The UNITY Formalism

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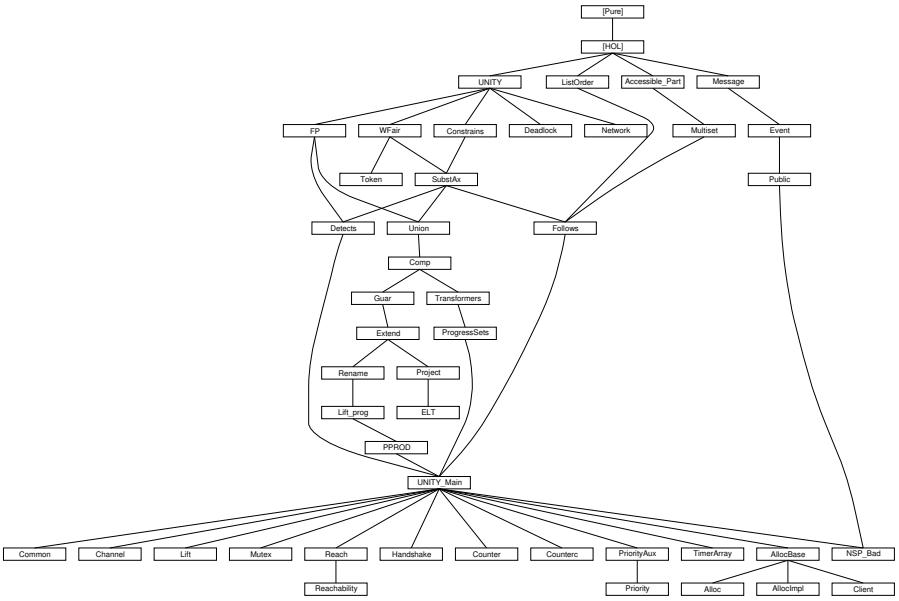

1 The Basic UNITY Theory

theory *UNITY* **imports** *Main* **begin**

typedef *(Program)*
```
  'a program = "{(init:: 'a set, acts :: ('a * 'a)set set,
                  allowed :: ('a * 'a)set set). Id ∈ acts & Id: allowed}"
```

  by *blast*

constdefs
```
  Acts :: "'a program => ('a * 'a)set set"
    "Acts F == (%(init, acts, allowed). acts) (Rep_Program F)"

  "constrains" :: "['a set, 'a set] => 'a program set"  (infixl "co"     60)
    "A co B == {F. ∀act ∈ Acts F. act``A ⊆ B}"

  unless  :: "['a set, 'a set] => 'a program set"  (infixl "unless" 60)
    "A unless B == (A-B) co (A ∪ B)"

  mk_program :: "('a set * ('a * 'a)set set * ('a * 'a)set set)
                  => 'a program"
    "mk_program == %(init, acts, allowed).
                      Abs_Program (init, insert Id acts, insert Id allowed)"

  Init :: "'a program => 'a set"
    "Init F == (%(init, acts, allowed). init) (Rep_Program F)"

  AllowedActs :: "'a program => ('a * 'a)set set"
    "AllowedActs F == (%(init, acts, allowed). allowed) (Rep_Program F)"

  Allowed :: "'a program => 'a program set"
    "Allowed F == {G. Acts G ⊆ AllowedActs F}"

  stable     :: "'a set => 'a program set"
    "stable A == A co A"

  strongest_rhs :: "['a program, 'a set] => 'a set"
    "strongest_rhs F A == Inter {B. F ∈ A co B}"

  invariant :: "'a set => 'a program set"
    "invariant A == {F. Init F ⊆ A} ∩ stable A"

  increasing :: "['a => 'b::{order}] => 'a program set"
    — Polymorphic in both states and the meaning of ≤
    "increasing f == ⋂z. stable {s. z ≤ f s}"
```

Perhaps equalities.ML shouldn't add this in the first place!

declare *image_Collect [simp del]*

1.0.1 The abstract type of programs

lemmas *program_typedef =*
```
     Rep_Program Rep_Program_inverse Abs_Program_inverse
```

```
    Program_def Init_def Acts_def AllowedActs_def mk_program_def

lemma Id_in_Acts [iff]: "Id ∈ Acts F"
apply (cut_tac x = F in Rep_Program)
apply (auto simp add: program_typedef)
done

lemma insert_Id_Acts [iff]: "insert Id (Acts F) = Acts F"
by (simp add: insert_absorb Id_in_Acts)

lemma Acts_nonempty [simp]: "Acts F ≠ {}"
by auto

lemma Id_in_AllowedActs [iff]: "Id ∈ AllowedActs F"
apply (cut_tac x = F in Rep_Program)
apply (auto simp add: program_typedef)
done

lemma insert_Id_AllowedActs [iff]: "insert Id (AllowedActs F) = AllowedActs
F"
by (simp add: insert_absorb Id_in_AllowedActs)
```

1.0.2 Inspectors for type "program"

```
lemma Init_eq [simp]: "Init (mk_program (init,acts,allowed)) = init"
by (simp add: program_typedef)

lemma Acts_eq [simp]: "Acts (mk_program (init,acts,allowed)) = insert Id
acts"
by (simp add: program_typedef)

lemma AllowedActs_eq [simp]:
    "AllowedActs (mk_program (init,acts,allowed)) = insert Id allowed"
by (simp add: program_typedef)
```

1.0.3 Equality for UNITY programs

```
lemma surjective_mk_program [simp]:
    "mk_program (Init F, Acts F, AllowedActs F) = F"
apply (cut_tac x = F in Rep_Program)
apply (auto simp add: program_typedef)
apply (drule_tac f = Abs_Program in arg_cong)+
apply (simp add: program_typedef insert_absorb)
done

lemma program_equalityI:
    "[| Init F = Init G; Acts F = Acts G; AllowedActs F = AllowedActs G |]

     ==> F = G"
apply (rule_tac t = F in surjective_mk_program [THEN subst])
apply (rule_tac t = G in surjective_mk_program [THEN subst], simp)
done

lemma program_equalityE:
    "[| F = G;
```

```
        [| Init F = Init G; Acts F = Acts G; AllowedActs F = AllowedActs G
|]
        ==> P |] ==> P"
by simp
```

```
lemma program_equality_iff:
     "(F=G) =
      (Init F = Init G & Acts F = Acts G &AllowedActs F = AllowedActs G)"
by (blast intro: program_equalityI program_equalityE)
```

1.0.4 co

```
lemma constrainsI:
    "(!!act s s'. [| act: Acts F;  (s,s') ∈ act;  s ∈ A |] ==> s': A')
     ==> F ∈ A co A'"
by (simp add: constrains_def, blast)
```

```
lemma constrainsD:
    "[| F ∈ A co A'; act: Acts F;  (s,s'): act;  s ∈ A |] ==> s': A'"
by (unfold constrains_def, blast)
```

```
lemma constrains_empty [iff]: "F ∈ {} co B"
by (unfold constrains_def, blast)
```

```
lemma constrains_empty2 [iff]: "(F ∈ A co {}) = (A={})"
by (unfold constrains_def, blast)
```

```
lemma constrains_UNIV [iff]: "(F ∈ UNIV co B) = (B = UNIV)"
by (unfold constrains_def, blast)
```

```
lemma constrains_UNIV2 [iff]: "F ∈ A co UNIV"
by (unfold constrains_def, blast)
```

monotonic in 2nd argument

```
lemma constrains_weaken_R:
    "[| F ∈ A co A'; A'<=B' |] ==> F ∈ A co B'"
by (unfold constrains_def, blast)
```

anti-monotonic in 1st argument

```
lemma constrains_weaken_L:
    "[| F ∈ A co A'; B ⊆ A |] ==> F ∈ B co A'"
by (unfold constrains_def, blast)
```

```
lemma constrains_weaken:
   "[| F ∈ A co A'; B ⊆ A; A'<=B' |] ==> F ∈ B co B'"
by (unfold constrains_def, blast)
```

1.0.5 Union

```
lemma constrains_Un:
    "[| F ∈ A co A'; F ∈ B co B' |] ==> F ∈ (A ∪ B) co (A' ∪ B')"
by (unfold constrains_def, blast)
```

```
lemma constrains_UN:
```

```
    "(!!i. i ∈ I ==> F ∈ (A i) co (A' i))
     ==> F ∈ (⋃i ∈ I. A i) co (⋃i ∈ I. A' i)"
by (unfold constrains_def, blast)

lemma constrains_Un_distrib: "(A ∪ B) co C = (A co C) ∩ (B co C)"
by (unfold constrains_def, blast)

lemma constrains_UN_distrib: "(⋃i ∈ I. A i) co B = (⋂i ∈ I. A i co B)"
by (unfold constrains_def, blast)

lemma constrains_Int_distrib: "C co (A ∩ B) = (C co A) ∩ (C co B)"
by (unfold constrains_def, blast)

lemma constrains_INT_distrib: "A co (⋂i ∈ I. B i) = (⋂i ∈ I. A co B i)"
by (unfold constrains_def, blast)
```

1.0.6 Intersection

```
lemma constrains_Int:
    "[| F ∈ A co A'; F ∈ B co B' |] ==> F ∈ (A ∩ B) co (A' ∩ B')"
by (unfold constrains_def, blast)

lemma constrains_INT:
    "(!!i. i ∈ I ==> F ∈ (A i) co (A' i))
     ==> F ∈ (⋂i ∈ I. A i) co (⋂i ∈ I. A' i)"
by (unfold constrains_def, blast)

lemma constrains_imp_subset: "F ∈ A co A' ==> A ⊆ A'"
by (unfold constrains_def, auto)
```

The reasoning is by subsets since "co" refers to single actions only. So this rule isn't that useful.

```
lemma constrains_trans:
    "[| F ∈ A co B; F ∈ B co C |] ==> F ∈ A co C"
by (unfold constrains_def, blast)

lemma constrains_cancel:
   "[| F ∈ A co (A' ∪ B); F ∈ B co B' |] ==> F ∈ A co (A' ∪ B')"
by (unfold constrains_def, clarify, blast)
```

1.0.7 unless

```
lemma unlessI: "F ∈ (A-B) co (A ∪ B) ==> F ∈ A unless B"
by (unfold unless_def, assumption)

lemma unlessD: "F ∈ A unless B ==> F ∈ (A-B) co (A ∪ B)"
by (unfold unless_def, assumption)
```

1.0.8 stable

```
lemma stableI: "F ∈ A co A ==> F ∈ stable A"
by (unfold stable_def, assumption)

lemma stableD: "F ∈ stable A ==> F ∈ A co A"
```

by *(unfold stable_def, assumption)*

lemma *stable_UNIV [simp]: "stable UNIV = UNIV"*
by *(unfold stable_def constrains_def, auto)*

1.0.9 Union

lemma *stable_Un:*
"[| F ∈ stable A; F ∈ stable A' |] ==> F ∈ stable (A ∪ A')"

apply *(unfold stable_def)*
apply *(blast intro: constrains_Un)*
done

lemma *stable_UN:*
"(!!i. i ∈ I ==> F ∈ stable (A i)) ==> F ∈ stable (⋃i ∈ I. A i)"
apply *(unfold stable_def)*
apply *(blast intro: constrains_UN)*
done

lemma *stable_Union:*
"(!!A. A ∈ X ==> F ∈ stable A) ==> F ∈ stable (⋃X)"
by *(unfold stable_def constrains_def, blast)*

1.0.10 Intersection

lemma *stable_Int:*
"[| F ∈ stable A; F ∈ stable A' |] ==> F ∈ stable (A ∩ A')"
apply *(unfold stable_def)*
apply *(blast intro: constrains_Int)*
done

lemma *stable_INT:*
"(!!i. i ∈ I ==> F ∈ stable (A i)) ==> F ∈ stable (⋂i ∈ I. A i)"
apply *(unfold stable_def)*
apply *(blast intro: constrains_INT)*
done

lemma *stable_Inter:*
"(!!A. A ∈ X ==> F ∈ stable A) ==> F ∈ stable (⋂X)"
by *(unfold stable_def constrains_def, blast)*

lemma *stable_constrains_Un:*
"[| F ∈ stable C; F ∈ A co (C ∪ A') |] ==> F ∈ (C ∪ A) co (C ∪ A')"
by *(unfold stable_def constrains_def, blast)*

lemma *stable_constrains_Int:*
"[| F ∈ stable C; F ∈ (C ∩ A) co A' |] ==> F ∈ (C ∩ A) co (C ∩ A')"
by *(unfold stable_def constrains_def, blast)*

lemmas *stable_constrains_stable = stable_constrains_Int[THEN stableI, standard]*

1.0.11 invariant

lemma *invariantI: "[| Init F ⊆ A; F ∈ stable A |] ==> F ∈ invariant A"*
by *(simp add: invariant_def)*

Could also say *invariant A ∩ invariant B ⊆ invariant (A ∩ B)*

lemma *invariant_Int:*
"[| F ∈ invariant A; F ∈ invariant B |] ==> F ∈ invariant (A ∩ B)"
by *(auto simp add: invariant_def stable_Int)*

1.0.12 increasing

lemma *increasingD:*
"F ∈ increasing f ==> F ∈ stable {s. z ⊆ f s}"
by *(unfold increasing_def, blast)*

lemma *increasing_constant [iff]: "F ∈ increasing (%s. c)"*
by *(unfold increasing_def stable_def, auto)*

lemma *mono_increasing_o:*
"mono g ==> increasing f ⊆ increasing (g o f)"
apply *(unfold increasing_def stable_def constrains_def, auto)*
apply *(blast intro: monoD order_trans)*
done

lemma *strict_increasingD:*
"!!z::nat. F ∈ increasing f ==> F ∈ stable {s. z < f s}"
by *(simp add: increasing_def Suc_le_eq [symmetric])*

lemma *elimination:*
"[| ∀m ∈ M. F ∈ {s. s x = m} co (B m) |]
==> F ∈ {s. s x ∈ M} co (⋃m ∈ M. B m)"
by *(unfold constrains_def, blast)*

As above, but for the trivial case of a one-variable state, in which the state is identified with its one variable.

lemma *elimination_sing:*
"(∀m ∈ M. F ∈ {m} co (B m)) ==> F ∈ M co (⋃m ∈ M. B m)"
by *(unfold constrains_def, blast)*

1.0.13 Theoretical Results from Section 6

lemma *constrains_strongest_rhs:*
"F ∈ A co (strongest_rhs F A)"
by *(unfold constrains_def strongest_rhs_def, blast)*

lemma *strongest_rhs_is_strongest:*
"F ∈ A co B ==> strongest_rhs F A ⊆ B"
by *(unfold constrains_def strongest_rhs_def, blast)*

1.0.14 Ad-hoc set-theory rules

lemma *Un_Diff_Diff [simp]: "A ∪ B - (A - B) = B"*
by *blast*

lemma *Int_Union_Union: "Union(B) ∩ A = Union((%C. C ∩ A)‘B)"*
by *blast*

Needed for WF reasoning in WFair.ML

lemma *Image_less_than [simp]: "less_than `` {k} = greaterThan k"*
by *blast*

lemma *Image_inverse_less_than [simp]: "less_than^-1 `` {k} = lessThan k"*
by *blast*

1.1 Partial versus Total Transitions

constdefs

```
  totalize_act :: "('a * 'a)set => ('a * 'a)set"
    "totalize_act act == act ∪ diag (-(Domain act))"

  totalize :: "'a program => 'a program"
    "totalize F == mk_program (Init F,
                               totalize_act ‘ Acts F,
                               AllowedActs F)"

  mk_total_program :: "('a set * ('a * 'a)set set * ('a * 'a)set set)
                   => 'a program"
    "mk_total_program args == totalize (mk_program args)"

  all_total :: "'a program => bool"
    "all_total F == ∀act ∈ Acts F. Domain act = UNIV"
```

lemma *insert_Id_image_Acts: "f Id = Id ==> insert Id (f‘Acts F) = f ‘ Acts F"*
by *(blast intro: sym [THEN image_eqI])*

1.1.1 Basic properties

lemma *totalize_act_Id [simp]: "totalize_act Id = Id"*
by *(simp add: totalize_act_def)*

lemma *Domain_totalize_act [simp]: "Domain (totalize_act act) = UNIV"*
by *(auto simp add: totalize_act_def)*

lemma *Init_totalize [simp]: "Init (totalize F) = Init F"*
by *(unfold totalize_def, auto)*

lemma *Acts_totalize [simp]: "Acts (totalize F) = (totalize_act ‘ Acts F)"*
by *(simp add: totalize_def insert_Id_image_Acts)*

lemma *AllowedActs_totalize [simp]: "AllowedActs (totalize F) = AllowedActs F"*
by *(simp add: totalize_def)*

lemma *totalize_constrains_iff [simp]: "(totalize F ∈ A co B) = (F ∈ A co B)"*
by *(simp add: totalize_def totalize_act_def constrains_def, blast)*

lemma *totalize_stable_iff [simp]: "(totalize F ∈ stable A) = (F ∈ stable A)"*
by *(simp add: stable_def)*

lemma *totalize_invariant_iff [simp]:*
"(totalize F ∈ invariant A) = (F ∈ invariant A)"
by *(simp add: invariant_def)*

lemma *all_total_totalize: "all_total (totalize F)"*
by *(simp add: totalize_def all_total_def)*

lemma *Domain_iff_totalize_act: "(Domain act = UNIV) = (totalize_act act = act)"*
by *(force simp add: totalize_act_def)*

lemma *all_total_imp_totalize: "all_total F ==> (totalize F = F)"*
apply *(simp add: all_total_def totalize_def)*
apply *(rule program_equalityI)*
apply *(simp_all add: Domain_iff_totalize_act image_def)*
done

lemma *all_total_iff_totalize: "all_total F = (totalize F = F)"*
apply *(rule iffI)*
apply *(erule all_total_imp_totalize)*
apply *(erule subst)*
apply *(rule all_total_totalize)*
done

lemma *mk_total_program_constrains_iff [simp]:*
"(mk_total_program args ∈ A co B) = (mk_program args ∈ A co B)"
by *(simp add: mk_total_program_def)*

1.2 Rules for Lazy Definition Expansion

They avoid expanding the full program, which is a large expression

lemma *def_prg_Init:*
"F == mk_total_program (init,acts,allowed) ==> Init F = init"
by *(simp add: mk_total_program_def)*

lemma *def_prg_Acts:*
"F == mk_total_program (init,acts,allowed)
==> Acts F = insert Id (totalize_act ‘ acts)"
by *(simp add: mk_total_program_def)*

lemma *def_prg_AllowedActs:*
"F == mk_total_program (init,acts,allowed)
==> AllowedActs F = insert Id allowed"
by *(simp add: mk_total_program_def)*

An action is expanded if a pair of states is being tested against it

```
lemma def_act_simp:
     "act == {(s,s'). P s s'} ==> ((s,s') ∈ act) = P s s'"
by (simp add: mk_total_program_def)
```

A set is expanded only if an element is being tested against it

```
lemma def_set_simp: "A == B ==> (x ∈ A) = (x ∈ B)"
by (simp add: mk_total_program_def)
```

1.2.1 Inspectors for type "program"

```
lemma Init_total_eq [simp]:
     "Init (mk_total_program (init,acts,allowed)) = init"
by (simp add: mk_total_program_def)

lemma Acts_total_eq [simp]:
    "Acts(mk_total_program(init,acts,allowed)) = insert Id (totalize_act`acts)"
by (simp add: mk_total_program_def)

lemma AllowedActs_total_eq [simp]:
     "AllowedActs (mk_total_program (init,acts,allowed)) = insert Id allowed"
by (auto simp add: mk_total_program_def)

end
```

2 Fixed Point of a Program

```
theory FP imports UNITY begin

constdefs

  FP_Orig :: "'a program => 'a set"
    "FP_Orig F == Union{A. ALL B. F : stable (A Int B)}"

  FP :: "'a program => 'a set"
    "FP F == {s. F : stable {s}}"

lemma stable_FP_Orig_Int: "F : stable (FP_Orig F Int B)"
apply (simp only: FP_Orig_def stable_def Int_Union2)
apply (blast intro: constrains_UN)
done

lemma FP_Orig_weakest:
    "(!!B. F : stable (A Int B)) ==> A <= FP_Orig F"
by (simp add: FP_Orig_def stable_def, blast)

lemma stable_FP_Int: "F : stable (FP F Int B)"
apply (subgoal_tac "FP F Int B = (UN x:B. FP F Int {x}) ")
prefer 2 apply blast
apply (simp (no_asm_simp) add: Int_insert_right)
apply (simp add: FP_def stable_def)
apply (rule constrains_UN)
apply (simp (no_asm))
```

done

lemma *FP_equivalence: "FP F = FP_Orig F"*
apply *(rule equalityI)*
apply *(rule stable_FP_Int [THEN FP_Orig_weakest])*
apply *(simp add: FP_Orig_def FP_def, clarify)*
apply *(drule_tac x = "{x}"* **in** *spec)*
apply *(simp add: Int_insert_right)*
done

lemma *FP_weakest:*
"(!!B. F : stable (A Int B)) ==> A <= FP F"
by *(simp add: FP_equivalence FP_Orig_weakest)*

lemma *Compl_FP:*
"-(FP F) = (UN act: Acts F. -{s. act``{s} <= {s}})"
by *(simp add: FP_def stable_def constrains_def, blast)*

lemma *Diff_FP: "A - (FP F) = (UN act: Acts F. A - {s. act``{s} <= {s}})"*
by *(simp add: Diff_eq Compl_FP)*

lemma *totalize_FP [simp]: "FP (totalize F) = FP F"*
by *(simp add: FP_def)*

end

3 Progress

theory *WFair* **imports** *UNITY* **begin**

The original version of this theory was based on weak fairness. (Thus, the entire UNITY development embodied this assumption, until February 2003.) Weak fairness states that if a command is enabled continuously, then it is eventually executed. Ernie Cohen suggested that I instead adopt unconditional fairness: every command is executed infinitely often.

In fact, Misra's paper on "Progress" seems to be ambiguous about the correct interpretation, and says that the two forms of fairness are equivalent. They differ only on their treatment of partial transitions, which under unconditional fairness behave magically. That is because if there are partial transitions then there may be no fair executions, making all leads-to properties hold vacuously.

Unconditional fairness has some great advantages. By distinguishing partial transitions from total ones that are the identity on part of their domain, it is more expressive. Also, by simplifying the definition of the transient property, it simplifies many proofs. A drawback is that some laws only hold under the assumption that all transitions are total. The best-known of these is the impossibility law for leads-to.

constdefs

— This definition specifies conditional fairness. The rest of the theory is generic to all forms of fairness. To get weak fairness, conjoin the inclusion below with $A \subseteq$ *Domain act*, which specifies that the action is enabled over all of *A*.

```
  transient :: "'a set => 'a program set"
    "transient A == {F. ∃act∈Acts F. act``A ⊆ -A}"

  ensures :: "['a set, 'a set] => 'a program set"       (infixl "ensures" 60)
    "A ensures B == (A-B co A ∪ B) ∩ transient (A-B)"
```

consts

```
  leads :: "'a program => ('a set * 'a set) set"
```
— LEADS-TO constant for the inductive definition

inductive *"leads F"*
intros

```
    Basis:  "F ∈ A ensures B ==> (A,B) ∈ leads F"

    Trans:  "[| (A,B) ∈ leads F;  (B,C) ∈ leads F |] ==> (A,C) ∈ leads F"

    Union:  "∀A ∈ S. (A,B) ∈ leads F ==> (Union S, B) ∈ leads F"
```

constdefs

```
  leadsTo :: "['a set, 'a set] => 'a program set"    (infixl "leadsTo" 60)
```
— visible version of the LEADS-TO relation
```
    "A leadsTo B == {F. (A,B) ∈ leads F}"

  wlt :: "['a program, 'a set] => 'a set"
```
— predicate transformer: the largest set that leads to *B*
```
    "wlt F B == Union {A. F ∈ A leadsTo B}"
```

syntax *(xsymbols)*
```
  "op leadsTo" :: "['a set, 'a set] => 'a program set" (infixl "⟼" 60)
```

3.1 transient

lemma *stable_transient:*
```
    "[| F ∈ stable A; F ∈ transient A |] ==> ∃act∈Acts F. A ⊆ - (Domain
act)"
apply (simp add: stable_def constrains_def transient_def, clarify)
apply (rule rev_bexI, auto)
done

lemma stable_transient_empty:
    "[| F ∈ stable A; F ∈ transient A; all_total F |] ==> A = {}"
apply (drule stable_transient, assumption)
apply (simp add: all_total_def)
done

lemma transient_strengthen:
    "[| F ∈ transient A; B ⊆ A |] ==> F ∈ transient B"
apply (unfold transient_def, clarify)
```

apply *(blast intro!: rev_bexI)*
done

lemma *transientI:*
"[| act: Acts F; act``A ⊆ -A |] ==> F ∈ transient A"
by *(unfold transient_def, blast)*

lemma *transientE:*
"[| F ∈ transient A;
!!act. [| act: Acts F; act``A ⊆ -A |] ==> P |]
==> P"
by *(unfold transient_def, blast)*

lemma *transient_empty [simp]: "transient {} = UNIV"*
by *(unfold transient_def, auto)*

This equation recovers the notion of weak fairness. A totalized program satisfies a transient assertion just if the original program contains a suitable action that is also enabled.

lemma *totalize_transient_iff:*
"(totalize F ∈ transient A) = (∃act∈Acts F. A ⊆ Domain act & act``A ⊆ -A)"
apply *(simp add: totalize_def totalize_act_def transient_def*
Un_Image Un_subset_iff, safe)
apply *(blast intro!: rev_bexI)+*
done

lemma *totalize_transientI:*
"[| act: Acts F; A ⊆ Domain act; act``A ⊆ -A |]
==> totalize F ∈ transient A"
by *(simp add: totalize_transient_iff, blast)*

3.2 ensures

lemma *ensuresI:*
"[| F ∈ (A-B) co (A ∪ B); F ∈ transient (A-B) |] ==> F ∈ A ensures B"
by *(unfold ensures_def, blast)*

lemma *ensuresD:*
"F ∈ A ensures B ==> F ∈ (A-B) co (A ∪ B) & F ∈ transient (A-B)"
by *(unfold ensures_def, blast)*

lemma *ensures_weaken_R:*
"[| F ∈ A ensures A'; A'<=B' |] ==> F ∈ A ensures B'"
apply *(unfold ensures_def)*
apply *(blast intro: constrains_weaken transient_strengthen)*
done

The L-version (precondition strengthening) fails, but we have this

lemma *stable_ensures_Int:*
"[| F ∈ stable C; F ∈ A ensures B |]
==> F ∈ (C ∩ A) ensures (C ∩ B)"
apply *(unfold ensures_def)*

```
apply (auto simp add: ensures_def Int_Un_distrib [symmetric] Diff_Int_distrib
[symmetric])
prefer 2 apply (blast intro: transient_strengthen)
apply (blast intro: stable_constrains_Int constrains_weaken)
done

lemma stable_transient_ensures:
     "[| F ∈ stable A;  F ∈ transient C;  A ⊆ B ∪ C |] ==> F ∈ A ensures
B"
apply (simp add: ensures_def stable_def)
apply (blast intro: constrains_weaken transient_strengthen)
done

lemma ensures_eq: "(A ensures B) = (A unless B) ∩ transient (A-B)"
by (simp (no_asm) add: ensures_def unless_def)
```

3.3 leadsTo

```
lemma leadsTo_Basis [intro]: "F ∈ A ensures B ==> F ∈ A leadsTo B"
apply (unfold leadsTo_def)
apply (blast intro: leads.Basis)
done

lemma leadsTo_Trans:
     "[| F ∈ A leadsTo B;  F ∈ B leadsTo C |] ==> F ∈ A leadsTo C"
apply (unfold leadsTo_def)
apply (blast intro: leads.Trans)
done

lemma leadsTo_Basis':
     "[| F ∈ A co A ∪ B; F ∈ transient A |] ==> F ∈ A leadsTo B"
apply (drule_tac B = "A-B" in constrains_weaken_L)
apply (drule_tac [2] B = "A-B" in transient_strengthen)
apply (rule_tac [3] ensuresI [THEN leadsTo_Basis])
apply (blast+)
done

lemma transient_imp_leadsTo: "F ∈ transient A ==> F ∈ A leadsTo (-A)"
by (simp (no_asm_simp) add: leadsTo_Basis ensuresI Compl_partition)
```

Useful with cancellation, disjunction

```
lemma leadsTo_Un_duplicate: "F ∈ A leadsTo (A' ∪ A') ==> F ∈ A leadsTo
A'"
by (simp add: Un_ac)

lemma leadsTo_Un_duplicate2:
     "F ∈ A leadsTo (A' ∪ C ∪ C) ==> F ∈ A leadsTo (A' ∪ C)"
by (simp add: Un_ac)
```

The Union introduction rule as we should have liked to state it

```
lemma leadsTo_Union:
    "(!!A. A ∈ S ==> F ∈ A leadsTo B) ==> F ∈ (Union S) leadsTo B"
apply (unfold leadsTo_def)
apply (blast intro: leads.Union)
```

done

lemma *leadsTo_Union_Int:*
"(!!A. A ∈ S ==> F ∈ (A ∩ C) leadsTo B) ==> F ∈ (Union S ∩ C) leadsTo B"
apply *(unfold leadsTo_def)*
apply *(simp only: Int_Union_Union)*
apply *(blast intro: leads.Union)*
done

lemma *leadsTo_UN:*
"(!!i. i ∈ I ==> F ∈ (A i) leadsTo B) ==> F ∈ (⋃i ∈ I. A i) leadsTo B"
apply *(subst Union_image_eq [symmetric])*
apply *(blast intro: leadsTo_Union)*
done

Binary union introduction rule

lemma *leadsTo_Un:*
"[| F ∈ A leadsTo C; F ∈ B leadsTo C |] ==> F ∈ (A ∪ B) leadsTo C"
apply *(subst Un_eq_Union)*
apply *(blast intro: leadsTo_Union)*
done

lemma *single_leadsTo_I:*
"(!!x. x ∈ A ==> F ∈ {x} leadsTo B) ==> F ∈ A leadsTo B"
by *(subst UN_singleton [symmetric], rule leadsTo_UN, blast)*

The INDUCTION rule as we should have liked to state it

lemma *leadsTo_induct:*
"[| F ∈ za leadsTo zb;
!!A B. F ∈ A ensures B ==> P A B;
!!A B C. [| F ∈ A leadsTo B; P A B; F ∈ B leadsTo C; P B C |]
==> P A C;
!!B S. ∀A ∈ S. F ∈ A leadsTo B & P A B ==> P (Union S) B
|] ==> P za zb"
apply *(unfold leadsTo_def)*
apply *(drule CollectD, erule leads.induct)*
apply *(blast+)*
done

lemma *subset_imp_ensures: "A ⊆ B ==> F ∈ A ensures B"*
by *(unfold ensures_def constrains_def transient_def, blast)*

lemmas *subset_imp_leadsTo = subset_imp_ensures [THEN leadsTo_Basis, standard]*

lemmas *leadsTo_refl = subset_refl [THEN subset_imp_leadsTo, standard]*

lemmas *empty_leadsTo = empty_subsetI [THEN subset_imp_leadsTo, standard, simp]*

lemmas *leadsTo_UNIV = subset_UNIV [THEN subset_imp_leadsTo, standard, simp]*

Lemma is the weak version: can't see how to do it in one step

lemma *leadsTo_induct_pre_lemma:*
```
  "[| F ∈ za leadsTo zb;
      P zb;
      !!A B. [| F ∈ A ensures B;  P B |] ==> P A;
      !!S. ∀A ∈ S. P A ==> P (Union S)
   |] ==> P za"
```

by induction on this formula

apply *(subgoal_tac "P zb --> P za")*

now solve first subgoal: this formula is sufficient

apply *(blast intro: leadsTo_refl)*
apply *(erule leadsTo_induct)*
apply *(blast+)*
done

lemma *leadsTo_induct_pre:*
```
  "[| F ∈ za leadsTo zb;
      P zb;
      !!A B. [| F ∈ A ensures B;  F ∈ B leadsTo zb;  P B |] ==> P A;
      !!S. ∀A ∈ S. F ∈ A leadsTo zb & P A ==> P (Union S)
   |] ==> P za"
```
apply *(subgoal_tac "F ∈ za leadsTo zb & P za")*
apply *(erule conjunct2)*
apply *(erule leadsTo_induct_pre_lemma)*
prefer *3* **apply** *(blast intro: leadsTo_Union)*
prefer *2* **apply** *(blast intro: leadsTo_Trans)*
apply *(blast intro: leadsTo_refl)*
done

lemma *leadsTo_weaken_R: "[| F ∈ A leadsTo A'; A'<=B' |] ==> F ∈ A leadsTo B'"*
by *(blast intro: subset_imp_leadsTo leadsTo_Trans)*

lemma *leadsTo_weaken_L [rule_format]:*
```
     "[| F ∈ A leadsTo A'; B ⊆ A |] ==> F ∈ B leadsTo A'"
```
by *(blast intro: leadsTo_Trans subset_imp_leadsTo)*

