *simp*
from x **show** $\bigwedge a.\ a \cdot x \in U$ **by** *simp*
qed

The sum of two subspaces is again a subspace.

lemma *sum-subspace* [*intro?*]:
includes *subspace* U E + *vectorspace* E + *subspace* V E

```
  shows U + V ⊴ E
proof
  have 0 ∈ U + V
  proof
    show 0 ∈ U ..
    show 0 ∈ V ..
    show (0::'a) = 0 + 0 by simp
  qed
  thus U + V ≠ {} by blast
  show U + V ⊆ E
  proof
    fix x assume x ∈ U + V
    then obtain u v where x: x = u + v and
      u: u ∈ U and v: v ∈ V ..
    have U ⊴ E . with u have u ∈ E ..
    moreover have V ⊴ E . with v have v ∈ E ..
    ultimately show x ∈ E using x by simp
  qed
  fix x y assume x: x ∈ U + V and y: y ∈ U + V
  show x + y ∈ U + V
  proof −
    from x obtain ux vx where x = ux + vx and ux ∈ U and vx ∈ V ..
    moreover
    from y obtain uy vy where y = uy + vy and uy ∈ U and vy ∈ V ..
    ultimately
    have ux + uy ∈ U
      and vx + vy ∈ V
      and x + y = (ux + uy) + (vx + vy)
      using x y by (simp-all add: add-ac)
    thus ?thesis ..
  qed
  fix a show a · x ∈ U + V
  proof −
    from x obtain u v where x = u + v and u ∈ U and v ∈ V ..
    hence a · u ∈ U and a · v ∈ V
      and a · x = (a · u) + (a · v) by (simp-all add: distrib)
    thus ?thesis ..
  qed
qed
```

The sum of two subspaces is a vectorspace.

```
lemma sum-vs [intro?]:
    U ⊴ E ⟹ V ⊴ E ⟹ vectorspace E ⟹ vectorspace (U + V)
  by (rule subspace.vectorspace) (rule sum-subspace)
```

4.4 Direct sums

The sum of U and V is called *direct*, iff the zero element is the only common element of U and V. For every element x of the direct sum of U and V the decomposition in $x = u + v$ with $u \in U$ and $v \in V$ is unique.

```
lemma decomp:
  includes vectorspace E + subspace U E + subspace V E
  assumes direct: U ∩ V = {0}
```

```
    and u1: u1 ∈ U and u2: u2 ∈ U
    and v1: v1 ∈ V and v2: v2 ∈ V
    and sum: u1 + v1 = u2 + v2
  shows u1 = u2 ∧ v1 = v2
proof
  have U: vectorspace U by (rule subspace.vectorspace)
  have V: vectorspace V by (rule subspace.vectorspace)
  from u1 u2 v1 v2 and sum have eq: u1 − u2 = v2 − v1
    by (simp add: add-diff-swap)
  from u1 u2 have u: u1 − u2 ∈ U
    by (rule vectorspace.diff-closed [OF U])
  with eq have v': v2 − v1 ∈ U by (simp only:)
  from v2 v1 have v: v2 − v1 ∈ V
    by (rule vectorspace.diff-closed [OF V])
  with eq have u': u1 − u2 ∈ V by (simp only:)

  show u1 = u2
  proof (rule add-minus-eq)
    show u1 ∈ E ..
    show u2 ∈ E ..
    from u u' and direct show u1 − u2 = 0 by force
  qed
  show v1 = v2
  proof (rule add-minus-eq [symmetric])
    show v1 ∈ E ..
    show v2 ∈ E ..
    from v v' and direct show v2 − v1 = 0 by force
  qed
qed
```

An application of the previous lemma will be used in the proof of the Hahn-Banach Theorem (see page 40): for any element $y + a \cdot x_0$ of the direct sum of a vectorspace H and the linear closure of x_0 the components $y \in H$ and a are uniquely determined.

```
lemma decomp-H':
  includes vectorspace E + subspace H E
  assumes y1: y1 ∈ H and y2: y2 ∈ H
    and x': x' ∉ H  x' ∈ E  x' ≠ 0
    and eq: y1 + a1 · x' = y2 + a2 · x'
  shows y1 = y2 ∧ a1 = a2
proof
  have c: y1 = y2 ∧ a1 · x' = a2 · x'
  proof (rule decomp)
    show a1 · x' ∈ lin x' ..
    show a2 · x' ∈ lin x' ..
    show H ∩ lin x' = {0}
    proof
      show H ∩ lin x' ⊆ {0}
      proof
        fix x assume x: x ∈ H ∩ lin x'
        then obtain a where xx': x = a · x'
          by blast
        have x = 0
        proof cases
```

assume $a = 0$
with xx' **and** x' **show** *?thesis* **by** *simp*
next
assume a: $a \neq 0$
from x **have** $x \in H$ **..**
with xx' **have** $inverse\ a \cdot a \cdot x' \in H$ **by** *simp*
with a **and** x' **have** $x' \in H$ **by** (*simp add*: *mult-assoc2*)
thus *?thesis* **by** *contradiction*
qed
thus $x \in \{0\}$ **..**
qed
show $\{0\} \subseteq H \cap lin\ x'$
proof −
have $0 \in H$ **..**
moreover have $0 \in lin\ x'$ **..**
ultimately show *?thesis* **by** *blast*
qed
qed
show $lin\ x' \trianglelefteq E$ **..**
qed
thus $y1 = y2$ **..**
from c **have** $a1 \cdot x' = a2 \cdot x'$ **..**
with x' **show** $a1 = a2$ **by** (*simp add*: *mult-right-cancel*)
qed

Since for any element $y + a \cdot x'$ of the direct sum of a vectorspace H and the linear closure of x' the components $y \in H$ and a are unique, it follows from $y \in H$ that $a = 0$.

lemma *decomp-H'-H*:
includes *vectorspace* E + *subspace* H E
assumes t: $t \in H$
and x': $x' \notin H$ $x' \in E$ $x' \neq 0$
shows $(SOME\ (y, a).\ t = y + a \cdot x' \wedge y \in H) = (t, 0)$
proof (*rule*, *simp-all only*: *split-paired-all split-conv*)
from t x' **show** $t = t + 0 \cdot x' \wedge t \in H$ **by** *simp*
fix y **and** a **assume** ya: $t = y + a \cdot x' \wedge y \in H$
have $y = t \wedge a = 0$
proof (*rule decomp-H'*)
from ya x' **show** $y + a \cdot x' = t + 0 \cdot x'$ **by** *simp*
from ya **show** $y \in H$ **..**
qed
with t x' **show** $(y, a) = (y + a \cdot x', 0)$ **by** *simp*
qed

The components $y \in H$ and a in $y + a \cdot x'$ are unique, so the function h' defined by $h'\ (y + a \cdot x') = h\ y + a \cdot \xi$ is definite.

lemma *h'-definite*:
includes *var H*
assumes *h'-def*:
$h' \equiv (\lambda x.\ let\ (y, a) = SOME\ (y, a).\ (x = y + a \cdot x' \wedge y \in H)$
$in\ (h\ y) + a * xi)$
and x: $x = y + a \cdot x'$
includes *vectorspace* E + *subspace* H E

assumes y: $y \in H$
and x': $x' \notin H$ $x' \in E$ $x' \neq 0$
shows $h'\ x = h\ y + a * xi$
proof $-$
from $x\ y\ x'$ **have** $x \in H + lin\ x'$ **by** *auto*
have $\exists!p.\ (\lambda(y,\ a).\ x = y + a \cdot x' \wedge y \in H)\ p$ (**is** $\exists!p.\ ?P\ p$)
proof
from $x\ y$ **show** $\exists\, p.\ ?P\ p$ **by** *blast*
fix $p\ q$ **assume** p: $?P\ p$ **and** q: $?P\ q$
show $p = q$
proof $-$
from p **have** xp: $x = fst\ p + snd\ p \cdot x' \wedge fst\ p \in H$
by (*cases* p) *simp*
from q **have** xq: $x = fst\ q + snd\ q \cdot x' \wedge fst\ q \in H$
by (*cases* q) *simp*
have $fst\ p = fst\ q \wedge snd\ p = snd\ q$
proof (*rule decomp-H'*)
from xp **show** $fst\ p \in H$ **..**
from xq **show** $fst\ q \in H$ **..**
from xp **and** xq **show** $fst\ p + snd\ p \cdot x' = fst\ q + snd\ q \cdot x'$
by *simp*
apply-end *assumption+*
qed
thus *?thesis* **by** (*cases* p, *cases* q) *simp*
qed
qed
hence eq: $(SOME\ (y,\ a).\ x = y + a \cdot x' \wedge y \in H) = (y,\ a)$
by (*rule some1-equality*) (*simp add*: $x\ y$)
with *h'-def* **show** $h'\ x = h\ y + a * xi$ **by** (*simp add*: *Let-def*)
qed

end

5 Normed vector spaces

theory *NormedSpace* **imports** *Subspace* **begin**

5.1 Quasinorms

A *seminorm* $\|\cdot\|$ is a function on a real vector space into the reals that has the following properties: it is positive definite, absolute homogenous and subadditive.

locale *norm-syntax* =
fixes *norm* :: *'a* $\Rightarrow$ *real* $(\|\text{-}\|)$

locale *seminorm* = *var* V + *norm-syntax* +
assumes *ge-zero* [*iff?*]: $x \in V \Longrightarrow 0 \leq \|x\|$
and *abs-homogenous* [*iff?*]: $x \in V \Longrightarrow \|a \cdot x\| = |a| * \|x\|$
and *subadditive* [*iff?*]: $x \in V \Longrightarrow y \in V \Longrightarrow \|x + y\| \leq \|x\| + \|y\|$

declare *seminorm.intro* [*intro?*]

lemma (**in** *seminorm*) *diff-subadditive*:
  includes *vectorspace*
  shows $x \in V \Longrightarrow y \in V \Longrightarrow \|x - y\| \leq \|x\| + \|y\|$
proof −
  assume *x*: $x \in V$ **and** *y*: $y \in V$
  hence $x - y = x + - 1 \cdot y$
    by (*simp add*: *diff-eq2 negate-eq2a*)
  also from *x y* **have** $\|\ldots\| \leq \|x\| + \|- 1 \cdot y\|$
    by (*simp add*: *subadditive*)
  also from *y* **have** $\|- 1 \cdot y\| = |- 1| * \|y\|$
    by (*rule abs-homogenous*)
  also have $\ldots = \|y\|$ **by** *simp*
  finally show *?thesis* **.**
qed

lemma (**in** *seminorm*) *minus*:
  includes *vectorspace*
  shows $x \in V \Longrightarrow \|- x\| = \|x\|$
proof −
  assume *x*: $x \in V$
  hence $- x = - 1 \cdot x$ **by** (*simp only*: *negate-eq1*)
  also from *x* **have** $\|\ldots\| = |- 1| * \|x\|$
    by (*rule abs-homogenous*)
  also have $\ldots = \|x\|$ **by** *simp*
  finally show *?thesis* **.**
qed