Distributes over binary unions

lemma *leadsTo_Un_distrib:*
```
     "F ∈ (A ∪ B) leadsTo C  =  (F ∈ A leadsTo C & F ∈ B leadsTo C)"
```
by *(blast intro: leadsTo_Un leadsTo_weaken_L)*

lemma *leadsTo_UN_distrib:*
```
     "F ∈ (⋃i ∈ I. A i) leadsTo B  =  (∀i ∈ I. F ∈ (A i) leadsTo B)"
```
by *(blast intro: leadsTo_UN leadsTo_weaken_L)*

lemma *leadsTo_Union_distrib:*
```
     "F ∈ (Union S) leadsTo B  =  (∀A ∈ S. F ∈ A leadsTo B)"
```
by *(blast intro: leadsTo_Union leadsTo_weaken_L)*

lemma *leadsTo_weaken:*

```
     "[| F ∈ A leadsTo A'; B ⊆ A; A'<=B' |] ==> F ∈ B leadsTo B'"
by (blast intro: leadsTo_weaken_R leadsTo_weaken_L leadsTo_Trans)
```

Set difference: maybe combine with *leadsTo_weaken_L*??

```
lemma leadsTo_Diff:
     "[| F ∈ (A-B) leadsTo C; F ∈ B leadsTo C |]   ==> F ∈ A leadsTo C"
by (blast intro: leadsTo_Un leadsTo_weaken)

lemma leadsTo_UN_UN:
   "(!! i. i ∈ I ==> F ∈ (A i) leadsTo (A' i))
    ==> F ∈ (⋃i ∈ I. A i) leadsTo (⋃i ∈ I. A' i)"
apply (simp only: Union_image_eq [symmetric])
apply (blast intro: leadsTo_Union leadsTo_weaken_R)
done
```

Binary union version

```
lemma leadsTo_Un_Un:
     "[| F ∈ A leadsTo A'; F ∈ B leadsTo B' |]
      ==> F ∈ (A ∪ B) leadsTo (A' ∪ B')"
by (blast intro: leadsTo_Un leadsTo_weaken_R)



lemma leadsTo_cancel2:
     "[| F ∈ A leadsTo (A' ∪ B); F ∈ B leadsTo B' |]
      ==> F ∈ A leadsTo (A' ∪ B')"
by (blast intro: leadsTo_Un_Un subset_imp_leadsTo leadsTo_Trans)

lemma leadsTo_cancel_Diff2:
     "[| F ∈ A leadsTo (A' ∪ B); F ∈ (B-A') leadsTo B' |]
      ==> F ∈ A leadsTo (A' ∪ B')"
apply (rule leadsTo_cancel2)
prefer 2 apply assumption
apply (simp_all (no_asm_simp))
done

lemma leadsTo_cancel1:
     "[| F ∈ A leadsTo (B ∪ A'); F ∈ B leadsTo B' |]
    ==> F ∈ A leadsTo (B' ∪ A')"
apply (simp add: Un_commute)
apply (blast intro!: leadsTo_cancel2)
done

lemma leadsTo_cancel_Diff1:
     "[| F ∈ A leadsTo (B ∪ A'); F ∈ (B-A') leadsTo B' |]
    ==> F ∈ A leadsTo (B' ∪ A')"
apply (rule leadsTo_cancel1)
prefer 2 apply assumption
apply (simp_all (no_asm_simp))
done
```

The impossibility law

```
lemma leadsTo_empty: "[|F ∈ A leadsTo {}; all_total F|] ==> A={}"
```

```
apply (erule leadsTo_induct_pre)
apply (simp_all add: ensures_def constrains_def transient_def all_total_def,
clarify)
apply (drule bspec, assumption)+
apply blast
done
```

3.4 PSP: Progress-Safety-Progress

Special case of PSP: Misra's "stable conjunction"

```
lemma psp_stable:
   "[| F ∈ A leadsTo A'; F ∈ stable B |]
    ==> F ∈ (A ∩ B) leadsTo (A' ∩ B)"
apply (unfold stable_def)
apply (erule leadsTo_induct)
prefer 3 apply (blast intro: leadsTo_Union_Int)
prefer 2 apply (blast intro: leadsTo_Trans)
apply (rule leadsTo_Basis)
apply (simp add: ensures_def Diff_Int_distrib2 [symmetric] Int_Un_distrib2
[symmetric])
apply (blast intro: transient_strengthen constrains_Int)
done

lemma psp_stable2:
   "[| F ∈ A leadsTo A'; F ∈ stable B |] ==> F ∈ (B ∩ A) leadsTo (B ∩ A')"
by (simp add: psp_stable Int_ac)

lemma psp_ensures:
   "[| F ∈ A ensures A'; F ∈ B co B' |]
    ==> F ∈ (A ∩ B') ensures ((A' ∩ B) ∪ (B' - B))"
apply (unfold ensures_def constrains_def, clarify)
apply (blast intro: transient_strengthen)
done

lemma psp:
     "[| F ∈ A leadsTo A'; F ∈ B co B' |]
      ==> F ∈ (A ∩ B') leadsTo ((A' ∩ B) ∪ (B' - B))"
apply (erule leadsTo_induct)
  prefer 3 apply (blast intro: leadsTo_Union_Int)
```

Basis case

```
 apply (blast intro: psp_ensures)
```

Transitivity case has a delicate argument involving "cancellation"

```
apply (rule leadsTo_Un_duplicate2)
apply (erule leadsTo_cancel_Diff1)
apply (simp add: Int_Diff Diff_triv)
apply (blast intro: leadsTo_weaken_L dest: constrains_imp_subset)
done

lemma psp2:
     "[| F ∈ A leadsTo A'; F ∈ B co B' |]
    ==> F ∈ (B' ∩ A) leadsTo ((B ∩ A') ∪ (B' - B))"
```

```
by (simp (no_asm_simp) add: psp Int_ac)

lemma psp_unless:
   "[| F ∈ A leadsTo A';  F ∈ B unless B' |]
    ==> F ∈ (A ∩ B) leadsTo ((A' ∩ B) ∪ B')"

apply (unfold unless_def)
apply (drule psp, assumption)
apply (blast intro: leadsTo_weaken)
done
```

3.5 Proving the induction rules

```
lemma leadsTo_wf_induct_lemma:
     "[| wf r;
         ∀m. F ∈ (A ∩ f-‘{m}) leadsTo
                    ((A ∩ f-‘(r^-1 ‘‘ {m})) ∪ B) |]
      ==> F ∈ (A ∩ f-‘{m}) leadsTo B"
apply (erule_tac a = m in wf_induct)
apply (subgoal_tac "F ∈ (A ∩ (f -‘ (r^-1 ‘‘ {x}))) leadsTo B")
 apply (blast intro: leadsTo_cancel1 leadsTo_Un_duplicate)
apply (subst vimage_eq_UN)
apply (simp only: UN_simps [symmetric])
apply (blast intro: leadsTo_UN)
done



lemma leadsTo_wf_induct:
     "[| wf r;
         ∀m. F ∈ (A ∩ f-‘{m}) leadsTo
                    ((A ∩ f-‘(r^-1 ‘‘ {m})) ∪ B) |]
      ==> F ∈ A leadsTo B"
apply (rule_tac t = A in subst)
 defer 1
 apply (rule leadsTo_UN)
 apply (erule leadsTo_wf_induct_lemma)
 apply assumption
apply fast
done


lemma bounded_induct:
     "[| wf r;
         ∀m ∈ I. F ∈ (A ∩ f-‘{m}) leadsTo
                      ((A ∩ f-‘(r^-1 ‘‘ {m})) ∪ B) |]
      ==> F ∈ A leadsTo ((A - (f-‘I)) ∪ B)"
apply (erule leadsTo_wf_induct, safe)
apply (case_tac "m ∈ I")
apply (blast intro: leadsTo_weaken)
apply (blast intro: subset_imp_leadsTo)
done
```

```
lemma lessThan_induct:
     "[| !!m::nat. F ∈ (A ∩ f-`{m}) leadsTo ((A ∩ f-`{..<m}) ∪ B) |]
      ==> F ∈ A leadsTo B"
apply (rule wf_less_than [THEN leadsTo_wf_induct])
apply (simp (no_asm_simp))
apply blast
done

lemma lessThan_bounded_induct:
     "!!l::nat. [| ∀m ∈ greaterThan l.
           F ∈ (A ∩ f-`{m}) leadsTo ((A ∩ f-`(lessThan m)) ∪ B) |]
      ==> F ∈ A leadsTo ((A ∩ (f-`(atMost l))) ∪ B)"
apply (simp only: Diff_eq [symmetric] vimage_Compl Compl_greaterThan [symmetric])
apply (rule wf_less_than [THEN bounded_induct])
apply (simp (no_asm_simp))
done

lemma greaterThan_bounded_induct:
     "(!!l::nat. ∀m ∈ lessThan l.
                F ∈ (A ∩ f-`{m}) leadsTo ((A ∩ f-`(greaterThan m)) ∪ B))
      ==> F ∈ A leadsTo ((A ∩ (f-`(atLeast l))) ∪ B)"
apply (rule_tac f = f and f1 = "%k. l - k"
       in wf_less_than [THEN wf_inv_image, THEN leadsTo_wf_induct])
apply (simp (no_asm) add: inv_image_def Image_singleton)
apply clarify
apply (case_tac "m<l")
 apply (blast intro: leadsTo_weaken_R diff_less_mono2)
apply (blast intro: not_leE subset_imp_leadsTo)
done
```

3.6 wlt

Misra's property W3

```
lemma wlt_leadsTo: "F ∈ (wlt F B) leadsTo B"
apply (unfold wlt_def)
apply (blast intro!: leadsTo_Union)
done

lemma leadsTo_subset: "F ∈ A leadsTo B ==> A ⊆ wlt F B"
apply (unfold wlt_def)
apply (blast intro!: leadsTo_Union)
done
```

Misra's property W2

```
lemma leadsTo_eq_subset_wlt: "F ∈ A leadsTo B = (A ⊆ wlt F B)"
by (blast intro!: leadsTo_subset wlt_leadsTo [THEN leadsTo_weaken_L])
```

Misra's property W4

```
lemma wlt_increasing: "B ⊆ wlt F B"
apply (simp (no_asm_simp) add: leadsTo_eq_subset_wlt [symmetric] subset_imp_leadsTo)
done
```

Used in the Trans case below

```
lemma lemma1:
   "[| B ⊆ A2;
       F ∈ (A1 - B) co (A1 ∪ B);
       F ∈ (A2 - C) co (A2 ∪ C) |]
    ==> F ∈ (A1 ∪ A2 - C) co (A1 ∪ A2 ∪ C)"
by (unfold constrains_def, clarify,  blast)
```

Lemma (1,2,3) of Misra's draft book, Chapter 4, "Progress"

```
lemma leadsTo_123:
     "F ∈ A leadsTo A'
      ==> ∃B. A ⊆ B & F ∈ B leadsTo A' & F ∈ (B-A') co (B ∪ A')"
apply (erule leadsTo_induct)
```

Basis

```
  apply (blast dest: ensuresD)
```

Trans

```
 apply clarify
 apply (rule_tac x = "Ba ∪ Bb" in exI)
 apply (blast intro: lemma1 leadsTo_Un_Un leadsTo_cancel1 leadsTo_Un_duplicate)
```

Union

```
apply (clarify dest!: ball_conj_distrib [THEN iffD1] bchoice)
apply (rule_tac x = "⋃A ∈ S. f A" in exI)
apply (auto intro: leadsTo_UN)

apply (rule_tac I1=S and A1="%i. f i - B" and A'1="%i. f i ∪ B"
       in constrains_UN [THEN constrains_weaken], auto)
done
```

Misra's property W5

```
lemma wlt_constrains_wlt: "F ∈ (wlt F B - B) co (wlt F B)"
proof -
  from wlt_leadsTo [of F B, THEN leadsTo_123]
  show ?thesis
  proof (elim exE conjE)

    fix C
    assume wlt: "wlt F B ⊆ C"
       and lt:  "F ∈ C leadsTo B"
       and co:  "F ∈ C - B co C ∪ B"
    have eq: "C = wlt F B"
    proof -
      from lt and wlt show ?thesis
           by (blast dest: leadsTo_eq_subset_wlt [THEN iffD1])
    qed
    from co show ?thesis by (simp add: eq wlt_increasing Un_absorb2)
  qed
qed
```

3.7 Completion: Binary and General Finite versions

```
lemma completion_lemma :
     "[| W = wlt F (B' ∪ C);
         F ∈ A leadsTo (A' ∪ C);  F ∈ A' co (A' ∪ C);
         F ∈ B leadsTo (B' ∪ C);  F ∈ B' co (B' ∪ C) |]
    ==> F ∈ (A ∩ B) leadsTo ((A' ∩ B') ∪ C)"
apply (subgoal_tac "F ∈ (W-C) co (W ∪ B' ∪ C) ")
 prefer 2
 apply (blast intro: wlt_constrains_wlt [THEN [2] constrains_Un,
                                         THEN constrains_weaken])
apply (subgoal_tac "F ∈ (W-C) co W")
 prefer 2
 apply (simp add: wlt_increasing Un_assoc Un_absorb2)
apply (subgoal_tac "F ∈ (A ∩ W - C) leadsTo (A' ∩ W ∪ C) ")
 prefer 2 apply (blast intro: wlt_leadsTo psp [THEN leadsTo_weaken])

apply (subgoal_tac "F ∈ (A' ∩ W ∪ C) leadsTo (A' ∩ B' ∪ C) ")
 prefer 2
 apply (rule leadsTo_Un_duplicate2)
 apply (blast intro: leadsTo_Un_Un wlt_leadsTo
                     [THEN psp2, THEN leadsTo_weaken] leadsTo_refl)
apply (drule leadsTo_Diff)
apply (blast intro: subset_imp_leadsTo)
apply (subgoal_tac "A ∩ B ⊆ A ∩ W")
 prefer 2
 apply (blast dest!: leadsTo_subset intro!: subset_refl [THEN Int_mono])
apply (blast intro: leadsTo_Trans subset_imp_leadsTo)
done

lemmas completion = completion_lemma [OF refl]

lemma finite_completion_lemma:
     "finite I ==> (∀i ∈ I. F ∈ (A i) leadsTo (A' i ∪ C)) -->
                   (∀i ∈ I. F ∈ (A' i) co (A' i ∪ C)) -->
                   F ∈ (⋂i ∈ I. A i) leadsTo ((⋂i ∈ I. A' i) ∪ C)"
apply (erule finite_induct, auto)
apply (rule completion)
   prefer 4
   apply (simp only: INT_simps [symmetric])
   apply (rule constrains_INT, auto)
done

lemma finite_completion:
     "[| finite I;
         !!i. i ∈ I ==> F ∈ (A i) leadsTo (A' i ∪ C);
         !!i. i ∈ I ==> F ∈ (A' i) co (A' i ∪ C) |]
      ==> F ∈ (⋂i ∈ I. A i) leadsTo ((⋂i ∈ I. A' i) ∪ C)"
by (blast intro: finite_completion_lemma [THEN mp, THEN mp])

lemma stable_completion:
     "[| F ∈ A leadsTo A';  F ∈ stable A';
         F ∈ B leadsTo B';  F ∈ stable B' |]
    ==> F ∈ (A ∩ B) leadsTo (A' ∩ B')"
```

```
apply (unfold stable_def)
apply (rule_tac C1 = "{}" in completion [THEN leadsTo_weaken_R])
apply (force+)
done

lemma finite_stable_completion:
     "[| finite I;
         !!i. i ∈ I ==> F ∈ (A i) leadsTo (A' i);
         !!i. i ∈ I ==> F ∈ stable (A' i) |]
      ==> F ∈ (⋂i ∈ I. A i) leadsTo (⋂i ∈ I. A' i)"
apply (unfold stable_def)
apply (rule_tac C1 = "{}" in finite_completion [THEN leadsTo_weaken_R])
apply (simp_all (no_asm_simp))
apply blast+
done

end
```

4 Weak Safety

```
theory Constrains imports UNITY begin

consts traces :: "['a set, ('a * 'a)set set] => ('a * 'a list) set"



inductive "traces init acts"
  intros

    Init:  "s ∈ init ==> (s,[]) ∈ traces init acts"

    Acts:  "[| act: acts;  (s,evs) ∈ traces init acts;  (s,s'): act |]
            ==> (s', s#evs) ∈ traces init acts"


consts reachable :: "'a program => 'a set"

inductive "reachable F"
  intros
    Init:  "s ∈ Init F ==> s ∈ reachable F"

    Acts:  "[| act: Acts F;  s ∈ reachable F;  (s,s'): act |]
            ==> s' ∈ reachable F"

constdefs
  Constrains :: "['a set, 'a set] => 'a program set"  (infixl "Co" 60)
    "A Co B == {F. F ∈ (reachable F ∩ A)  co  B}"

  Unless  :: "['a set, 'a set] => 'a program set"     (infixl "Unless" 60)
    "A Unless B == (A-B) Co (A ∪ B)"

  Stable     :: "'a set => 'a program set"
```

```
    "Stable A == A Co A"


  Always :: "'a set => 'a program set"
    "Always A == {F. Init F ⊆ A} ∩ Stable A"


  Increasing :: "['a => 'b::{order}] => 'a program set"
    "Increasing f == ⋂z. Stable {s. z ≤ f s}"
```

4.1 traces and reachable

lemma *reachable_equiv_traces:*
```
     "reachable F = {s. ∃evs. (s,evs) ∈ traces (Init F) (Acts F)}"
```
apply *safe*
apply *(erule_tac [2] traces.induct)*
apply *(erule reachable.induct)*
apply *(blast intro: reachable.intros traces.intros)+*
done

lemma *Init_subset_reachable: "Init F ⊆ reachable F"*
by *(blast intro: reachable.intros)*

lemma *stable_reachable [intro!,simp]:*
```
     "Acts G ⊆ Acts F ==> G ∈ stable (reachable F)"
```
by *(blast intro: stableI constrainsI reachable.intros)*

lemma *invariant_reachable: "F ∈ invariant (reachable F)"*
apply *(simp add: invariant_def)*
apply *(blast intro: reachable.intros)*
done

lemma *invariant_includes_reachable: "F ∈ invariant A ==> reachable F ⊆ A"*
apply *(simp add: stable_def constrains_def invariant_def)*
apply *(rule subsetI)*
apply *(erule reachable.induct)*
apply *(blast intro: reachable.intros)+*
done

4.2 Co

lemmas *constrains_reachable_Int =*
```
    subset_refl [THEN stable_reachable [unfolded stable_def],
                 THEN constrains_Int, standard]
```

lemma *Constrains_eq_constrains:*
```
     "A Co B = {F. F ∈ (reachable F  ∩  A) co (reachable F  ∩  B)}"
```
apply *(unfold Constrains_def)*
apply *(blast dest: constrains_reachable_Int intro: constrains_weaken)*
done

lemma *constrains_imp_Constrains: "F ∈ A co A' ==> F ∈ A Co A'"*
apply *(unfold Constrains_def)*
apply *(blast intro: constrains_weaken_L)*
done

lemma *stable_imp_Stable: "F ∈ stable A ==> F ∈ Stable A"*
apply *(unfold stable_def Stable_def)*
apply *(erule constrains_imp_Constrains)*
done

lemma *ConstrainsI:*
"(!!act s s'. [| act: Acts F; (s,s') ∈ act; s ∈ A |] ==> s': A')
==> F ∈ A Co A'"
apply *(rule constrains_imp_Constrains)*
apply *(blast intro: constrainsI)*
done

lemma *Constrains_empty [iff]: "F ∈ {} Co B"*
by *(unfold Constrains_def constrains_def, blast)*

lemma *Constrains_UNIV [iff]: "F ∈ A Co UNIV"*
by *(blast intro: ConstrainsI)*

lemma *Constrains_weaken_R:*
"[| F ∈ A Co A'; A'<=B' |] ==> F ∈ A Co B'"
apply *(unfold Constrains_def)*
apply *(blast intro: constrains_weaken_R)*
done

lemma *Constrains_weaken_L:*
"[| F ∈ A Co A'; B ⊆ A |] ==> F ∈ B Co A'"
apply *(unfold Constrains_def)*
apply *(blast intro: constrains_weaken_L)*
done

lemma *Constrains_weaken:*
"[| F ∈ A Co A'; B ⊆ A; A'<=B' |] ==> F ∈ B Co B'"
apply *(unfold Constrains_def)*
apply *(blast intro: constrains_weaken)*
done

lemma *Constrains_Un:*
"[| F ∈ A Co A'; F ∈ B Co B' |] ==> F ∈ (A ∪ B) Co (A' ∪ B')"
apply *(unfold Constrains_def)*
apply *(blast intro: constrains_Un [THEN constrains_weaken])*
done

lemma *Constrains_UN:*
assumes *Co: "!!i. i ∈ I ==> F ∈ (A i) Co (A' i)"*
shows *"F ∈ (⋃i ∈ I. A i) Co (⋃i ∈ I. A' i)"*
apply *(unfold Constrains_def)*
apply *(rule CollectI)*

apply *(rule Co [unfolded Constrains_def, THEN CollectD, THEN constrains_UN,*

THEN constrains_weaken], auto)
done

lemma *Constrains_Int:*
"[| F ∈ A Co A'; F ∈ B Co B' |] ==> F ∈ (A ∩ B) Co (A' ∩ B')"
apply *(unfold Constrains_def)*
apply *(blast intro: constrains_Int [THEN constrains_weaken])*
done

lemma *Constrains_INT:*
assumes *Co: "!!i. i ∈ I ==> F ∈ (A i) Co (A' i)"*
shows *"F ∈ (⋂i ∈ I. A i) Co (⋂i ∈ I. A' i)"*
apply *(unfold Constrains_def)*
apply *(rule CollectI)*
apply *(rule Co [unfolded Constrains_def, THEN CollectD, THEN constrains_INT,*

THEN constrains_weaken], auto)
done

lemma *Constrains_imp_subset: "F ∈ A Co A' ==> reachable F ∩ A ⊆ A'"*
by *(simp add: constrains_imp_subset Constrains_def)*

lemma *Constrains_trans: "[| F ∈ A Co B; F ∈ B Co C |] ==> F ∈ A Co C"*
apply *(simp add: Constrains_eq_constrains)*
apply *(blast intro: constrains_trans constrains_weaken)*
done

lemma *Constrains_cancel:*
"[| F ∈ A Co (A' ∪ B); F ∈ B Co B' |] ==> F ∈ A Co (A' ∪ B')"
by *(simp add: Constrains_eq_constrains constrains_def, blast)*

4.3 Stable

lemma *Stable_eq: "[| F ∈ Stable A; A = B |] ==> F ∈ Stable B"*
by *blast*

lemma *Stable_eq_stable: "(F ∈ Stable A) = (F ∈ stable (reachable F ∩ A))"*
by *(simp add: Stable_def Constrains_eq_constrains stable_def)*

lemma *StableI: "F ∈ A Co A ==> F ∈ Stable A"*
by *(unfold Stable_def, assumption)*

lemma *StableD: "F ∈ Stable A ==> F ∈ A Co A"*
by *(unfold Stable_def, assumption)*

lemma *Stable_Un:*
"[| F ∈ Stable A; F ∈ Stable A' |] ==> F ∈ Stable (A ∪ A')"
apply *(unfold Stable_def)*
apply *(blast intro: Constrains_Un)*
done

lemma *Stable_Int:*
"[| F ∈ Stable A; F ∈ Stable A' |] ==> F ∈ Stable (A ∩ A')"
apply *(unfold Stable_def)*
apply *(blast intro: Constrains_Int)*
done

lemma *Stable_Constrains_Un:*
"[| F ∈ Stable C; F ∈ A Co (C ∪ A') |]
==> F ∈ (C ∪ A) Co (C ∪ A')"
apply *(unfold Stable_def)*
apply *(blast intro: Constrains_Un [THEN Constrains_weaken])*
done

lemma *Stable_Constrains_Int:*
"[| F ∈ Stable C; F ∈ (C ∩ A) Co A' |]
==> F ∈ (C ∩ A) Co (C ∩ A')"
apply *(unfold Stable_def)*
apply *(blast intro: Constrains_Int [THEN Constrains_weaken])*
done

lemma *Stable_UN:*
"(!!i. i ∈ I ==> F ∈ Stable (A i)) ==> F ∈ Stable (⋃i ∈ I. A i)"
by *(simp add: Stable_def Constrains_UN)*

lemma *Stable_INT:*
"(!!i. i ∈ I ==> F ∈ Stable (A i)) ==> F ∈ Stable (⋂i ∈ I. A i)"
by *(simp add: Stable_def Constrains_INT)*

lemma *Stable_reachable:* "F ∈ Stable (reachable F)"
by *(simp add: Stable_eq_stable)*

4.4 Increasing

lemma *IncreasingD:*
"F ∈ Increasing f ==> F ∈ Stable {s. x ≤ f s}"
by *(unfold Increasing_def, blast)*

lemma *mono_Increasing_o:*
"mono g ==> Increasing f ⊆ Increasing (g o f)"
apply *(simp add: Increasing_def Stable_def Constrains_def stable_def*
constrains_def)
apply *(blast intro: monoD order_trans)*
done

lemma *strict_IncreasingD:*
"!!z::nat. F ∈ Increasing f ==> F ∈ Stable {s. z < f s}"
by *(simp add: Increasing_def Suc_le_eq [symmetric])*

lemma *increasing_imp_Increasing:*
"F ∈ increasing f ==> F ∈ Increasing f"
apply *(unfold increasing_def Increasing_def)*
apply *(blast intro: stable_imp_Stable)*
done

lemmas *Increasing_constant =*
increasing_constant [THEN increasing_imp_Increasing, standard, iff]

4.5 The Elimination Theorem

lemma *Elimination:*
"[| ∀m. F ∈ {s. s x = m} Co (B m) |]
==> F ∈ {s. s x ∈ M} Co (⋃m ∈ M. B m)"
by *(unfold Constrains_def constrains_def, blast)*

lemma *Elimination_sing:*
"(∀m. F ∈ {m} Co (B m)) ==> F ∈ M Co (⋃m ∈ M. B m)"
by *(unfold Constrains_def constrains_def, blast)*

4.6 Specialized laws for handling Always

lemma *AlwaysI: "[| Init F ⊆ A; F ∈ Stable A |] ==> F ∈ Always A"*
by *(simp add: Always_def)*

lemma *AlwaysD: "F ∈ Always A ==> Init F ⊆ A & F ∈ Stable A"*
by *(simp add: Always_def)*

lemmas *AlwaysE = AlwaysD [THEN conjE, standard]*
lemmas *Always_imp_Stable = AlwaysD [THEN conjunct2, standard]*

lemma *Always_includes_reachable: "F ∈ Always A ==> reachable F ⊆ A"*
apply *(simp add: Stable_def Constrains_def constrains_def Always_def)*
apply *(rule subsetI)*
apply *(erule reachable.induct)*
apply *(blast intro: reachable.intros)+*
done

lemma *invariant_imp_Always:*
"F ∈ invariant A ==> F ∈ Always A"
apply *(unfold Always_def invariant_def Stable_def stable_def)*
apply *(blast intro: constrains_imp_Constrains)*
done

lemmas *Always_reachable =*
invariant_reachable [THEN invariant_imp_Always, standard]

lemma *Always_eq_invariant_reachable:*
"Always A = {F. F ∈ invariant (reachable F ∩ A)}"
apply *(simp add: Always_def invariant_def Stable_def Constrains_eq_constrains*
stable_def)
apply *(blast intro: reachable.intros)*
done

lemma *Always_eq_includes_reachable: "Always A = {F. reachable F ⊆ A}"*

by *(auto dest: invariant_includes_reachable simp add: Int_absorb2 invariant_reachable Always_eq_invariant_reachable)*

lemma *Always_UNIV_eq [simp]: "Always UNIV = UNIV"*
by *(auto simp add: Always_eq_includes_reachable)*

lemma *UNIV_AlwaysI: "UNIV ⊆ A ==> F ∈ Always A"*
by *(auto simp add: Always_eq_includes_reachable)*

lemma *Always_eq_UN_invariant: "Always A = (⋃I ∈ Pow A. invariant I)"*
apply *(simp add: Always_eq_includes_reachable)*
apply *(blast intro: invariantI Init_subset_reachable [THEN subsetD]*
invariant_includes_reachable [THEN subsetD])
done

lemma *Always_weaken: "[| F ∈ Always A; A ⊆ B |] ==> F ∈ Always B"*
by *(auto simp add: Always_eq_includes_reachable)*

4.7 "Co" rules involving Always

lemma *Always_Constrains_pre:*
"F ∈ Always INV ==> (F ∈ (INV ∩ A) Co A') = (F ∈ A Co A')"
by *(simp add: Always_includes_reachable [THEN Int_absorb2] Constrains_def*

Int_assoc [symmetric])

lemma *Always_Constrains_post:*
"F ∈ Always INV ==> (F ∈ A Co (INV ∩ A')) = (F ∈ A Co A')"
by *(simp add: Always_includes_reachable [THEN Int_absorb2]*
Constrains_eq_constrains Int_assoc [symmetric])

lemmas *Always_ConstrainsI = Always_Constrains_pre [THEN iffD1, standard]*

lemmas *Always_ConstrainsD = Always_Constrains_post [THEN iffD2, standard]*

lemma *Always_Constrains_weaken:*
"[| F ∈ Always C; F ∈ A Co A';
C ∩ B ⊆ A; C ∩ A' ⊆ B' |]
==> F ∈ B Co B'"
apply *(rule Always_ConstrainsI, assumption)*
apply *(drule Always_ConstrainsD, assumption)*
apply *(blast intro: Constrains_weaken)*
done

lemma *Always_Int_distrib: "Always (A ∩ B) = Always A ∩ Always B"*
by *(auto simp add: Always_eq_includes_reachable)*

lemma *Always_INT_distrib: "Always (INTER I A) = (⋂i ∈ I. Always (A i))"*

by *(auto simp add: Always_eq_includes_reachable)*

lemma *Always_Int_I:*
"[| F ∈ Always A; F ∈ Always B |] ==> F ∈ Always (A ∩ B)"
by *(simp add: Always_Int_distrib)*

lemma *Always_Compl_Un_eq:*
"F ∈ Always A ==> (F ∈ Always (-A ∪ B)) = (F ∈ Always B)"
by *(auto simp add: Always_eq_includes_reachable)*

lemmas *Always_thin = thin_rl [of "F ∈ Always A", standard]*

4.8 Totalize

lemma *reachable_imp_reachable_tot:*
"s ∈ reachable F ==> s ∈ reachable (totalize F)"
apply *(erule reachable.induct)*
apply *(rule reachable.Init)*
apply *simp*
apply *(rule_tac act = "totalize_act act" in reachable.Acts)*
apply *(auto simp add: totalize_act_def)*
done

lemma *reachable_tot_imp_reachable:*
"s ∈ reachable (totalize F) ==> s ∈ reachable F"
apply *(erule reachable.induct)*
apply *(rule reachable.Init, simp)*
apply *(force simp add: totalize_act_def intro: reachable.Acts)*
done

lemma *reachable_tot_eq [simp]: "reachable (totalize F) = reachable F"*
by *(blast intro: reachable_imp_reachable_tot reachable_tot_imp_reachable)*

lemma *totalize_Constrains_iff [simp]: "(totalize F ∈ A Co B) = (F ∈ A Co B)"*
by *(simp add: Constrains_def)*

lemma *totalize_Stable_iff [simp]: "(totalize F ∈ Stable A) = (F ∈ Stable A)"*
by *(simp add: Stable_def)*

lemma *totalize_Always_iff [simp]: "(totalize F ∈ Always A) = (F ∈ Always A)"*
by *(simp add: Always_def)*

end

5 Weak Progress

theory *SubstAx* **imports** *WFair Constrains* **begin**

constdefs
```
  Ensures :: "['a set, 'a set] => 'a program set"    (infixl "Ensures" 60)
   "A Ensures B == {F. F ∈ (reachable F ∩ A) ensures B}"

  LeadsTo :: "['a set, 'a set] => 'a program set"    (infixl "LeadsTo" 60)
   "A LeadsTo B == {F. F ∈ (reachable F ∩ A) leadsTo B}"
```

syntax *(xsymbols)*
```
  "op LeadsTo" :: "['a set, 'a set] => 'a program set" (infixl " ⟼w " 60)
```

Resembles the previous definition of LeadsTo

lemma *LeadsTo_eq_leadsTo:*
```
    "A LeadsTo B = {F. F ∈ (reachable F ∩ A) leadsTo (reachable F ∩ B)}"
```
apply *(unfold LeadsTo_def)*
apply *(blast dest: psp_stable2 intro: leadsTo_weaken)*
done

5.1 Specialized laws for handling invariants

lemma *Always_LeadsTo_pre:*
```
    "F ∈ Always INV ==> (F ∈ (INV ∩ A) LeadsTo A') = (F ∈ A LeadsTo A')"
```
by *(simp add: LeadsTo_def Always_eq_includes_reachable Int_absorb2*
Int_assoc [symmetric])

lemma *Always_LeadsTo_post:*
```
    "F ∈ Always INV ==> (F ∈ A LeadsTo (INV ∩ A')) = (F ∈ A LeadsTo A')"
```
by *(simp add: LeadsTo_eq_leadsTo Always_eq_includes_reachable Int_absorb2*

Int_assoc [symmetric])

lemmas *Always_LeadsToI = Always_LeadsTo_pre [THEN iffD1, standard]*

lemmas *Always_LeadsToD = Always_LeadsTo_post [THEN iffD2, standard]*

5.2 Introduction rules: Basis, Trans, Union

lemma *leadsTo_imp_LeadsTo: "F ∈ A leadsTo B ==> F ∈ A LeadsTo B"*
apply *(simp add: LeadsTo_def)*
apply *(blast intro: leadsTo_weaken_L)*
done

lemma *LeadsTo_Trans:*
```
    "[| F ∈ A LeadsTo B;  F ∈ B LeadsTo C |] ==> F ∈ A LeadsTo C"
```
apply *(simp add: LeadsTo_eq_leadsTo)*
apply *(blast intro: leadsTo_Trans)*
done

lemma *LeadsTo_Union:*

```
    "(!!A. A ∈ S ==> F ∈ A LeadsTo B) ==> F ∈ (Union S) LeadsTo B"
apply (simp add: LeadsTo_def)
apply (subst Int_Union)
apply (blast intro: leadsTo_UN)
done
```

5.3 Derived rules

```
lemma LeadsTo_UNIV [simp]: "F ∈ A LeadsTo UNIV"
by (simp add: LeadsTo_def)
```

Useful with cancellation, disjunction

```
lemma LeadsTo_Un_duplicate:
    "F ∈ A LeadsTo (A' ∪ A') ==> F ∈ A LeadsTo A'"
by (simp add: Un_ac)

lemma LeadsTo_Un_duplicate2:
    "F ∈ A LeadsTo (A' ∪ C ∪ C) ==> F ∈ A LeadsTo (A' ∪ C)"
by (simp add: Un_ac)

lemma LeadsTo_UN:
    "(!!i. i ∈ I ==> F ∈ (A i) LeadsTo B) ==> F ∈ (⋃i ∈ I. A i) LeadsTo
B"
apply (simp only: Union_image_eq [symmetric])
apply (blast intro: LeadsTo_Union)
done
```

Binary union introduction rule

```
lemma LeadsTo_Un:
    "[| F ∈ A LeadsTo C; F ∈ B LeadsTo C |] ==> F ∈ (A ∪ B) LeadsTo C"
apply (subst Un_eq_Union)
apply (blast intro: LeadsTo_Union)
done
```

Lets us look at the starting state

```
lemma single_LeadsTo_I:
    "(!!s. s ∈ A ==> F ∈ {s} LeadsTo B) ==> F ∈ A LeadsTo B"
by (subst UN_singleton [symmetric], rule LeadsTo_UN, blast)

lemma subset_imp_LeadsTo: "A ⊆ B ==> F ∈ A LeadsTo B"
apply (simp add: LeadsTo_def)
apply (blast intro: subset_imp_leadsTo)
done

lemmas empty_LeadsTo = empty_subsetI [THEN subset_imp_LeadsTo, standard,
simp]

lemma LeadsTo_weaken_R [rule_format]:
    "[| F ∈ A LeadsTo A';  A' ⊆ B' |] ==> F ∈ A LeadsTo B'"
apply (simp add: LeadsTo_def)
apply (blast intro: leadsTo_weaken_R)
done
```

lemma *LeadsTo_weaken_L [rule_format]:*
```
"[| F ∈ A LeadsTo A';  B ⊆ A |]
 ==> F ∈ B LeadsTo A'"
```
apply *(simp add: LeadsTo_def)*
apply *(blast intro: leadsTo_weaken_L)*
done

lemma *LeadsTo_weaken:*
```
"[| F ∈ A LeadsTo A';
    B  ⊆ A;   A' ⊆ B' |]
 ==> F ∈ B LeadsTo B'"
```
by *(blast intro: LeadsTo_weaken_R LeadsTo_weaken_L LeadsTo_Trans)*

lemma *Always_LeadsTo_weaken:*
```
"[| F ∈ Always C;  F ∈ A LeadsTo A';
    C ∩ B ⊆ A;   C ∩ A' ⊆ B' |]
 ==> F ∈ B LeadsTo B'"
```
by *(blast dest: Always_LeadsToI intro: LeadsTo_weaken intro: Always_LeadsToD)*

lemma *LeadsTo_Un_post: "F ∈ A LeadsTo B ==> F ∈ (A ∪ B) LeadsTo B"*
by *(blast intro: LeadsTo_Un subset_imp_LeadsTo)*

lemma *LeadsTo_Trans_Un:*
```
"[| F ∈ A LeadsTo B;  F ∈ B LeadsTo C |]
 ==> F ∈ (A ∪ B) LeadsTo C"
```
by *(blast intro: LeadsTo_Un subset_imp_LeadsTo LeadsTo_weaken_L LeadsTo_Trans)*

lemma *LeadsTo_Un_distrib:*
```
"(F ∈ (A ∪ B) LeadsTo C)  = (F ∈ A LeadsTo C & F ∈ B LeadsTo C)"
```
by *(blast intro: LeadsTo_Un LeadsTo_weaken_L)*

lemma *LeadsTo_UN_distrib:*
```
"(F ∈ (⋃i ∈ I. A i) LeadsTo B)  =  (∀i ∈ I. F ∈ (A i) LeadsTo B)"
```
by *(blast intro: LeadsTo_UN LeadsTo_weaken_L)*

lemma *LeadsTo_Union_distrib:*
```
"(F ∈ (Union S) LeadsTo B)  =  (∀A ∈ S. F ∈ A LeadsTo B)"
```
by *(blast intro: LeadsTo_Union LeadsTo_weaken_L)*

lemma *LeadsTo_Basis: "F ∈ A Ensures B ==> F ∈ A LeadsTo B"*
by *(simp add: Ensures_def LeadsTo_def leadsTo_Basis)*

lemma *EnsuresI:*
```
"[| F ∈ (A-B) Co (A ∪ B);  F ∈ transient (A-B) |]
 ==> F ∈ A Ensures B"
```
apply *(simp add: Ensures_def Constrains_eq_constrains)*

```
apply (blast intro: ensuresI constrains_weaken transient_strengthen)
done

lemma Always_LeadsTo_Basis:
     "[| F ∈ Always INV;
         F ∈ (INV ∩ (A-A')) Co (A ∪ A');
         F ∈ transient (INV ∩ (A-A')) |]
  ==> F ∈ A LeadsTo A'"
apply (rule Always_LeadsToI, assumption)
apply (blast intro: EnsuresI LeadsTo_Basis Always_ConstrainsD [THEN Constrains_weaken]
transient_strengthen)
done
```

Set difference: maybe combine with `leadsTo_weaken_L`?? This is the most useful form of the "disjunction" rule

```
lemma LeadsTo_Diff:
     "[| F ∈ (A-B) LeadsTo C;  F ∈ (A ∩ B) LeadsTo C |]
      ==> F ∈ A LeadsTo C"
by (blast intro: LeadsTo_Un LeadsTo_weaken)


lemma LeadsTo_UN_UN:
     "(!! i. i ∈ I ==> F ∈ (A i) LeadsTo (A' i))
      ==> F ∈ (⋃i ∈ I. A i) LeadsTo (⋃i ∈ I. A' i)"
apply (simp only: Union_image_eq [symmetric])
apply (blast intro: LeadsTo_Union LeadsTo_weaken_R)
done
```

Version with no index set

```
lemma LeadsTo_UN_UN_noindex:
     "(!!i. F ∈ (A i) LeadsTo (A' i)) ==> F ∈ (⋃i. A i) LeadsTo (⋃i. A'
i)"
by (blast intro: LeadsTo_UN_UN)
```

Version with no index set

```
lemma all_LeadsTo_UN_UN:
     "∀i. F ∈ (A i) LeadsTo (A' i)
      ==> F ∈ (⋃i. A i) LeadsTo (⋃i. A' i)"
by (blast intro: LeadsTo_UN_UN)
```

Binary union version

```
lemma LeadsTo_Un_Un:
     "[| F ∈ A LeadsTo A'; F ∈ B LeadsTo B' |]
            ==> F ∈ (A ∪ B) LeadsTo (A' ∪ B')"
by (blast intro: LeadsTo_Un LeadsTo_weaken_R)


lemma LeadsTo_cancel2:
     "[| F ∈ A LeadsTo (A' ∪ B); F ∈ B LeadsTo B' |]
      ==> F ∈ A LeadsTo (A' ∪ B')"
by (blast intro: LeadsTo_Un_Un subset_imp_LeadsTo LeadsTo_Trans)
```

```
lemma LeadsTo_cancel_Diff2:
     "[| F ∈ A LeadsTo (A' ∪ B); F ∈ (B-A') LeadsTo B' |]
      ==> F ∈ A LeadsTo (A' ∪ B')"
apply (rule LeadsTo_cancel2)
prefer 2 apply assumption
apply (simp_all (no_asm_simp))
done

lemma LeadsTo_cancel1:
     "[| F ∈ A LeadsTo (B ∪ A'); F ∈ B LeadsTo B' |]
      ==> F ∈ A LeadsTo (B' ∪ A')"
apply (simp add: Un_commute)
apply (blast intro!: LeadsTo_cancel2)
done

lemma LeadsTo_cancel_Diff1:
     "[| F ∈ A LeadsTo (B ∪ A'); F ∈ (B-A') LeadsTo B' |]
      ==> F ∈ A LeadsTo (B' ∪ A')"
apply (rule LeadsTo_cancel1)
prefer 2 apply assumption
apply (simp_all (no_asm_simp))
done
```

The impossibility law

The set "A" may be non-empty, but it contains no reachable states

```
lemma LeadsTo_empty: "[|F ∈ A LeadsTo {}; all_total F|] ==> F ∈ Always (-A)"
apply (simp add: LeadsTo_def Always_eq_includes_reachable)
apply (drule leadsTo_empty, auto)
done
```

5.4 PSP: Progress-Safety-Progress

Special case of PSP: Misra's "stable conjunction"

```
lemma PSP_Stable:
     "[| F ∈ A LeadsTo A';  F ∈ Stable B |]
      ==> F ∈ (A ∩ B) LeadsTo (A' ∩ B)"
apply (simp add: LeadsTo_eq_leadsTo Stable_eq_stable)
apply (drule psp_stable, assumption)
apply (simp add: Int_ac)
done

lemma PSP_Stable2:
     "[| F ∈ A LeadsTo A'; F ∈ Stable B |]
      ==> F ∈ (B ∩ A) LeadsTo (B ∩ A')"
by (simp add: PSP_Stable Int_ac)

lemma PSP:
     "[| F ∈ A LeadsTo A'; F ∈ B Co B' |]
      ==> F ∈ (A ∩ B') LeadsTo ((A' ∩ B) ∪ (B' - B))"
apply (simp add: LeadsTo_def Constrains_eq_constrains)
apply (blast dest: psp intro: leadsTo_weaken)
```

```
done

lemma PSP2:
     "[| F ∈ A LeadsTo A'; F ∈ B Co B' |]
      ==> F ∈ (B' ∩ A) LeadsTo ((B ∩ A') ∪ (B' - B))"
by (simp add: PSP Int_ac)

lemma PSP_Unless:
     "[| F ∈ A LeadsTo A'; F ∈ B Unless B' |]
      ==> F ∈ (A ∩ B) LeadsTo ((A' ∩ B) ∪ B')"
apply (unfold Unless_def)
apply (drule PSP, assumption)
apply (blast intro: LeadsTo_Diff LeadsTo_weaken subset_imp_LeadsTo)
done


lemma Stable_transient_Always_LeadsTo:
     "[| F ∈ Stable A;  F ∈ transient C;
         F ∈ Always (-A ∪ B ∪ C) |] ==> F ∈ A LeadsTo B"
apply (erule Always_LeadsTo_weaken)
apply (rule LeadsTo_Diff)
   prefer 2
   apply (erule
          transient_imp_leadsTo [THEN leadsTo_imp_LeadsTo, THEN PSP_Stable2])
   apply (blast intro: subset_imp_LeadsTo)+
done
```

5.5 Induction rules

```
lemma LeadsTo_wf_induct:
     "[| wf r;
         ∀m. F ∈ (A ∩ f-`{m}) LeadsTo
                    ((A ∩ f-`(r^-1 `` {m})) ∪ B) |]
      ==> F ∈ A LeadsTo B"
apply (simp add: LeadsTo_eq_leadsTo)
apply (erule leadsTo_wf_induct)
apply (blast intro: leadsTo_weaken)
done


lemma Bounded_induct:
     "[| wf r;
         ∀m ∈ I. F ∈ (A ∩ f-`{m}) LeadsTo
                      ((A ∩ f-`(r^-1 `` {m})) ∪ B) |]
      ==> F ∈ A LeadsTo ((A - (f-`I)) ∪ B)"
apply (erule LeadsTo_wf_induct, safe)
apply (case_tac "m ∈ I")
apply (blast intro: LeadsTo_weaken)
apply (blast intro: subset_imp_LeadsTo)
done


lemma LessThan_induct:
     "(!!m::nat. F ∈ (A ∩ f-`{m}) LeadsTo ((A ∩ f-`(lessThan m)) ∪ B))
```

```
     ==> F ∈ A LeadsTo B"
by (rule wf_less_than [THEN LeadsTo_wf_induct], auto)
```

Integer version. Could generalize from 0 to any lower bound

```
lemma integ_0_le_induct:
     "[| F ∈ Always {s. (0::int) ≤ f s};
         !! z. F ∈ (A ∩ {s. f s = z}) LeadsTo
                   ((A ∩ {s. f s < z}) ∪ B) |]
      ==> F ∈ A LeadsTo B"
apply (rule_tac f = "nat o f" in LessThan_induct)
apply (simp add: vimage_def)
apply (rule Always_LeadsTo_weaken, assumption+)
apply (auto simp add: nat_eq_iff nat_less_iff)
done

lemma LessThan_bounded_induct:
     "!!l::nat. ∀m ∈ greaterThan l.
                   F ∈ (A ∩ f-‘{m}) LeadsTo ((A ∩ f-‘(lessThan m)) ∪ B)
           ==> F ∈ A LeadsTo ((A ∩ (f-‘(atMost l))) ∪ B)"
apply (simp only: Diff_eq [symmetric] vimage_Compl
                  Compl_greaterThan [symmetric])
apply (rule wf_less_than [THEN Bounded_induct], simp)
done

lemma GreaterThan_bounded_induct:
     "!!l::nat. ∀m ∈ lessThan l.
                F ∈ (A ∩ f-‘{m}) LeadsTo ((A ∩ f-‘(greaterThan m)) ∪ B)
      ==> F ∈ A LeadsTo ((A ∩ (f-‘(atLeast l))) ∪ B)"
apply (rule_tac f = f and f1 = "%k. l - k"
       in wf_less_than [THEN wf_inv_image, THEN LeadsTo_wf_induct])
apply (simp add: inv_image_def Image_singleton, clarify)
apply (case_tac "m<l")
 apply (blast intro: LeadsTo_weaken_R diff_less_mono2)
apply (blast intro: not_leE subset_imp_LeadsTo)
done
```

5.6 Completion: Binary and General Finite versions

```
lemma Completion:
     "[| F ∈ A LeadsTo (A' ∪ C);  F ∈ A' Co (A' ∪ C);
         F ∈ B LeadsTo (B' ∪ C);  F ∈ B' Co (B' ∪ C) |]
      ==> F ∈ (A ∩ B) LeadsTo ((A' ∩ B') ∪ C)"
apply (simp add: LeadsTo_eq_leadsTo Constrains_eq_constrains Int_Un_distrib)
apply (blast intro: completion leadsTo_weaken)
done

lemma Finite_completion_lemma:
     "finite I
      ==> (∀i ∈ I. F ∈ (A i) LeadsTo (A' i ∪ C)) -->
          (∀i ∈ I. F ∈ (A' i) Co (A' i ∪ C)) -->
          F ∈ (⋂i ∈ I. A i) LeadsTo ((⋂i ∈ I. A' i) ∪ C)"
apply (erule finite_induct, auto)
apply (rule Completion)
   prefer 4
```

```
   apply (simp only: INT_simps [symmetric])
   apply (rule Constrains_INT, auto)
done

lemma Finite_completion:
     "[| finite I;
         !!i. i ∈ I ==> F ∈ (A i) LeadsTo (A' i ∪ C);
         !!i. i ∈ I ==> F ∈ (A' i) Co (A' i ∪ C) |]
      ==> F ∈ (⋂i ∈ I. A i) LeadsTo ((⋂i ∈ I. A' i) ∪ C)"
by (blast intro: Finite_completion_lemma [THEN mp, THEN mp])

lemma Stable_completion:
     "[| F ∈ A LeadsTo A';  F ∈ Stable A';
         F ∈ B LeadsTo B';  F ∈ Stable B' |]
      ==> F ∈ (A ∩ B) LeadsTo (A' ∩ B')"
apply (unfold Stable_def)
apply (rule_tac C1 = "{}" in Completion [THEN LeadsTo_weaken_R])
apply (force+)
done

lemma Finite_stable_completion:
     "[| finite I;
         !!i. i ∈ I ==> F ∈ (A i) LeadsTo (A' i);
         !!i. i ∈ I ==> F ∈ Stable (A' i) |]
      ==> F ∈ (⋂i ∈ I. A i) LeadsTo (⋂i ∈ I. A' i)"
apply (unfold Stable_def)
apply (rule_tac C1 = "{}" in Finite_completion [THEN LeadsTo_weaken_R])
apply (simp_all, blast+)
done

end
```

6 The Detects Relation

```
theory Detects imports FP SubstAx begin

consts
   op_Detects  :: "['a set, 'a set] => 'a program set"  (infixl "Detects" 60)
   op_Equality :: "['a set, 'a set] => 'a set"          (infixl "<==>" 60)

defs
  Detects_def:  "A Detects B == (Always (-A ∪ B)) ∩ (B LeadsTo A)"
  Equality_def: "A <==> B == (-A ∪ B) ∩ (A ∪ -B)"




lemma Always_at_FP:
     "[|F ∈ A LeadsTo B; all_total F|] ==> F ∈ Always (-((FP F) ∩ A ∩ -B))"
apply (rule LeadsTo_empty)
apply (subgoal_tac "F ∈ (FP F ∩ A ∩ - B) LeadsTo (B ∩ (FP F ∩ -B))")
apply (subgoal_tac [2] " (FP F ∩ A ∩ - B) = (A ∩ (FP F ∩ -B))")
```

```
apply (subgoal_tac "(B ∩ (FP F ∩ -B)) = {}")
apply auto
apply (blast intro: PSP_Stable stable_imp_Stable stable_FP_Int)
done


lemma Detects_Trans:
     "[| F ∈ A Detects B; F ∈ B Detects C |] ==> F ∈ A Detects C"
apply (unfold Detects_def Int_def)
apply (simp (no_asm))
apply safe
apply (rule_tac [2] LeadsTo_Trans, auto)
apply (subgoal_tac "F ∈ Always ((-A ∪ B) ∩ (-B ∪ C))")
 apply (blast intro: Always_weaken)
apply (simp add: Always_Int_distrib)
done

lemma Detects_refl: "F ∈ A Detects A"
apply (unfold Detects_def)
apply (simp (no_asm) add: Un_commute Compl_partition subset_imp_LeadsTo)
done

lemma Detects_eq_Un: "(A<==>B) = (A ∩ B) ∪ (-A ∩ -B)"
by (unfold Equality_def, blast)


lemma Detects_antisym:
     "[| F ∈ A Detects B;  F ∈ B Detects A|] ==> F ∈ Always (A <==> B)"
apply (unfold Detects_def Equality_def)
apply (simp add: Always_Int_I Un_commute)
done




lemma Detects_Always:
     "[|F ∈ A Detects B; all_total F|] ==> F ∈ Always (-(FP F) ∪ (A <==>
B))"
apply (unfold Detects_def Equality_def)
apply (simp add: Un_Int_distrib Always_Int_distrib)
apply (blast dest: Always_at_FP intro: Always_weaken)
done



lemma Detects_Imp_LeadstoEQ:
     "F ∈ A Detects B ==> F ∈ UNIV LeadsTo (A <==> B)"
apply (unfold Detects_def Equality_def)
apply (rule_tac B = B in LeadsTo_Diff)
 apply (blast intro: Always_LeadsToI subset_imp_LeadsTo)
apply (blast intro: Always_LeadsTo_weaken)
done
```

end

7 Unions of Programs

theory *Union* **imports** *SubstAx FP* **begin**

constdefs

```
  ok :: "['a program, 'a program] => bool"      (infixl "ok" 65)
    "F ok G == Acts F ⊆ AllowedActs G &
               Acts G ⊆ AllowedActs F"


  OK  :: "['a set, 'a => 'b program] => bool"
    "OK I F == (∀i ∈ I. ∀j ∈ I-{i}. Acts (F i) ⊆ AllowedActs (F j))"

  JOIN  :: "['a set, 'a => 'b program] => 'b program"
    "JOIN I F == mk_program (⋂i ∈ I. Init (F i), ⋃i ∈ I. Acts (F i),
                             ⋂i ∈ I. AllowedActs (F i))"

  Join :: "['a program, 'a program] => 'a program"      (infixl "Join" 65)
    "F Join G == mk_program (Init F ∩ Init G, Acts F ∪ Acts G,
                             AllowedActs F ∩ AllowedActs G)"

  SKIP :: "'a program"
    "SKIP == mk_program (UNIV, {}, UNIV)"


  safety_prop :: "'a program set => bool"
    "safety_prop X == SKIP: X & (∀G. Acts G ⊆ UNION X Acts --> G ∈ X)"
```

syntax

```
  "@JOIN1"     :: "[pttrns, 'b set] => 'b set"         ("(3JN _./ _)" 10)
  "@JOIN"      :: "[pttrn, 'a set, 'b set] => 'b set"  ("(3JN _:_./ _)" 10)
```

translations

```
  "JN x : A. B"   == "JOIN A (%x. B)"
  "JN x y. B"   == "JN x. JN y. B"
  "JN x. B"     == "JOIN UNIV (%x. B)"
```

syntax *(xsymbols)*

```
  SKIP     :: "'a program"                              ("⊥")
  Join     :: "['a program, 'a program] => 'a program"  (infixl "⊔" 65)
  "@JOIN1" :: "[pttrns, 'b set] => 'b set"              ("(3⨆ _./ _)" 10)
  "@JOIN"  :: "[pttrn, 'a set, 'b set] => 'b set"       ("(3⨆ _∈_./ _)" 10)
```

7.1 SKIP

lemma *Init_SKIP [simp]: "Init SKIP = UNIV"*
by *(simp add: SKIP_def)*

lemma *Acts_SKIP [simp]: "Acts SKIP = {Id}"*
by *(simp add: SKIP_def)*

lemma *AllowedActs_SKIP [simp]: "AllowedActs SKIP = UNIV"*
by *(auto simp add: SKIP_def)*

lemma *reachable_SKIP [simp]: "reachable SKIP = UNIV"*
by *(force elim: reachable.induct intro: reachable.intros)*

7.2 SKIP and safety properties

lemma *SKIP_in_constrains_iff [iff]: "(SKIP ∈ A co B) = (A ⊆ B)"*
by *(unfold constrains_def, auto)*

lemma *SKIP_in_Constrains_iff [iff]: "(SKIP ∈ A Co B) = (A ⊆ B)"*
by *(unfold Constrains_def, auto)*

lemma *SKIP_in_stable [iff]: "SKIP ∈ stable A"*
by *(unfold stable_def, auto)*

declare *SKIP_in_stable [THEN stable_imp_Stable, iff]*

7.3 Join

lemma *Init_Join [simp]: "Init (F⊔G) = Init F ∩ Init G"*
by *(simp add: Join_def)*

lemma *Acts_Join [simp]: "Acts (F⊔G) = Acts F ∪ Acts G"*
by *(auto simp add: Join_def)*

lemma *AllowedActs_Join [simp]:*
"AllowedActs (F⊔G) = AllowedActs F ∩ AllowedActs G"
by *(auto simp add: Join_def)*

7.4 JN

lemma *JN_empty [simp]: "(⨆i∈{}. F i) = SKIP"*
by *(unfold JOIN_def SKIP_def, auto)*

lemma *JN_insert [simp]: "(⨆i ∈ insert a I. F i) = (F a)⊔(⨆i ∈ I. F i)"*
apply *(rule program_equalityI)*
apply *(auto simp add: JOIN_def Join_def)*
done

lemma *Init_JN [simp]: "Init (⨆i ∈ I. F i) = (⋂i ∈ I. Init (F i))"*
by *(simp add: JOIN_def)*

lemma *Acts_JN [simp]: "Acts (⨆i ∈ I. F i) = insert Id (⋃i ∈ I. Acts (F i))"*
by *(auto simp add: JOIN_def)*

lemma *AllowedActs_JN [simp]:*
"AllowedActs (⨆i ∈ I. F i) = (⋂i ∈ I. AllowedActs (F i))"

by *(auto simp add: JOIN_def)*

lemma *JN_cong [cong]:*
"[| I=J; !!i. i ∈ J ==> F i = G i |] ==> (⨆i ∈ I. F i) = (⨆i ∈ J. G i)"
by *(simp add: JOIN_def)*

7.5 Algebraic laws

lemma *Join_commute: "F⊔G = G⊔F"*
by *(simp add: Join_def Un_commute Int_commute)*

lemma *Join_assoc: "(F⊔G)⊔H = F⊔(G⊔H)"*
by *(simp add: Un_ac Join_def Int_assoc insert_absorb)*

lemma *Join_left_commute: "A⊔(B⊔C) = B⊔(A⊔C)"*
by *(simp add: Un_ac Int_ac Join_def insert_absorb)*

lemma *Join_SKIP_left [simp]: "SKIP⊔F = F"*
apply *(unfold Join_def SKIP_def)*
apply *(rule program_equalityI)*
apply *(simp_all (no_asm) add: insert_absorb)*
done

lemma *Join_SKIP_right [simp]: "F⊔SKIP = F"*
apply *(unfold Join_def SKIP_def)*
apply *(rule program_equalityI)*
apply *(simp_all (no_asm) add: insert_absorb)*
done

lemma *Join_absorb [simp]: "F⊔F = F"*
apply *(unfold Join_def)*
apply *(rule program_equalityI, auto)*
done

lemma *Join_left_absorb: "F⊔(F⊔G) = F⊔G"*
apply *(unfold Join_def)*
apply *(rule program_equalityI, auto)*
done

lemmas *Join_ac = Join_assoc Join_left_absorb Join_commute Join_left_commute*

7.6 Laws Governing ⨆

lemma *JN_absorb: "k ∈ I ==> F k⊔(⨆i ∈ I. F i) = (⨆i ∈ I. F i)"*
by *(auto intro!: program_equalityI)*

lemma *JN_Un: "(⨆i ∈ I ∪ J. F i) = ((⨆i ∈ I. F i)⊔(⨆i ∈ J. F i))"*
by *(auto intro!: program_equalityI)*

lemma *JN_constant: "(⨆i ∈ I. c) = (if I={} then SKIP else c)"*
by *(rule program_equalityI, auto)*

lemma *JN_Join_distrib:*
```
"(⨆i ∈ I. F i⊔G i) = (⨆i ∈ I. F i) ⊔ (⨆i ∈ I. G i)"
```
by *(auto intro!: program_equalityI)*

lemma *JN_Join_miniscope:*
```
"i ∈ I ==> (⨆i ∈ I. F i⊔G) = ((⨆i ∈ I. F i)⊔G)"
```
by *(auto simp add: JN_Join_distrib JN_constant)*

lemma *JN_Join_diff: "i ∈ I ==> F i⊔JOIN (I - {i}) F = JOIN I F"*
apply *(unfold JOIN_def Join_def)*
apply *(rule program_equalityI, auto)*
done

7.7 Safety: co, stable, FP

lemma *JN_constrains:*
```
"i ∈ I ==> (⨆i ∈ I. F i) ∈ A co B = (∀i ∈ I. F i ∈ A co B)"
```
by *(simp add: constrains_def JOIN_def, blast)*

lemma *Join_constrains [simp]:*
```
"(F⊔G ∈ A co B) = (F ∈ A co B & G ∈ A co B)"
```
by *(auto simp add: constrains_def Join_def)*

lemma *Join_unless [simp]:*
```
"(F⊔G ∈ A unless B) = (F ∈ A unless B & G ∈ A unless B)"
```
by *(simp add: Join_constrains unless_def)*

lemma *Join_constrains_weaken:*
```
"[| F ∈ A co A';  G ∈ B co B' |]
 ==> F⊔G ∈ (A ∩ B) co (A' ∪ B')"
```
by *(simp, blast intro: constrains_weaken)*

lemma *JN_constrains_weaken:*
```
"[| ∀i ∈ I. F i ∈ A i co A' i;  i ∈ I |]
 ==> (⨆i ∈ I. F i) ∈ (⋂i ∈ I. A i) co (⋃i ∈ I. A' i)"
```
apply *(simp (no_asm_simp) add: JN_constrains)*
apply *(blast intro: constrains_weaken)*
done

lemma *JN_stable: "(⨆i ∈ I. F i) ∈ stable A = (∀i ∈ I. F i ∈ stable A)"*
by *(simp add: stable_def constrains_def JOIN_def)*

lemma *invariant_JN_I:*
```
"[| !!i. i ∈ I ==> F i ∈ invariant A;  i ∈ I |]
  ==> (⨆i ∈ I. F i) ∈ invariant A"
```
by *(simp add: invariant_def JN_stable, blast)*

lemma *Join_stable [simp]:*

```
     "(F⊔G ∈ stable A) =
      (F ∈ stable A & G ∈ stable A)"
by (simp add: stable_def)

lemma Join_increasing [simp]:
     "(F⊔G ∈ increasing f) =
      (F ∈ increasing f & G ∈ increasing f)"
by (simp add: increasing_def Join_stable, blast)

lemma invariant_JoinI:
     "[| F ∈ invariant A; G ∈ invariant A |]
      ==> F⊔G ∈ invariant A"
by (simp add: invariant_def, blast)

lemma FP_JN: "FP (⋃i ∈ I. F i) = (⋂i ∈ I. FP (F i))"
by (simp add: FP_def JN_stable INTER_def)
```

7.8 Progress: transient, ensures

```
lemma JN_transient:
     "i ∈ I ==>
    (⋃i ∈ I. F i) ∈ transient A = (∃i ∈ I. F i ∈ transient A)"
by (auto simp add: transient_def JOIN_def)

lemma Join_transient [simp]:
     "F⊔G ∈ transient A =
      (F ∈ transient A | G ∈ transient A)"
by (auto simp add: bex_Un transient_def Join_def)

lemma Join_transient_I1: "F ∈ transient A ==> F⊔G ∈ transient A"
by (simp add: Join_transient)

lemma Join_transient_I2: "G ∈ transient A ==> F⊔G ∈ transient A"
by (simp add: Join_transient)


lemma JN_ensures:
     "i ∈ I ==>
      (⋃i ∈ I. F i) ∈ A ensures B =
      ((∀i ∈ I. F i ∈ (A-B) co (A ∪ B)) & (∃i ∈ I. F i ∈ A ensures B))"
by (auto simp add: ensures_def JN_constrains JN_transient)

lemma Join_ensures:
     "F⊔G ∈ A ensures B =
      (F ∈ (A-B) co (A ∪ B) & G ∈ (A-B) co (A ∪ B) &
       (F ∈ transient (A-B) | G ∈ transient (A-B)))"
by (auto simp add: ensures_def Join_transient)

lemma stable_Join_constrains:
    "[| F ∈ stable A;  G ∈ A co A' |]
     ==> F⊔G ∈ A co A'"
apply (unfold stable_def constrains_def Join_def)
apply (simp add: ball_Un, blast)
done
```

```
lemma stable_Join_Always1:
     "[| F ∈ stable A;  G ∈ invariant A |] ==> F⊔G ∈ Always A"
apply (simp (no_asm_use) add: Always_def invariant_def Stable_eq_stable)
apply (force intro: stable_Int)
done


lemma stable_Join_Always2:
     "[| F ∈ invariant A;  G ∈ stable A |] ==> F⊔G ∈ Always A"
apply (subst Join_commute)
apply (blast intro: stable_Join_Always1)
done

lemma stable_Join_ensures1:
     "[| F ∈ stable A;  G ∈ A ensures B |] ==> F⊔G ∈ A ensures B"
apply (simp (no_asm_simp) add: Join_ensures)
apply (simp add: stable_def ensures_def)
apply (erule constrains_weaken, auto)
done


lemma stable_Join_ensures2:
     "[| F ∈ A ensures B;  G ∈ stable A |] ==> F⊔G ∈ A ensures B"
apply (subst Join_commute)
apply (blast intro: stable_Join_ensures1)
done
```