5.2 Norms

A *norm* $\|\cdot\|$ is a seminorm that maps only the *0* vector to *0*.

locale *norm* = *seminorm* +
  assumes *zero-iff* [*iff*]: $x \in V \Longrightarrow (\|x\| = 0) = (x = 0)$

5.3 Normed vector spaces

A vector space together with a norm is called a *normed space*.

locale *normed-vectorspace* = *vectorspace* + *norm*

declare *normed-vectorspace.intro* [*intro?*]

lemma (**in** *normed-vectorspace*) *gt-zero* [*intro?*]:
  $x \in V \Longrightarrow x \neq 0 \Longrightarrow 0 < \|x\|$
proof −
  assume *x*: $x \in V$ **and** *neq*: $x \neq 0$
  from *x* **have** $0 \leq \|x\|$ **..**
  also have [*symmetric*]: $\ldots \neq 0$
  proof
    assume $\|x\| = 0$
    with *x* **have** $x = 0$ **by** *simp*
    with *neq* **show** *False* **by** *contradiction*
  qed
  finally show *?thesis* **.**
qed

Any subspace of a normed vector space is again a normed vectorspace.

lemma *subspace-normed-vs* [*intro?*]:
  includes *subspace F E* + *normed-vectorspace E*
  shows *normed-vectorspace F norm*
proof
  show *vectorspace F* **by** (*rule vectorspace*)
  have *seminorm E norm* **. with** *subset* **show** *seminorm F norm*
    by (*simp add*: *seminorm-def*)
  have *norm-axioms E norm* **. with** *subset* **show** *norm-axioms F norm*
    by (*simp add*: *norm-axioms-def*)
qed

end

6 Linearforms

theory *Linearform* **imports** *VectorSpace* **begin**

A *linear form* is a function on a vector space into the reals that is additive and multiplicative.

locale *linearform* = *var V* + *var f* +
  assumes *add* [*iff*]: $x \in V \Longrightarrow y \in V \Longrightarrow f\ (x + y) = f\,x + f\,y$
    and *mult* [*iff*]: $x \in V \Longrightarrow f\ (a \cdot x) = a * f\,x$

declare *linearform.intro* [*intro?*]

lemma (**in** *linearform*) *neg* [*iff*]:
  includes *vectorspace*
  shows $x \in V \Longrightarrow f\ (-\ x) = -\ f\,x$
proof −
  assume *x*: $x \in V$
  hence $f\ (-\ x) = f\ ((-\ 1) \cdot x)$ **by** (*simp add*: *negate-eq1*)
  also from *x* **have** ... $= (-\ 1) * (f\,x)$ **by** (*rule mult*)
  also from *x* **have** ... $= -\ (f\,x)$ **by** *simp*
  finally show *?thesis* **.**
qed

lemma (**in** *linearform*) *diff* [*iff*]:
  includes *vectorspace*
  shows $x \in V \Longrightarrow y \in V \Longrightarrow f\ (x - y) = f\,x - f\,y$
proof −
  assume *x*: $x \in V$ **and** *y*: $y \in V$
  hence $x - y = x + -\ y$ **by** (*rule diff-eq1*)
  also have f ... $= f\,x + f\ (-\ y)$ **by** (*rule add*) (*simp-all add*: *x y*)
  also from - *y* **have** $f\ (-\ y) = -\ f\,y$ **by** (*rule neg*)
  finally show *?thesis* **by** *simp*
qed

Every linear form yields *0* for the *0* vector.

lemma (**in** *linearform*) *zero* [*iff*]:
  includes *vectorspace*

shows $f\ 0 = 0$
proof $-$
have $f\ 0 = f\ (0 - 0)$ **by** *simp*
also have $\ldots = f\ 0 - f\ 0$ **by** (*rule diff*) *simp-all*
also have $\ldots = 0$ **by** *simp*
finally show *?thesis* **.**
qed

end

7 An order on functions

theory *FunctionOrder* **imports** *Subspace Linearform* **begin**

7.1 The graph of a function

We define the *graph* of a (real) function f with domain F as the set

$$\{(x, f\ x).\ x \in F\}$$

So we are modeling partial functions by specifying the domain and the mapping function. We use the term "function" also for its graph.

types $'a\ graph = ('a \times real)\ set$

constdefs
$graph :: 'a\ set \Rightarrow ('a \Rightarrow real) \Rightarrow 'a\ graph$
$graph\ F\ f \equiv \{(x, f\ x) \mid x.\ x \in F\}$

lemma *graphI* [*intro*]: $x \in F \Longrightarrow (x, f\ x) \in graph\ F\ f$
by (*unfold graph-def*) *blast*

lemma *graphI2* [*intro?*]: $x \in F \Longrightarrow \exists\, t \in graph\ F\ f.\ t = (x, f\ x)$
by (*unfold graph-def*) *blast*

lemma *graphE* [*elim?*]:
$(x, y) \in graph\ F\ f \Longrightarrow (x \in F \Longrightarrow y = f\ x \Longrightarrow C) \Longrightarrow C$
by (*unfold graph-def*) *blast*

7.2 Functions ordered by domain extension

A function h' is an extension of h, iff the graph of h is a subset of the graph of h'.

lemma *graph-extI*:
$(\bigwedge x.\ x \in H \Longrightarrow h\ x = h'\ x) \Longrightarrow H \subseteq H'$
$\Longrightarrow graph\ H\ h \subseteq graph\ H'\ h'$
by (*unfold graph-def*) *blast*

lemma *graph-extD1* [*dest?*]:
$graph\ H\ h \subseteq graph\ H'\ h' \Longrightarrow x \in H \Longrightarrow h\ x = h'\ x$
by (*unfold graph-def*) *blast*

lemma *graph-extD2* [*dest?*]:
$graph\ H\ h \subseteq graph\ H'\ h' \Longrightarrow H \subseteq H'$
by (*unfold graph-def*) *blast*

7.3 Domain and function of a graph

The inverse functions to *graph* are *domain* and *funct*.

constdefs
$domain :: 'a\ graph \Rightarrow 'a\ set$
$domain\ g \equiv \{x.\ \exists\, y.\ (x,\ y) \in g\}$

$funct :: 'a\ graph \Rightarrow ('a \Rightarrow real)$
$funct\ g \equiv \lambda x.\ (SOME\ y.\ (x,\ y) \in g)$

The following lemma states that g is the graph of a function if the relation induced by g is unique.

lemma *graph-domain-funct*:
assumes *uniq*: $\bigwedge x\ y\ z.\ (x,\ y) \in g \Longrightarrow (x,\ z) \in g \Longrightarrow z = y$
shows $graph\ (domain\ g)\ (funct\ g) = g$
proof (*unfold domain-def funct-def graph-def*, *auto*)
fix $a\ b$ **assume** $(a,\ b) \in g$
show $(a,\ SOME\ y.\ (a,\ y) \in g) \in g$ **by** (*rule someI2*)
show $\exists\, y.\ (a,\ y) \in g$ **..**
show $b = (SOME\ y.\ (a,\ y) \in g)$
proof (*rule some-equality* [*symmetric*])
fix y **assume** $(a,\ y) \in g$
show $y = b$ **by** (*rule uniq*)
qed
qed

7.4 Norm-preserving extensions of a function

Given a linear form f on the space F and a seminorm p on E. The set of all linear extensions of f, to superspaces H of F, which are bounded by p, is defined as follows.

constdefs
norm-pres-extensions ::
$'a::\{plus,\ minus,\ zero\}\ set \Rightarrow ('a \Rightarrow real) \Rightarrow 'a\ set \Rightarrow ('a \Rightarrow real)$
$\Rightarrow 'a\ graph\ set$
norm-pres-extensions $E\ p\ F\ f$
$\equiv \{g.\ \exists\, H\ h.\ g = graph\ H\ h$
$\wedge\ linearform\ H\ h$
$\wedge\ H \trianglelefteq E$
$\wedge\ F \trianglelefteq H$
$\wedge\ graph\ F\ f \subseteq graph\ H\ h$
$\wedge\ (\forall\, x \in H.\ h\ x \le p\ x)\}$

lemma *norm-pres-extensionE* [*elim*]:
$g \in$ *norm-pres-extensions* $E\ p\ F\ f$
$\Longrightarrow (\bigwedge H\ h.\ g = graph\ H\ h \Longrightarrow linearform\ H\ h$
$\Longrightarrow H \trianglelefteq E \Longrightarrow F \trianglelefteq H \Longrightarrow graph\ F\ f \subseteq graph\ H\ h$
$\Longrightarrow \forall\, x \in H.\ h\ x \le p\ x \Longrightarrow C) \Longrightarrow C$

by (*unfold norm-pres-extensions-def*) *blast*

lemma *norm-pres-extensionI2* [*intro*]:
linearform H h $\Longrightarrow$ *H* $\trianglelefteq$ *E* $\Longrightarrow$ *F* $\trianglelefteq$ *H*
$\Longrightarrow$ *graph F f* $\subseteq$ *graph H h* $\Longrightarrow$ $\forall\, x \in H.\ h\ x \le p\ x$
$\Longrightarrow$ *graph H h* $\in$ *norm-pres-extensions E p F f*
by (*unfold norm-pres-extensions-def*) *blast*

lemma *norm-pres-extensionI*:
$\exists\, H\ h.\ g =$ *graph H h*
$\wedge$ *linearform H h*
$\wedge$ $H \trianglelefteq E$
$\wedge$ $F \trianglelefteq H$
$\wedge$ *graph F f* $\subseteq$ *graph H h*
$\wedge$ $(\forall\, x \in H.\ h\ x \le p\ x) \Longrightarrow g \in$ *norm-pres-extensions E p F f*
by (*unfold norm-pres-extensions-def*) *blast*

end

8 The norm of a function

theory *FunctionNorm* **imports** *NormedSpace FunctionOrder* **begin**

8.1 Continuous linear forms

A linear form f on a normed vector space $(V, \|\cdot\|)$ is *continuous*, iff it is bounded, i.e.

$$\exists\, c \in R.\ \forall\, x \in V.\ |f\ x| \le c \cdot \|x\|$$

In our application no other functions than linear forms are considered, so we can define continuous linear forms as bounded linear forms:

locale *continuous* = *var V* + *norm-syntax* + *linearform* +
assumes *bounded*: $\exists\, c.\ \forall\, x \in V.\ |f\ x| \le c * \|x\|$

declare *continuous.intro* [*intro?*] *continuous-axioms.intro* [*intro?*]

lemma *continuousI* [*intro*]:
includes *norm-syntax* + *linearform*
assumes *r*: $\bigwedge x.\ x \in V \Longrightarrow |f\ x| \le c * \|x\|$
shows *continuous V norm f*
proof
show *linearform V f* **.**
from *r* **have** $\exists\, c.\ \forall\, x{\in}V.\ |f\ x| \le c * \|x\|$ **by** *blast*
then show *continuous-axioms V norm f* **..**
qed

8.2 The norm of a linear form

The least real number c for which holds

$$\forall\, x \in V.\ |f\ x| \le c \cdot \|x\|$$

is called the *norm* of *f*.