7.9 the ok and OK relations

```
lemma ok_SKIP1 [iff]: "SKIP ok F"
by (simp add: ok_def)

lemma ok_SKIP2 [iff]: "F ok SKIP"
by (simp add: ok_def)

lemma ok_Join_commute:
     "(F ok G & (F⊔G) ok H) = (G ok H & F ok (G⊔H))"
by (auto simp add: ok_def)

lemma ok_commute: "(F ok G) = (G ok F)"
by (auto simp add: ok_def)

lemmas ok_sym = ok_commute [THEN iffD1, standard]

lemma ok_iff_OK:
     "OK {(0::int,F),(1,G),(2,H)} snd = (F ok G & (F⊔G) ok H)"
apply (simp add: Ball_def conj_disj_distribR ok_def Join_def OK_def insert_absorb
              all_conj_distrib)
apply blast
done

lemma ok_Join_iff1 [iff]: "F ok (G⊔H) = (F ok G & F ok H)"
```

by *(auto simp add: ok_def)*

lemma *ok_Join_iff2 [iff]: "(G⊔H) ok F = (G ok F & H ok F)"*
by *(auto simp add: ok_def)*

lemma *ok_Join_commute_I: "[| F ok G; (F⊔G) ok H |] ==> F ok (G⊔H)"*
by *(auto simp add: ok_def)*

lemma *ok_JN_iff1 [iff]: "F ok (JOIN I G) = (∀i ∈ I. F ok G i)"*
by *(auto simp add: ok_def)*

lemma *ok_JN_iff2 [iff]: "(JOIN I G) ok F = (∀i ∈ I. G i ok F)"*
by *(auto simp add: ok_def)*

lemma *OK_iff_ok: "OK I F = (∀i ∈ I. ∀j ∈ I-{i}. (F i) ok (F j))"*
by *(auto simp add: ok_def OK_def)*

lemma *OK_imp_ok: "[| OK I F; i ∈ I; j ∈ I; i ≠ j|] ==> (F i) ok (F j)"*
by *(auto simp add: OK_iff_ok)*

7.10 Allowed

lemma *Allowed_SKIP [simp]: "Allowed SKIP = UNIV"*
by *(auto simp add: Allowed_def)*

lemma *Allowed_Join [simp]: "Allowed (F⊔G) = Allowed F ∩ Allowed G"*
by *(auto simp add: Allowed_def)*

lemma *Allowed_JN [simp]: "Allowed (JOIN I F) = (⋂i ∈ I. Allowed (F i))"*
by *(auto simp add: Allowed_def)*

lemma *ok_iff_Allowed: "F ok G = (F ∈ Allowed G & G ∈ Allowed F)"*
by *(simp add: ok_def Allowed_def)*

lemma *OK_iff_Allowed: "OK I F = (∀i ∈ I. ∀j ∈ I-{i}. F i ∈ Allowed(F j))"*
by *(auto simp add: OK_iff_ok ok_iff_Allowed)*

7.11 *safety_prop*, for reasoning about given instances of "ok"

lemma *safety_prop_Acts_iff:*
"safety_prop X ==> (Acts G ⊆ insert Id (UNION X Acts)) = (G ∈ X)"
by *(auto simp add: safety_prop_def)*

lemma *safety_prop_AllowedActs_iff_Allowed:*
"safety_prop X ==> (UNION X Acts ⊆ AllowedActs F) = (X ⊆ Allowed F)"
by *(auto simp add: Allowed_def safety_prop_Acts_iff [symmetric])*

lemma *Allowed_eq:*
"safety_prop X ==> Allowed (mk_program (init, acts, UNION X Acts)) = X"
by *(simp add: Allowed_def safety_prop_Acts_iff)*

lemma *safety_prop_constrains [iff]: "safety_prop (A co B) = (A ⊆ B)"*

by *(simp add: safety_prop_def constrains_def, blast)*

lemma *safety_prop_stable [iff]: "safety_prop (stable A)"*
by *(simp add: stable_def)*

lemma *safety_prop_Int [simp]:*
"[| safety_prop X; safety_prop Y |] ==> safety_prop (X ∩ Y)"
by *(simp add: safety_prop_def, blast)*

lemma *safety_prop_INTER1 [simp]:*
"(!!i. safety_prop (X i)) ==> safety_prop (⋂i. X i)"
by *(auto simp add: safety_prop_def, blast)*

lemma *safety_prop_INTER [simp]:*
"(!!i. i ∈ I ==> safety_prop (X i)) ==> safety_prop (⋂i ∈ I. X i)"
by *(auto simp add: safety_prop_def, blast)*

lemma *def_prg_Allowed:*
"[| F == mk_program (init, acts, UNION X Acts) ; safety_prop X |]
==> Allowed F = X"
by *(simp add: Allowed_eq)*

lemma *Allowed_totalize [simp]: "Allowed (totalize F) = Allowed F"*
by *(simp add: Allowed_def)*

lemma *def_total_prg_Allowed:*
"[| F == mk_total_program (init, acts, UNION X Acts) ; safety_prop X |]

==> Allowed F = X"
by *(simp add: mk_total_program_def def_prg_Allowed)*

lemma *def_UNION_ok_iff:*
"[| F == mk_program(init,acts,UNION X Acts); safety_prop X |]
==> F ok G = (G ∈ X & acts ⊆ AllowedActs G)"
by *(auto simp add: ok_def safety_prop_Acts_iff)*

The union of two total programs is total.

lemma *totalize_Join: "totalize F⊔totalize G = totalize (F⊔G)"*
by *(simp add: program_equalityI totalize_def Join_def image_Un)*

lemma *all_total_Join: "[|all_total F; all_total G|] ==> all_total (F⊔G)"*
by *(simp add: all_total_def, blast)*

lemma *totalize_JN: "(⨆i ∈ I. totalize (F i)) = totalize(⨆i ∈ I. F i)"*
by *(simp add: program_equalityI totalize_def JOIN_def image_UN)*

lemma *all_total_JN: "(!!i. i∈I ==> all_total (F i)) ==> all_total(⨆i∈I.*
F i)"
by *(simp add: all_total_iff_totalize totalize_JN [symmetric])*

end

8 Composition: Basic Primitives

theory *Comp* **imports** *Union* **begin**

instance *program :: (type) ord* **..**

defs
```
component_def:          "F ≤ H == ∃G. F⊔G = H"
strict_component_def:   "(F < (H::'a program)) == (F ≤ H & F ≠ H)"
```

constdefs
```
component_of :: "'a program =>'a program=> bool"
                                    (infixl "component'_of" 50)
"F component_of H == ∃G. F ok G & F⊔G = H"

strict_component_of :: "'a program⇒'a program=> bool"
                                    (infixl "strict'_component'_of" 50)
"F strict_component_of H == F component_of H & F≠H"

preserves :: "('a=>'b) => 'a program set"
  "preserves v == ⋂z. stable {s. v s = z}"

localize  :: "('a=>'b) => 'a program => 'a program"
"localize v F == mk_program(Init F, Acts F,
                            AllowedActs F ∩ (⋃G ∈ preserves v. Acts G))"

funPair      :: "['a => 'b, 'a => 'c, 'a] => 'b * 'c"
"funPair f g == %x. (f x, g x)"
```

8.1 The component relation

lemma *componentI: "H ≤ F | H ≤ G ==> H ≤ (F⊔G)"*
apply *(unfold component_def, auto)*
apply *(rule_tac x = "G⊔Ga" in exI)*
apply *(rule_tac [2] x = "G⊔F" in exI)*
apply *(auto simp add: Join_ac)*
done

lemma *component_eq_subset:*
```
     "(F ≤ G) =
      (Init G ⊆ Init F & Acts F ⊆ Acts G & AllowedActs G ⊆ AllowedActs F)"
```
apply *(unfold component_def)*
apply *(force intro!: exI program_equalityI)*
done

lemma *component_SKIP [iff]: "SKIP ≤ F"*
apply *(unfold component_def)*
apply *(force intro: Join_SKIP_left)*
done

lemma *component_refl [iff]: "F ≤ (F :: 'a program)"*
apply *(unfold component_def)*
apply *(blast intro: Join_SKIP_right)*

done

lemma *SKIP_minimal: "F ≤ SKIP ==> F = SKIP"*
by *(auto intro!: program_equalityI simp add: component_eq_subset)*

lemma *component_Join1: "F ≤ (F⊔G)"*
by *(unfold component_def, blast)*

lemma *component_Join2: "G ≤ (F⊔G)"*
apply *(unfold component_def)*
apply *(simp add: Join_commute, blast)*
done

lemma *Join_absorb1: "F ≤ G ==> F⊔G = G"*
by *(auto simp add: component_def Join_left_absorb)*

lemma *Join_absorb2: "G ≤ F ==> F⊔G = F"*
by *(auto simp add: Join_ac component_def)*

lemma *JN_component_iff: "((JOIN I F) ≤ H) = (∀i ∈ I. F i ≤ H)"*
by *(simp add: component_eq_subset, blast)*

lemma *component_JN: "i ∈ I ==> (F i) ≤ (⨆i ∈ I. (F i))"*
apply *(unfold component_def)*
apply *(blast intro: JN_absorb)*
done

lemma *component_trans: "[| F ≤ G; G ≤ H |] ==> F ≤ (H :: 'a program)"*
apply *(unfold component_def)*
apply *(blast intro: Join_assoc [symmetric])*
done

lemma *component_antisym: "[| F ≤ G; G ≤ F |] ==> F = (G :: 'a program)"*
apply *(simp (no_asm_use) add: component_eq_subset)*
apply *(blast intro!: program_equalityI)*
done

lemma *Join_component_iff: "((F⊔G) ≤ H) = (F ≤ H & G ≤ H)"*
by *(simp add: component_eq_subset, blast)*

lemma *component_constrains: "[| F ≤ G; G ∈ A co B |] ==> F ∈ A co B"*
by *(auto simp add: constrains_def component_eq_subset)*

lemma *component_stable: "[| F ≤ G; G ∈ stable A |] ==> F ∈ stable A"*
by *(auto simp add: stable_def component_constrains)*

lemmas *program_less_le = strict_component_def [THEN meta_eq_to_obj_eq]*

8.2 The preserves property

lemma *preservesI: "(!!z. F ∈ stable {s. v s = z}) ==> F ∈ preserves v"*
by *(unfold preserves_def, blast)*

lemma *preserves_imp_eq:*
```
"[| F ∈ preserves v;  act ∈ Acts F;  (s,s') ∈ act |] ==> v s = v s'"
```
by *(unfold preserves_def stable_def constrains_def, force)*

lemma *Join_preserves [iff]:*
```
"(F⊔G ∈ preserves v) = (F ∈ preserves v & G ∈ preserves v)"
```
by *(unfold preserves_def, auto)*

lemma *JN_preserves [iff]:*
```
"(JOIN I F ∈ preserves v) = (∀i ∈ I. F i ∈ preserves v)"
```
by *(simp add: JN_stable preserves_def, blast)*

lemma *SKIP_preserves [iff]: "SKIP ∈ preserves v"*
by *(auto simp add: preserves_def)*

lemma *funPair_apply [simp]: "(funPair f g) x = (f x, g x)"*
by *(simp add: funPair_def)*

lemma *preserves_funPair: "preserves (funPair v w) = preserves v ∩ preserves w"*
by *(auto simp add: preserves_def stable_def constrains_def, blast)*

declare *preserves_funPair [THEN eqset_imp_iff, iff]*

lemma *funPair_o_distrib: "(funPair f g) o h = funPair (f o h) (g o h)"*
by *(simp add: funPair_def o_def)*

lemma *fst_o_funPair [simp]: "fst o (funPair f g) = f"*
by *(simp add: funPair_def o_def)*

lemma *snd_o_funPair [simp]: "snd o (funPair f g) = g"*
by *(simp add: funPair_def o_def)*

lemma *subset_preserves_o: "preserves v ⊆ preserves (w o v)"*
by *(force simp add: preserves_def stable_def constrains_def)*

lemma *preserves_subset_stable: "preserves v ⊆ stable {s. P (v s)}"*
apply *(auto simp add: preserves_def stable_def constrains_def)*
apply *(rename_tac s' s)*
apply *(subgoal_tac "v s = v s'")*
apply *(force+)*
done

lemma *preserves_subset_increasing: "preserves v ⊆ increasing v"*
by *(auto simp add: preserves_subset_stable [THEN subsetD] increasing_def)*

lemma *preserves_id_subset_stable: "preserves id ⊆ stable A"*
by *(force simp add: preserves_def stable_def constrains_def)*

```
lemma safety_prop_preserves [iff]: "safety_prop (preserves v)"
by (auto intro: safety_prop_INTER1 simp add: preserves_def)



lemma stable_localTo_stable2:
     "[| F ∈ stable {s. P (v s) (w s)};
         G ∈ preserves v;  G ∈ preserves w |]
      ==> F⊔G ∈ stable {s. P (v s) (w s)}"
apply simp
apply (subgoal_tac "G ∈ preserves (funPair v w) ")
 prefer 2 apply simp
apply (drule_tac P1 = "split ?Q" in preserves_subset_stable [THEN subsetD],

       auto)
done

lemma Increasing_preserves_Stable:
     "[| F ∈ stable {s. v s ≤ w s};  G ∈ preserves v; F⊔G ∈ Increasing w
|]
      ==> F⊔G ∈ Stable {s. v s ≤ w s}"
apply (auto simp add: stable_def Stable_def Increasing_def Constrains_def
all_conj_distrib)
apply (blast intro: constrains_weaken)

apply (auto simp add: preserves_def stable_def constrains_def)

apply (erule_tac V = "∀act ∈ Acts F. ?P act" in thin_rl)
apply (erule_tac V = "∀z. ∀act ∈ Acts F. ?P z act" in thin_rl)
apply (subgoal_tac "v x = v xa")
 apply auto
apply (erule order_trans, blast)
done



lemma component_of_imp_component: "F component_of H ==> F ≤ H"
by (unfold component_def component_of_def, blast)



lemma component_of_refl [simp]: "F component_of F"
apply (unfold component_of_def)
apply (rule_tac x = SKIP in exI, auto)
done

lemma component_of_SKIP [simp]: "SKIP component_of F"
by (unfold component_of_def, auto)

lemma component_of_trans:
     "[| F component_of G; G component_of H |] ==> F component_of H"
apply (unfold component_of_def)
```

apply *(blast intro: Join_assoc [symmetric])*
done

lemmas *strict_component_of_eq =*
strict_component_of_def [THEN meta_eq_to_obj_eq, standard]

lemma *localize_Init_eq [simp]: "Init (localize v F) = Init F"*
by *(simp add: localize_def)*

lemma *localize_Acts_eq [simp]: "Acts (localize v F) = Acts F"*
by *(simp add: localize_def)*

lemma *localize_AllowedActs_eq [simp]:*
"AllowedActs (localize v F) = AllowedActs F ∩ (⋃G ∈ preserves v. Acts G)"
by *(unfold localize_def, auto)*

end

9 Guarantees Specifications

theory *Guar* **imports** *Comp* **begin**

instance *program :: (type) order*
by *(intro_classes,*
(assumption | rule component_refl component_trans component_antisym
program_less_le)+)

Existential and Universal properties. I formalize the two-program case, proving equivalence with Chandy and Sanders's n-ary definitions

constdefs

ex_prop :: "'a program set => bool"
"ex_prop X == ∀F G. F ok G -->F ∈ X | G ∈ X --> (F⊔G) ∈ X"

strict_ex_prop :: "'a program set => bool"
"strict_ex_prop X == ∀F G. F ok G --> (F ∈ X | G ∈ X) = (F⊔G ∈ X)"

uv_prop :: "'a program set => bool"
"uv_prop X == SKIP ∈ X & (∀F G. F ok G --> F ∈ X & G ∈ X --> (F⊔G) ∈ X)"

strict_uv_prop :: "'a program set => bool"
"strict_uv_prop X ==
SKIP ∈ X & (∀F G. F ok G --> (F ∈ X & G ∈ X) = (F⊔G ∈ X))"

Guarantees properties

constdefs

guar :: "['a program set, 'a program set] => 'a program set"
*(**infixl** "guarantees" 55)*

```
  "X guarantees Y == {F. ∀G. F ok G --> F⊔G ∈ X --> F⊔G ∈ Y}"


   wg :: "['a program, 'a program set] =>  'a program set"
  "wg F Y == Union({X. F ∈ X guarantees Y})"


   wx :: "('a program) set => ('a program)set"
   "wx X == Union({Y. Y ⊆ X & ex_prop Y})"


  welldef :: "'a program set"
  "welldef == {F. Init F ≠ {}}"

  refines :: "['a program, 'a program, 'a program set] => bool"
                        ("(3_ refines _ wrt _)" [10,10,10] 10)
  "G refines F wrt X ==
     ∀H. (F ok H & G ok H & F⊔H ∈ welldef ∩ X) -->
         (G⊔H ∈ welldef ∩ X)"

  iso_refines :: "['a program, 'a program, 'a program set] => bool"
                              ("(3_ iso'_refines _ wrt _)" [10,10,10] 10)
  "G iso_refines F wrt X ==
   F ∈ welldef ∩ X --> G ∈ welldef ∩ X"


lemma OK_insert_iff:
     "(OK (insert i I) F) =
      (if i ∈ I then OK I F else OK I F & (F i ok JOIN I F))"
by (auto intro: ok_sym simp add: OK_iff_ok)
```

9.1 Existential Properties

```
lemma ex1 [rule_format]:
 "[| ex_prop X; finite GG |] ==>
     GG ∩ X ≠ {}--> OK GG (%G. G) --> (⨆G ∈ GG. G) ∈ X"
apply (unfold ex_prop_def)
apply (erule finite_induct)
apply (auto simp add: OK_insert_iff Int_insert_left)
done


lemma ex2:
     "∀GG. finite GG & GG ∩ X ≠ {} --> OK GG (%G. G) -->(⨆G ∈ GG. G):X

      ==> ex_prop X"
apply (unfold ex_prop_def, clarify)
apply (drule_tac x = "{F,G}" in spec)
apply (auto dest: ok_sym simp add: OK_iff_ok)
done
```

lemma *ex_prop_finite:*
"ex_prop X =
(∀GG. finite GG & GG ∩ X ≠ {} & OK GG (%G. G)--> (⨆G ∈ GG. G) ∈ X)"
by *(blast intro: ex1 ex2)*

lemma *ex_prop_equiv:*
"ex_prop X = (∀G. G ∈ X = (∀H. (G component_of H) --> H ∈ X))"
apply *auto*
apply *(unfold ex_prop_def component_of_def, safe, blast, blast)*
apply *(subst Join_commute)*
apply *(drule ok_sym, blast)*
done

9.2 Universal Properties

lemma *uv1 [rule_format]:*
"[| uv_prop X; finite GG |]
==> GG ⊆ X & OK GG (%G. G) --> (⨆G ∈ GG. G) ∈ X"
apply *(unfold uv_prop_def)*
apply *(erule finite_induct)*
apply *(auto simp add: Int_insert_left OK_insert_iff)*
done

lemma *uv2:*
"∀GG. finite GG & GG ⊆ X & OK GG (%G. G) --> (⨆G ∈ GG. G) ∈ X
==> uv_prop X"
apply *(unfold uv_prop_def)*
apply *(rule conjI)*
apply *(drule_tac x = "{}"* **in** *spec)*
prefer *2*
apply *clarify*
apply *(drule_tac x = "{F,G}"* **in** *spec)*
apply *(auto dest: ok_sym simp add: OK_iff_ok)*
done

lemma *uv_prop_finite:*
"uv_prop X =
(∀GG. finite GG & GG ⊆ X & OK GG (%G. G) --> (⨆G ∈ GG. G): X)"
by *(blast intro: uv1 uv2)*

9.3 Guarantees

lemma *guaranteesI:*
"(!!G. [| F ok G; F⊔G ∈ X |] ==> F⊔G ∈ Y) ==> F ∈ X guarantees Y"
by *(simp add: guar_def component_def)*

lemma *guaranteesD:*
"[| F ∈ X guarantees Y; F ok G; F⊔G ∈ X |] ==> F⊔G ∈ Y"
by *(unfold guar_def component_def, blast)*

lemma *component_guaranteesD:*
"[| F ∈ X guarantees Y; F⨆G = H; H ∈ X; F ok G |] ==> H ∈ Y"
by *(unfold guar_def, blast)*

lemma *guarantees_weaken:*
"[| F ∈ X guarantees X'; Y ⊆ X; X' ⊆ Y' |] ==> F ∈ Y guarantees Y'"
by *(unfold guar_def, blast)*

lemma *subset_imp_guarantees_UNIV:* "X ⊆ Y ==> X guarantees Y = UNIV"
by *(unfold guar_def, blast)*

lemma *subset_imp_guarantees:* "X ⊆ Y ==> F ∈ X guarantees Y"
by *(unfold guar_def, blast)*

lemma *ex_prop_imp:* "ex_prop Y ==> (Y = UNIV guarantees Y)"
apply *(simp (no_asm_use) add: guar_def ex_prop_equiv)*
apply *safe*
apply *(drule_tac x = x* **in** *spec)*
apply *(drule_tac [2] x = x* **in** *spec)*
apply *(drule_tac [2] sym)*
apply *(auto simp add: component_of_def)*
done

lemma *guarantees_imp:* "(Y = UNIV guarantees Y) ==> ex_prop(Y)"
by *(auto simp add: guar_def ex_prop_equiv component_of_def dest: sym)*

lemma *ex_prop_equiv2:* "(ex_prop Y) = (Y = UNIV guarantees Y)"
apply *(rule iffI)*
apply *(rule ex_prop_imp)*
apply *(auto simp add: guarantees_imp)*
done

9.4 Distributive Laws. Re-Orient to Perform Miniscoping

lemma *guarantees_UN_left:*
"(⋃i ∈ I. X i) guarantees Y = (⋂i ∈ I. X i guarantees Y)"
by *(unfold guar_def, blast)*

lemma *guarantees_Un_left:*
"(X ∪ Y) guarantees Z = (X guarantees Z) ∩ (Y guarantees Z)"
by *(unfold guar_def, blast)*

lemma *guarantees_INT_right:*
"X guarantees (⋂i ∈ I. Y i) = (⋂i ∈ I. X guarantees Y i)"
by *(unfold guar_def, blast)*

lemma *guarantees_Int_right:*
"Z guarantees (X ∩ Y) = (Z guarantees X) ∩ (Z guarantees Y)"
by *(unfold guar_def, blast)*

lemma *guarantees_Int_right_I:*
```
     "[| F ∈ Z guarantees X;  F ∈ Z guarantees Y |]
     ==> F ∈ Z guarantees (X ∩ Y)"
```
by *(simp add: guarantees_Int_right)*

lemma *guarantees_INT_right_iff:*
```
     "(F ∈ X guarantees (INTER I Y)) = (∀i∈I. F ∈ X guarantees (Y i))"
```
by *(simp add: guarantees_INT_right)*

lemma *shunting: "(X guarantees Y) = (UNIV guarantees (-X ∪ Y))"*
by *(unfold guar_def, blast)*

lemma *contrapositive: "(X guarantees Y) = -Y guarantees -X"*
by *(unfold guar_def, blast)*

lemma *combining1:*
```
    "[| F ∈ V guarantees X;  F ∈ (X ∩ Y) guarantees Z |]
     ==> F ∈ (V ∩ Y) guarantees Z"
```
by *(unfold guar_def, blast)*

lemma *combining2:*
```
    "[| F ∈ V guarantees (X ∪ Y);  F ∈ Y guarantees Z |]
     ==> F ∈ V guarantees (X ∪ Z)"
```
by *(unfold guar_def, blast)*

lemma *all_guarantees:*
```
     "∀i∈I. F ∈ X guarantees (Y i) ==> F ∈ X guarantees (⋂i ∈ I. Y i)"
```
by *(unfold guar_def, blast)*

lemma *ex_guarantees:*
```
     "∃i∈I. F ∈ X guarantees (Y i) ==> F ∈ X guarantees (⋃i ∈ I. Y i)"
```
by *(unfold guar_def, blast)*

9.5 Guarantees: Additional Laws (by lcp)

lemma *guarantees_Join_Int:*
```
    "[| F ∈ U guarantees V;  G ∈ X guarantees Y; F ok G |]
     ==> F⊔G ∈ (U ∩ X) guarantees (V ∩ Y)"
```
apply *(simp add: guar_def, safe)*
apply *(simp add: Join_assoc)*
apply *(subgoal_tac "F⊔G⊔Ga = G⊔(F⊔Ga) ")*
apply *(simp add: ok_commute)*
apply *(simp add: Join_ac)*
done

lemma *guarantees_Join_Un:*
```
    "[| F ∈ U guarantees V;  G ∈ X guarantees Y; F ok G |]
     ==> F⊔G ∈ (U ∪ X) guarantees (V ∪ Y)"
```

```
apply (simp add: guar_def, safe)
 apply (simp add: Join_assoc)
apply (subgoal_tac "F⊔G⊔Ga = G⊔(F⊔Ga) ")
 apply (simp add: ok_commute)
apply (simp add: Join_ac)
done

lemma guarantees_JN_INT:
     "[| ∀i∈I. F i ∈ X i guarantees Y i;  OK I F |]
      ==> (JOIN I F) ∈ (INTER I X) guarantees (INTER I Y)"
apply (unfold guar_def, auto)
apply (drule bspec, assumption)
apply (rename_tac "i")
apply (drule_tac x = "JOIN (I-{i}) F⊔G" in spec)
apply (auto intro: OK_imp_ok
            simp add: Join_assoc [symmetric] JN_Join_diff JN_absorb)
done

lemma guarantees_JN_UN:
    "[| ∀i∈I. F i ∈ X i guarantees Y i;  OK I F |]
     ==> (JOIN I F) ∈ (UNION I X) guarantees (UNION I Y)"
apply (unfold guar_def, auto)
apply (drule bspec, assumption)
apply (rename_tac "i")
apply (drule_tac x = "JOIN (I-{i}) F⊔G" in spec)
apply (auto intro: OK_imp_ok
            simp add: Join_assoc [symmetric] JN_Join_diff JN_absorb)
done
```

9.6 Guarantees Laws for Breaking Down the Program (by lcp)

```
lemma guarantees_Join_I1:
     "[| F ∈ X guarantees Y;  F ok G |] ==> F⊔G ∈ X guarantees Y"
by (simp add: guar_def Join_assoc)

lemma guarantees_Join_I2:
     "[| G ∈ X guarantees Y;  F ok G |] ==> F⊔G ∈ X guarantees Y"
apply (simp add: Join_commute [of _ G] ok_commute [of _ G])
apply (blast intro: guarantees_Join_I1)
done

lemma guarantees_JN_I:
     "[| i ∈ I;  F i ∈ X guarantees Y;  OK I F |]
      ==> (⨆i ∈ I. (F i)) ∈ X guarantees Y"
apply (unfold guar_def, clarify)
apply (drule_tac x = "JOIN (I-{i}) F⊔G" in spec)
apply (auto intro: OK_imp_ok
            simp add: JN_Join_diff JN_Join_diff Join_assoc [symmetric])
done
```

```
lemma Join_welldef_D1: "F⊔G ∈ welldef ==> F ∈ welldef"
by (unfold welldef_def, auto)

lemma Join_welldef_D2: "F⊔G ∈ welldef ==> G ∈ welldef"
by (unfold welldef_def, auto)



lemma refines_refl: "F refines F wrt X"
by (unfold refines_def, blast)


lemma refines_trans:
     "[| H refines G wrt X;  G refines F wrt X |] ==> H refines F wrt X"
apply (simp add: refines_def)
oops


lemma strict_ex_refine_lemma:
     "strict_ex_prop X
      ==> (∀H. F ok H & G ok H & F⊔H ∈ X --> G⊔H ∈ X)
              = (F ∈ X --> G ∈ X)"
by (unfold strict_ex_prop_def, auto)

lemma strict_ex_refine_lemma_v:
     "strict_ex_prop X
      ==> (∀H. F ok H & G ok H & F⊔H ∈ welldef & F⊔H ∈ X --> G⊔H ∈ X) =

          (F ∈ welldef ∩ X --> G ∈ X)"
apply (unfold strict_ex_prop_def, safe)
apply (erule_tac x = SKIP and P = "%H. ?PP H --> ?RR H" in allE)
apply (auto dest: Join_welldef_D1 Join_welldef_D2)
done

lemma ex_refinement_thm:
     "[| strict_ex_prop X;
         ∀H. F ok H & G ok H & F⊔H ∈ welldef ∩ X --> G⊔H ∈ welldef |]
      ==> (G refines F wrt X) = (G iso_refines F wrt X)"
apply (rule_tac x = SKIP in allE, assumption)
apply (simp add: refines_def iso_refines_def strict_ex_refine_lemma_v)
done


lemma strict_uv_refine_lemma:
     "strict_uv_prop X ==>
      (∀H. F ok H & G ok H & F⊔H ∈ X --> G⊔H ∈ X) = (F ∈ X --> G ∈ X)"
by (unfold strict_uv_prop_def, blast)

lemma strict_uv_refine_lemma_v:
     "strict_uv_prop X
      ==> (∀H. F ok H & G ok H & F⊔H ∈ welldef & F⊔H ∈ X --> G⊔H ∈ X) =

          (F ∈ welldef ∩ X --> G ∈ X)"
apply (unfold strict_uv_prop_def, safe)
```

```
apply (erule_tac x = SKIP and P = "%H. ?PP H --> ?RR H" in allE)
apply (auto dest: Join_welldef_D1 Join_welldef_D2)
done

lemma uv_refinement_thm:
     "[| strict_uv_prop X;
         ∀H. F ok H & G ok H & F⊔H ∈ welldef ∩ X -->
             G⊔H ∈ welldef |]
      ==> (G refines F wrt X) = (G iso_refines F wrt X)"
apply (rule_tac x = SKIP in allE, assumption)
apply (simp add: refines_def iso_refines_def strict_uv_refine_lemma_v)
done


lemma guarantees_equiv:
    "(F ∈ X guarantees Y) = (∀H. H ∈ X ⟶ (F component_of H ⟶ H ∈ Y))"
by (unfold guar_def component_of_def, auto)

lemma wg_weakest: "!!X. F∈ (X guarantees Y) ==> X ⊆ (wg F Y)"
by (unfold wg_def, auto)

lemma wg_guarantees: "F∈ ((wg F Y) guarantees Y)"
by (unfold wg_def guar_def, blast)

lemma wg_equiv: "(H ∈ wg F X) = (F component_of H --> H ∈ X)"
by (simp add: guarantees_equiv wg_def, blast)

lemma component_of_wg: "F component_of H ==> (H ∈ wg F X) = (H ∈ X)"
by (simp add: wg_equiv)

lemma wg_finite:
    "∀FF. finite FF & FF ∩ X ≠ {} --> OK FF (%F. F)
          --> (∀F∈FF. ((⨆F ∈ FF. F): wg F X) = ((⨆F ∈ FF. F):X))"
apply clarify
apply (subgoal_tac "F component_of (⨆F ∈ FF. F) ")
apply (drule_tac X = X in component_of_wg, simp)
apply (simp add: component_of_def)
apply (rule_tac x = "⨆F ∈ (FF-{F}) . F" in exI)
apply (auto intro: JN_Join_diff dest: ok_sym simp add: OK_iff_ok)
done

lemma wg_ex_prop: "ex_prop X ==> (F ∈ X) = (∀H. H ∈ wg F X)"
apply (simp (no_asm_use) add: ex_prop_equiv wg_equiv)
apply blast
done



lemma wx_subset: "(wx X)<=X"
by (unfold wx_def, auto)

lemma wx_ex_prop: "ex_prop (wx X)"
apply (simp add: wx_def ex_prop_equiv cong: bex_cong, safe, blast)
apply force
```

```
done

lemma wx_weakest: "∀Z. Z<= X --> ex_prop Z --> Z ⊆ wx X"
by (auto simp add: wx_def)


lemma wx'_ex_prop: "ex_prop({F. ∀G. F ok G --> F⊔G ∈ X})"
apply (unfold ex_prop_def, safe)
 apply (drule_tac x = "G⊔Ga" in spec)
 apply (force simp add: ok_Join_iff1 Join_assoc)
apply (drule_tac x = "F⊔Ga" in spec)
apply (simp add: ok_Join_iff1 ok_commute  Join_ac)
done
```

Equivalence with the other definition of wx

```
lemma wx_equiv: "wx X = {F. ∀G. F ok G --> (F⊔G) ∈ X}"
apply (unfold wx_def, safe)
 apply (simp add: ex_prop_def, blast)
apply (simp (no_asm))
apply (rule_tac x = "{F. ∀G. F ok G --> F⊔G ∈ X}" in exI, safe)
apply (rule_tac [2] wx'_ex_prop)
apply (drule_tac x = SKIP in spec)+
apply auto
done
```