For non-trivial vector spaces $V \neq \{0\}$ the norm can be defined as

$$\|f\| = \sup x \neq 0.\ |f\,x| \ /\ \|x\|$$

For the case $V = \{0\}$ the supremum would be taken from an empty set. Since $\mathbb{R}$ is unbounded, there would be no supremum. To avoid this situation it must be guaranteed that there is an element in this set. This element must be $\{\} \geq 0$ so that *fn-norm* has the norm properties. Furthermore it does not have to change the norm in all other cases, so it must be 0, as all other elements are $\{\} \geq 0$.

Thus we define the set B where the supremum is taken from as follows:

$$\{0\} \cup \{|f\,x| \ /\ \|x\|.\ x \neq 0 \wedge x \in F\}$$

fn-norm is equal to the supremum of B, if the supremum exists (otherwise it is undefined).

locale *fn-norm* = *norm-syntax* +
  fixes B **defines** $B\ V\,f \equiv \{0\} \cup \{|f\,x| \ /\ \|x\| \mid x.\ x \neq 0 \wedge x \in V\}$
  fixes *fn-norm* ($\|$-$\|$-- [0, 1000] 999)
  defines $\|f\|$-$V \equiv \bigsqcup(B\ V\,f)$

lemma (**in** *fn-norm*) *B-not-empty* [*intro*]: $0 \in B\ V\,f$
  by (*simp add*: *B-def*)

The following lemma states that every continuous linear form on a normed space $(V, \|\cdot\|)$ has a function norm.

lemma (**in** *normed-vectorspace*) *fn-norm-works*:
  includes *fn-norm* + *continuous*
  shows *lub* $(B\ V\,f)$ $(\|f\|$-$V)$
proof −

The existence of the supremum is shown using the completeness of the reals. Completeness means, that every non-empty bounded set of reals has a supremum.

  have $\exists\,a.$ *lub* $(B\ V\,f)\ a$
  proof (*rule real-complete*)

First we have to show that B is non-empty:

    have $0 \in B\ V\,f$ **..**
    thus $\exists\,x.\ x \in B\ V\,f$ **..**

Then we have to show that B is bounded:

    show $\exists\,c.\ \forall\,y \in B\ V\,f.\ y \leq c$
    proof −

We know that f is bounded by some value c.

      from *bounded* **obtain** c **where** c: $\forall\,x \in V.\ |f\,x| \leq c * \|x\|$ **..**

To prove the thesis, we have to show that there is some b, such that $y \leq b$ for all $y \in B$. Due to the definition of B there are two cases.

      def $b \equiv$ *max* c 0

```
      have ∀ y ∈ B V f. y ≤ b
      proof
        fix y assume y: y ∈ B V f
        show y ≤ b
        proof cases
          assume y = 0
          thus ?thesis by (unfold b-def) arith
        next
```

The second case is $y = |f\ x|\ /\ \|x\|$ for some $x \in V$ with $x \neq 0$.

```
          assume y ≠ 0
          with y obtain x where y-rep: y = |f x| * inverse ‖x‖
              and x: x ∈ V and neq: x ≠ 0
            by (auto simp add: B-def real-divide-def)
          from x neq have gt: 0 < ‖x‖ ..
```

The thesis follows by a short calculation using the fact that f is bounded.

```
          note y-rep
          also have |f x| * inverse ‖x‖ ≤ (c * ‖x‖) * inverse ‖x‖
          proof (rule mult-right-mono)
            from c show |f x| ≤ c * ‖x‖ ..
            from gt have 0 < inverse ‖x‖
              by (rule positive-imp-inverse-positive)
            thus 0 ≤ inverse ‖x‖ by (rule order-less-imp-le)
          qed
          also have . . . = c * (‖x‖ * inverse ‖x‖)
            by (rule real-mult-assoc)
          also
          from gt have ‖x‖ ≠ 0 by simp
          hence ‖x‖ * inverse ‖x‖ = 1 by simp
          also have c * 1 ≤ b by (simp add: b-def le-maxI1)
          finally show y ≤ b .
        qed
      qed
      thus ?thesis ..
    qed
  qed
  then show ?thesis by (unfold fn-norm-def) (rule the-lubI-ex)
qed

lemma (in normed-vectorspace) fn-norm-ub [iff?]:
  includes fn-norm + continuous
  assumes b: b ∈ B V f
  shows b ≤ ‖f‖-V
proof −
  have lub (B V f) (‖f‖-V)
    by (unfold B-def fn-norm-def) (rule fn-norm-works [OF continuous.intro])
  from this and b show ?thesis ..
qed

lemma (in normed-vectorspace) fn-norm-leastB:
  includes fn-norm + continuous
  assumes b: ⋀b. b ∈ B V f ⟹ b ≤ y
  shows ‖f‖-V ≤ y
```

proof −
  have *lub* (*B V f*) ($\|f\|$-*V*)
    by (*unfold B-def fn-norm-def*) (*rule fn-norm-works* [*OF continuous.intro*])
  from *this* **and** *b* **show** *?thesis* **..**
qed

The norm of a continuous function is always $\geq$ *0*.

lemma (**in** *normed-vectorspace*) *fn-norm-ge-zero* [*iff*]:
  includes *fn-norm* + *continuous*
  shows $0 \leq \|f\|$-*V*
proof −

The function norm is defined as the supremum of B. So it is $\geq$ *0* if all elements in B are $\geq$ *0*, provided the supremum exists and B is not empty.

  have *lub* (*B V f*) ($\|f\|$-*V*)
    by (*unfold B-def fn-norm-def*) (*rule fn-norm-works* [*OF continuous.intro*])
  moreover have $0 \in B\ V f$ **..**
  ultimately show *?thesis* **..**
qed

The fundamental property of function norms is:

$$|f\,x| \leq \|f\| \cdot \|x\|$$

lemma (**in** *normed-vectorspace*) *fn-norm-le-cong*:
  includes *fn-norm* + *continuous* + *linearform*
  assumes *x*: $x \in V$
  shows $|f\,x| \leq \|f\|$-$V * \|x\|$
proof *cases*
  assume $x = 0$
  then have $|f\,x| = |f\ 0|$ **by** *simp*
  also have $f\ 0 = 0$ **..**
  also have $|\ldots| = 0$ **by** *simp*
  also have *a*: $0 \leq \|f\|$-*V*
      by (*unfold B-def fn-norm-def*)
        (*rule fn-norm-ge-zero* [*OF continuous.intro*])
    have $0 \leq$ *norm x* **..**
  with *a* **have** $0 \leq \|f\|$-$V * \|x\|$ **by** (*simp add*: *zero-le-mult-iff*)
  finally show $|f\,x| \leq \|f\|$-$V * \|x\|$ **.**
next
  assume $x \neq 0$
  with *x* **have** *neq*: $\|x\| \neq 0$ **by** *simp*
  then have $|f\,x| = (|f\,x| *$ *inverse* $\|x\|) * \|x\|$ **by** *simp*
  also have $\ldots \leq\ \|f\|$-$V * \|x\|$
  proof (*rule mult-right-mono*)
    from *x* **show** $0 \leq \|x\|$ **..**
    from *x* **and** *neq* **have** $|f\,x| *$ *inverse* $\|x\| \in B\ V f$
      by (*auto simp add*: *B-def real-divide-def*)
    then show $|f\,x| *$ *inverse* $\|x\| \leq \|f\|$-*V*
      by (*unfold B-def fn-norm-def*) (*rule fn-norm-ub* [*OF continuous.intro*])
  qed
  finally show *?thesis* **.**
qed

The function norm is the least positive real number for which the following inequation holds:

$$|f\ x| \leq c \cdot \|x\|$$

lemma (**in** *normed-vectorspace*) *fn-norm-least* [*intro?*]:
  includes *fn-norm* + *continuous*
  assumes *ineq*: $\forall x \in V.\ |f\ x| \leq c * \|x\|$ **and** *ge*: $0 \leq c$
  shows $\|f\|$-$V \leq c$
proof (*rule fn-norm-leastB* [*folded B-def fn-norm-def*])
  fix *b* **assume** *b*: $b \in B\ V\ f$
  show $b \leq c$
  proof *cases*
    assume $b = 0$
    with *ge* **show** *?thesis* **by** *simp*
  next
    assume $b \neq 0$
    with *b* **obtain** *x* **where** *b-rep*: $b = |f\ x| * inverse\ \|x\|$
      and *x-neq*: $x \neq 0$ **and** *x*: $x \in V$
    by (*auto simp add*: *B-def real-divide-def*)
    note *b-rep*
    also have $|f\ x| * inverse\ \|x\| \leq (c * \|x\|) * inverse\ \|x\|$
    proof (*rule mult-right-mono*)
      have $0 < \|x\|$ **..**
      then show $0 \leq inverse\ \|x\|$ **by** *simp*
      from *ineq* **and** *x* **show** $|f\ x| \leq c * \|x\|$ **..**
    qed
    also have $\ldots = c$
    proof −
      from *x-neq* **and** *x* **have** $\|x\| \neq 0$ **by** *simp*
      then show *?thesis* **by** *simp*
    qed
    finally show *?thesis* **.**
  qed
qed (*simp-all*! *add*: *continuous-def*)

end

9 Zorn's Lemma

theory *ZornLemma* **imports** *Zorn* **begin**

Zorn's Lemmas states: if every linear ordered subset of an ordered set S has an upper bound in S, then there exists a maximal element in S. In our application, S is a set of sets ordered by set inclusion. Since the union of a chain of sets is an upper bound for all elements of the chain, the conditions of Zorn's lemma can be modified: if S is non-empty, it suffices to show that for every non-empty chain c in S the union of c also lies in S.

theorem *Zorn's-Lemma*:
  assumes *r*: $\bigwedge c.\ c \in chain\ S \Longrightarrow \exists x.\ x \in c \Longrightarrow \bigcup c \in S$
    and *aS*: $a \in S$
  shows $\exists y \in S.\ \forall z \in S.\ y \subseteq z \longrightarrow y = z$

proof (*rule Zorn-Lemma2*)[1]
  show $\forall\, c \in chain\ S.\ \exists\, y \in S.\ \forall\, z \in c.\ z \subseteq y$
  proof
    fix c **assume** $c \in chain\ S$
    show $\exists\, y \in S.\ \forall\, z \in c.\ z \subseteq y$
    proof *cases*

If c is an empty chain, then every element in S is an upper bound of c.

      assume $c = \{\}$
      with *aS* **show** *?thesis* **by** *fast*

If c is non-empty, then $\bigcup c$ is an upper bound of c, lying in S.

    next
      assume c: $c \neq \{\}$
      show *?thesis*
      proof
        show $\forall\, z \in c.\ z \subseteq \bigcup c$ **by** *fast*
        show $\bigcup c \in S$
        proof (*rule r*)
          from c **show** $\exists\, x.\ x \in c$ **by** *fast*
        qed
      qed
    qed
  qed
qed

end

[1]See http://isabelle.in.tum.de/library/HOL/HOL-Complex/Zorn.html

Part II

Lemmas for the Proof

10 The supremum w.r.t. the function order

theory *HahnBanachSupLemmas* **imports** *FunctionNorm ZornLemma* **begin**

This section contains some lemmas that will be used in the proof of the Hahn-Banach Theorem. In this section the following context is presumed. Let E be a real vector space with a seminorm p on E. F is a subspace of E and f a linear form on F. We consider a chain c of norm-preserving extensions of f, such that $\bigcup c = \mathit{graph}\ H\ h$. We will show some properties about the limit function h, i.e. the supremum of the chain c.

Let c be a chain of norm-preserving extensions of the function f and let *graph H h* be the supremum of c. Every element in H is member of one of the elements of the chain.

lemmas $[\mathit{dest?}] = \mathit{chainD}$
lemmas $\mathit{chainE2}\ [\mathit{elim?}] = \mathit{chainD2}\ [\mathit{elim\text{-}format}, \mathit{standard}]$

lemma *some-H'h't*:
  assumes M: $M = \mathit{norm\text{-}pres\text{-}extensions}\ E\ p\ F\ f$
    and cM: $c \in \mathit{chain}\ M$
    and u: $\mathit{graph}\ H\ h = \bigcup c$
    and x: $x \in H$
  shows $\exists H'\ h'.\ \mathit{graph}\ H'\ h' \in c$
    $\wedge (x,\ h\ x) \in \mathit{graph}\ H'\ h'$
    $\wedge\ \mathit{linearform}\ H'\ h' \wedge H' \trianglelefteq E$
    $\wedge\ F \trianglelefteq H' \wedge \mathit{graph}\ F\ f \subseteq \mathit{graph}\ H'\ h'$
    $\wedge\ (\forall x \in H'.\ h'\ x \leq p\ x)$
proof $-$
  from x **have** $(x,\ h\ x) \in \mathit{graph}\ H\ h$ **..**
  also from u **have** $\ldots = \bigcup c$ **.**
  finally obtain g **where** gc: $g \in c$ **and** gh: $(x,\ h\ x) \in g$ **by** *blast*

  from cM **have** $c \subseteq M$ **..**
  with gc **have** $g \in M$ **..**
  also from M **have** $\ldots = \mathit{norm\text{-}pres\text{-}extensions}\ E\ p\ F\ f$ **.**
  finally obtain H' **and** h' **where** g: $g = \mathit{graph}\ H'\ h'$
    and $*$: $\mathit{linearform}\ H'\ h'\quad H' \trianglelefteq E\quad F \trianglelefteq H'$
      $\mathit{graph}\ F\ f \subseteq \mathit{graph}\ H'\ h'\quad \forall x \in H'.\ h'\ x \leq p\ x$ **..**

  from gc **and** g **have** $\mathit{graph}\ H'\ h' \in c$ **by** (*simp only*:)
  moreover from gh **and** g **have** $(x,\ h\ x) \in \mathit{graph}\ H'\ h'$ **by** (*simp only*:)
  ultimately show *?thesis* **using** $*$ **by** *blast*
qed

Let c be a chain of norm-preserving extensions of the function f and let *graph H h* be the supremum of c. Every element in the domain H of the supremum function is member of the domain H' of some function h', such that h extends h'.

lemma *some-H'h'*:
  assumes M: $M = \textit{norm-pres-extensions}\ E\ p\ F\ f$
    and cM: $c \in \textit{chain}\ M$
    and u: $\textit{graph}\ H\ h = \bigcup c$
    and x: $x \in H$
  shows $\exists H'\ h'.\ x \in H' \wedge \textit{graph}\ H'\ h' \subseteq \textit{graph}\ H\ h$
    $\wedge\ \textit{linearform}\ H'\ h' \wedge H' \trianglelefteq E \wedge F \trianglelefteq H'$
    $\wedge\ \textit{graph}\ F\ f \subseteq \textit{graph}\ H'\ h' \wedge (\forall x \in H'.\ h'\ x \leq p\ x)$
proof −
  from $M\ cM\ u\ x$ **obtain** $H'\ h'$ **where**
      $x\text{-}hx$: $(x,\ h\ x) \in \textit{graph}\ H'\ h'$
    and c: $\textit{graph}\ H'\ h' \in c$
    and $*$: $\textit{linearform}\ H'\ h'\quad H' \trianglelefteq E\quad F \trianglelefteq H'$
      $\textit{graph}\ F\ f \subseteq \textit{graph}\ H'\ h'\quad \forall x \in H'.\ h'\ x \leq p\ x$
    by (*rule some-H'h't* [*elim-format*]) *blast*
  from *x-hx* **have** $x \in H'$ **..**
  moreover from *cM u c* **have** $\textit{graph}\ H'\ h' \subseteq \textit{graph}\ H\ h$
    by (*simp only*: *chain-ball-Union-upper*)
  ultimately show *?thesis* **using** $*$ **by** *blast*
qed

Any two elements x and y in the domain H of the supremum function h are both in the domain H' of some function h', such that h extends h'.

lemma *some-H'h'2*:
  assumes M: $M = \textit{norm-pres-extensions}\ E\ p\ F\ f$
    and cM: $c \in \textit{chain}\ M$
    and u: $\textit{graph}\ H\ h = \bigcup c$
    and x: $x \in H$
    and y: $y \in H$
  shows $\exists H'\ h'.\ x \in H' \wedge y \in H'$
    $\wedge\ \textit{graph}\ H'\ h' \subseteq \textit{graph}\ H\ h$
    $\wedge\ \textit{linearform}\ H'\ h' \wedge H' \trianglelefteq E \wedge F \trianglelefteq H'$
    $\wedge\ \textit{graph}\ F\ f \subseteq \textit{graph}\ H'\ h' \wedge (\forall x \in H'.\ h'\ x \leq p\ x)$
proof −

y is in the domain H'' of some function h'', such that h extends h''.

  from $M\ cM\ u$ **and** y **obtain** $H'\ h'$ **where**
      $y\text{-}hy$: $(y,\ h\ y) \in \textit{graph}\ H'\ h'$
    and c': $\textit{graph}\ H'\ h' \in c$
    and $*$:
      $\textit{linearform}\ H'\ h'\quad H' \trianglelefteq E\quad F \trianglelefteq H'$
      $\textit{graph}\ F\ f \subseteq \textit{graph}\ H'\ h'\quad \forall x \in H'.\ h'\ x \leq p\ x$
    by (*rule some-H'h't* [*elim-format*]) *blast*

x is in the domain H' of some function h', such that h extends h'.

  from $M\ cM\ u$ **and** x **obtain** $H''\ h''$ **where**
      $x\text{-}hx$: $(x,\ h\ x) \in \textit{graph}\ H''\ h''$
    and c'': $\textit{graph}\ H''\ h'' \in c$
    and $**$:
      $\textit{linearform}\ H''\ h''\quad H'' \trianglelefteq E\quad F \trianglelefteq H''$
      $\textit{graph}\ F\ f \subseteq \textit{graph}\ H''\ h''\quad \forall x \in H''.\ h''\ x \leq p\ x$
    by (*rule some-H'h't* [*elim-format*]) *blast*

Since both h' and h'' are elements of the chain, h'' is an extension of h' or vice versa. Thus both x and y are contained in the greater one.

from *cM* **have** *graph* H'' $h'' \subseteq$ *graph* H' h' $\vee$ *graph* H' $h' \subseteq$ *graph* H'' h''
  (**is** *?case1* $\vee$ *?case2*) **..**
then show *?thesis*
proof
  assume *?case1*
  have $(x, h\ x) \in$ *graph* H'' h'' **.**
  also have ... $\subseteq$ *graph* H' h' **.**
  finally have *xh*:$(x, h\ x) \in$ *graph* H' h' **.**
  then have $x \in H'$ **..**
  moreover from *y-hy* **have** $y \in H'$ **..**
  moreover from *cM u* **and** c' **have** *graph* H' $h' \subseteq$ *graph* H h
    by (*simp only*: *chain-ball-Union-upper*)
  ultimately show *?thesis* **using** $*$ **by** *blast*
next
  assume *?case2*
  from *x-hx* **have** $x \in H''$ **..**
  moreover {
    from *y-hy* **have** $(y, h\ y) \in$ *graph* H' h' **.**
    also have $\ldots \subseteq$ *graph* H'' h'' **.**
    finally have $(y, h\ y) \in$ *graph* H'' h'' **.**
  } then have $y \in H''$ **..**
  moreover from *cM u* **and** c'' **have** *graph* H'' $h'' \subseteq$ *graph* H h
    by (*simp only*: *chain-ball-Union-upper*)
  ultimately show *?thesis* **using** $**$ **by** *blast*
qed
qed

The relation induced by the graph of the supremum of a chain c is definite, i. e. t is the graph of a function.

lemma *sup-definite*:
  assumes *M-def*: $M \equiv$ *norm-pres-extensions* E p F f
    and *cM*: $c \in$ *chain* M
    and *xy*: $(x, y) \in \bigcup c$
    and *xz*: $(x, z) \in \bigcup c$
  shows $z = y$
proof $-$
  from *cM* **have** *c*: $c \subseteq M$ **..**
  from *xy* **obtain** *G1* **where** *xy'*: $(x, y) \in$ *G1* **and** *G1*: *G1* $\in c$ **..**
  from *xz* **obtain** *G2* **where** *xz'*: $(x, z) \in$ *G2* **and** *G2*: *G2* $\in c$ **..**

  from *G1 c* **have** *G1* $\in M$ **..**
  then obtain *H1 h1* **where** *G1-rep*: *G1* $=$ *graph H1 h1*
    by (*unfold M-def*) *blast*

  from *G2 c* **have** *G2* $\in M$ **..**
  then obtain *H2 h2* **where** *G2-rep*: *G2* $=$ *graph H2 h2*
    by (*unfold M-def*) *blast*