Propositions 7 to 11 are about this second definition of wx. They are the same as the ones proved for the first definition of wx, by equivalence

```
lemma guarantees_wx_eq: "(X guarantees Y) = wx(-X ∪ Y)"
by (simp add: guar_def wx_equiv)



lemma stable_guarantees_Always:
     "Init F ⊆ A ==> F ∈ (stable A) guarantees (Always A)"
apply (rule guaranteesI)
apply (simp add: Join_commute)
apply (rule stable_Join_Always1)
 apply (simp_all add: invariant_def Join_stable)
done

lemma constrains_guarantees_leadsTo:
     "F ∈ transient A ==> F ∈ (A co A ∪ B) guarantees (A leadsTo (B-A))"
apply (rule guaranteesI)
apply (rule leadsTo_Basis')
 apply (drule constrains_weaken_R)
  prefer 2 apply assumption
 apply blast
apply (blast intro: Join_transient_I1)
done

end
```

10 Extending State Sets

theory *Extend* **imports** *Guar* **begin**

constdefs

```
  Restrict :: "[ 'a set, ('a*'b) set] => ('a*'b) set"
    "Restrict A r == r ∩ (A <*> UNIV)"

  good_map :: "['a*'b => 'c] => bool"
    "good_map h == surj h & (∀x y. fst (inv h (h (x,y))) = x)"


  extend_set :: "['a*'b => 'c, 'a set] => 'c set"
    "extend_set h A == h ` (A <*> UNIV)"

  project_set :: "['a*'b => 'c, 'c set] => 'a set"
    "project_set h C == {x. ∃y. h(x,y) ∈ C}"

  extend_act :: "['a*'b => 'c, ('a*'a) set] => ('c*'c) set"
    "extend_act h == %act. ⋃(s,s') ∈ act. ⋃y. {(h(s,y), h(s',y))}"

  project_act :: "['a*'b => 'c, ('c*'c) set] => ('a*'a) set"
    "project_act h act == {(x,x'). ∃y y'. (h(x,y), h(x',y')) ∈ act}"

  extend :: "['a*'b => 'c, 'a program] => 'c program"
    "extend h F == mk_program (extend_set h (Init F),
                               extend_act h ` Acts F,
                               project_act h -` AllowedActs F)"


  project :: "['a*'b => 'c, 'c set, 'c program] => 'a program"
    "project h C F ==
       mk_program (project_set h (Init F),
                   project_act h ` Restrict C ` Acts F,
                   {act. Restrict (project_set h C) act :
                         project_act h ` Restrict C ` AllowedActs F})"
```

locale *Extend* =
  fixes *f* :: "'c => 'a"
    and *g* :: "'c => 'b"
    and *h* :: "'a*'b => 'c"
    and *slice* :: "['c set, 'b] => 'a set"
  assumes
    good_h: "good_map h"
  defines *f_def*: "f z == fst (inv h z)"
      and *g_def*: "g z == snd (inv h z)"
      and *slice_def*: "slice Z y == {x. h(x,y) ∈ Z}"

10.1 Restrict

lemma *Restrict_iff [iff]*: "((x,y): Restrict A r) = ((x,y): r & x ∈ A)"
by *(unfold Restrict_def, blast)*

```
lemma Restrict_UNIV [simp]: "Restrict UNIV = id"
apply (rule ext)
apply (auto simp add: Restrict_def)
done

lemma Restrict_empty [simp]: "Restrict {} r = {}"
by (auto simp add: Restrict_def)

lemma Restrict_Int [simp]: "Restrict A (Restrict B r) = Restrict (A ∩ B)
r"
by (unfold Restrict_def, blast)

lemma Restrict_triv: "Domain r ⊆ A ==> Restrict A r = r"
by (unfold Restrict_def, auto)

lemma Restrict_subset: "Restrict A r ⊆ r"
by (unfold Restrict_def, auto)

lemma Restrict_eq_mono:
     "[| A ⊆ B;  Restrict B r = Restrict B s |]
      ==> Restrict A r = Restrict A s"
by (unfold Restrict_def, blast)

lemma Restrict_imageI:
     "[| s ∈ RR;  Restrict A r = Restrict A s |]
      ==> Restrict A r ∈ Restrict A ‘ RR"
by (unfold Restrict_def image_def, auto)

lemma Domain_Restrict [simp]: "Domain (Restrict A r) = A ∩ Domain r"
by blast

lemma Image_Restrict [simp]: "(Restrict A r) `` B = r `` (A ∩ B)"
by blast


lemma good_mapI:
     assumes surj_h: "surj h"
         and prem:   "!! x x' y y'. h(x,y) = h(x',y') ==> x=x'"
     shows "good_map h"
apply (simp add: good_map_def)
apply (safe intro!: surj_h)
apply (rule prem)
apply (subst surjective_pairing [symmetric])
apply (subst surj_h [THEN surj_f_inv_f])
apply (rule refl)
done

lemma good_map_is_surj: "good_map h ==> surj h"
by (unfold good_map_def, auto)


lemma fst_inv_equalityI:
     assumes surj_h: "surj h"
```

```
        and prem:   "!! x y. g (h(x,y)) = x"
    shows "fst (inv h z) = g z"
apply (unfold inv_def)
apply (rule_tac y1 = z in surj_h [THEN surjD, THEN exE])
apply (rule someI2)
apply (drule_tac [2] f = g in arg_cong)
apply (auto simp add: prem)
done
```

10.2 Trivial properties of f, g, h

```
lemma (in Extend) f_h_eq [simp]: "f(h(x,y)) = x"
by (simp add: f_def good_h [unfolded good_map_def, THEN conjunct2])

lemma (in Extend) h_inject1 [dest]: "h(x,y) = h(x',y') ==> x=x'"
apply (drule_tac f = f in arg_cong)
apply (simp add: f_def good_h [unfolded good_map_def, THEN conjunct2])
done

lemma (in Extend) h_f_g_equiv: "h(f z, g z) == z"
by (simp add: f_def g_def
           good_h [unfolded good_map_def, THEN conjunct1, THEN surj_f_inv_f])

lemma (in Extend) h_f_g_eq: "h(f z, g z) = z"
by (simp add: h_f_g_equiv)


lemma (in Extend) split_extended_all:
    "(!!z. PROP P z) == (!!u y. PROP P (h (u, y)))"
proof
   assume allP: "⋀z. PROP P z"
   fix u y
   show "PROP P (h (u, y))" by (rule allP)
 next
   assume allPh: "⋀u y. PROP P (h(u,y))"
   fix z
   have Phfgz: "PROP P (h (f z, g z))" by (rule allPh)
   show "PROP P z" by (rule Phfgz [unfolded h_f_g_equiv])
qed
```

10.3 `extend_set`: basic properties

```
lemma project_set_iff [iff]:
    "(x ∈ project_set h C) = (∃y. h(x,y) ∈ C)"
by (simp add: project_set_def)

lemma extend_set_mono: "A ⊆ B ==> extend_set h A ⊆ extend_set h B"
by (unfold extend_set_def, blast)

lemma (in Extend) mem_extend_set_iff [iff]: "z ∈ extend_set h A = (f z ∈
A)"
apply (unfold extend_set_def)
apply (force intro: h_f_g_eq [symmetric])
done
```

lemma (**in** *Extend*) *extend_set_strict_mono [iff]:*
    "(extend_set h A ⊆ extend_set h B) = (A ⊆ B)"
by *(unfold extend_set_def, force)*

lemma *extend_set_empty [simp]: "extend_set h {} = {}"*
by *(unfold extend_set_def, auto)*

lemma (**in** *Extend*) *extend_set_eq_Collect: "extend_set h {s. P s} = {s. P(f s)}"*
by *auto*

lemma (**in** *Extend*) *extend_set_sing: "extend_set h {x} = {s. f s = x}"*
by *auto*

lemma (**in** *Extend*) *extend_set_inverse [simp]:*
    "project_set h (extend_set h C) = C"
by *(unfold extend_set_def, auto)*

lemma (**in** *Extend*) *extend_set_project_set:*
    "C ⊆ extend_set h (project_set h C)"
apply *(unfold extend_set_def)*
apply *(auto simp add: split_extended_all, blast)*
done

lemma (**in** *Extend*) *inj_extend_set: "inj (extend_set h)"*
apply *(rule inj_on_inverseI)*
apply *(rule extend_set_inverse)*
done

lemma (**in** *Extend*) *extend_set_UNIV_eq [simp]: "extend_set h UNIV = UNIV"*
apply *(unfold extend_set_def)*
apply *(auto simp add: split_extended_all)*
done

10.4 *project_set*: basic properties

lemma (**in** *Extend*) *project_set_eq: "project_set h C = f ‘ C"*
by *(auto intro: f_h_eq [symmetric] simp add: split_extended_all)*

lemma (**in** *Extend*) *project_set_I: "!!z. z ∈ C ==> f z ∈ project_set h C"*
by *(auto simp add: split_extended_all)*

10.5 More laws

lemma (**in** *Extend*) *project_set_extend_set_Int:*
    "project_set h ((extend_set h A) ∩ B) = A ∩ (project_set h B)"
by *auto*

lemma (**in** *Extend*) *project_set_extend_set_Un:*
    "project_set h ((extend_set h A) ∪ B) = A ∪ (project_set h B)"
by *auto*

lemma *project_set_Int_subset:*
```
"project_set h (A ∩ B) ⊆ (project_set h A) ∩ (project_set h B)"
```
by *auto*

lemma (**in** *Extend*) *extend_set_Un_distrib:*
```
"extend_set h (A ∪ B) = extend_set h A ∪ extend_set h B"
```
by *auto*

lemma (**in** *Extend*) *extend_set_Int_distrib:*
```
"extend_set h (A ∩ B) = extend_set h A ∩ extend_set h B"
```
by *auto*

lemma (**in** *Extend*) *extend_set_INT_distrib:*
```
"extend_set h (INTER A B) = (⋂x ∈ A. extend_set h (B x))"
```
by *auto*

lemma (**in** *Extend*) *extend_set_Diff_distrib:*
```
"extend_set h (A - B) = extend_set h A - extend_set h B"
```
by *auto*

lemma (**in** *Extend*) *extend_set_Union:*
```
"extend_set h (Union A) = (⋃X ∈ A. extend_set h X)"
```
by *blast*

lemma (**in** *Extend*) *extend_set_subset_Compl_eq:*
```
"(extend_set h A ⊆ - extend_set h B) = (A ⊆ - B)"
```
by *(unfold extend_set_def, auto)*

10.6 `extend_act`

lemma (**in** *Extend*) *mem_extend_act_iff [iff]:*
```
"((h(s,y), h(s',y)) ∈ extend_act h act) = ((s, s') ∈ act)"
```
by *(unfold extend_act_def, auto)*

lemma (**in** *Extend*) *extend_act_D:*
```
"(z, z') ∈ extend_act h act ==> (f z, f z') ∈ act"
```
by *(unfold extend_act_def, auto)*

lemma (**in** *Extend*) *extend_act_inverse [simp]:*
```
"project_act h (extend_act h act) = act"
```
by *(unfold extend_act_def project_act_def, blast)*

lemma (**in** *Extend*) *project_act_extend_act_restrict [simp]:*
```
"project_act h (Restrict C (extend_act h act)) =
 Restrict (project_set h C) act"
```
by *(unfold extend_act_def project_act_def, blast)*

lemma (**in** *Extend*) *subset_extend_act_D:*
```
"act' ⊆ extend_act h act ==> project_act h act' ⊆ act"
```
by *(unfold extend_act_def project_act_def, force)*

lemma (**in** *Extend*) *inj_extend_act: "inj (extend_act h)"*

```
apply (rule inj_on_inverseI)
apply (rule extend_act_inverse)
done

lemma (in Extend) extend_act_Image [simp]:
     "extend_act h act `` (extend_set h A) = extend_set h (act `` A)"
by (unfold extend_set_def extend_act_def, force)

lemma (in Extend) extend_act_strict_mono [iff]:
     "(extend_act h act' ⊆ extend_act h act) = (act'<=act)"
by (unfold extend_act_def, auto)

declare (in Extend) inj_extend_act [THEN inj_eq, iff]


lemma Domain_extend_act:
    "Domain (extend_act h act) = extend_set h (Domain act)"
by (unfold extend_set_def extend_act_def, force)

lemma (in Extend) extend_act_Id [simp]:
    "extend_act h Id = Id"
apply (unfold extend_act_def)
apply (force intro: h_f_g_eq [symmetric])
done

lemma (in Extend) project_act_I:
     "!!z z'. (z, z') ∈ act ==> (f z, f z') ∈ project_act h act"
apply (unfold project_act_def)
apply (force simp add: split_extended_all)
done

lemma (in Extend) project_act_Id [simp]: "project_act h Id = Id"
by (unfold project_act_def, force)

lemma (in Extend) Domain_project_act:
  "Domain (project_act h act) = project_set h (Domain act)"
apply (unfold project_act_def)
apply (force simp add: split_extended_all)
done
```

10.7 extend

Basic properties

```
lemma Init_extend [simp]:
     "Init (extend h F) = extend_set h (Init F)"
by (unfold extend_def, auto)

lemma Init_project [simp]:
     "Init (project h C F) = project_set h (Init F)"
by (unfold project_def, auto)

lemma (in Extend) Acts_extend [simp]:
     "Acts (extend h F) = (extend_act h ` Acts F)"
```

```
by (simp add: extend_def insert_Id_image_Acts)

lemma (in Extend) AllowedActs_extend [simp]:
     "AllowedActs (extend h F) = project_act h -‘ AllowedActs F"
by (simp add: extend_def insert_absorb)

lemma Acts_project [simp]:
     "Acts(project h C F) = insert Id (project_act h ‘ Restrict C ‘ Acts F)"
by (auto simp add: project_def image_iff)

lemma (in Extend) AllowedActs_project [simp]:
     "AllowedActs(project h C F) =
        {act. Restrict (project_set h C) act
               ∈ project_act h ‘ Restrict C ‘ AllowedActs F}"
apply (simp (no_asm) add: project_def image_iff)
apply (subst insert_absorb)
apply (auto intro!: bexI [of _ Id] simp add: project_act_def)
done

lemma (in Extend) Allowed_extend:
     "Allowed (extend h F) = project h UNIV -‘ Allowed F"
apply (simp (no_asm) add: AllowedActs_extend Allowed_def)
apply blast
done

lemma (in Extend) extend_SKIP [simp]: "extend h SKIP = SKIP"
apply (unfold SKIP_def)
apply (rule program_equalityI, auto)
done

lemma project_set_UNIV [simp]: "project_set h UNIV = UNIV"
by auto

lemma project_set_Union:
     "project_set h (Union A) = (⋃X ∈ A. project_set h X)"
by blast



lemma (in Extend) project_act_Restrict_subset:
     "project_act h (Restrict C act) ⊆
      Restrict (project_set h C) (project_act h act)"
by (auto simp add: project_act_def)

lemma (in Extend) project_act_Restrict_Id_eq:
     "project_act h (Restrict C Id) = Restrict (project_set h C) Id"
by (auto simp add: project_act_def)

lemma (in Extend) project_extend_eq:
     "project h C (extend h F) =
      mk_program (Init F, Restrict (project_set h C) ‘ Acts F,
                  {act. Restrict (project_set h C) act
                         ∈ project_act h ‘ Restrict C ‘
                                   (project_act h -‘ AllowedActs F)})"
```

```
apply (rule program_equalityI)
  apply simp
 apply (simp add: image_eq_UN)
apply (simp add: project_def)
done

lemma (in Extend) extend_inverse [simp]:
     "project h UNIV (extend h F) = F"
apply (simp (no_asm_simp) add: project_extend_eq image_eq_UN
         subset_UNIV [THEN subset_trans, THEN Restrict_triv])
apply (rule program_equalityI)
apply (simp_all (no_asm))
apply (subst insert_absorb)
apply (simp (no_asm) add: bexI [of _ Id])
apply auto
apply (rename_tac "act")
apply (rule_tac x = "extend_act h act" in bexI, auto)
done

lemma (in Extend) inj_extend: "inj (extend h)"
apply (rule inj_on_inverseI)
apply (rule extend_inverse)
done

lemma (in Extend) extend_Join [simp]:
     "extend h (F⊔G) = extend h F⊔extend h G"
apply (rule program_equalityI)
apply (simp (no_asm) add: extend_set_Int_distrib)
apply (simp add: image_Un, auto)
done

lemma (in Extend) extend_JN [simp]:
     "extend h (JOIN I F) = (⨆i ∈ I. extend h (F i))"
apply (rule program_equalityI)
  apply (simp (no_asm) add: extend_set_INT_distrib)
 apply (simp add: image_UN, auto)
done



lemma (in Extend) extend_mono: "F ≤ G ==> extend h F ≤ extend h G"
by (force simp add: component_eq_subset)

lemma (in Extend) project_mono: "F ≤ G ==> project h C F ≤ project h C
G"
by (simp add: component_eq_subset, blast)

lemma (in Extend) all_total_extend: "all_total F ==> all_total (extend h
F)"
by (simp add: all_total_def Domain_extend_act)
```

10.8 Safety: co, stable

```
lemma (in Extend) extend_constrains:
```

```
     "(extend h F ∈ (extend_set h A) co (extend_set h B)) =
      (F ∈ A co B)"
by (simp add: constrains_def)

lemma (in Extend) extend_stable:
     "(extend h F ∈ stable (extend_set h A)) = (F ∈ stable A)"
by (simp add: stable_def extend_constrains)

lemma (in Extend) extend_invariant:
     "(extend h F ∈ invariant (extend_set h A)) = (F ∈ invariant A)"
by (simp add: invariant_def extend_stable)


lemma (in Extend) extend_constrains_project_set:
     "extend h F ∈ A co B ==> F ∈ (project_set h A) co (project_set h B)"
by (auto simp add: constrains_def, force)

lemma (in Extend) extend_stable_project_set:
     "extend h F ∈ stable A ==> F ∈ stable (project_set h A)"
by (simp add: stable_def extend_constrains_project_set)
```

10.9 Weak safety primitives: Co, Stable

```
lemma (in Extend) reachable_extend_f:
     "p ∈ reachable (extend h F) ==> f p ∈ reachable F"
apply (erule reachable.induct)
apply (auto intro: reachable.intros simp add: extend_act_def image_iff)
done

lemma (in Extend) h_reachable_extend:
     "h(s,y) ∈ reachable (extend h F) ==> s ∈ reachable F"
by (force dest!: reachable_extend_f)

lemma (in Extend) reachable_extend_eq:
     "reachable (extend h F) = extend_set h (reachable F)"
apply (unfold extend_set_def)
apply (rule equalityI)
apply (force intro: h_f_g_eq [symmetric] dest!: reachable_extend_f, clarify)
apply (erule reachable.induct)
apply (force intro: reachable.intros)+
done

lemma (in Extend) extend_Constrains:
     "(extend h F ∈ (extend_set h A) Co (extend_set h B)) =
      (F ∈ A Co B)"
by (simp add: Constrains_def reachable_extend_eq extend_constrains
              extend_set_Int_distrib [symmetric])

lemma (in Extend) extend_Stable:
     "(extend h F ∈ Stable (extend_set h A)) = (F ∈ Stable A)"
by (simp add: Stable_def extend_Constrains)

lemma (in Extend) extend_Always:
     "(extend h F ∈ Always (extend_set h A)) = (F ∈ Always A)"
```

by *(simp (no_asm_simp) add: Always_def extend_Stable)*

lemma *project_act_mono:*
"D ⊆ C ==>
project_act h (Restrict D act) ⊆ project_act h (Restrict C act)"
by *(auto simp add: project_act_def)*

lemma *(in Extend) project_constrains_mono:*
"[| D ⊆ C; project h C F ∈ A co B |] ==> project h D F ∈ A co B"
apply *(auto simp add: constrains_def)*
apply *(drule project_act_mono, blast)*
done

lemma *(in Extend) project_stable_mono:*
"[| D ⊆ C; project h C F ∈ stable A |] ==> project h D F ∈ stable A"
by *(simp add: stable_def project_constrains_mono)*

lemma *(in Extend) project_constrains:*
"(project h C F ∈ A co B) =
(F ∈ (C ∩ extend_set h A) co (extend_set h B) & A ⊆ B)"
apply *(unfold constrains_def)*
apply *(auto intro!: project_act_I simp add: ball_Un)*
apply *(force intro!: project_act_I dest!: subsetD)*

apply *(unfold project_act_def)*
apply *(force dest!: subsetD)*
done

lemma *(in Extend) project_stable:*
"(project h UNIV F ∈ stable A) = (F ∈ stable (extend_set h A))"
apply *(unfold stable_def)*
apply *(simp (no_asm) add: project_constrains)*
done

lemma *(in Extend) project_stable_I:*
"F ∈ stable (extend_set h A) ==> project h C F ∈ stable A"
apply *(drule project_stable [THEN iffD2])*
apply *(blast intro: project_stable_mono)*
done

lemma *(in Extend) Int_extend_set_lemma:*
"A ∩ extend_set h ((project_set h A) ∩ B) = A ∩ extend_set h B"
by *(auto simp add: split_extended_all)*

lemma *project_constrains_project_set:*
"G ∈ C co B ==> project h C G ∈ project_set h C co project_set h B"
by *(simp add: constrains_def project_def project_act_def, blast)*

lemma *project_stable_project_set:*
"G ∈ stable C ==> project h C G ∈ stable (project_set h C)"
by *(simp add: stable_def project_constrains_project_set)*

10.10 Progress: transient, ensures

lemma *(in Extend) extend_transient:*
"(extend h F ∈ transient (extend_set h A)) = (F ∈ transient A)"
by *(auto simp add: transient_def extend_set_subset_Compl_eq Domain_extend_act)*

lemma *(in Extend) extend_ensures:*
"(extend h F ∈ (extend_set h A) ensures (extend_set h B)) =
(F ∈ A ensures B)"
by *(simp add: ensures_def extend_constrains extend_transient*
extend_set_Un_distrib [symmetric] extend_set_Diff_distrib [symmetric])

lemma *(in Extend) leadsTo_imp_extend_leadsTo:*
"F ∈ A leadsTo B
==> extend h F ∈ (extend_set h A) leadsTo (extend_set h B)"
apply *(erule leadsTo_induct)*
apply *(simp add: leadsTo_Basis extend_ensures)*
apply *(blast intro: leadsTo_Trans)*
apply *(simp add: leadsTo_UN extend_set_Union)*
done

10.11 Proving the converse takes some doing!

lemma *(in Extend) slice_iff [iff]: "(x ∈ slice C y) = (h(x,y) ∈ C)"*
by *(simp (no_asm) add: slice_def)*

lemma *(in Extend) slice_Union: "slice (Union S) y = (⋃x ∈ S. slice x y)"*
by *auto*

lemma *(in Extend) slice_extend_set: "slice (extend_set h A) y = A"*
by *auto*

lemma *(in Extend) project_set_is_UN_slice:*
"project_set h A = (⋃y. slice A y)"
by *auto*

lemma *(in Extend) extend_transient_slice:*
"extend h F ∈ transient A ==> F ∈ transient (slice A y)"
by *(unfold transient_def, auto)*

lemma *(in Extend) extend_constrains_slice:*
"extend h F ∈ A co B ==> F ∈ (slice A y) co (slice B y)"
by *(auto simp add: constrains_def)*

lemma *(in Extend) extend_ensures_slice:*
"extend h F ∈ A ensures B ==> F ∈ (slice A y) ensures (project_set h
B)"
apply *(auto simp add: ensures_def extend_constrains extend_transient)*

```
apply (erule_tac [2] extend_transient_slice [THEN transient_strengthen])
apply (erule extend_constrains_slice [THEN constrains_weaken], auto)
done

lemma (in Extend) leadsTo_slice_project_set:
     "∀y. F ∈ (slice B y) leadsTo CU ==> F ∈ (project_set h B) leadsTo CU"
apply (simp (no_asm) add: project_set_is_UN_slice)
apply (blast intro: leadsTo_UN)
done

lemma (in Extend) extend_leadsTo_slice [rule_format]:
     "extend h F ∈ AU leadsTo BU
      ==> ∀y. F ∈ (slice AU y) leadsTo (project_set h BU)"
apply (erule leadsTo_induct)
  apply (blast intro: extend_ensures_slice leadsTo_Basis)
 apply (blast intro: leadsTo_slice_project_set leadsTo_Trans)
apply (simp add: leadsTo_UN slice_Union)
done

lemma (in Extend) extend_leadsTo:
     "(extend h F ∈ (extend_set h A) leadsTo (extend_set h B)) =
      (F ∈ A leadsTo B)"
apply safe
apply (erule_tac [2] leadsTo_imp_extend_leadsTo)
apply (drule extend_leadsTo_slice)
apply (simp add: slice_extend_set)
done

lemma (in Extend) extend_LeadsTo:
     "(extend h F ∈ (extend_set h A) LeadsTo (extend_set h B)) =
      (F ∈ A LeadsTo B)"
by (simp add: LeadsTo_def reachable_extend_eq extend_leadsTo
              extend_set_Int_distrib [symmetric])
```

10.12 preserves

```
lemma (in Extend) project_preserves_I:
     "G ∈ preserves (v o f) ==> project h C G ∈ preserves v"
by (auto simp add: preserves_def project_stable_I extend_set_eq_Collect)


lemma (in Extend) project_preserves_id_I:
     "G ∈ preserves f ==> project h C G ∈ preserves id"
by (simp add: project_preserves_I)

lemma (in Extend) extend_preserves:
     "(extend h G ∈ preserves (v o f)) = (G ∈ preserves v)"
by (auto simp add: preserves_def extend_stable [symmetric]
                   extend_set_eq_Collect)

lemma (in Extend) inj_extend_preserves: "inj h ==> (extend h G ∈ preserves
g)"
by (auto simp add: preserves_def extend_def extend_act_def stable_def
                   constrains_def g_def)
```

10.13 Guarantees

```
lemma (in Extend) project_extend_Join:
     "project h UNIV ((extend h F)⊔G) = F⊔(project h UNIV G)"
apply (rule program_equalityI)
  apply (simp add: project_set_extend_set_Int)
 apply (simp add: image_eq_UN UN_Un, auto)
done

lemma (in Extend) extend_Join_eq_extend_D:
     "(extend h F)⊔G = extend h H ==> H = F⊔(project h UNIV G)"
apply (drule_tac f = "project h UNIV" in arg_cong)
apply (simp add: project_extend_Join)
done




lemma (in Extend) ok_extend_imp_ok_project:
     "extend h F ok G ==> F ok project h UNIV G"
apply (auto simp add: ok_def)
apply (drule subsetD)
apply (auto intro!: rev_image_eqI)
done

lemma (in Extend) ok_extend_iff: "(extend h F ok extend h G) = (F ok G)"
apply (simp add: ok_def, safe)
apply (force+)
done

lemma (in Extend) OK_extend_iff: "OK I (%i. extend h (F i)) = (OK I F)"
apply (unfold OK_def, safe)
apply (drule_tac x = i in bspec)
apply (drule_tac [2] x = j in bspec)
apply (force+)
done

lemma (in Extend) guarantees_imp_extend_guarantees:
     "F ∈ X guarantees Y ==>
      extend h F ∈ (extend h ` X) guarantees (extend h ` Y)"
apply (rule guaranteesI, clarify)
apply (blast dest: ok_extend_imp_ok_project extend_Join_eq_extend_D
                   guaranteesD)
done

lemma (in Extend) extend_guarantees_imp_guarantees:
     "extend h F ∈ (extend h ` X) guarantees (extend h ` Y)
      ==> F ∈ X guarantees Y"
apply (auto simp add: guar_def)
apply (drule_tac x = "extend h G" in spec)
apply (simp del: extend_Join
            add: extend_Join [symmetric] ok_extend_iff
                 inj_extend [THEN inj_image_mem_iff])
done
```

```
lemma (in Extend) extend_guarantees_eq:
     "(extend h F ∈ (extend h ‘ X) guarantees (extend h ‘ Y)) =
      (F ∈ X guarantees Y)"
by (blast intro: guarantees_imp_extend_guarantees
                 extend_guarantees_imp_guarantees)

end
```

11 Renaming of State Sets

```
theory Rename imports Extend begin

constdefs

  rename :: "['a => 'b, 'a program] => 'b program"
    "rename h == extend (%(x,u::unit). h x)"

declare image_inv_f_f [simp] image_surj_f_inv_f [simp]

declare Extend.intro [simp,intro]

lemma good_map_bij [simp,intro]: "bij h ==> good_map (%(x,u). h x)"
apply (rule good_mapI)
apply (unfold bij_def inj_on_def surj_def, auto)
done

lemma fst_o_inv_eq_inv: "bij h ==> fst (inv (%(x,u). h x) s) = inv h s"
apply (unfold bij_def split_def, clarify)
apply (subgoal_tac "surj (%p. h (fst p))")
 prefer 2 apply (simp add: surj_def)
apply (erule injD)
apply (simp (no_asm_simp) add: surj_f_inv_f)
apply (erule surj_f_inv_f)
done

lemma mem_rename_set_iff: "bij h ==> z ∈ h‘A = (inv h z ∈ A)"
by (force simp add: bij_is_inj bij_is_surj [THEN surj_f_inv_f])


lemma extend_set_eq_image [simp]: "extend_set (%(x,u). h x) A = h‘A"
by (force simp add: extend_set_def)

lemma Init_rename [simp]: "Init (rename h F) = h‘(Init F)"
by (simp add: rename_def)
```

11.1 inverse properties

```
lemma extend_set_inv:
     "bij h
      ==> extend_set (%(x,u::'c). inv h x) = project_set (%(x,u::'c). h x)"
apply (unfold bij_def)
```

```
apply (rule ext)
apply (force simp add: extend_set_def project_set_def surj_f_inv_f)
done



lemma bij_extend_act_eq_project_act: "bij h
      ==> extend_act (%(x,u::'c). h x) = project_act (%(x,u::'c). inv h x)"
apply (rule ext)
apply (force simp add: extend_act_def project_act_def bij_def surj_f_inv_f)
done

lemma bij_extend_act: "bij h ==> bij (extend_act (%(x,u::'c). h x))"
apply (rule bijI)
apply (rule Extend.inj_extend_act)
apply (auto simp add: bij_extend_act_eq_project_act)
apply (rule surjI)
apply (rule Extend.extend_act_inverse)
apply (blast intro: bij_imp_bij_inv good_map_bij)
done

lemma bij_project_act: "bij h ==> bij (project_act (%(x,u::'c). h x))"
apply (frule bij_imp_bij_inv [THEN bij_extend_act])
apply (simp add: bij_extend_act_eq_project_act bij_imp_bij_inv inv_inv_eq)
done

lemma bij_inv_project_act_eq: "bij h ==> inv (project_act (%(x,u::'c). inv
h x)) =
                project_act (%(x,u::'c). h x)"
apply (simp (no_asm_simp) add: bij_extend_act_eq_project_act [symmetric])
apply (rule surj_imp_inv_eq)
apply (blast intro: bij_extend_act bij_is_surj)
apply (simp (no_asm_simp) add: Extend.extend_act_inverse)
done

lemma extend_inv: "bij h
      ==> extend (%(x,u::'c). inv h x) = project (%(x,u::'c). h x) UNIV"
apply (frule bij_imp_bij_inv)
apply (rule ext)
apply (rule program_equalityI)
  apply (simp (no_asm_simp) add: extend_set_inv)
 apply (simp add: Extend.project_act_Id Extend.Acts_extend
          insert_Id_image_Acts bij_extend_act_eq_project_act inv_inv_eq)
apply (simp add: Extend.AllowedActs_extend Extend.AllowedActs_project
            bij_project_act bij_vimage_eq_inv_image bij_inv_project_act_eq)
done

lemma rename_inv_rename [simp]: "bij h ==> rename (inv h) (rename h F) =
F"
by (simp add: rename_def extend_inv Extend.extend_inverse)

lemma rename_rename_inv [simp]: "bij h ==> rename h (rename (inv h) F) =
F"
apply (frule bij_imp_bij_inv)
```

```
apply (erule inv_inv_eq [THEN subst], erule rename_inv_rename)
done

lemma rename_inv_eq: "bij h ==> rename (inv h) = inv (rename h)"
by (rule inv_equality [symmetric], auto)



lemma bij_extend: "bij h ==> bij (extend (%(x,u::'c). h x))"
apply (rule bijI)
apply (blast intro: Extend.inj_extend)
apply (rule_tac f = "extend (% (x,u) . inv h x)" in surjI)
apply (subst (1 2) inv_inv_eq [of h, symmetric], assumption+)
apply (simp add: bij_imp_bij_inv extend_inv [of "inv h"])
apply (simp add: inv_inv_eq)
apply (rule Extend.extend_inverse)
apply (simp add: bij_imp_bij_inv)
done

lemma bij_project: "bij h ==> bij (project (%(x,u::'c). h x) UNIV)"
apply (subst extend_inv [symmetric])
apply (auto simp add: bij_imp_bij_inv bij_extend)
done

lemma inv_project_eq:
     "bij h
      ==> inv (project (%(x,u::'c). h x) UNIV) = extend (%(x,u::'c). h x)"
apply (rule inj_imp_inv_eq)
apply (erule bij_project [THEN bij_is_inj])
apply (simp (no_asm_simp) add: Extend.extend_inverse)
done

lemma Allowed_rename [simp]:
     "bij h ==> Allowed (rename h F) = rename h ` Allowed F"
apply (simp (no_asm_simp) add: rename_def Extend.Allowed_extend)
apply (subst bij_vimage_eq_inv_image)
apply (rule bij_project, blast)
apply (simp (no_asm_simp) add: inv_project_eq)
done

lemma bij_rename: "bij h ==> bij (rename h)"
apply (simp (no_asm_simp) add: rename_def bij_extend)
done
lemmas surj_rename = bij_rename [THEN bij_is_surj, standard]

lemma inj_rename_imp_inj: "inj (rename h) ==> inj h"
apply (unfold inj_on_def, auto)
apply (drule_tac x = "mk_program ({x}, {}, {})" in spec)
apply (drule_tac x = "mk_program ({y}, {}, {})" in spec)
apply (auto simp add: program_equality_iff rename_def extend_def)
done

lemma surj_rename_imp_surj: "surj (rename h) ==> surj h"
apply (unfold surj_def, auto)
```

apply *(drule_tac x = "mk_program ({y}, {}, {})" in spec)*
apply *(auto simp add: program_equality_iff rename_def extend_def)*
done

lemma *bij_rename_imp_bij: "bij (rename h) ==> bij h"*
apply *(unfold bij_def)*
apply *(simp (no_asm_simp) add: inj_rename_imp_inj surj_rename_imp_surj)*
done

lemma *bij_rename_iff [simp]: "bij (rename h) = bij h"*
by *(blast intro: bij_rename bij_rename_imp_bij)*