G_1 is contained in G_2 or vice versa, since both G_1 and G_2 are members of c.

from *cM G1 G2* **have** *G1* $\subseteq$ *G2* $\vee$ *G2* $\subseteq$ *G1* (**is** *?case1* $\vee$ *?case2*) **..**
then show *?thesis*

proof
  assume *?case1*
  with xy' *G2-rep* **have** $(x, y) \in$ *graph H2 h2* **by** *blast*
  hence $y = h2\ x$ **..**
  also
  from xz' *G2-rep* **have** $(x, z) \in$ *graph H2 h2* **by** (*simp only*:)
  hence $z = h2\ x$ **..**
  finally show *?thesis* **.**
next
  assume *?case2*
  with xz' *G1-rep* **have** $(x, z) \in$ *graph H1 h1* **by** *blast*
  hence $z = h1\ x$ **..**
  also
  from xy' *G1-rep* **have** $(x, y) \in$ *graph H1 h1* **by** (*simp only*:)
  hence $y = h1\ x$ **..**
  finally show *?thesis* **..**
qed
qed

The limit function h is linear. Every element x in the domain of h is in the domain of a function h' in the chain of norm preserving extensions. Furthermore, h is an extension of h' so the function values of x are identical for h' and h. Finally, the function h' is linear by construction of M.

lemma *sup-lf*:
  assumes M: $M =$ *norm-pres-extensions E p F f*
  and cM: $c \in$ *chain M*
  and u: *graph H h* $= \bigcup c$
  shows *linearform H h*
proof
  fix $x\ y$ **assume** x: $x \in H$ **and** y: $y \in H$
  with $M\ cM\ u$ **obtain** $H'\ h'$ **where**
    x': $x \in H'$ **and** y': $y \in H'$
    and b: *graph* $H'\ h' \subseteq$ *graph H h*
    and *linearform*: *linearform* $H'\ h'$
    and *subspace*: $H' \trianglelefteq E$
  by (*rule some-H'h'2* [*elim-format*]) *blast*

  show $h\ (x + y) = h\ x + h\ y$
  proof −
    from *linearform* $x'\ y'$ **have** $h'\ (x + y) = h'\ x + h'\ y$
      by (*rule linearform.add*)
    also from $b\ x'$ **have** $h'\ x = h\ x$ **..**
    also from $b\ y'$ **have** $h'\ y = h\ y$ **..**
    also from *subspace* $x'\ y'$ **have** $x + y \in H'$
      by (*rule subspace.add-closed*)
    with b **have** $h'\ (x + y) = h\ (x + y)$ **..**
    finally show *?thesis* **.**
  qed
next
  fix $x\ a$ **assume** x: $x \in H$
  with $M\ cM\ u$ **obtain** $H'\ h'$ **where**
    x': $x \in H'$
    and b: *graph* $H'\ h' \subseteq$ *graph H h*

and *linearform*: *linearform* H' h'
and *subspace*: $H' \trianglelefteq E$
by (*rule some-H'h'* [*elim-format*]) *blast*

show $h\ (a \cdot x) = a * h\ x$
proof −
from *linearform* x' **have** $h'\ (a \cdot x) = a * h'\ x$
by (*rule linearform.mult*)
also from $b\ x'$ **have** $h'\ x = h\ x$ **..**
also from *subspace* x' **have** $a \cdot x \in H'$
by (*rule subspace.mult-closed*)
with b **have** $h'\ (a \cdot x) = h\ (a \cdot x)$ **..**
finally show *?thesis* **.**
qed
qed

The limit of a non-empty chain of norm preserving extensions of *f* is an extension of *f*, since every element of the chain is an extension of *f* and the supremum is an extension for every element of the chain.

lemma *sup-ext*:
assumes *graph*: *graph* $H\ h = \bigcup c$
and M: M = *norm-pres-extensions* $E\ p\ F\ f$
and *cM*: $c \in$ *chain* M
and *ex*: $\exists x.\ x \in c$
shows *graph* $F\ f \subseteq$ *graph* $H\ h$
proof −
from *ex* **obtain** x **where** *xc*: $x \in c$ **..**
from *cM* **have** $c \subseteq M$ **..**
with *xc* **have** $x \in M$ **..**
with M **have** $x \in$ *norm-pres-extensions* $E\ p\ F\ f$
by (*simp only*:)
then obtain $G\ g$ **where** x = *graph* $G\ g$ **and** *graph* $F\ f \subseteq$ *graph* $G\ g$ **..**
then have *graph* $F\ f \subseteq x$ **by** (*simp only*:)
also from *xc* **have** $\ldots \subseteq \bigcup c$ **by** *blast*
also from *graph* **have** $\ldots$ = *graph* $H\ h$ **..**
finally show *?thesis* **.**
qed

The domain H of the limit function is a superspace of F, since F is a subset of H. The existence of the *0* element in F and the closure properties follow from the fact that F is a vector space.

lemma *sup-supF*:
assumes *graph*: *graph* $H\ h = \bigcup c$
and M: M = *norm-pres-extensions* $E\ p\ F\ f$
and *cM*: $c \in$ *chain* M
and *ex*: $\exists x.\ x \in c$
and *FE*: $F \trianglelefteq E$
shows $F \trianglelefteq H$
proof
from *FE* **show** $F \neq \{\}$ **by** (*rule subspace.non-empty*)
from *graph M cM ex* **have** *graph* $F\ f \subseteq$ *graph* $H\ h$ **by** (*rule sup-ext*)
then show $F \subseteq H$ **..**

fix x y **assume** $x \in F$ **and** $y \in F$
with FE **show** $x + y \in F$ **by** (*rule subspace.add-closed*)
next
fix x a **assume** $x \in F$
with FE **show** $a \cdot x \in F$ **by** (*rule subspace.mult-closed*)
qed

The domain H of the limit function is a subspace of E.

lemma *sup-subE*:
assumes *graph*: $graph\ H\ h = \bigcup c$
and M: $M = norm\text{-}pres\text{-}extensions\ E\ p\ F\ f$
and cM: $c \in chain\ M$
and ex: $\exists\, x.\ x \in c$
and FE: $F \trianglelefteq E$
and E: $vectorspace\ E$
shows $H \trianglelefteq E$
proof
show $H \neq \{\}$
proof −
from FE E **have** $0 \in F$ **by** (*rule subspace.zero*)
also from *graph* M cM ex FE **have** $F \trianglelefteq H$ **by** (*rule sup-supF*)
then have $F \subseteq H$ **..**
finally show *?thesis* **by** *blast*
qed
show $H \subseteq E$
proof
fix x **assume** $x \in H$
with M cM *graph*
obtain H' h' **where** x: $x \in H'$ **and** $H'E$: $H' \trianglelefteq E$
by (*rule some-H'h'* [*elim-format*]) *blast*
from $H'E$ **have** $H' \subseteq E$ **..**
with x **show** $x \in E$ **..**
qed
fix x y **assume** x: $x \in H$ **and** y: $y \in H$
show $x + y \in H$
proof −
from M cM *graph* x y **obtain** H' h' **where**
x': $x \in H'$ **and** y': $y \in H'$ **and** $H'E$: $H' \trianglelefteq E$
and *graphs*: $graph\ H'\ h' \subseteq graph\ H\ h$
by (*rule some-H'h'2* [*elim-format*]) *blast*
from $H'E$ x' y' **have** $x + y \in H'$
by (*rule subspace.add-closed*)
also from *graphs* **have** $H' \subseteq H$ **..**
finally show *?thesis* **.**
qed
next
fix x a **assume** x: $x \in H$
show $a \cdot x \in H$
proof −
from M cM *graph* x
obtain H' h' **where** x': $x \in H'$ **and** $H'E$: $H' \trianglelefteq E$
and *graphs*: $graph\ H'\ h' \subseteq graph\ H\ h$
by (*rule some-H'h'* [*elim-format*]) *blast*
from $H'E$ x' **have** $a \cdot x \in H'$ **by** (*rule subspace.mult-closed*)

also from *graphs* **have** $H' \subseteq H$ **..**
finally show *?thesis* **.**
qed
qed

The limit function is bounded by the norm p as well, since all elements in the chain are bounded by p.

lemma *sup-norm-pres*:
assumes *graph*: $graph\ H\ h = \bigcup c$
and M: $M = norm\text{-}pres\text{-}extensions\ E\ p\ F\ f$
and cM: $c \in chain\ M$
shows $\forall x \in H.\ h\ x \leq p\ x$
proof
fix x **assume** $x \in H$
with *M cM graph* **obtain** $H'\ h'$ **where** x': $x \in H'$
and *graphs*: $graph\ H'\ h' \subseteq graph\ H\ h$
and a: $\forall x \in H'.\ h'\ x \leq p\ x$
by (*rule some-H'h'* [*elim-format*]) *blast*
from *graphs* x' **have** [*symmetric*]: $h'\ x = h\ x$ **..**
also from $a\ x'$ **have** $h'\ x \leq p\ x$ **..**
finally show $h\ x \leq p\ x$ **.**
qed

The following lemma is a property of linear forms on real vector spaces. It will be used for the lemma *abs-HahnBanach* (see page 47). For real vector spaces the following inequations are equivalent:

$$\forall x \in H.\ |h\ x| \leq p\ x \quad \text{and} \quad \forall x \in H.\ h\ x \leq p\ x$$

lemma *abs-ineq-iff*:
includes *subspace H E + vectorspace E + seminorm E p + linearform H h*
shows $(\forall x \in H.\ |h\ x| \leq p\ x) = (\forall x \in H.\ h\ x \leq p\ x)$ (**is** $?L = ?R$)
proof
have H: *vectorspace* H **..**
{
assume l: $?L$
show $?R$
proof
fix x **assume** x: $x \in H$
have $h\ x \leq |h\ x|$ **by** *arith*
also from $l\ x$ **have** $\ldots \leq p\ x$ **..**
finally show $h\ x \leq p\ x$ **.**
qed
next
assume r: $?R$
show $?L$
proof
fix x **assume** x: $x \in H$
show $\bigwedge a\ b :: real.\ -\ a \leq b \Longrightarrow b \leq a \Longrightarrow |b| \leq a$
by *arith*
have *linearform H h* **.**
from *this H x* **have** $-\ h\ x = h\ (-\ x)$ **by** (*rule linearform.neg* [*symmetric*])
also

from $H\ x$ **have** $-\ x \in H$ **by** (*rule vectorspace.neg-closed*)
with r **have** $h\ (-\ x) \leq p\ (-\ x)$ **..**
also have $\ldots = p\ x$
proof (*rule seminorm.minus*)
from x **show** $x \in E$ **..**
qed
finally have $-\ h\ x \leq p\ x$ **.**
then show $-\ p\ x \leq h\ x$ **by** *simp*
from $r\ x$ **show** $h\ x \leq p\ x$ **..**
qed
}
qed

end

11 Extending non-maximal functions

theory *HahnBanachExtLemmas* **imports** *FunctionNorm* **begin**

In this section the following context is presumed. Let E be a real vector space with a seminorm q on E. F is a subspace of E and f a linear function on F. We consider a subspace H of E that is a superspace of F and a linear form h on H. H is a not equal to E and x_0 is an element in $E - H$. H is extended to the direct sum $H' = H + lin\ x_0$, so for any $x \in H'$ the decomposition of $x = y + a \cdot x$ with $y \in H$ is unique. h' is defined on H' by $h'\ x = h\ y + a \cdot \xi$ for a certain ξ.
Subsequently we show some properties of this extension h' of h.