11.2 the lattice operations

lemma *rename_SKIP [simp]: "bij h ==> rename h SKIP = SKIP"*
by *(simp add: rename_def Extend.extend_SKIP)*

lemma *rename_Join [simp]:*
"bij h ==> rename h (F Join G) = rename h F Join rename h G"
by *(simp add: rename_def Extend.extend_Join)*

lemma *rename_JN [simp]:*
"bij h ==> rename h (JOIN I F) = (⨆i ∈ I. rename h (F i))"
by *(simp add: rename_def Extend.extend_JN)*

11.3 Strong Safety: co, stable

lemma *rename_constrains:*
"bij h ==> (rename h F ∈ (h‘A) co (h‘B)) = (F ∈ A co B)"
apply *(unfold rename_def)*
apply *(subst extend_set_eq_image [symmetric])+*
apply *(erule good_map_bij [THEN Extend.intro, THEN Extend.extend_constrains])*
done

lemma *rename_stable:*
"bij h ==> (rename h F ∈ stable (h‘A)) = (F ∈ stable A)"
apply *(simp add: stable_def rename_constrains)*
done

lemma *rename_invariant:*
"bij h ==> (rename h F ∈ invariant (h‘A)) = (F ∈ invariant A)"
apply *(simp add: invariant_def rename_stable bij_is_inj [THEN inj_image_subset_iff])*
done

lemma *rename_increasing:*
"bij h ==> (rename h F ∈ increasing func) = (F ∈ increasing (func o h))"
apply *(simp add: increasing_def rename_stable [symmetric] bij_image_Collect_eq bij_is_surj [THEN surj_f_inv_f])*
done

11.4 Weak Safety: Co, Stable

lemma *reachable_rename_eq:*

```
    "bij h ==> reachable (rename h F) = h ‘ (reachable F)"
apply (simp add: rename_def Extend.reachable_extend_eq)
done

lemma rename_Constrains:
    "bij h ==> (rename h F ∈ (h‘A) Co (h‘B)) = (F ∈ A Co B)"
by (simp add: Constrains_def reachable_rename_eq rename_constrains
              bij_is_inj image_Int [symmetric])

lemma rename_Stable:
    "bij h ==> (rename h F ∈ Stable (h‘A)) = (F ∈ Stable A)"
by (simp add: Stable_def rename_Constrains)

lemma rename_Always: "bij h ==> (rename h F ∈ Always (h‘A)) = (F ∈ Always
A)"
by (simp add: Always_def rename_Stable bij_is_inj [THEN inj_image_subset_iff])

lemma rename_Increasing:
    "bij h ==> (rename h F ∈ Increasing func) = (F ∈ Increasing (func o
h))"
by (simp add: Increasing_def rename_Stable [symmetric] bij_image_Collect_eq

             bij_is_surj [THEN surj_f_inv_f])
```

11.5 Progress: transient, ensures

```
lemma rename_transient:
    "bij h ==> (rename h F ∈ transient (h‘A)) = (F ∈ transient A)"
apply (unfold rename_def)
apply (subst extend_set_eq_image [symmetric])
apply (erule good_map_bij [THEN Extend.intro, THEN Extend.extend_transient])
done

lemma rename_ensures:
    "bij h ==> (rename h F ∈ (h‘A) ensures (h‘B)) = (F ∈ A ensures B)"
apply (unfold rename_def)
apply (subst extend_set_eq_image [symmetric])+
apply (erule good_map_bij [THEN Extend.intro, THEN Extend.extend_ensures])
done

lemma rename_leadsTo:
    "bij h ==> (rename h F ∈ (h‘A) leadsTo (h‘B)) = (F ∈ A leadsTo B)"
apply (unfold rename_def)
apply (subst extend_set_eq_image [symmetric])+
apply (erule good_map_bij [THEN Extend.intro, THEN Extend.extend_leadsTo])
done

lemma rename_LeadsTo:
    "bij h ==> (rename h F ∈ (h‘A) LeadsTo (h‘B)) = (F ∈ A LeadsTo B)"
apply (unfold rename_def)
apply (subst extend_set_eq_image [symmetric])+
apply (erule good_map_bij [THEN Extend.intro, THEN Extend.extend_LeadsTo])
done
```

```
lemma rename_rename_guarantees_eq:
     "bij h ==> (rename h F ∈ (rename h ‘ X) guarantees
                              (rename h ‘ Y)) =
                (F ∈ X guarantees Y)"
apply (unfold rename_def)
apply (subst good_map_bij [THEN Extend.intro, THEN Extend.extend_guarantees_eq
[symmetric]], assumption)
apply (simp (no_asm_simp) add: fst_o_inv_eq_inv o_def)
done

lemma rename_guarantees_eq_rename_inv:
     "bij h ==> (rename h F ∈ X guarantees Y) =
                (F ∈ (rename (inv h) ‘ X) guarantees
                     (rename (inv h) ‘ Y))"
apply (subst rename_rename_guarantees_eq [symmetric], assumption)
apply (simp add: image_eq_UN o_def bij_is_surj [THEN surj_f_inv_f])
done

lemma rename_preserves:
     "bij h ==> (rename h G ∈ preserves v) = (G ∈ preserves (v o h))"
apply (subst good_map_bij [THEN Extend.intro, THEN Extend.extend_preserves
[symmetric]], assumption)
apply (simp add: o_def fst_o_inv_eq_inv rename_def bij_is_surj [THEN surj_f_inv_f])
done

lemma ok_rename_iff [simp]: "bij h ==> (rename h F ok rename h G) = (F ok
G)"
by (simp add: Extend.ok_extend_iff rename_def)

lemma OK_rename_iff [simp]: "bij h ==> OK I (%i. rename h (F i)) = (OK I
F)"
by (simp add: Extend.OK_extend_iff rename_def)
```

11.6 "image" versions of the rules, for lifting "guarantees" properties

```
lemmas bij_eq_rename = surj_rename [THEN surj_f_inv_f, symmetric]

lemma rename_image_constrains:
     "bij h ==> rename h ‘ (A co B) = (h ‘ A) co (h‘B)"
apply auto
 defer 1
 apply (rename_tac F)
 apply (subgoal_tac "∃G. F = rename h G")
 apply (auto intro!: bij_eq_rename simp add: rename_constrains)
done

lemma rename_image_stable: "bij h ==> rename h ‘ stable A = stable (h ‘ A)"
apply auto
 defer 1
 apply (rename_tac F)
 apply (subgoal_tac "∃G. F = rename h G")
 apply (auto intro!: bij_eq_rename simp add: rename_stable)
done
```

```
lemma rename_image_increasing:
     "bij h ==> rename h ` increasing func = increasing (func o inv h)"
apply auto
 defer 1
 apply (rename_tac F)
 apply (subgoal_tac "∃G. F = rename h G")
 apply (auto intro!: bij_eq_rename simp add: rename_increasing o_def bij_is_inj)

done

lemma rename_image_invariant:
     "bij h ==> rename h ` invariant A = invariant (h ` A)"
apply auto
 defer 1
 apply (rename_tac F)
 apply (subgoal_tac "∃G. F = rename h G")
 apply (auto intro!: bij_eq_rename simp add: rename_invariant)
done

lemma rename_image_Constrains:
     "bij h ==> rename h ` (A Co B) = (h ` A) Co (h`B)"
apply auto
 defer 1
 apply (rename_tac F)
 apply (subgoal_tac "∃G. F = rename h G")
 apply (auto intro!: bij_eq_rename simp add: rename_Constrains)
done

lemma rename_image_preserves:
     "bij h ==> rename h ` preserves v = preserves (v o inv h)"
by (simp add: o_def rename_image_stable preserves_def bij_image_INT
              bij_image_Collect_eq)

lemma rename_image_Stable:
     "bij h ==> rename h ` Stable A = Stable (h ` A)"
apply auto
 defer 1
 apply (rename_tac F)
 apply (subgoal_tac "∃G. F = rename h G")
 apply (auto intro!: bij_eq_rename simp add: rename_Stable)
done

lemma rename_image_Increasing:
     "bij h ==> rename h ` Increasing func = Increasing (func o inv h)"
apply auto
 defer 1
 apply (rename_tac F)
 apply (subgoal_tac "∃G. F = rename h G")
 apply (auto intro!: bij_eq_rename simp add: rename_Increasing o_def bij_is_inj)
done

lemma rename_image_Always: "bij h ==> rename h ` Always A = Always (h ` A)"
apply auto
```

```
 defer 1
 apply (rename_tac F)
 apply (subgoal_tac "∃G. F = rename h G")
 apply (auto intro!: bij_eq_rename simp add: rename_Always)
done

lemma rename_image_leadsTo:
     "bij h ==> rename h ‘ (A leadsTo B) = (h ‘ A) leadsTo (h‘B)"
apply auto
 defer 1
 apply (rename_tac F)
 apply (subgoal_tac "∃G. F = rename h G")
 apply (auto intro!: bij_eq_rename simp add: rename_leadsTo)
done

lemma rename_image_LeadsTo:
     "bij h ==> rename h ‘ (A LeadsTo B) = (h ‘ A) LeadsTo (h‘B)"
apply auto
 defer 1
 apply (rename_tac F)
 apply (subgoal_tac "∃G. F = rename h G")
 apply (auto intro!: bij_eq_rename simp add: rename_LeadsTo)
done

end
```

12 Replication of Components

```
theory Lift_prog imports Rename begin

constdefs

  insert_map :: "[nat, 'b, nat=>'b] => (nat=>'b)"
    "insert_map i z f k == if k<i then f k
                           else if k=i then z
                           else f(k - 1)"

  delete_map :: "[nat, nat=>'b] => (nat=>'b)"
    "delete_map i g k == if k<i then g k else g (Suc k)"

  lift_map :: "[nat, 'b * ((nat=>'b) * 'c)] => (nat=>'b) * 'c"
    "lift_map i == %(s,(f,uu)). (insert_map i s f, uu)"

  drop_map :: "[nat, (nat=>'b) * 'c] => 'b * ((nat=>'b) * 'c)"
    "drop_map i == %(g, uu). (g i, (delete_map i g, uu))"

  lift_set :: "[nat, ('b * ((nat=>'b) * 'c)) set] => ((nat=>'b) * 'c) set"
    "lift_set i A == lift_map i ‘ A"

  lift :: "[nat, ('b * ((nat=>'b) * 'c)) program] => ((nat=>'b) * 'c) program"
    "lift i == rename (lift_map i)"
```

```
  sub :: "['a, 'a=>'b] => 'b"
    "sub == %i f. f i"


declare insert_map_def [simp] delete_map_def [simp]

lemma insert_map_inverse: "delete_map i (insert_map i x f) = f"
by (rule ext, simp)

lemma insert_map_delete_map_eq: "(insert_map i x (delete_map i g)) = g(i:=x)"
apply (rule ext)
apply (auto split add: nat_diff_split)
done
```

12.1 Injectiveness proof

```
lemma insert_map_inject1: "(insert_map i x f) = (insert_map i y g) ==> x=y"
by (drule_tac x = i in fun_cong, simp)

lemma insert_map_inject2: "(insert_map i x f) = (insert_map i y g) ==> f=g"
apply (drule_tac f = "delete_map i" in arg_cong)
apply (simp add: insert_map_inverse)
done

lemma insert_map_inject':
     "(insert_map i x f) = (insert_map i y g) ==> x=y & f=g"
by (blast dest: insert_map_inject1 insert_map_inject2)

lemmas insert_map_inject = insert_map_inject' [THEN conjE, elim!]


lemma lift_map_eq_iff [iff]:
     "(lift_map i (s,(f,uu)) = lift_map i' (s',(f',uu')))
      = (uu = uu' & insert_map i s f = insert_map i' s' f')"
by (unfold lift_map_def, auto)


lemma drop_map_lift_map_eq [simp]: "!!s. drop_map i (lift_map i s) = s"
apply (unfold lift_map_def drop_map_def)
apply (force intro: insert_map_inverse)
done

lemma inj_lift_map: "inj (lift_map i)"
apply (unfold lift_map_def)
apply (rule inj_onI, auto)
done
```

12.2 Surjectiveness proof

```
lemma lift_map_drop_map_eq [simp]: "!!s. lift_map i (drop_map i s) = s"
apply (unfold lift_map_def drop_map_def)
apply (force simp add: insert_map_delete_map_eq)
done
```

lemma *drop_map_inject [dest!]: "(drop_map i s) = (drop_map i s') ==> s=s'"*
by *(drule_tac f = "lift_map i"* **in** *arg_cong, simp)*

lemma *surj_lift_map: "surj (lift_map i)"*
apply *(rule surjI)*
apply *(rule lift_map_drop_map_eq)*
done

lemma *bij_lift_map [iff]: "bij (lift_map i)"*
by *(simp add: bij_def inj_lift_map surj_lift_map)*

lemma *inv_lift_map_eq [simp]: "inv (lift_map i) = drop_map i"*
by *(rule inv_equality, auto)*

lemma *inv_drop_map_eq [simp]: "inv (drop_map i) = lift_map i"*
by *(rule inv_equality, auto)*

lemma *bij_drop_map [iff]: "bij (drop_map i)"*
by *(simp del: inv_lift_map_eq add: inv_lift_map_eq [symmetric] bij_imp_bij_inv)*

lemma *sub_apply [simp]: "sub i f = f i"*
by *(simp add: sub_def)*

lemma *all_total_lift: "all_total F ==> all_total (lift i F)"*
by *(simp add: lift_def rename_def Extend.all_total_extend)*

lemma *insert_map_upd_same: "(insert_map i t f)(i := s) = insert_map i s f"*
by *(rule ext, auto)*

lemma *insert_map_upd:*
"(insert_map j t f)(i := s) =
(if i=j then insert_map i s f
else if i<j then insert_map j t (f(i:=s))
else insert_map j t (f(i - Suc 0 := s)))"
apply *(rule ext)*
apply *(simp split add: nat_diff_split)*

This simplification is VERY slow

done

lemma *insert_map_eq_diff:*
"[| insert_map i s f = insert_map j t g; i≠j |]
==> ∃g'. insert_map i s' f = insert_map j t g'"
apply *(subst insert_map_upd_same [symmetric])*
apply *(erule ssubst)*
apply *(simp only: insert_map_upd if_False split: split_if, blast)*
done

lemma *lift_map_eq_diff:*
"[| lift_map i (s,(f,uu)) = lift_map j (t,(g,vv)); i≠j |]
==> ∃g'. lift_map i (s',(f,uu)) = lift_map j (t,(g',vv))"
apply *(unfold lift_map_def, auto)*

apply *(blast dest: insert_map_eq_diff)*
done

12.3 The Operator *lift_set*

lemma *lift_set_empty [simp]: "lift_set i {} = {}"*
by *(unfold lift_set_def, auto)*

lemma *lift_set_iff: "(lift_map i x ∈ lift_set i A) = (x ∈ A)"*
apply *(unfold lift_set_def)*
apply *(rule inj_lift_map [THEN inj_image_mem_iff])*
done

lemma *lift_set_iff2 [iff]:*
"((f,uu) ∈ lift_set i A) = ((f i, (delete_map i f, uu)) ∈ A)"
by *(simp add: lift_set_def mem_rename_set_iff drop_map_def)*

lemma *lift_set_mono: "A ⊆ B ==> lift_set i A ⊆ lift_set i B"*
apply *(unfold lift_set_def)*
apply *(erule image_mono)*
done

lemma *lift_set_Un_distrib: "lift_set i (A ∪ B) = lift_set i A ∪ lift_set i B"*
by *(simp add: lift_set_def image_Un)*

lemma *lift_set_Diff_distrib: "lift_set i (A-B) = lift_set i A - lift_set i B"*
apply *(unfold lift_set_def)*
apply *(rule inj_lift_map [THEN image_set_diff])*
done

12.4 The Lattice Operations

lemma *bij_lift [iff]: "bij (lift i)"*
by *(simp add: lift_def)*

lemma *lift_SKIP [simp]: "lift i SKIP = SKIP"*
by *(simp add: lift_def)*

lemma *lift_Join [simp]: "lift i (F Join G) = lift i F Join lift i G"*
by *(simp add: lift_def)*

lemma *lift_JN [simp]: "lift j (JOIN I F) = (⨆i ∈ I. lift j (F i))"*
by *(simp add: lift_def)*

12.5 Safety: constrains, stable, invariant

lemma *lift_constrains:*
"(lift i F ∈ (lift_set i A) co (lift_set i B)) = (F ∈ A co B)"
by *(simp add: lift_def lift_set_def rename_constrains)*

lemma *lift_stable:*
"(lift i F ∈ stable (lift_set i A)) = (F ∈ stable A)"
by *(simp add: lift_def lift_set_def rename_stable)*

lemma *lift_invariant:*
"(lift i F ∈ invariant (lift_set i A)) = (F ∈ invariant A)"
by *(simp add: lift_def lift_set_def rename_invariant)*

lemma *lift_Constrains:*
"(lift i F ∈ (lift_set i A) Co (lift_set i B)) = (F ∈ A Co B)"
by *(simp add: lift_def lift_set_def rename_Constrains)*

lemma *lift_Stable:*
"(lift i F ∈ Stable (lift_set i A)) = (F ∈ Stable A)"
by *(simp add: lift_def lift_set_def rename_Stable)*

lemma *lift_Always:*
"(lift i F ∈ Always (lift_set i A)) = (F ∈ Always A)"
by *(simp add: lift_def lift_set_def rename_Always)*

12.6 Progress: transient, ensures

lemma *lift_transient:*
"(lift i F ∈ transient (lift_set i A)) = (F ∈ transient A)"
by *(simp add: lift_def lift_set_def rename_transient)*

lemma *lift_ensures:*
"(lift i F ∈ (lift_set i A) ensures (lift_set i B)) =
(F ∈ A ensures B)"
by *(simp add: lift_def lift_set_def rename_ensures)*

lemma *lift_leadsTo:*
"(lift i F ∈ (lift_set i A) leadsTo (lift_set i B)) =
(F ∈ A leadsTo B)"
by *(simp add: lift_def lift_set_def rename_leadsTo)*

lemma *lift_LeadsTo:*
"(lift i F ∈ (lift_set i A) LeadsTo (lift_set i B)) =
(F ∈ A LeadsTo B)"
by *(simp add: lift_def lift_set_def rename_LeadsTo)*

lemma *lift_lift_guarantees_eq:*
"(lift i F ∈ (lift i ‘ X) guarantees (lift i ‘ Y)) =
(F ∈ X guarantees Y)"
apply *(unfold lift_def)*
apply *(subst bij_lift_map [THEN rename_rename_guarantees_eq, symmetric])*
apply *(simp add: o_def)*
done

lemma *lift_guarantees_eq_lift_inv:*
"(lift i F ∈ X guarantees Y) =

```
     (F ∈ (rename (drop_map i) ‘ X) guarantees (rename (drop_map i) ‘ Y))"
by (simp add: bij_lift_map [THEN rename_guarantees_eq_rename_inv] lift_def)


lemma lift_preserves_snd_I: "F ∈ preserves snd ==> lift i F ∈ preserves
snd"
apply (drule_tac w1=snd in subset_preserves_o [THEN subsetD])
apply (simp add: lift_def rename_preserves)
apply (simp add: lift_map_def o_def split_def del: split_comp_eq)
done

lemma delete_map_eqE':
    "(delete_map i g) = (delete_map i g') ==> ∃x. g = g'(i:=x)"
apply (drule_tac f = "insert_map i (g i) " in arg_cong)
apply (simp add: insert_map_delete_map_eq)
apply (erule exI)
done

lemmas delete_map_eqE = delete_map_eqE' [THEN exE, elim!]

lemma delete_map_neq_apply:
    "[| delete_map j g = delete_map j g';  i≠j |] ==> g i = g' i"
by force


lemma vimage_o_fst_eq [simp]: "(f o fst) -‘ A = (f-‘A) <*> UNIV"
by auto

lemma vimage_sub_eq_lift_set [simp]:
    "(sub i -‘A) <*> UNIV = lift_set i (A <*> UNIV)"
by auto

lemma mem_lift_act_iff [iff]:
    "((s,s') ∈ extend_act (%(x,u::unit). lift_map i x) act) =
     ((drop_map i s, drop_map i s') ∈ act)"
apply (unfold extend_act_def, auto)
apply (rule bexI, auto)
done

lemma preserves_snd_lift_stable:
    "[| F ∈ preserves snd;  i≠j |]
     ==> lift j F ∈ stable (lift_set i (A <*> UNIV))"
apply (auto simp add: lift_def lift_set_def stable_def constrains_def
                      rename_def extend_def mem_rename_set_iff)
apply (auto dest!: preserves_imp_eq simp add: lift_map_def drop_map_def)
apply (drule_tac x = i in fun_cong, auto)
done


lemma constrains_imp_lift_constrains:
   "[| F i ∈ (A <*> UNIV) co (B <*> UNIV);
       F j ∈ preserves snd |]
```

```
    ==> lift j (F j) ∈ (lift_set i (A <*> UNIV)) co (lift_set i (B <*> UNIV))"
apply (case_tac "i=j")
apply (simp add: lift_def lift_set_def rename_constrains)
apply (erule preserves_snd_lift_stable[THEN stableD, THEN constrains_weaken_R],
       assumption)
apply (erule constrains_imp_subset [THEN lift_set_mono])
done


lemma lift_map_image_Times:
     "lift_map i ‘ (A <*> UNIV) =
      (⋃s ∈ A. ⋃f. {insert_map i s f}) <*> UNIV"
apply (auto intro!: bexI image_eqI simp add: lift_map_def)
apply (rule split_conv [symmetric])
done

lemma lift_preserves_eq:
     "(lift i F ∈ preserves v) = (F ∈ preserves (v o lift_map i))"
by (simp add: lift_def rename_preserves)


lemma lift_preserves_sub:
     "F ∈ preserves snd
      ==> lift i F ∈ preserves (v o sub j o fst) =
          (if i=j then F ∈ preserves (v o fst) else True)"
apply (drule subset_preserves_o [THEN subsetD])
apply (simp add: lift_preserves_eq o_def drop_map_lift_map_eq)
apply (auto cong del: if_weak_cong
       simp add: lift_map_def eq_commute split_def o_def simp del:split_comp_eq)
done
```

12.7 Lemmas to Handle Function Composition (o) More Consistently

```
lemma o_equiv_assoc: "f o g = h ==> f' o f o g = f' o h"
by (simp add: expand_fun_eq o_def)

lemma o_equiv_apply: "f o g = h ==> ∀x. f(g x) = h x"
by (simp add: expand_fun_eq o_def)

lemma fst_o_lift_map: "sub i o fst o lift_map i = fst"
apply (rule ext)
apply (auto simp add: o_def lift_map_def sub_def)
done

lemma snd_o_lift_map: "snd o lift_map i = snd o snd"
apply (rule ext)
apply (auto simp add: o_def lift_map_def)
done
```

12.8 More lemmas about extend and project

They could be moved to theory Extend or Project

```
lemma extend_act_extend_act:
     "extend_act h' (extend_act h act) =
      extend_act (%(x,(y,y')). h'(h(x,y),y')) act"
apply (auto elim!: rev_bexI simp add: extend_act_def, blast)
done

lemma project_act_project_act:
     "project_act h (project_act h' act) =
      project_act (%(x,(y,y')). h'(h(x,y),y')) act"
by (auto elim!: rev_bexI simp add: project_act_def)

lemma project_act_extend_act:
     "project_act h (extend_act h' act) =
        {(x,x'). ∃s s' y y' z. (s,s') ∈ act &
                 h(x,y) = h'(s,z) & h(x',y') = h'(s',z)}"
by (simp add: extend_act_def project_act_def, blast)
```

12.9 OK and "lift"

```
lemma act_in_UNION_preserves_fst:
     "act ⊆ {(x,x'). fst x = fst x'} ==> act ∈ UNION (preserves fst) Acts"
apply (rule_tac a = "mk_program (UNIV,{act},UNIV) " in UN_I)
apply (auto simp add: preserves_def stable_def constrains_def)
done

lemma UNION_OK_lift_I:
     "[| ∀i ∈ I. F i ∈ preserves snd;
         ∀i ∈ I. UNION (preserves fst) Acts ⊆ AllowedActs (F i) |]
      ==> OK I (%i. lift i (F i))"
apply (auto simp add: OK_def lift_def rename_def Extend.Acts_extend)
apply (simp add: Extend.AllowedActs_extend project_act_extend_act)
apply (rename_tac "act")
apply (subgoal_tac
       "{(x, x'). ∃s f u s' f' u'.
                   ((s, f, u), s', f', u') ∈ act &
                   lift_map j x = lift_map i (s, f, u) &
                   lift_map j x' = lift_map i (s', f', u') }
                ⊆ { (x,x') . fst x = fst x'}")
apply (blast intro: act_in_UNION_preserves_fst, clarify)
apply (drule_tac x = j in fun_cong)+
apply (drule_tac x = i in bspec, assumption)
apply (frule preserves_imp_eq, auto)
done

lemma OK_lift_I:
     "[| ∀i ∈ I. F i ∈ preserves snd;
         ∀i ∈ I. preserves fst ⊆ Allowed (F i) |]
      ==> OK I (%i. lift i (F i))"
by (simp add: safety_prop_AllowedActs_iff_Allowed UNION_OK_lift_I)

lemma Allowed_lift [simp]: "Allowed (lift i F) = lift i ` (Allowed F)"
by (simp add: lift_def Allowed_rename)

lemma lift_image_preserves:
```

```
    "lift i ` preserves v = preserves (v o drop_map i)"
by (simp add: rename_image_preserves lift_def inv_lift_map_eq)

end
```

```
theory PPROD imports Lift_prog begin

constdefs

  PLam  :: "[nat set, nat => ('b * ((nat=>'b) * 'c)) program]
            => ((nat=>'b) * 'c) program"
    "PLam I F == ⨆i ∈ I. lift i (F i)"

syntax
  "@PLam" :: "[pttrn, nat set, 'b set] => (nat => 'b) set"
             ("(3plam _:_./ _)" 10)

translations
  "plam x : A. B"   == "PLam A (%x. B)"


lemma Init_PLam [simp]: "Init (PLam I F) = (⋂i ∈ I. lift_set i (Init (F
i)))"
by (simp add: PLam_def lift_def lift_set_def)

lemma PLam_empty [simp]: "PLam {} F = SKIP"
by (simp add: PLam_def)

lemma PLam_SKIP [simp]: "(plam i : I. SKIP) = SKIP"
by (simp add: PLam_def lift_SKIP JN_constant)

lemma PLam_insert: "PLam (insert i I) F = (lift i (F i)) Join (PLam I F)"
by (unfold PLam_def, auto)

lemma PLam_component_iff: "((PLam I F) ≤ H) = (∀i ∈ I. lift i (F i) ≤
H)"
by (simp add: PLam_def JN_component_iff)

lemma component_PLam: "i ∈ I ==> lift i (F i) ≤ (PLam I F)"
apply (unfold PLam_def)

apply (fast intro: component_JN)
done


lemma PLam_constrains:
     "[| i ∈ I;  ∀j. F j ∈ preserves snd |]
      ==> (PLam I F ∈ (lift_set i (A <*> UNIV)) co
```

```
                    (lift_set i (B <*> UNIV)))  =
          (F i ∈ (A <*> UNIV) co (B <*> UNIV))"
apply (simp add: PLam_def JN_constrains)
apply (subst insert_Diff [symmetric], assumption)
apply (simp add: lift_constrains)
apply (blast intro: constrains_imp_lift_constrains)
done

lemma PLam_stable:
     "[| i ∈ I;  ∀j. F j ∈ preserves snd |]
      ==> (PLam I F ∈ stable (lift_set i (A <*> UNIV))) =
          (F i ∈ stable (A <*> UNIV))"
by (simp add: stable_def PLam_constrains)

lemma PLam_transient:
     "i ∈ I ==>
    PLam I F ∈ transient A = (∃i ∈ I. lift i (F i) ∈ transient A)"
by (simp add: JN_transient PLam_def)
```

This holds because the `F j` cannot change `lift_set i`

```
lemma PLam_ensures:
     "[| i ∈ I;  F i ∈ (A <*> UNIV) ensures (B <*> UNIV);
         ∀j. F j ∈ preserves snd |]
      ==> PLam I F ∈ lift_set i (A <*> UNIV) ensures lift_set i (B <*> UNIV)"
apply (simp add: ensures_def PLam_constrains PLam_transient
              lift_set_Un_distrib [symmetric] lift_set_Diff_distrib [symmetric]
              Times_Un_distrib1 [symmetric] Times_Diff_distrib1 [symmetric])
apply (rule rev_bexI, assumption)
apply (simp add: lift_transient)
done

lemma PLam_leadsTo_Basis:
     "[| i ∈ I;
         F i ∈ ((A <*> UNIV) - (B <*> UNIV)) co
               ((A <*> UNIV) ∪ (B <*> UNIV));
         F i ∈ transient ((A <*> UNIV) - (B <*> UNIV));
         ∀j. F j ∈ preserves snd |]
      ==> PLam I F ∈ lift_set i (A <*> UNIV) leadsTo lift_set i (B <*> UNIV)"
by (rule PLam_ensures [THEN leadsTo_Basis], rule_tac [2] ensuresI)



lemma invariant_imp_PLam_invariant:
     "[| F i ∈ invariant (A <*> UNIV);  i ∈ I;
         ∀j. F j ∈ preserves snd |]
      ==> PLam I F ∈ invariant (lift_set i (A <*> UNIV))"
by (auto simp add: PLam_stable invariant_def)


lemma PLam_preserves_fst [simp]:
     "∀j. F j ∈ preserves snd
      ==> (PLam I F ∈ preserves (v o sub j o fst)) =
```

```
        (if j ∈ I then F j ∈ preserves (v o fst) else True)"
by (simp add: PLam_def lift_preserves_sub)

lemma PLam_preserves_snd [simp,intro]:
     "∀j. F j ∈ preserves snd ==> PLam I F ∈ preserves snd"
by (simp add: PLam_def lift_preserves_snd_I)
```

This rule looks unsatisfactory because it refers to *lift*. One must use *lift_guarantees_eq_lift_inv* to rewrite the first subgoal and something like *lift_preserves_sub* to rewrite the third. However there's no obvious alternative for the third premise.

```
lemma guarantees_PLam_I:
    "[| lift i (F i): X guarantees Y;  i ∈ I;
        OK I (%i. lift i (F i)) |]
     ==> (PLam I F) ∈ X guarantees Y"
apply (unfold PLam_def)
apply (simp add: guarantees_JN_I)
done

lemma Allowed_PLam [simp]:
     "Allowed (PLam I F) = (⋂i ∈ I. lift i ` Allowed(F i))"
by (simp add: PLam_def)


lemma PLam_preserves [simp]:
     "(PLam I F) ∈ preserves v = (∀i ∈ I. F i ∈ preserves (v o lift_map
i))"
by (simp add: PLam_def lift_def rename_preserves)
```

end

13 The Prefix Ordering on Lists

theory *ListOrder* **imports** *Main* **begin**

```
consts
  genPrefix :: "('a * 'a)set => ('a list * 'a list)set"

inductive "genPrefix(r)"
 intros
   Nil:     "([],[]) : genPrefix(r)"

   prepend: "[| (xs,ys) : genPrefix(r);  (x,y) : r |] ==>
             (x#xs, y#ys) : genPrefix(r)"

   append:  "(xs,ys) : genPrefix(r) ==> (xs, ys@zs) : genPrefix(r)"
```

instance *list :: (type)ord* **..**

defs
```
  prefix_def:        "xs <= zs  ==  (xs,zs) : genPrefix Id"

  strict_prefix_def: "xs < zs  ==  xs <= zs & xs ~= (zs::'a list)"
```

constdefs
```
  Le :: "(nat*nat) set"
    "Le == {(x,y). x <= y}"

  Ge :: "(nat*nat) set"
    "Ge == {(x,y). y <= x}"
```

syntax
```
  pfixLe :: "[nat list, nat list] => bool"          (infixl "pfixLe" 50)
  pfixGe :: "[nat list, nat list] => bool"          (infixl "pfixGe" 50)
```

translations
```
  "xs pfixLe ys" == "(xs,ys) : genPrefix Le"

  "xs pfixGe ys" == "(xs,ys) : genPrefix Ge"
```