This lemma will be used to show the existence of a linear extension of f (see page 45). It is a consequence of the completeness of $\mathbb{R}$. To show

$$\exists \xi.\ \forall y \in F.\ a\ y \leq \xi \wedge \xi \leq b\ y$$

it suffices to show that

$$\forall u \in F.\ \forall v \in F.\ a\ u \leq b\ v$$

lemma *ex-xi*:
includes *vectorspace F*
assumes r: $\bigwedge u\ v.\ u \in F \Longrightarrow v \in F \Longrightarrow a\ u \leq b\ v$
shows $\exists\, xi{::}real.\ \forall y \in F.\ a\ y \leq xi \wedge xi \leq b\ y$
proof $-$

From the completeness of the reals follows: The set $S = \{a\ u.\ u \in F\}$ has a supremum, if it is non-empty and has an upper bound.

let $?S = \{a\ u \mid u.\ u \in F\}$
have $\exists\, xi.\ lub\ ?S\ xi$
proof (*rule real-complete*)
have $a\ 0 \in ?S$ **by** *blast*
then show $\exists X.\ X \in ?S$ **..**
have $\forall y \in ?S.\ y \leq b\ 0$
proof

```
      fix y assume y: y ∈ ?S
      then obtain u where u: u ∈ F and y: y = a u by blast
      from u and zero have a u ≤ b 0 by (rule r)
      with y show y ≤ b 0 by (simp only:)
    qed
    then show ∃ u. ∀ y ∈ ?S. y ≤ u ..
  qed
  then obtain xi where xi: lub ?S xi ..
  {
    fix y assume y ∈ F
    then have a y ∈ ?S by blast
    with xi have a y ≤ xi by (rule lub.upper)
  } moreover {
    fix y assume y: y ∈ F
    from xi have xi ≤ b y
    proof (rule lub.least)
      fix au assume au ∈ ?S
      then obtain u where u: u ∈ F and au: au = a u by blast
      from u y have a u ≤ b y by (rule r)
      with au show au ≤ b y by (simp only:)
    qed
  } ultimately show ∃ xi. ∀ y ∈ F. a y ≤ xi ∧ xi ≤ b y by blast
qed
```

The function h' is defined as a $h'\ x = h\ y + a \cdot \xi$ where $x = y + a \cdot \xi$ is a linear extension of h to H'.

```
lemma h'-lf:
  includes var H + var h + var E
  assumes h'-def: h' ≡ λx. let (y, a) =
      SOME (y, a). x = y + a · x0 ∧ y ∈ H in h y + a ∗ xi
    and H'-def: H' ≡ H + lin x0
    and HE: H ⊴ E
  includes linearform H h
  assumes x0: x0 ∉ H  x0 ∈ E  x0 ≠ 0
  includes vectorspace E
  shows linearform H' h'
proof
  have H': vectorspace H'
  proof (unfold H'-def)
    have x0 ∈ E .
    then have lin x0 ⊴ E ..
    with HE show vectorspace (H + lin x0) ..
  qed
  {
    fix x1 x2 assume x1: x1 ∈ H' and x2: x2 ∈ H'
    show h' (x1 + x2) = h' x1 + h' x2
    proof -
      from H' x1 x2 have x1 + x2 ∈ H'
        by (rule vectorspace.add-closed)
      with x1 x2 obtain y y1 y2 a a1 a2 where
            x1x2: x1 + x2 = y + a · x0 and y: y ∈ H
          and x1-rep: x1 = y1 + a1 · x0 and y1: y1 ∈ H
          and x2-rep: x2 = y2 + a2 · x0 and y2: y2 ∈ H
```

```
  by (unfold H'-def sum-def lin-def) blast

 have ya: y1 + y2 = y ∧ a1 + a2 = a using - HE - y x0
 proof (rule decomp-H')     from HE y1 y2 show y1 + y2 ∈ H
    by (rule subspace.add-closed)
  from x0 and HE y y1 y2
  have x0 ∈ E  y ∈ E  y1 ∈ E  y2 ∈ E by auto
  with x1-rep x2-rep have (y1 + y2) + (a1 + a2) · x0 = x1 + x2
    by (simp add: add-ac add-mult-distrib2)
  also note x1x2
  finally show (y1 + y2) + (a1 + a2) · x0 = y + a · x0 .
 qed

 from h'-def x1x2 - HE y x0
 have h' (x1 + x2) = h y + a * xi
  by (rule h'-definite)
 also have ... = h (y1 + y2) + (a1 + a2) * xi
  by (simp only: ya)
 also from y1 y2 have h (y1 + y2) = h y1 + h y2
  by simp
 also have ... + (a1 + a2) * xi = (h y1 + a1 * xi) + (h y2 + a2 * xi)
  by (simp add: left-distrib)
 also from h'-def x1-rep - HE y1 x0
 have h y1 + a1 * xi = h' x1
  by (rule h'-definite [symmetric])
 also from h'-def x2-rep - HE y2 x0
 have h y2 + a2 * xi = h' x2
  by (rule h'-definite [symmetric])
 finally show ?thesis .
qed
next
 fix x1 c assume x1: x1 ∈ H'
 show h' (c · x1) = c * (h' x1)
 proof -
  from H' x1 have ax1: c · x1 ∈ H'
   by (rule vectorspace.mult-closed)
  with x1 obtain y a y1 a1 where
      cx1-rep: c · x1 = y + a · x0 and y: y ∈ H
    and x1-rep: x1 = y1 + a1 · x0 and y1: y1 ∈ H
   by (unfold H'-def sum-def lin-def) blast

  have ya: c · y1 = y ∧ c * a1 = a using - HE - y x0
  proof (rule decomp-H')
   from HE y1 show c · y1 ∈ H
    by (rule subspace.mult-closed)
   from x0 and HE y y1
   have x0 ∈ E  y ∈ E  y1 ∈ E by auto
   with x1-rep have c · y1 + (c * a1) · x0 = c · x1
    by (simp add: mult-assoc add-mult-distrib1)
   also note cx1-rep
   finally show c · y1 + (c * a1) · x0 = y + a · x0 .
  qed

  from h'-def cx1-rep - HE y x0 have h' (c · x1) = h y + a * xi
```

```
      by (rule h'-definite)
    also have ... = h (c · y1) + (c ∗ a1) ∗ xi
      by (simp only: ya)
    also from y1 have h (c · y1) = c ∗ h y1
      by simp
    also have ... + (c ∗ a1) ∗ xi = c ∗ (h y1 + a1 ∗ xi)
      by (simp only: right-distrib)
    also from h'-def x1-rep - HE y1 x0 have h y1 + a1 ∗ xi = h' x1
      by (rule h'-definite [symmetric])
    finally show ?thesis .
  qed
 }
qed
```

The linear extension h' of h is bounded by the seminorm p.

```
lemma h'-norm-pres:
  includes var H + var h + var E
  assumes h'-def: h' ≡ λx. let (y, a) =
      SOME (y, a). x = y + a · x0 ∧ y ∈ H in h y + a ∗ xi
    and H'-def: H' ≡ H + lin x0
    and x0: x0 ∉ H  x0 ∈ E  x0 ≠ 0
  includes vectorspace E + subspace H E + seminorm E p + linearform H h
  assumes a: ∀ y ∈ H. h y ≤ p y
    and a': ∀ y ∈ H. − p (y + x0) − h y ≤ xi ∧ xi ≤ p (y + x0) − h y
  shows ∀ x ∈ H'. h' x ≤ p x
proof
  fix x assume x': x ∈ H'
  show h' x ≤ p x
  proof −
    from a' have a1: ∀ ya ∈ H. − p (ya + x0) − h ya ≤ xi
      and a2: ∀ ya ∈ H. xi ≤ p (ya + x0) − h ya by auto
    from x' obtain y a where
        x-rep: x = y + a · x0 and y: y ∈ H
      by (unfold H'-def sum-def lin-def) blast
    from y have y': y ∈ E ..
    from y have ay: inverse a · y ∈ H by simp

    from h'-def x-rep - - y x0 have h' x = h y + a ∗ xi
      by (rule h'-definite)
    also have ... ≤ p (y + a · x0)
    proof (rule linorder-cases)
      assume z: a = 0
      then have h y + a ∗ xi = h y by simp
      also from a y have ... ≤ p y ..
      also from x0 y' z have p y = p (y + a · x0) by simp
      finally show ?thesis .
    next
```

In the case $a < 0$, we use a_1 with ya taken as y / a:

```
      assume lz: a < 0 hence nz: a ≠ 0 by simp
      from a1 ay
      have − p (inverse a · y + x0) − h (inverse a · y) ≤ xi ..
      with lz have a ∗ xi ≤
```

```
        a * (− p (inverse a · y + x0) − h (inverse a · y))
      by (simp add: mult-left-mono-neg order-less-imp-le)

    also have ... =
        − a * (p (inverse a · y + x0)) − a * (h (inverse a · y))
      by (simp add: right-diff-distrib)
    also from lz x0 y' have − a * (p (inverse a · y + x0)) =
        p (a · (inverse a · y + x0))
      by (simp add: abs-homogenous)
    also from nz x0 y' have ... = p (y + a · x0)
      by (simp add: add-mult-distrib1 mult-assoc [symmetric])
    also from nz y have a * (h (inverse a · y)) =  h y
      by simp
    finally have a * xi ≤ p (y + a · x0) − h y .
    then show ?thesis by simp
  next
```

In the case $a > 0$, we use a_2 with *ya* taken as y / a:

```
    assume gz: 0 < a hence nz: a ≠ 0 by simp
    from a2 ay
    have xi ≤ p (inverse a · y + x0) − h (inverse a · y) ..
    with gz have a * xi ≤
        a * (p (inverse a · y + x0) − h (inverse a · y))
      by simp
    also have ... = a * p (inverse a · y + x0) − a * h (inverse a · y)
      by (simp add: right-diff-distrib)
    also from gz x0 y'
    have a * p (inverse a · y + x0) = p (a · (inverse a · y + x0))
      by (simp add: abs-homogenous)
    also from nz x0 y' have ... = p (y + a · x0)
      by (simp add: add-mult-distrib1 mult-assoc [symmetric])
    also from nz y have a * h (inverse a · y) = h y
      by simp
    finally have a * xi ≤ p (y + a · x0) − h y .
    then show ?thesis by simp
  qed
  also from x-rep have ... = p x by (simp only:)
  finally show ?thesis .
 qed
qed

end
```

Part III

The Main Proof

12 The Hahn-Banach Theorem

theory *HahnBanach* **imports** *HahnBanachLemmas* **begin**

We present the proof of two different versions of the Hahn-Banach Theorem, closely following [1, §36].

12.1 The Hahn-Banach Theorem for vector spaces

Hahn-Banach Theorem. Let F be a subspace of a real vector space E, let p be a semi-norm on E, and f be a linear form defined on F such that f is bounded by p, i.e. $\forall\, x \in F.\ f\, x \le p\, x$. Then f can be extended to a linear form h on E such that h is norm-preserving, i.e. h is also bounded by p.

Proof Sketch.

1. Define M as the set of norm-preserving extensions of f to subspaces of E. The linear forms in M are ordered by domain extension.
2. We show that every non-empty chain in M has an upper bound in M.
3. With Zorn's Lemma we conclude that there is a maximal function g in M.
4. The domain H of g is the whole space E, as shown by classical contradiction:
   - Assuming g is not defined on whole E, it can still be extended in a norm-preserving way to a super-space H' of H.
   - Thus g can not be maximal. Contradiction!

theorem *HahnBanach*:
  includes *vectorspace* E + *subspace* F E + *seminorm* E p + *linearform* F f
  assumes *fp*: $\forall\, x \in F.\ f\, x \le p\, x$
  shows $\exists\, h.\ linearform\ E\ h \wedge (\forall\, x \in F.\ h\ x = f\ x) \wedge (\forall\, x \in E.\ h\ x \le p\ x)$
    — Let E be a vector space, F a subspace of E, p a seminorm on E,
    — and f a linear form on F such that f is bounded by p,
    — then f can be extended to a linear form h on E in a norm-preserving way.
proof −
  def $M \equiv$ *norm-pres-extensions* E p F f
  hence M: $M = \dots$ **by** (*simp only*:)
  have E: *vectorspace* E **.**
  have F: *vectorspace* F **..**
  {
    fix c **assume** *cM*: $c \in chain\ M$ **and** *ex*: $\exists\, x.\ x \in c$
    have $\bigcup c \in M$
      — Show that every non-empty chain c of M has an upper bound in M:
      — $\bigcup c$ is greater than any element of the chain c, so it suffices to show $\bigcup c \in M$.

proof (*unfold M-def*, *rule norm-pres-extensionI*)
let *?H* = *domain* ($\bigcup$ *c*)
let *?h* = *funct* ($\bigcup$ *c*)

have *a*: *graph ?H ?h* = $\bigcup$ *c*
proof (*rule graph-domain-funct*)
fix $x\ y\ z$ **assume** $(x, y) \in \bigcup c$ **and** $(x, z) \in \bigcup c$
with *M-def cM* **show** $z = y$ **by** (*rule sup-definite*)
qed
moreover from *M cM a* **have** *linearform ?H ?h*
by (*rule sup-lf*)
moreover from *a M cM ex* **have** *?H* $\trianglelefteq$ *E*
by (*rule sup-subE*)
moreover from *a M cM ex* **have** *F* $\trianglelefteq$ *?H*
by (*rule sup-supF*)
moreover from *a M cM ex* **have** *graph F f* $\subseteq$ *graph ?H ?h*
by (*rule sup-ext*)
moreover from *a M cM* **have** $\forall x \in$ *?H*. *?h* $x \le p\ x$
by (*rule sup-norm-pres*)
ultimately show $\exists H\ h.\ \bigcup c =$ *graph H h*
$\wedge$ *linearform H h*
$\wedge$ $H \trianglelefteq E$
$\wedge$ $F \trianglelefteq H$
$\wedge$ *graph F f* $\subseteq$ *graph H h*
$\wedge$ $(\forall x \in H.\ h\ x \le p\ x)$ **by** *blast*
qed
}
hence $\exists g \in M.\ \forall x \in M.\ g \subseteq x \longrightarrow g = x$
— With Zorn's Lemma we can conclude that there is a maximal element in M.
proof (*rule Zorn's-Lemma*)
— We show that M is non-empty:
show *graph F f* $\in M$
proof (*unfold M-def*, *rule norm-pres-extensionI2*)
show *linearform F f* **.**
show $F \trianglelefteq E$ **.**
from *F* **show** $F \trianglelefteq F$ **by** (*rule vectorspace.subspace-refl*)
show *graph F f* $\subseteq$ *graph F f* **..**
show $\forall x{\in}F.\ f\ x \le p\ x$ **.**
qed
qed
then obtain *g* **where** *gM*: $g \in M$ **and** $\forall x \in M.\ g \subseteq x \longrightarrow g = x$
by *blast*
from *gM* [*unfolded M-def*] **obtain** *H h* **where**
g-rep: *g* = *graph H h*
and *linearform*: *linearform H h*
and *HE*: $H \trianglelefteq E$ **and** *FH*: $F \trianglelefteq H$
and *graphs*: *graph F f* $\subseteq$ *graph H h*
and *hp*: $\forall x \in H.\ h\ x \le p\ x$ **..**
— g is a norm-preserving extension of f, in other words:
— g is the graph of some linear form h defined on a subspace H of E,
— and h is an extension of f that is again bounded by p.
from *HE* **have** *H*: *vectorspace H*
by (*rule subspace.vectorspace*)

```
have HE-eq: H = E
 — We show that h is defined on whole E by classical contradiction.
proof (rule classical)
 assume neq: H ≠ E
  — Assume h is not defined on whole E. Then show that h can be extended
  — in a norm-preserving way to a function h′ with the graph g′.
 have ∃ g′ ∈ M. g ⊆ g′ ∧ g ≠ g′
 proof −
  from HE have H ⊆ E ..
  with neq obtain x′ where x′E: x′ ∈ E and x′ ∉ H by blast
  obtain x′: x′ ≠ 0
  proof
   show x′ ≠ 0
   proof
    assume x′ = 0
    with H have x′ ∈ H by (simp only: vectorspace.zero)
    then show False by contradiction
   qed
  qed

  def H′ ≡ H + lin x′
   — Define H′ as the direct sum of H and the linear closure of x′.
  have HH′: H ⊴ H′
  proof (unfold H′-def)
   have vectorspace (lin x′) ..
   with H show H ⊴ H + lin x′ ..
  qed

  obtain xi where
   ∀ y ∈ H. − p (y + x′) − h y ≤ xi
     ∧ xi ≤ p (y + x′) − h y
   — Pick a real number ξ that fulfills certain inequations; this will
   — be used to establish that h′ is a norm-preserving extension of h.
  proof −
   from H have ∃ xi. ∀ y ∈ H. − p (y + x′) − h y ≤ xi
     ∧ xi ≤ p (y + x′) − h y
   proof (rule ex-xi)
    fix u v assume u: u ∈ H and v: v ∈ H
    with HE have uE: u ∈ E and vE: v ∈ E by auto
    from H u v linearform have h v − h u = h (v − u)
     by (simp add: linearform.diff)
    also from hp and H u v have . . . ≤ p (v − u)
     by (simp only: vectorspace.diff-closed)
    also from x′E uE vE have v − u = x′ + − x′ + v + − u
     by (simp add: diff-eq1)
    also from x′E uE vE have . . . = v + x′ + − (u + x′)
     by (simp add: add-ac)
    also from x′E uE vE have . . . = (v + x′) − (u + x′)
     by (simp add: diff-eq1)
    also from x′E uE vE E have p . . . ≤ p (v + x′) + p (u + x′)
     by (simp add: diff-subadditive)
    finally have h v − h u ≤ p (v + x′) + p (u + x′) .
    then show − p (u + x′) − h u ≤ p (v + x′) − h v by simp
```

```
  qed
  then show ?thesis ..
qed

def h' ≡ λx. let (y, a) =
    SOME (y, a). x = y + a · x' ∧ y ∈ H in h y + a * xi
  — Define the extension h' of h to H' using ξ.

have g ⊆ graph H' h' ∧ g ≠ graph H' h'
  — h' is an extension of h ...
proof
  show g ⊆ graph H' h'
  proof −
    have graph H h ⊆ graph H' h'
    proof (rule graph-extI)
      fix t assume t: t ∈ H
      have (SOME (y, a). t = y + a · x' ∧ y ∈ H) = (t, 0)
        by (rule decomp-H'-H)
      with h'-def show h t = h' t by (simp add: Let-def)
    next
      from HH' show H ⊆ H' ..
    qed
    with g-rep show ?thesis by (simp only:)
  qed

  show g ≠ graph H' h'
  proof −
    have graph H h ≠ graph H' h'
    proof
      assume eq: graph H h = graph H' h'
      have x' ∈ H'
      proof (unfold H'-def, rule)
        from H show 0 ∈ H by (rule vectorspace.zero)
        from x'E show x' ∈ lin x' by (rule x-lin-x)
        from x'E show x' = 0 + x' by simp
      qed
      hence (x', h' x') ∈ graph H' h' ..
      with eq have (x', h' x') ∈ graph H h by (simp only:)
      hence x' ∈ H ..
      thus False by contradiction
    qed
    with g-rep show ?thesis by simp
  qed
qed
moreover have graph H' h' ∈ M
  — and h' is norm-preserving.
proof (unfold M-def)
  show graph H' h' ∈ norm-pres-extensions E p F f
  proof (rule norm-pres-extensionI2)
    show linearform H' h' by (rule h'-lf)
    show H' ⊴ E
    proof (unfold H'-def, rule)
      show H ⊴ E .
```