13.1 preliminary lemmas

lemma *Nil_genPrefix [iff]: "([], xs) : genPrefix r"*
by *(cut_tac genPrefix.Nil [THEN genPrefix.append], auto)*

lemma *genPrefix_length_le: "(xs,ys) : genPrefix r ==> length xs <= length ys"*
by *(erule genPrefix.induct, auto)*

lemma *cdlemma:*
```
     "[| (xs', ys'): genPrefix r |]
      ==> (ALL x xs. xs' = x#xs --> (EX y ys. ys' = y#ys & (x,y) : r & (xs, ys) : genPrefix r))"
```
apply *(erule genPrefix.induct, blast, blast)*
apply *(force intro: genPrefix.append)*
done

lemma *cons_genPrefixE [elim!]:*
```
     "[| (x#xs, zs): genPrefix r;
         !!y ys. [| zs = y#ys;  (x,y) : r;  (xs, ys) : genPrefix r |] ==> P

      |] ==> P"
```
by *(drule cdlemma, simp, blast)*

lemma *Cons_genPrefix_Cons [iff]:*
```
     "((x#xs,y#ys) : genPrefix r) = ((x,y) : r & (xs,ys) : genPrefix r)"
```
by *(blast intro: genPrefix.prepend)*

13.2 genPrefix is a partial order

lemma *refl_genPrefix: "reflexive r ==> reflexive (genPrefix r)"*

apply *(unfold refl_def, auto)*
apply *(induct_tac "x")*
prefer *2* **apply** *(blast intro: genPrefix.prepend)*
apply *(blast intro: genPrefix.Nil)*
done

lemma *genPrefix_refl [simp]: "reflexive r ==> (l,l) : genPrefix r"*
by *(erule reflD [OF refl_genPrefix UNIV_I])*

lemma *genPrefix_mono: "r<=s ==> genPrefix r <= genPrefix s"*
apply *clarify*
apply *(erule genPrefix.induct)*
apply *(auto intro: genPrefix.append)*
done

lemma *append_genPrefix [rule_format]:*
"ALL zs. (xs @ ys, zs) : genPrefix r --> (xs, zs) : genPrefix r"
by *(induct_tac "xs", auto)*

lemma *genPrefix_trans_O [rule_format]:*
"(x, y) : genPrefix r
==> ALL z. (y,z) : genPrefix s --> (x, z) : genPrefix (s O r)"
apply *(erule genPrefix.induct)*
prefer *3* **apply** *(blast dest: append_genPrefix)*
prefer *2* **apply** *(blast intro: genPrefix.prepend, blast)*
done

lemma *genPrefix_trans [rule_format]:*
"[| (x,y) : genPrefix r; (y,z) : genPrefix r; trans r |]
==> (x,z) : genPrefix r"
apply *(rule trans_O_subset [THEN genPrefix_mono, THEN subsetD])*
apply *assumption*
apply *(blast intro: genPrefix_trans_O)*
done

lemma *prefix_genPrefix_trans [rule_format]:*
"[| x<=y; (y,z) : genPrefix r |] ==> (x, z) : genPrefix r"
apply *(unfold prefix_def)*
apply *(subst R_O_Id [symmetric], erule genPrefix_trans_O, assumption)*
done

lemma *genPrefix_prefix_trans [rule_format]:*
"[| (x,y) : genPrefix r; y<=z |] ==> (x,z) : genPrefix r"
apply *(unfold prefix_def)*
apply *(subst Id_O_R [symmetric], erule genPrefix_trans_O, assumption)*

done

lemma *trans_genPrefix: "trans r ==> trans (genPrefix r)"*
by *(blast intro: transI genPrefix_trans)*

lemma *genPrefix_antisym [rule_format]:*
"[| (xs,ys) : genPrefix r; antisym r |]
==> (ys,xs) : genPrefix r --> xs = ys"
apply *(erule genPrefix.induct)*

Base case

apply *blast*

prepend case

apply *(simp add: antisym_def)*

append case is the hardest

apply *clarify*
apply *(subgoal_tac "length zs = 0", force)*
apply *(drule genPrefix_length_le)+*
apply *(simp del: length_0_conv)*
done

lemma *antisym_genPrefix: "antisym r ==> antisym (genPrefix r)"*
by *(blast intro: antisymI genPrefix_antisym)*

13.3 recursion equations

lemma *genPrefix_Nil [simp]: "((xs, []) : genPrefix r) = (xs = [])"*
apply *(induct_tac "xs")*
prefer *2* **apply** *blast*
apply *simp*
done

lemma *same_genPrefix_genPrefix [simp]:*
"reflexive r ==> ((xs@ys, xs@zs) : genPrefix r) = ((ys,zs) : genPrefix r)"
apply *(unfold refl_def)*
apply *(induct_tac "xs")*
apply *(simp_all (no_asm_simp))*
done

lemma *genPrefix_Cons:*
"((xs, y#ys) : genPrefix r) =
(xs=[] | (EX z zs. xs=z#zs & (z,y) : r & (zs,ys) : genPrefix r))"
by *(case_tac "xs", auto)*

lemma *genPrefix_take_append:*
"[| reflexive r; (xs,ys) : genPrefix r |]
==> (xs@zs, take (length xs) ys @ zs) : genPrefix r"
apply *(erule genPrefix.induct)*

```
apply (frule_tac [3] genPrefix_length_le)
apply (simp_all (no_asm_simp) add: diff_is_0_eq [THEN iffD2])
done

lemma genPrefix_append_both:
     "[| reflexive r;  (xs,ys) : genPrefix r;  length xs = length ys |]
      ==>  (xs@zs, ys @ zs) : genPrefix r"
apply (drule genPrefix_take_append, assumption)
apply (simp add: take_all)
done



lemma append_cons_eq: "xs @ y # ys = (xs @ [y]) @ ys"
by auto

lemma aolemma:
     "[| (xs,ys) : genPrefix r;  reflexive r |]
      ==> length xs < length ys --> (xs @ [ys ! length xs], ys) : genPrefix
r"
apply (erule genPrefix.induct)
  apply blast
 apply simp
```

Append case is hardest

```
apply simp
apply (frule genPrefix_length_le [THEN le_imp_less_or_eq])
apply (erule disjE)
apply (simp_all (no_asm_simp) add: neq_Nil_conv nth_append)
apply (blast intro: genPrefix.append, auto)
apply (subst append_cons_eq, fast intro: genPrefix_append_both genPrefix.append)
done

lemma append_one_genPrefix:
     "[| (xs,ys) : genPrefix r;  length xs < length ys;  reflexive r |]
      ==> (xs @ [ys ! length xs], ys) : genPrefix r"
by (blast intro: aolemma [THEN mp])



lemma genPrefix_imp_nth [rule_format]:
     "ALL i ys. i < length xs
                --> (xs, ys) : genPrefix r --> (xs ! i, ys ! i) : r"
apply (induct_tac "xs", auto)
apply (case_tac "i", auto)
done

lemma nth_imp_genPrefix [rule_format]:
     "ALL ys. length xs <= length ys
      --> (ALL i. i < length xs --> (xs ! i, ys ! i) : r)
      --> (xs, ys) : genPrefix r"
apply (induct_tac "xs")
apply (simp_all (no_asm_simp) add: less_Suc_eq_0_disj all_conj_distrib)
```

```
apply clarify
apply (case_tac "ys")
apply (force+)
done

lemma genPrefix_iff_nth:
     "((xs,ys) : genPrefix r) =
      (length xs <= length ys & (ALL i. i < length xs --> (xs!i, ys!i) : r))"
apply (blast intro: genPrefix_length_le genPrefix_imp_nth nth_imp_genPrefix)
done
```

13.4 The type of lists is partially ordered

```
declare reflexive_Id [iff]
        antisym_Id [iff]
        trans_Id [iff]

lemma prefix_refl [iff]: "xs <= (xs::'a list)"
by (simp add: prefix_def)

lemma prefix_trans: "!!xs::'a list. [| xs <= ys; ys <= zs |] ==> xs <= zs"
apply (unfold prefix_def)
apply (blast intro: genPrefix_trans)
done

lemma prefix_antisym: "!!xs::'a list. [| xs <= ys; ys <= xs |] ==> xs = ys"
apply (unfold prefix_def)
apply (blast intro: genPrefix_antisym)
done

lemma prefix_less_le: "!!xs::'a list. (xs < zs) = (xs <= zs & xs ~= zs)"
by (unfold strict_prefix_def, auto)

instance list :: (type) order
  by (intro_classes,
      (assumption | rule prefix_refl prefix_trans prefix_antisym
                    prefix_less_le)+)


lemma set_mono: "xs <= ys ==> set xs <= set ys"
apply (unfold prefix_def)
apply (erule genPrefix.induct, auto)
done




lemma Nil_prefix [iff]: "[] <= xs"
apply (unfold prefix_def)
apply (simp add: Nil_genPrefix)
done

lemma prefix_Nil [simp]: "(xs <= []) = (xs = [])"
apply (unfold prefix_def)
```

```
apply (simp add: genPrefix_Nil)
done

lemma Cons_prefix_Cons [simp]: "(x#xs <= y#ys) = (x=y & xs<=ys)"
by (simp add: prefix_def)

lemma same_prefix_prefix [simp]: "(xs@ys <= xs@zs) = (ys <= zs)"
by (simp add: prefix_def)

lemma append_prefix [iff]: "(xs@ys <= xs) = (ys <= [])"
by (insert same_prefix_prefix [of xs ys "[]"], simp)

lemma prefix_appendI [simp]: "xs <= ys ==> xs <= ys@zs"
apply (unfold prefix_def)
apply (erule genPrefix.append)
done

lemma prefix_Cons:
   "(xs <= y#ys) = (xs=[] | (? zs. xs=y#zs & zs <= ys))"
by (simp add: prefix_def genPrefix_Cons)

lemma append_one_prefix:
  "[| xs <= ys; length xs < length ys |] ==> xs @ [ys ! length xs] <= ys"
apply (unfold prefix_def)
apply (simp add: append_one_genPrefix)
done

lemma prefix_length_le: "xs <= ys ==> length xs <= length ys"
apply (unfold prefix_def)
apply (erule genPrefix_length_le)
done

lemma splemma: "xs<=ys ==> xs~=ys --> length xs < length ys"
apply (unfold prefix_def)
apply (erule genPrefix.induct, auto)
done

lemma strict_prefix_length_less: "xs < ys ==> length xs < length ys"
apply (unfold strict_prefix_def)
apply (blast intro: splemma [THEN mp])
done

lemma mono_length: "mono length"
by (blast intro: monoI prefix_length_le)


lemma prefix_iff: "(xs <= zs) = (EX ys. zs = xs@ys)"
apply (unfold prefix_def)
apply (auto simp add: genPrefix_iff_nth nth_append)
apply (rule_tac x = "drop (length xs) zs" in exI)
apply (rule nth_equalityI)
apply (simp_all (no_asm_simp) add: nth_append)
done
```

```
lemma prefix_snoc [simp]: "(xs <= ys@[y]) = (xs = ys@[y] | xs <= ys)"
apply (simp add: prefix_iff)
apply (rule iffI)
 apply (erule exE)
 apply (rename_tac "zs")
 apply (rule_tac xs = zs in rev_exhaust)
  apply simp
 apply clarify
 apply (simp del: append_assoc add: append_assoc [symmetric], force)
done

lemma prefix_append_iff:
     "(xs <= ys@zs) = (xs <= ys | (? us. xs = ys@us & us <= zs))"
apply (rule_tac xs = zs in rev_induct)
 apply force
apply (simp del: append_assoc add: append_assoc [symmetric], force)
done


lemma common_prefix_linear [rule_format]:
     "!!zs::'a list. xs <= zs --> ys <= zs --> xs <= ys | ys <= xs"
by (rule_tac xs = zs in rev_induct, auto)
```

13.5 pfixLe, pfixGe: properties inherited from the translations

```
lemma reflexive_Le [iff]: "reflexive Le"
by (unfold refl_def Le_def, auto)

lemma antisym_Le [iff]: "antisym Le"
by (unfold antisym_def Le_def, auto)

lemma trans_Le [iff]: "trans Le"
by (unfold trans_def Le_def, auto)

lemma pfixLe_refl [iff]: "x pfixLe x"
by simp

lemma pfixLe_trans: "[| x pfixLe y; y pfixLe z |] ==> x pfixLe z"
by (blast intro: genPrefix_trans)

lemma pfixLe_antisym: "[| x pfixLe y; y pfixLe x |] ==> x = y"
by (blast intro: genPrefix_antisym)

lemma prefix_imp_pfixLe: "xs<=ys ==> xs pfixLe ys"
apply (unfold prefix_def Le_def)
apply (blast intro: genPrefix_mono [THEN [2] rev_subsetD])
done

lemma reflexive_Ge [iff]: "reflexive Ge"
by (unfold refl_def Ge_def, auto)

lemma antisym_Ge [iff]: "antisym Ge"
by (unfold antisym_def Ge_def, auto)
```

```
lemma trans_Ge [iff]: "trans Ge"
by (unfold trans_def Ge_def, auto)

lemma pfixGe_refl [iff]: "x pfixGe x"
by simp

lemma pfixGe_trans: "[| x pfixGe y; y pfixGe z |] ==> x pfixGe z"
by (blast intro: genPrefix_trans)

lemma pfixGe_antisym: "[| x pfixGe y; y pfixGe x |] ==> x = y"
by (blast intro: genPrefix_antisym)

lemma prefix_imp_pfixGe: "xs<=ys ==> xs pfixGe ys"
apply (unfold prefix_def Ge_def)
apply (blast intro: genPrefix_mono [THEN [2] rev_subsetD])
done

end
```

14 The accessible part of a relation

```
theory Accessible_Part
imports Main
begin
```

14.1 Inductive definition

Inductive definition of the accessible part `acc r` of a relation; see also [**?**].

```
consts
  acc :: "('a × 'a) set => 'a set"
inductive "acc r"
  intros
    accI: "(!!y. (y, x) ∈ r ==> y ∈ acc r) ==> x ∈ acc r"

syntax
  termi :: "('a × 'a) set => 'a set"
translations
  "termi r" == "acc (r^-1)"
```

14.2 Induction rules

```
theorem acc_induct:
  "a ∈ acc r ==>
    (!!x. x ∈ acc r ==> ∀y. (y, x) ∈ r --> P y ==> P x) ==> P a"
proof -
  assume major: "a ∈ acc r"
  assume hyp: "!!x. x ∈ acc r ==> ∀y. (y, x) ∈ r --> P y ==> P x"
  show ?thesis
    apply (rule major [THEN acc.induct])
    apply (rule hyp)
     apply (rule accI)
```

```
     apply fast
    apply fast
    done
qed

theorems acc_induct_rule = acc_induct [rule_format, induct set: acc]

theorem acc_downward: "b ∈ acc r ==> (a, b) ∈ r ==> a ∈ acc r"
  apply (erule acc.elims)
  apply fast
  done

lemma acc_downwards_aux: "(b, a) ∈ r* ==> a ∈ acc r --> b ∈ acc r"
  apply (erule rtrancl_induct)
   apply blast
  apply (blast dest: acc_downward)
  done

theorem acc_downwards: "a ∈ acc r ==> (b, a) ∈ r* ==> b ∈ acc r"
  apply (blast dest: acc_downwards_aux)
  done

theorem acc_wfI: "∀x. x ∈ acc r ==> wf r"
  apply (rule wfUNIVI)
  apply (induct_tac P x rule: acc_induct)
   apply blast
  apply blast
  done

theorem acc_wfD: "wf r ==> x ∈ acc r"
  apply (erule wf_induct)
  apply (rule accI)
  apply blast
  done

theorem wf_acc_iff: "wf r = (∀x. x ∈ acc r)"
  apply (blast intro: acc_wfI dest: acc_wfD)
  done

end
```

15 Multisets

```
theory Multiset
imports Accessible_Part
begin
```

15.1 The type of multisets

```
typedef 'a multiset = "{f::'a => nat. finite {x . 0 < f x}}"
proof
  show "(λx. 0::nat) ∈ ?multiset" by simp
```

qed

lemmas *multiset_typedef [simp] =*
Abs_multiset_inverse Rep_multiset_inverse Rep_multiset
and *[simp] = Rep_multiset_inject [symmetric]*

constdefs
Mempty :: "'a multiset" ("{#}")
"{#} == Abs_multiset (λa. 0)"

single :: "'a => 'a multiset" ("{#_#}")
"{#a#} == Abs_multiset (λb. if b = a then 1 else 0)"

count :: "'a multiset => 'a => nat"
"count == Rep_multiset"

MCollect :: "'a multiset => ('a => bool) => 'a multiset"
"MCollect M P == Abs_multiset (λx. if P x then Rep_multiset M x else 0)"

syntax
"_Melem" :: "'a => 'a multiset => bool" ("(_/ :# _)" [50, 51] 50)
"_MCollect" :: "pttrn => 'a multiset => bool => 'a multiset" ("(1{# _
: _./ _#})")
translations
"a :# M" == "0 < count M a"
"{#x:M. P#}" == "MCollect M (λx. P)"

constdefs
set_of :: "'a multiset => 'a set"
"set_of M == {x. x :# M}"

instance *multiset :: (type) "{plus, minus, zero}"* **..**

defs *(***overloaded***)*
union_def: "M + N == Abs_multiset (λa. Rep_multiset M a + Rep_multiset N
a)"
diff_def: "M - N == Abs_multiset (λa. Rep_multiset M a - Rep_multiset N
a)"
Zero_multiset_def [simp]: "0 == {#}"
size_def: "size M == setsum (count M) (set_of M)"

constdefs
multiset_inter :: "'a multiset ⇒ 'a multiset ⇒ 'a multiset" **(infixl** *"#∩"*
70)
"multiset_inter A B ≡ A - (A - B)"

Preservation of the representing set *multiset*.

lemma *const0_in_multiset [simp]: "(λa. 0) ∈ multiset"*
by *(simp add: multiset_def)*

lemma *only1_in_multiset [simp]: "(λb. if b = a then 1 else 0) ∈ multiset"*
by *(simp add: multiset_def)*

lemma *union_preserves_multiset [simp]:*

```
    "M ∈ multiset ==> N ∈ multiset ==> (λa. M a + N a) ∈ multiset"
  apply (simp add: multiset_def)
  apply (drule (1) finite_UnI)
  apply (simp del: finite_Un add: Un_def)
  done

lemma diff_preserves_multiset [simp]:
    "M ∈ multiset ==> (λa. M a - N a) ∈ multiset"
  apply (simp add: multiset_def)
  apply (rule finite_subset)
   apply auto
  done
```

15.2 Algebraic properties of multisets

15.2.1 Union

```
lemma union_empty [simp]: "M + {#} = M ∧ {#} + M = M"
  by (simp add: union_def Mempty_def)

lemma union_commute: "M + N = N + (M::'a multiset)"
  by (simp add: union_def add_ac)

lemma union_assoc: "(M + N) + K = M + (N + (K::'a multiset))"
  by (simp add: union_def add_ac)

lemma union_lcomm: "M + (N + K) = N + (M + (K::'a multiset))"
proof -
  have "M + (N + K) = (N + K) + M"
    by (rule union_commute)
  also have "... = N + (K + M)"
    by (rule union_assoc)
  also have "K + M = M + K"
    by (rule union_commute)
  finally show ?thesis .
qed

lemmas union_ac = union_assoc union_commute union_lcomm

instance multiset :: (type) comm_monoid_add
proof
  fix a b c :: "'a multiset"
  show "(a + b) + c = a + (b + c)" by (rule union_assoc)
  show "a + b = b + a" by (rule union_commute)
  show "0 + a = a" by simp
qed
```

15.2.2 Difference

```
lemma diff_empty [simp]: "M - {#} = M ∧ {#} - M = {#}"
  by (simp add: Mempty_def diff_def)

lemma diff_union_inverse2 [simp]: "M + {#a#} - {#a#} = M"
  by (simp add: union_def diff_def)
```

15.2.3 Count of elements

lemma *count_empty [simp]: "count {#} a = 0"*
by *(simp add: count_def Mempty_def)*

lemma *count_single [simp]: "count {#b#} a = (if b = a then 1 else 0)"*
by *(simp add: count_def single_def)*

lemma *count_union [simp]: "count (M + N) a = count M a + count N a"*
by *(simp add: count_def union_def)*

lemma *count_diff [simp]: "count (M - N) a = count M a - count N a"*
by *(simp add: count_def diff_def)*

15.2.4 Set of elements

lemma *set_of_empty [simp]: "set_of {#} = {}"*
by *(simp add: set_of_def)*

lemma *set_of_single [simp]: "set_of {#b#} = {b}"*
by *(simp add: set_of_def)*

lemma *set_of_union [simp]: "set_of (M + N) = set_of M ∪ set_of N"*
by *(auto simp add: set_of_def)*

lemma *set_of_eq_empty_iff [simp]: "(set_of M = {}) = (M = {#})"*
by *(auto simp add: set_of_def Mempty_def count_def expand_fun_eq)*

lemma *mem_set_of_iff [simp]: "(x ∈ set_of M) = (x :# M)"*
by *(auto simp add: set_of_def)*

15.2.5 Size

lemma *size_empty [simp]: "size {#} = 0"*
by *(simp add: size_def)*

lemma *size_single [simp]: "size {#b#} = 1"*
by *(simp add: size_def)*

lemma *finite_set_of [iff]: "finite (set_of M)"*
using *Rep_multiset [of M]*
by *(simp add: multiset_def set_of_def count_def)*

lemma *setsum_count_Int:*
"finite A ==> setsum (count N) (A ∩ set_of N) = setsum (count N) A"
apply *(erule finite_induct)*
apply *simp*
apply *(simp add: Int_insert_left set_of_def)*
done

lemma *size_union [simp]: "size (M + N::'a multiset) = size M + size N"*
apply *(unfold size_def)*
apply *(subgoal_tac "count (M + N) = (λa. count M a + count N a)")*
prefer *2*
apply *(rule ext, simp)*

```
  apply (simp (no_asm_simp) add: setsum_Un_nat setsum_addf setsum_count_Int)
  apply (subst Int_commute)
  apply (simp (no_asm_simp) add: setsum_count_Int)
  done

lemma size_eq_0_iff_empty [iff]: "(size M = 0) = (M = {#})"
  apply (unfold size_def Mempty_def count_def, auto)
  apply (simp add: set_of_def count_def expand_fun_eq)
  done

lemma size_eq_Suc_imp_elem: "size M = Suc n ==> ∃a. a :# M"
  apply (unfold size_def)
  apply (drule setsum_SucD, auto)
  done
```

15.2.6 Equality of multisets

```
lemma multiset_eq_conv_count_eq: "(M = N) = (∀a. count M a = count N a)"
  by (simp add: count_def expand_fun_eq)

lemma single_not_empty [simp]: "{#a#} ≠ {#} ∧ {#} ≠ {#a#}"
  by (simp add: single_def Mempty_def expand_fun_eq)

lemma single_eq_single [simp]: "({#a#} = {#b#}) = (a = b)"
  by (auto simp add: single_def expand_fun_eq)

lemma union_eq_empty [iff]: "(M + N = {#}) = (M = {#} ∧ N = {#})"
  by (auto simp add: union_def Mempty_def expand_fun_eq)

lemma empty_eq_union [iff]: "({#} = M + N) = (M = {#} ∧ N = {#})"
  by (auto simp add: union_def Mempty_def expand_fun_eq)

lemma union_right_cancel [simp]: "(M + K = N + K) = (M = (N::'a multiset))"
  by (simp add: union_def expand_fun_eq)

lemma union_left_cancel [simp]: "(K + M = K + N) = (M = (N::'a multiset))"
  by (simp add: union_def expand_fun_eq)

lemma union_is_single:
    "(M + N = {#a#}) = (M = {#a#} ∧ N={#} ∨ M = {#} ∧ N = {#a#})"
  apply (simp add: Mempty_def single_def union_def add_is_1 expand_fun_eq)
  apply blast
  done

lemma single_is_union:
     "({#a#} = M + N) = ({#a#} = M ∧ N = {#} ∨ M = {#} ∧ {#a#} = N)"
  apply (unfold Mempty_def single_def union_def)
  apply (simp add: add_is_1 one_is_add expand_fun_eq)
  apply (blast dest: sym)
  done

lemma add_eq_conv_diff:
  "(M + {#a#} = N + {#b#}) =
   (M = N ∧ a = b ∨ M = N - {#a#} + {#b#} ∧ N = M - {#b#} + {#a#})"
```

```
  apply (unfold single_def union_def diff_def)
  apply (simp (no_asm) add: expand_fun_eq)
  apply (rule conjI, force, safe, simp_all)
  apply (simp add: eq_sym_conv)
  done

declare Rep_multiset_inject [symmetric, simp del]
```

15.2.7 Intersection

```
lemma multiset_inter_count:
    "count (A #∩ B) x = min (count A x) (count B x)"
  by (simp add: multiset_inter_def min_def)

lemma multiset_inter_commute: "A #∩ B = B #∩ A"
  by (simp add: multiset_eq_conv_count_eq multiset_inter_count
    min_max.below_inf.inf_commute)

lemma multiset_inter_assoc: "A #∩ (B #∩ C) = A #∩ B #∩ C"
  by (simp add: multiset_eq_conv_count_eq multiset_inter_count
    min_max.below_inf.inf_assoc)

lemma multiset_inter_left_commute: "A #∩ (B #∩ C) = B #∩ (A #∩ C)"
  by (simp add: multiset_eq_conv_count_eq multiset_inter_count min_def)

lemmas multiset_inter_ac =
  multiset_inter_commute
  multiset_inter_assoc
  multiset_inter_left_commute

lemma multiset_union_diff_commute: "B #∩ C = {#} ⟹ A + B - C = A - C +
B"
  apply (simp add: multiset_eq_conv_count_eq multiset_inter_count min_def
    split: split_if_asm)
  apply clarsimp
  apply (erule_tac x = a in allE)
  apply auto
  done
```

15.3 Induction over multisets

```
lemma setsum_decr:
  "finite F ==> (0::nat) < f a ==>
    setsum (f (a := f a - 1)) F = (if a∈F then setsum f F - 1 else setsum
f F)"
  apply (erule finite_induct, auto)
  apply (drule_tac a = a in mk_disjoint_insert, auto)
  done

lemma rep_multiset_induct_aux:
  assumes "P (λa. (0::nat))"
    and "!!f b. f ∈ multiset ==> P f ==> P (f (b := f b + 1))"
  shows "∀f. f ∈ multiset --> setsum f {x. 0 < f x} = n --> P f"
proof -
```

```
  note premises = prems [unfolded multiset_def]
  show ?thesis
    apply (unfold multiset_def)
    apply (induct_tac n, simp, clarify)
     apply (subgoal_tac "f = (λa.0)")
      apply simp
      apply (rule premises)
     apply (rule ext, force, clarify)
    apply (frule setsum_SucD, clarify)
    apply (rename_tac a)
    apply (subgoal_tac "finite {x. 0 < (f (a := f a - 1)) x}")
     prefer 2
     apply (rule finite_subset)
      prefer 2
      apply assumption
     apply simp
     apply blast
    apply (subgoal_tac "f = (f (a := f a - 1))(a := (f (a := f a - 1)) a +
1)")
     prefer 2
     apply (rule ext)
     apply (simp (no_asm_simp))
     apply (erule ssubst, rule premises, blast)
    apply (erule allE, erule impE, erule_tac [2] mp, blast)
    apply (simp (no_asm_simp) add: setsum_decr del: fun_upd_apply One_nat_def)
    apply (subgoal_tac "{x. x ≠ a --> 0 < f x} = {x. 0 < f x}")
     prefer 2
     apply blast
    apply (subgoal_tac "{x. x ≠ a ∧ 0 < f x} = {x. 0 < f x} - {a}")
     prefer 2
     apply blast
    apply (simp add: le_imp_diff_is_add setsum_diff1_nat cong: conj_cong)
    done
qed

theorem rep_multiset_induct:
  "f ∈ multiset ==> P (λa. 0) ==>
    (!!f b. f ∈ multiset ==> P f ==> P (f (b := f b + 1))) ==> P f"
  using rep_multiset_induct_aux by blast

theorem multiset_induct [induct type: multiset]:
  assumes prem1: "P {#}"
    and prem2: "!!M x. P M ==> P (M + {#x#})"
  shows "P M"
proof -
  note defns = union_def single_def Mempty_def
  show ?thesis
    apply (rule Rep_multiset_inverse [THEN subst])
    apply (rule Rep_multiset [THEN rep_multiset_induct])
     apply (rule prem1 [unfolded defns])
    apply (subgoal_tac "f(b := f b + 1) = (λa. f a + (if a=b then 1 else 0))")
     prefer 2
     apply (simp add: expand_fun_eq)
    apply (erule ssubst)
```

```
    apply (erule Abs_multiset_inverse [THEN subst])
    apply (erule prem2 [unfolded defns, simplified])
    done
qed

lemma MCollect_preserves_multiset:
    "M ∈ multiset ==> (λx. if P x then M x else 0) ∈ multiset"
  apply (simp add: multiset_def)
  apply (rule finite_subset, auto)
  done

lemma count_MCollect [simp]:
    "count {# x:M. P x #} a = (if P a then count M a else 0)"
  by (simp add: count_def MCollect_def MCollect_preserves_multiset)

lemma set_of_MCollect [simp]: "set_of {# x:M. P x #} = set_of M ∩ {x. P
x}"
  by (auto simp add: set_of_def)

lemma multiset_partition: "M = {# x:M. P x #} + {# x:M. ¬ P x #}"
  by (subst multiset_eq_conv_count_eq, auto)

lemma add_eq_conv_ex:
  "(M + {#a#} = N + {#b#}) =
    (M = N ∧ a = b ∨ (∃K. M = K + {#b#} ∧ N = K + {#a#}))"
  by (auto simp add: add_eq_conv_diff)

declare multiset_typedef [simp del]
```