show *vectorspace E* **.**
from $x'E$ **show** $lin\ x' \trianglelefteq E$ **..**
qed
have $F \trianglelefteq H$ **.**
from *H this HH'* **show** FH': $F \trianglelefteq H'$
by (*rule vectorspace.subspace-trans*)
show $graph\ F\ f \subseteq graph\ H'\ h'$
proof (*rule graph-extI*)
fix x **assume** x: $x \in F$
with *graphs* **have** $f\ x = h\ x$ **..**
also have $\ldots = h\ x + 0 * xi$ **by** *simp*
also have $\ldots = (let\ (y,\ a) = (x,\ 0)\ in\ h\ y + a * xi)$
by (*simp add*: *Let-def*)
also have $(x,\ 0) =$
$(SOME\ (y,\ a).\ x = y + a \cdot x' \wedge y \in H)$
proof (*rule decomp-H'-H* [*symmetric*])
from *FH x* **show** $x \in H$ **..**
from x' **show** $x' \neq 0$ **.**
qed
also have
$(let\ (y,\ a) = (SOME\ (y,\ a).\ x = y + a \cdot x' \wedge y \in H)$
$in\ h\ y + a * xi) = h'\ x$ **by** (*simp only*: *h'-def*)
finally show $f\ x = h'\ x$ **.**
next
from FH' **show** $F \subseteq H'$ **..**
qed
show $\forall x \in H'.\ h'\ x \leq p\ x$ **by** (*rule h'-norm-pres*)
qed
qed
ultimately show *?thesis* **..**
qed
hence $\neg (\forall x \in M.\ g \subseteq x \longrightarrow g = x)$ **by** *simp*
— So the graph g of h cannot be maximal. Contradiction!

then show $H = E$ **by** *contradiction*
qed

from *HE-eq* **and** *linearform* **have** *linearform E h*
by (*simp only*:)
moreover have $\forall x \in F.\ h\ x = f\ x$
proof
fix x **assume** $x \in F$
with *graphs* **have** $f\ x = h\ x$ **..**
then show $h\ x = f\ x$ **..**
qed
moreover from *HE-eq* **and** *hp* **have** $\forall x \in E.\ h\ x \leq p\ x$
by (*simp only*:)
ultimately show *?thesis* **by** *blast*
qed

12.2 Alternative formulation

The following alternative formulation of the Hahn-Banach Theorem uses the fact that for a real linear form f and a seminorm p the following inequations are equivalent:[2]

$$\forall x \in H.\ |h\ x| \leq p\ x \quad \text{and} \quad \forall x \in H.\ h\ x \leq p\ x$$

theorem *abs-HahnBanach*:
  includes *vectorspace E* + *subspace F E* + *linearform F f* + *seminorm E p*
  assumes *fp*: $\forall x \in F.\ |f\ x| \leq p\ x$
  shows $\exists g.$ *linearform E g*
    $\wedge\ (\forall x \in F.\ g\ x = f\ x)$
    $\wedge\ (\forall x \in E.\ |g\ x| \leq p\ x)$
proof −
  have $\exists g.$ *linearform E g* $\wedge\ (\forall x \in F.\ g\ x = f\ x)$
    $\wedge\ (\forall x \in E.\ g\ x \leq p\ x)$
  proof (*rule HahnBanach*)
    show $\forall x \in F.\ f\ x \leq p\ x$
      by (*rule abs-ineq-iff* [*THEN iffD1*])
  qed
  then obtain *g* **where** ∗ : *linearform E g* $\forall x \in F.\ g\ x = f\ x$
      and $\forall x \in E.\ g\ x \leq p\ x$ **by** *blast*
  have $\forall x \in E.\ |g\ x| \leq p\ x$
  proof (*rule abs-ineq-iff* [*THEN iffD2*])
    show $E \trianglelefteq E$ **..**
  qed
  with ∗ **show** *?thesis* **by** *blast*
qed

12.3 The Hahn-Banach Theorem for normed spaces

Every continuous linear form f on a subspace F of a norm space E, can be extended to a continuous linear form g on E such that $\|f\| = \|g\|$.

theorem *norm-HahnBanach*:
  includes *normed-vectorspace E* + *subspace F E* + *linearform F f* + *fn-norm* + *continuous F norm* (‖-‖) *f*
  shows $\exists g.$ *linearform E g*
    $\wedge$ *continuous E norm g*
    $\wedge\ (\forall x \in F.\ g\ x = f\ x)$
    $\wedge\ \|g\|\text{-}E = \|f\|\text{-}F$
proof −
  have *E*: *vectorspace E* **.**
  have *E-norm*: *normed-vectorspace E norm* **..**
  have *FE*: $F \trianglelefteq E$ **.**
  have *F*: *vectorspace F* **..**
  have *linearform*: *linearform F f* **.**
  have *F-norm*: *normed-vectorspace F norm*
    by (*rule subspace-normed-vs* [*OF E-norm*])
  have *ge-zero*: $0 \leq \|f\|\text{-}F$
    by (*rule normed-vectorspace.fn-norm-ge-zero*
      [*OF F-norm continuous.intro*, *folded B-def fn-norm-def*])

We define a function p on E as follows: $p\ x = \|f\| \cdot \|x\|$

  def $p \equiv \lambda x.\ \|f\|\text{-}F * \|x\|$

[2] This was shown in lemma *abs-ineq-iff* (see page 37).

p is a seminorm on E:

have q: *seminorm E p*
proof
  fix x y a **assume** x: $x \in E$ **and** y: $y \in E$

p is positive definite:

    have $0 \leq \|f\|$-F **by** (*rule ge-zero*)
    moreover from x **have** $0 \leq \|x\|$ **..**
  ultimately show $0 \leq p\ x$
    by (*simp add*: *p-def zero-le-mult-iff*)

p is absolutely homogenous:

  show $p\ (a \cdot x) = |a| * p\ x$
  proof −
    have $p\ (a \cdot x) = \|f\|$-$F * \|a \cdot x\|$ **by** (*simp only*: *p-def*)
    also from x **have** $\|a \cdot x\| = |a| * \|x\|$ **by** (*rule abs-homogenous*)
    also have $\|f\|$-$F * (|a| * \|x\|) = |a| * (\|f\|$-$F * \|x\|)$ **by** *simp*
    also have $\ldots = |a| * p\ x$ **by** (*simp only*: *p-def*)
    finally show *?thesis* **.**
  qed

Furthermore, p is subadditive:

  show $p\ (x + y) \leq p\ x + p\ y$
  proof −
    have $p\ (x + y) = \|f\|$-$F * \|x + y\|$ **by** (*simp only*: *p-def*)
    also have a: $0 \leq \|f\|$-F **by** (*rule ge-zero*)
    from x y **have** $\|x + y\| \leq \|x\| + \|y\|$ **..**
    with a **have** $\|f\|$-$F * \|x + y\| \leq \|f\|$-$F * (\|x\| + \|y\|)$
      by (*simp add*: *mult-left-mono*)
    also have $\ldots = \|f\|$-$F * \|x\| + \|f\|$-$F * \|y\|$ **by** (*simp only*: *right-distrib*)
    also have $\ldots = p\ x + p\ y$ **by** (*simp only*: *p-def*)
    finally show *?thesis* **.**
  qed
qed

f is bounded by p.

have $\forall x \in F.\ |f\ x| \leq p\ x$
proof
  fix x **assume** $x \in F$
  show $|f\ x| \leq p\ x$
    by (*unfold p-def*) (*rule normed-vectorspace.fn-norm-le-cong*
      [*OF F-norm continuous.intro*, *folded B-def fn-norm-def*])
qed

Using the fact that p is a seminorm and f is bounded by p we can apply the Hahn-Banach Theorem for real vector spaces. So f can be extended in a norm-preserving way to some function g on the whole vector space E.

with *E FE linearform q* **obtain** g **where**
      linearformE: *linearform E g*
    and a: $\forall x \in F.\ g\ x = f\ x$
    and b: $\forall x \in E.\ |g\ x| \leq p\ x$
  by (*rule abs-HahnBanach* [*elim-format*]) *iprover*

We furthermore have to show that g is also continuous:

have *g-cont*: *continuous E norm g* **using** *linearformE*
proof
  fix x **assume** $x \in E$
  with b **show** $|g\ x| \leq \|f\|\text{-}F * \|x\|$
    by (*simp only*: *p-def*)
qed

To complete the proof, we show that $\|g\| = \|f\|$.

have $\|g\|\text{-}E = \|f\|\text{-}F$
proof (*rule order-antisym*)

First we show $\|g\| \leq \|f\|$. The function norm $\|g\|$ is defined as the smallest $c \in \mathbb{R}$ such that

$$\forall\, x \in E.\ |g\ x| \leq c \cdot \|x\|$$

Furthermore holds

$$\forall\, x \in E.\ |g\ x| \leq \|f\| \cdot \|x\|$$

  have $\forall\, x \in E.\ |g\ x| \leq \|f\|\text{-}F * \|x\|$
  proof
    fix x **assume** $x \in E$
    with b **show** $|g\ x| \leq \|f\|\text{-}F * \|x\|$
      by (*simp only*: *p-def*)
  qed
  from *g-cont this ge-zero*
  show $\|g\|\text{-}E \leq \|f\|\text{-}F$
    by (*rule fn-norm-least* [*of g*, *folded B-def fn-norm-def*])

The other direction is achieved by a similar argument.

  show $\|f\|\text{-}F \leq \|g\|\text{-}E$
 proof (*rule normed-vectorspace.fn-norm-least* [*OF F-norm*, *folded B-def fn-norm-def*])
    show $\forall\, x \in F.\ |f\ x| \leq \|g\|\text{-}E * \|x\|$
    proof
      fix x **assume** x: $x \in F$
      from a **have** $g\ x = f\ x$ **..**
      hence $|f\ x| = |g\ x|$ **by** (*simp only*:)
      also from *g-cont*
      have $\ldots \leq \|g\|\text{-}E * \|x\|$
      proof (*rule fn-norm-le-cong* [*of g*, *folded B-def fn-norm-def*])
        from *FE x* **show** $x \in E$ **..**
      qed
      finally show $|f\ x| \leq \|g\|\text{-}E * \|x\|$ **.**
    qed
    show $0 \leq \|g\|\text{-}E$
      using *g-cont*
      by (*rule fn-norm-ge-zero* [*of g*, *folded B-def fn-norm-def*])
  next
    show *continuous F norm f* **by** (*rule continuous.intro*)
  qed
qed
with *linearformE a g-cont* **show** *?thesis* **by** *blast*

qed

end

References

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[3] B. Nowak and A. Trybulec. Hahn-Banach theorem. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.uwb.edu.pl/JFM/Vol5/hahnban.html.