15.4 Multiset orderings

15.4.1 Well-foundedness

```
constdefs
  mult1 :: "('a × 'a) set => ('a multiset × 'a multiset) set"
  "mult1 r ==
    {(N, M). ∃a M0 K. M = M0 + {#a#} ∧ N = M0 + K ∧
      (∀b. b :# K --> (b, a) ∈ r)}"

  mult :: "('a × 'a) set => ('a multiset × 'a multiset) set"
  "mult r == (mult1 r)⁺"

lemma not_less_empty [iff]: "(M, {#}) ∉ mult1 r"
  by (simp add: mult1_def)

lemma less_add: "(N, M0 + {#a#}) ∈ mult1 r ==>
    (∃M. (M, M0) ∈ mult1 r ∧ N = M + {#a#}) ∨
    (∃K. (∀b. b :# K --> (b, a) ∈ r) ∧ N = M0 + K)"
  (concl is "?case1 (mult1 r) ∨ ?case2")
proof (unfold mult1_def)
  let ?r = "λK a. ∀b. b :# K --> (b, a) ∈ r"
  let ?R = "λN M. ∃a M0 K. M = M0 + {#a#} ∧ N = M0 + K ∧ ?r K a"
  let ?case1 = "?case1 {(N, M). ?R N M}"
```

```
  assume "(N, M0 + {#a#}) ∈ {(N, M). ?R N M}"
  hence "∃a' M0' K.
      M0 + {#a#} = M0' + {#a'#} ∧ N = M0' + K ∧ ?r K a'" by simp
  thus "?case1 ∨ ?case2"
  proof (elim exE conjE)
    fix a' M0' K
    assume N: "N = M0' + K" and r: "?r K a'"
    assume "M0 + {#a#} = M0' + {#a'#}"
    hence "M0 = M0' ∧ a = a' ∨
        (∃K'. M0 = K' + {#a'#} ∧ M0' = K' + {#a#})"
      by (simp only: add_eq_conv_ex)
    thus ?thesis
    proof (elim disjE conjE exE)
      assume "M0 = M0'" "a = a'"
      with N r have "?r K a ∧ N = M0 + K" by simp
      hence ?case2 .. thus ?thesis ..
    next
      fix K'
      assume "M0' = K' + {#a#}"
      with N have n: "N = K' + K + {#a#}" by (simp add: union_ac)

      assume "M0 = K' + {#a'#}"
      with r have "?R (K' + K) M0" by blast
      with n have ?case1 by simp thus ?thesis ..
    qed
  qed
qed

lemma all_accessible: "wf r ==> ∀M. M ∈ acc (mult1 r)"
proof
  let ?R = "mult1 r"
  let ?W = "acc ?R"
  {
    fix M M0 a
    assume M0: "M0 ∈ ?W"
      and wf_hyp: "!!b. (b, a) ∈ r ==> (∀M ∈ ?W. M + {#b#} ∈ ?W)"
      and acc_hyp: "∀M. (M, M0) ∈ ?R --> M + {#a#} ∈ ?W"
    have "M0 + {#a#} ∈ ?W"
    proof (rule accI [of "M0 + {#a#}"])
      fix N
      assume "(N, M0 + {#a#}) ∈ ?R"
      hence "((∃M. (M, M0) ∈ ?R ∧ N = M + {#a#}) ∨
          (∃K. (∀b. b :# K --> (b, a) ∈ r) ∧ N = M0 + K))"
        by (rule less_add)
      thus "N ∈ ?W"
      proof (elim exE disjE conjE)
        fix M assume "(M, M0) ∈ ?R" and N: "N = M + {#a#}"
        from acc_hyp have "(M, M0) ∈ ?R --> M + {#a#} ∈ ?W" ..
        hence "M + {#a#} ∈ ?W" ..
        thus "N ∈ ?W" by (simp only: N)
      next
        fix K
        assume N: "N = M0 + K"
        assume "∀b. b :# K --> (b, a) ∈ r"
```

```
      have "?this --> M0 + K ∈ ?W" (is "?P K")
      proof (induct K)
        from M0 have "M0 + {#} ∈ ?W" by simp
        thus "?P {#}" ..

        fix K x assume hyp: "?P K"
        show "?P (K + {#x#})"
        proof
          assume a: "∀b. b :# (K + {#x#}) --> (b, a) ∈ r"
          hence "(x, a) ∈ r" by simp
          with wf_hyp have b: "∀M ∈ ?W. M + {#x#} ∈ ?W" by blast

          from a hyp have "M0 + K ∈ ?W" by simp
          with b have "(M0 + K) + {#x#} ∈ ?W" ..
          thus "M0 + (K + {#x#}) ∈ ?W" by (simp only: union_assoc)
        qed
      qed
      hence "M0 + K ∈ ?W" ..
      thus "N ∈ ?W" by (simp only: N)
    qed
  qed
} note tedious_reasoning = this

assume wf: "wf r"
fix M
show "M ∈ ?W"
proof (induct M)
  show "{#} ∈ ?W"
  proof (rule accI)
    fix b assume "(b, {#}) ∈ ?R"
    with not_less_empty show "b ∈ ?W" by contradiction
  qed

  fix M a assume "M ∈ ?W"
  from wf have "∀M ∈ ?W. M + {#a#} ∈ ?W"
  proof induct
    fix a
    assume "!!b. (b, a) ∈ r ==> (∀M ∈ ?W. M + {#b#} ∈ ?W)"
    show "∀M ∈ ?W. M + {#a#} ∈ ?W"
    proof
      fix M assume "M ∈ ?W"
      thus "M + {#a#} ∈ ?W"
        by (rule acc_induct) (rule tedious_reasoning)
    qed
  qed
  thus "M + {#a#} ∈ ?W" ..
qed
qed

theorem wf_mult1: "wf r ==> wf (mult1 r)"
  by (rule acc_wfI, rule all_accessible)

theorem wf_mult: "wf r ==> wf (mult r)"
  by (unfold mult_def, rule wf_trancl, rule wf_mult1)
```

15.4.2 Closure-free presentation

lemma *diff_union_single_conv: "a :# J ==> I + J - {#a#} = I + (J - {#a#})"*
by *(simp add: multiset_eq_conv_count_eq)*

One direction.

```
lemma mult_implies_one_step:
  "trans r ==> (M, N) ∈ mult r ==>
    ∃I J K. N = I + J ∧ M = I + K ∧ J ≠ {#} ∧
    (∀k ∈ set_of K. ∃j ∈ set_of J. (k, j) ∈ r)"
  apply (unfold mult_def mult1_def set_of_def)
  apply (erule converse_trancl_induct, clarify)
   apply (rule_tac x = M0 in exI, simp, clarify)
  apply (case_tac "a :# K")
   apply (rule_tac x = I in exI)
   apply (simp (no_asm))
   apply (rule_tac x = "(K - {#a#}) + Ka" in exI)
   apply (simp (no_asm_simp) add: union_assoc [symmetric])
   apply (drule_tac f = "λM. M - {#a#}" in arg_cong)
   apply (simp add: diff_union_single_conv)
   apply (simp (no_asm_use) add: trans_def)
   apply blast
  apply (subgoal_tac "a :# I")
   apply (rule_tac x = "I - {#a#}" in exI)
   apply (rule_tac x = "J + {#a#}" in exI)
   apply (rule_tac x = "K + Ka" in exI)
   apply (rule conjI)
    apply (simp add: multiset_eq_conv_count_eq split: nat_diff_split)
   apply (rule conjI)
    apply (drule_tac f = "λM. M - {#a#}" in arg_cong, simp)
    apply (simp add: multiset_eq_conv_count_eq split: nat_diff_split)
   apply (simp (no_asm_use) add: trans_def)
   apply blast
  apply (subgoal_tac "a :# (M0 + {#a#})")
   apply simp
  apply (simp (no_asm))
  done

lemma elem_imp_eq_diff_union: "a :# M ==> M = M - {#a#} + {#a#}"
by (simp add: multiset_eq_conv_count_eq)

lemma size_eq_Suc_imp_eq_union: "size M = Suc n ==> ∃a N. M = N + {#a#}"
  apply (erule size_eq_Suc_imp_elem [THEN exE])
  apply (drule elem_imp_eq_diff_union, auto)
  done

lemma one_step_implies_mult_aux:
  "trans r ==>
    ∀I J K. (size J = n ∧ J ≠ {#} ∧ (∀k ∈ set_of K. ∃j ∈ set_of J. (k,
j) ∈ r))
      --> (I + K, I + J) ∈ mult r"
  apply (induct_tac n, auto)
  apply (frule size_eq_Suc_imp_eq_union, clarify)
  apply (rename_tac "J'", simp)
```

```
  apply (erule notE, auto)
  apply (case_tac "J' = {#}")
   apply (simp add: mult_def)
   apply (rule r_into_trancl)
   apply (simp add: mult1_def set_of_def, blast)
```

Now we know $J' \neq \{\#\}$.

```
  apply (cut_tac M = K and P = "λx. (x, a) ∈ r" in multiset_partition)
  apply (erule_tac P = "∀k ∈ set_of K. ?P k" in rev_mp)
  apply (erule ssubst)
  apply (simp add: Ball_def, auto)
  apply (subgoal_tac
    "((I + {# x : K. (x, a) ∈ r #}) + {# x : K. (x, a) ∉ r #},
      (I + {# x : K. (x, a) ∈ r #}) + J') ∈ mult r")
   prefer 2
   apply force
  apply (simp (no_asm_use) add: union_assoc [symmetric] mult_def)
  apply (erule trancl_trans)
  apply (rule r_into_trancl)
  apply (simp add: mult1_def set_of_def)
  apply (rule_tac x = a in exI)
  apply (rule_tac x = "I + J'" in exI)
  apply (simp add: union_ac)
  done

lemma one_step_implies_mult:
  "trans r ==> J ≠ {#} ==> ∀k ∈ set_of K. ∃j ∈ set_of J. (k, j) ∈ r
    ==> (I + K, I + J) ∈ mult r"
  apply (insert one_step_implies_mult_aux, blast)
  done
```

15.4.3 Partial-order properties

```
instance multiset :: (type) ord ..

defs (overloaded)
  less_multiset_def: "M' < M == (M', M) ∈ mult {(x', x). x' < x}"
  le_multiset_def: "M' <= M == M' = M ∨ M' < (M::'a multiset)"

lemma trans_base_order: "trans {(x', x). x' < (x::'a::order)}"
  apply (unfold trans_def)
  apply (blast intro: order_less_trans)
  done
```

Irreflexivity.

```
lemma mult_irrefl_aux:
    "finite A ==> (∀x ∈ A. ∃y ∈ A. x < (y::'a::order)) --> A = {}"
  apply (erule finite_induct)
   apply (auto intro: order_less_trans)
  done

lemma mult_less_not_refl: "¬ M < (M::'a::order multiset)"
  apply (unfold less_multiset_def, auto)
```

```
  apply (drule trans_base_order [THEN mult_implies_one_step], auto)
  apply (drule finite_set_of [THEN mult_irrefl_aux [rule_format (no_asm)]])
  apply (simp add: set_of_eq_empty_iff)
  done

lemma mult_less_irrefl [elim!]: "M < (M::'a::order multiset) ==> R"
by (insert mult_less_not_refl, fast)
```

Transitivity.

```
theorem mult_less_trans: "K < M ==> M < N ==> K < (N::'a::order multiset)"
  apply (unfold less_multiset_def mult_def)
  apply (blast intro: trancl_trans)
  done
```

Asymmetry.

```
theorem mult_less_not_sym: "M < N ==> ¬ N < (M::'a::order multiset)"
  apply auto
  apply (rule mult_less_not_refl [THEN notE])
  apply (erule mult_less_trans, assumption)
  done

theorem mult_less_asym:
    "M < N ==> (¬ P ==> N < (M::'a::order multiset)) ==> P"
  by (insert mult_less_not_sym, blast)

theorem mult_le_refl [iff]: "M <= (M::'a::order multiset)"
by (unfold le_multiset_def, auto)
```

Anti-symmetry.

```
theorem mult_le_antisym:
    "M <= N ==> N <= M ==> M = (N::'a::order multiset)"
  apply (unfold le_multiset_def)
  apply (blast dest: mult_less_not_sym)
  done
```

Transitivity.

```
theorem mult_le_trans:
    "K <= M ==> M <= N ==> K <= (N::'a::order multiset)"
  apply (unfold le_multiset_def)
  apply (blast intro: mult_less_trans)
  done

theorem mult_less_le: "(M < N) = (M <= N ∧ M ≠ (N::'a::order multiset))"
by (unfold le_multiset_def, auto)
```

Partial order.

```
instance multiset :: (order) order
  apply intro_classes
     apply (rule mult_le_refl)
    apply (erule mult_le_trans, assumption)
   apply (erule mult_le_antisym, assumption)
  apply (rule mult_less_le)
  done
```

15.4.4 Monotonicity of multiset union

```
lemma mult1_union:
    "(B, D) ∈ mult1 r ==> trans r ==> (C + B, C + D) ∈ mult1 r"
  apply (unfold mult1_def, auto)
  apply (rule_tac x = a in exI)
  apply (rule_tac x = "C + M0" in exI)
  apply (simp add: union_assoc)
  done

lemma union_less_mono2: "B < D ==> C + B < C + (D::'a::order multiset)"
  apply (unfold less_multiset_def mult_def)
  apply (erule trancl_induct)
   apply (blast intro: mult1_union transI order_less_trans r_into_trancl)
  apply (blast intro: mult1_union transI order_less_trans r_into_trancl trancl_trans)
  done

lemma union_less_mono1: "B < D ==> B + C < D + (C::'a::order multiset)"
  apply (subst union_commute [of B C])
  apply (subst union_commute [of D C])
  apply (erule union_less_mono2)
  done

lemma union_less_mono:
    "A < C ==> B < D ==> A + B < C + (D::'a::order multiset)"
  apply (blast intro!: union_less_mono1 union_less_mono2 mult_less_trans)
  done

lemma union_le_mono:
    "A <= C ==> B <= D ==> A + B <= C + (D::'a::order multiset)"
  apply (unfold le_multiset_def)
  apply (blast intro: union_less_mono union_less_mono1 union_less_mono2)
  done

lemma empty_leI [iff]: "{#} <= (M::'a::order multiset)"
  apply (unfold le_multiset_def less_multiset_def)
  apply (case_tac "M = {#}")
   prefer 2
   apply (subgoal_tac "({#} + {#}, {#} + M) ∈ mult (Collect (split op <))")
    prefer 2
    apply (rule one_step_implies_mult)
      apply (simp only: trans_def, auto)
  done

lemma union_upper1: "A <= A + (B::'a::order multiset)"
proof -
  have "A + {#} <= A + B" by (blast intro: union_le_mono)
  thus ?thesis by simp
qed

lemma union_upper2: "B <= A + (B::'a::order multiset)"
by (subst union_commute, rule union_upper1)
```

15.5 Link with lists

```
consts
  multiset_of :: "'a list ⇒ 'a multiset"
primrec
  "multiset_of [] = {#}"
  "multiset_of (a # x) = multiset_of x + {# a #}"

lemma multiset_of_zero_iff[simp]: "(multiset_of x = {#}) = (x = [])"
  by (induct_tac x, auto)

lemma multiset_of_zero_iff_right[simp]: "({#} = multiset_of x) = (x = [])"
  by (induct_tac x, auto)

lemma set_of_multiset_of[simp]: "set_of(multiset_of x) = set x"
  by (induct_tac x, auto)

lemma mem_set_multiset_eq: "x ∈ set xs = (x :# multiset_of xs)"
  by (induct xs) auto

lemma multiset_of_append[simp]:
  "multiset_of (xs @ ys) = multiset_of xs + multiset_of ys"
  by (rule_tac x=ys in spec, induct_tac xs, auto simp: union_ac)

lemma surj_multiset_of: "surj multiset_of"
  apply (unfold surj_def, rule allI)
  apply (rule_tac M=y in multiset_induct, auto)
  apply (rule_tac x = "x # xa" in exI, auto)
  done

lemma set_count_greater_0: "set x = {a. 0 < count (multiset_of x) a}"
  by (induct_tac x, auto)

lemma distinct_count_atmost_1:
   "distinct x = (! a. count (multiset_of x) a = (if a ∈ set x then 1 else
0))"
   apply ( induct_tac x, simp, rule iffI, simp_all)
   apply (rule conjI)
   apply (simp_all add: set_of_multiset_of [THEN sym] del: set_of_multiset_of)
   apply (erule_tac x=a in allE, simp, clarify)
   apply (erule_tac x=aa in allE, simp)
   done

lemma multiset_of_eq_setD:
  "multiset_of xs = multiset_of ys ⟹ set xs = set ys"
  by (rule) (auto simp add:multiset_eq_conv_count_eq set_count_greater_0)

lemma set_eq_iff_multiset_of_eq_distinct:
  "⟦distinct x; distinct y⟧
   ⟹ (set x = set y) = (multiset_of x = multiset_of y)"
  by (auto simp: multiset_eq_conv_count_eq distinct_count_atmost_1)

lemma set_eq_iff_multiset_of_remdups_eq:
   "(set x = set y) = (multiset_of (remdups x) = multiset_of (remdups y))"
```

```
  apply (rule iffI)
  apply (simp add: set_eq_iff_multiset_of_eq_distinct[THEN iffD1])
  apply (drule distinct_remdups[THEN distinct_remdups
                      [THEN set_eq_iff_multiset_of_eq_distinct[THEN iffD2]]])
  apply simp
  done

lemma multiset_of_compl_union[simp]:
 "multiset_of [x∈xs. P x] + multiset_of [x∈xs. ¬P x] = multiset_of xs"
  by (induct xs) (auto simp: union_ac)

lemma count_filter:
  "count (multiset_of xs) x = length [y ∈ xs. y = x]"
  by (induct xs, auto)
```

15.6 Pointwise ordering induced by count

```
consts
  mset_le :: "['a multiset, 'a multiset] ⇒ bool"

syntax
  "_mset_le" :: "'a multiset ⇒ 'a multiset ⇒ bool"   ("_ ≤# _"  [50,51]
50)
translations
  "x ≤# y" == "mset_le x y"

defs
  mset_le_def: "xs ≤# ys == (∀a. count xs a ≤ count ys a)"

lemma mset_le_refl[simp]: "xs ≤# xs"
  by (unfold mset_le_def) auto

lemma mset_le_trans: "⟦ xs ≤# ys; ys ≤# zs ⟧ ⟹ xs ≤# zs"
  by (unfold mset_le_def) (fast intro: order_trans)

lemma mset_le_antisym: "⟦ xs≤# ys; ys ≤# xs⟧ ⟹ xs = ys"
  apply (unfold mset_le_def)
  apply (rule multiset_eq_conv_count_eq[THEN iffD2])
  apply (blast intro: order_antisym)
  done

lemma mset_le_exists_conv:
  "(xs ≤# ys) = (∃zs. ys = xs + zs)"
  apply (unfold mset_le_def, rule iffI, rule_tac x = "ys - xs" in exI)
  apply (auto intro: multiset_eq_conv_count_eq [THEN iffD2])
  done

lemma mset_le_mono_add_right_cancel[simp]: "(xs + zs ≤# ys + zs) = (xs ≤#
ys)"
  by (unfold mset_le_def) auto

lemma mset_le_mono_add_left_cancel[simp]: "(zs + xs ≤# zs + ys) = (xs ≤#
ys)"
  by (unfold mset_le_def) auto
```

```
lemma mset_le_mono_add: "⟦ xs ≤# ys; vs ≤# ws ⟧ ⟹ xs + vs ≤# ys + ws"
  apply (unfold mset_le_def)
  apply auto
  apply (erule_tac x=a in allE)+
  apply auto
  done

lemma mset_le_add_left[simp]: "xs ≤# xs + ys"
  by (unfold mset_le_def) auto

lemma mset_le_add_right[simp]: "ys ≤# xs + ys"
  by (unfold mset_le_def) auto

lemma multiset_of_remdups_le: "multiset_of (remdups x) ≤# multiset_of x"
  apply (induct x)
   apply auto
  apply (rule mset_le_trans)
   apply auto
  done

end
```

16 The Follows Relation of Charpentier and Sivilotte

```
theory Follows imports SubstAx ListOrder Multiset begin

constdefs

  Follows :: "['a => 'b::{order}, 'a => 'b::{order}] => 'a program set"
                (infixl "Fols" 65)
   "f Fols g == Increasing g ∩ Increasing f Int
                Always {s. f s ≤ g s} Int
                (⋂k. {s. k ≤ g s} LeadsTo {s. k ≤ f s})"



lemma mono_Always_o:
     "mono h ==> Always {s. f s ≤ g s} ⊆ Always {s. h (f s) ≤ h (g s)}"
apply (simp add: Always_eq_includes_reachable)
apply (blast intro: monoD)
done

lemma mono_LeadsTo_o:
     "mono (h::'a::order => 'b::order)
      ==> (⋂j. {s. j ≤ g s} LeadsTo {s. j ≤ f s}) ⊆
          (⋂k. {s. k ≤ h (g s)} LeadsTo {s. k ≤ h (f s)})"
apply auto
apply (rule single_LeadsTo_I)
apply (drule_tac x = "g s" in spec)
apply (erule LeadsTo_weaken)
apply (blast intro: monoD order_trans)+
```

done

lemma *Follows_constant [iff]: "F ∈ (%s. c) Fols (%s. c)"*
by *(simp add: Follows_def)*

lemma *mono_Follows_o: "mono h ==> f Fols g ⊆ (h o f) Fols (h o g)"*
by *(auto simp add: Follows_def mono_Increasing_o [THEN [2] rev_subsetD]*
mono_Always_o [THEN [2] rev_subsetD]
mono_LeadsTo_o [THEN [2] rev_subsetD, THEN INT_D])

lemma *mono_Follows_apply:*
"mono h ==> f Fols g ⊆ (%x. h (f x)) Fols (%x. h (g x))"
apply *(drule mono_Follows_o)*
apply *(force simp add: o_def)*
done

lemma *Follows_trans:*
"[| F ∈ f Fols g; F ∈ g Fols h |] ==> F ∈ f Fols h"
apply *(simp add: Follows_def)*
apply *(simp add: Always_eq_includes_reachable)*
apply *(blast intro: order_trans LeadsTo_Trans)*
done

16.1 Destruction rules

lemma *Follows_Increasing1: "F ∈ f Fols g ==> F ∈ Increasing f"*
by *(simp add: Follows_def)*

lemma *Follows_Increasing2: "F ∈ f Fols g ==> F ∈ Increasing g"*
by *(simp add: Follows_def)*

lemma *Follows_Bounded: "F ∈ f Fols g ==> F ∈ Always {s. f s ⊆ g s}"*
by *(simp add: Follows_def)*

lemma *Follows_LeadsTo:*
"F ∈ f Fols g ==> F ∈ {s. k ≤ g s} LeadsTo {s. k ≤ f s}"
by *(simp add: Follows_def)*

lemma *Follows_LeadsTo_pfixLe:*
"F ∈ f Fols g ==> F ∈ {s. k pfixLe g s} LeadsTo {s. k pfixLe f s}"
apply *(rule single_LeadsTo_I, clarify)*
apply *(drule_tac k="g s"* **in** *Follows_LeadsTo)*
apply *(erule LeadsTo_weaken)*
apply *blast*
apply *(blast intro: pfixLe_trans prefix_imp_pfixLe)*
done

lemma *Follows_LeadsTo_pfixGe:*
"F ∈ f Fols g ==> F ∈ {s. k pfixGe g s} LeadsTo {s. k pfixGe f s}"
apply *(rule single_LeadsTo_I, clarify)*
apply *(drule_tac k="g s"* **in** *Follows_LeadsTo)*
apply *(erule LeadsTo_weaken)*
apply *blast*
apply *(blast intro: pfixGe_trans prefix_imp_pfixGe)*

done

lemma *Always_Follows1:*
```
"[| F ∈ Always {s. f s = f' s}; F ∈ f Fols g |] ==> F ∈ f' Fols g"
```

apply *(simp add: Follows_def Increasing_def Stable_def, auto)*
apply *(erule_tac [3] Always_LeadsTo_weaken)*
apply *(erule_tac A = "{s. z ≤ f s}"* **and** *A' = "{s. z ≤ f s}"*
in *Always_Constrains_weaken, auto)*
apply *(drule Always_Int_I, assumption)*
apply *(force intro: Always_weaken)*
done

lemma *Always_Follows2:*
```
"[| F ∈ Always {s. g s = g' s}; F ∈ f Fols g |] ==> F ∈ f Fols g'"
```
apply *(simp add: Follows_def Increasing_def Stable_def, auto)*
apply *(erule_tac [3] Always_LeadsTo_weaken)*
apply *(erule_tac A = "{s. z ≤ g s}"* **and** *A' = "{s. z ≤ g s}"*
in *Always_Constrains_weaken, auto)*
apply *(drule Always_Int_I, assumption)*
apply *(force intro: Always_weaken)*
done

16.2 Union properties (with the subset ordering)

lemma *increasing_Un:*
```
"[| F ∈ increasing f;  F ∈ increasing g |]
 ==> F ∈ increasing (%s. (f s) ∪ (g s))"
```
apply *(simp add: increasing_def stable_def constrains_def, auto)*
apply *(drule_tac x = "f xa"* **in** *spec)*
apply *(drule_tac x = "g xa"* **in** *spec)*
apply *(blast dest!: bspec)*
done

lemma *Increasing_Un:*
```
"[| F ∈ Increasing f;  F ∈ Increasing g |]
 ==> F ∈ Increasing (%s. (f s) ∪ (g s))"
```
apply *(auto simp add: Increasing_def Stable_def Constrains_def*
stable_def constrains_def)
apply *(drule_tac x = "f xa"* **in** *spec)*
apply *(drule_tac x = "g xa"* **in** *spec)*
apply *(blast dest!: bspec)*
done

lemma *Always_Un:*
```
"[| F ∈ Always {s. f' s ≤ f s}; F ∈ Always {s. g' s ≤ g s} |]
 ==> F ∈ Always {s. f' s ∪ g' s ≤ f s ∪ g s}"
```
by *(simp add: Always_eq_includes_reachable, blast)*

lemma *Follows_Un_lemma:*
```
"[| F ∈ Increasing f; F ∈ Increasing g;
```

```
        F ∈ Increasing g'; F ∈ Always {s. f' s ≤ f s};
        ∀k. F ∈ {s. k ≤ f s} LeadsTo {s. k ≤ f' s} |]
     ==> F ∈ {s. k ≤ f s ∪ g s} LeadsTo {s. k ≤ f' s ∪ g s}"
apply (rule single_LeadsTo_I)
apply (drule_tac x = "f s" in IncreasingD)
apply (drule_tac x = "g s" in IncreasingD)
apply (rule LeadsTo_weaken)
apply (rule PSP_Stable)
apply (erule_tac x = "f s" in spec)
apply (erule Stable_Int, assumption, blast+)
done

lemma Follows_Un:
    "[| F ∈ f' Fols f;  F ∈ g' Fols g |]
     ==> F ∈ (%s. (f' s) ∪ (g' s)) Fols (%s. (f s) ∪ (g s))"
apply (simp add: Follows_def Increasing_Un Always_Un del: Un_subset_iff, auto)
apply (rule LeadsTo_Trans)
apply (blast intro: Follows_Un_lemma)

apply (blast intro: Follows_Un_lemma [THEN LeadsTo_weaken])
done
```

16.3 Multiset union properties (with the multiset ordering)

```
lemma increasing_union:
    "[| F ∈ increasing f;  F ∈ increasing g |]
     ==> F ∈ increasing (%s. (f s) + (g s :: ('a::order) multiset))"
apply (simp add: increasing_def stable_def constrains_def, auto)
apply (drule_tac x = "f xa" in spec)
apply (drule_tac x = "g xa" in spec)
apply (drule bspec, assumption)
apply (blast intro: union_le_mono order_trans)
done

lemma Increasing_union:
    "[| F ∈ Increasing f;  F ∈ Increasing g |]
     ==> F ∈ Increasing (%s. (f s) + (g s :: ('a::order) multiset))"
apply (auto simp add: Increasing_def Stable_def Constrains_def
                      stable_def constrains_def)
apply (drule_tac x = "f xa" in spec)
apply (drule_tac x = "g xa" in spec)
apply (drule bspec, assumption)
apply (blast intro: union_le_mono order_trans)
done

lemma Always_union:
     "[| F ∈ Always {s. f' s ≤ f s}; F ∈ Always {s. g' s ≤ g s} |]
      ==> F ∈ Always {s. f' s + g' s ≤ f s + (g s :: ('a::order) multiset)}"
apply (simp add: Always_eq_includes_reachable)
apply (blast intro: union_le_mono)
done
```

```
lemma Follows_union_lemma:
     "[| F ∈ Increasing f; F ∈ Increasing g;
         F ∈ Increasing g'; F ∈ Always {s. f' s ≤ f s};
         ∀k::('a::order) multiset.
           F ∈ {s. k ≤ f s} LeadsTo {s. k ≤ f' s} |]
      ==> F ∈ {s. k ≤ f s + g s} LeadsTo {s. k ≤ f' s + g s}"
apply (rule single_LeadsTo_I)
apply (drule_tac x = "f s" in IncreasingD)
apply (drule_tac x = "g s" in IncreasingD)
apply (rule LeadsTo_weaken)
apply (rule PSP_Stable)
apply (erule_tac x = "f s" in spec)
apply (erule Stable_Int, assumption, blast)
apply (blast intro: union_le_mono order_trans)
done


lemma Follows_union:
     "!!g g' ::'b => ('a::order) multiset.
        [| F ∈ f' Fols f;  F ∈ g' Fols g |]
        ==> F ∈ (%s. (f' s) + (g' s)) Fols (%s. (f s) + (g s))"
apply (simp add: Follows_def)
apply (simp add: Increasing_union Always_union, auto)
apply (rule LeadsTo_Trans)
apply (blast intro: Follows_union_lemma)

apply (simp add: union_commute)
apply (blast intro: Follows_union_lemma)
done

lemma Follows_setsum:
     "!!f ::['c,'b] => ('a::order) multiset.
        [| ∀i ∈ I. F ∈ f' i Fols f i;  finite I |]
        ==> F ∈ (%s. ∑i ∈ I. f' i s) Fols (%s. ∑i ∈ I. f i s)"
apply (erule rev_mp)
apply (erule finite_induct, simp)
apply (simp add: Follows_union)
done



lemma Increasing_imp_Stable_pfixGe:
     "F ∈ Increasing func ==> F ∈ Stable {s. h pfixGe (func s)}"
apply (simp add: Increasing_def Stable_def Constrains_def constrains_def)
apply (blast intro: trans_Ge [THEN trans_genPrefix, THEN transD]
                    prefix_imp_pfixGe)
done


lemma LeadsTo_le_imp_pfixGe:
     "∀z. F ∈ {s. z ≤ f s} LeadsTo {s. z ≤ g s}
      ==> F ∈ {s. z pfixGe f s} LeadsTo {s. z pfixGe g s}"
apply (rule single_LeadsTo_I)
apply (drule_tac x = "f s" in spec)
```

```
apply (erule LeadsTo_weaken)
 prefer 2
 apply (blast intro: trans_Ge [THEN trans_genPrefix, THEN transD]
                     prefix_imp_pfixGe, blast)
done

end
```

17 Predicate Transformers

theory *Transformers* **imports** *Comp* **begin**

17.1 Defining the Predicate Transformers *wp*, *awp* and *wens*

constdefs

```
  wp :: "[('a*'a) set, 'a set] => 'a set"
```
— Dijkstra's weakest-precondition operator (for an individual command)
```
    "wp act B == - (act^-1 `` (-B))"

  awp :: "['a program, 'a set] => 'a set"
```
— Dijkstra's weakest-precondition operator (for a program)
```
    "awp F B == (⋂act ∈ Acts F. wp act B)"

  wens :: "['a program, ('a*'a) set, 'a set] => 'a set"
```
— The weakest-ensures transformer
```
    "wens F act B == gfp(λX. (wp act B ∩ awp F (B ∪ X)) ∪ B)"
```

The fundamental theorem for wp

theorem *wp_iff: "(A <= wp act B) = (act `` A <= B)"*
by *(force simp add: wp_def)*

This lemma is a good deal more intuitive than the definition!

lemma *in_wp_iff: "(a ∈ wp act B) = (∀x. (a,x) ∈ act --> x ∈ B)"*
by *(simp add: wp_def, blast)*

lemma *Compl_Domain_subset_wp: "- (Domain act) ⊆ wp act B"*
by *(force simp add: wp_def)*

lemma *wp_empty [simp]: "wp act {} = - (Domain act)"*
by *(force simp add: wp_def)*

The identity relation is the skip action

lemma *wp_Id [simp]: "wp Id B = B"*
by *(simp add: wp_def)*

lemma *wp_totalize_act:*
"wp (totalize_act act) B = (wp act B ∩ Domain act) ∪ (B - Domain act)"
by *(simp add: wp_def totalize_act_def, blast)*

lemma *awp_subset: "(awp F A ⊆ A)"*
by *(force simp add: awp_def wp_def)*

lemma *awp_Int_eq: "awp F (A∩B) = awp F A ∩ awp F B"*
by *(simp add: awp_def wp_def, blast)*

The fundamental theorem for awp

theorem *awp_iff_constrains: "(A <= awp F B) = (F ∈ A co B)"*
by *(simp add: awp_def constrains_def wp_iff INT_subset_iff)*

lemma *awp_iff_stable: "(A ⊆ awp F A) = (F ∈ stable A)"*
by *(simp add: awp_iff_constrains stable_def)*

lemma *stable_imp_awp_ident: "F ∈ stable A ==> awp F A = A"*
apply *(rule equalityI [OF awp_subset])*
apply *(simp add: awp_iff_stable)*
done

lemma *wp_mono: "(A ⊆ B) ==> wp act A ⊆ wp act B"*
by *(simp add: wp_def, blast)*

lemma *awp_mono: "(A ⊆ B) ==> awp F A ⊆ awp F B"*
by *(simp add: awp_def wp_def, blast)*

lemma *wens_unfold:*
"wens F act B = (wp act B ∩ awp F (B ∪ wens F act B)) ∪ B"
apply *(simp add: wens_def)*
apply *(rule gfp_unfold)*
apply *(simp add: mono_def wp_def awp_def, blast)*
done

lemma *wens_Id [simp]: "wens F Id B = B"*
by *(simp add: wens_def gfp_def wp_def awp_def, blast)*

These two theorems justify the claim that *wens* returns the weakest assertion satisfying the ensures property

lemma *ensures_imp_wens: "F ∈ A ensures B ==> ∃act ∈ Acts F. A ⊆ wens F act B"*
apply *(simp add: wens_def ensures_def transient_def, clarify)*
apply *(rule rev_bexI, assumption)*
apply *(rule gfp_upperbound)*
apply *(simp add: constrains_def awp_def wp_def, blast)*
done

lemma *wens_ensures: "act ∈ Acts F ==> F ∈ (wens F act B) ensures B"*
by *(simp add: wens_def gfp_def constrains_def awp_def wp_def*
ensures_def transient_def, blast)

These two results constitute assertion (4.13) of the thesis

lemma *wens_mono: "(A ⊆ B) ==> wens F act A ⊆ wens F act B"*
apply *(simp add: wens_def wp_def awp_def)*
apply *(rule gfp_mono, blast)*
done

lemma *wens_weakening: "B ⊆ wens F act B"*
by *(simp add: wens_def gfp_def, blast)*

Assertion (6), or 4.16 in the thesis

```
lemma subset_wens: "A-B ⊆ wp act B ∩ awp F (B ∪ A) ==> A ⊆ wens F act
B"
apply (simp add: wens_def wp_def awp_def)
apply (rule gfp_upperbound, blast)
done
```

Assertion 4.17 in the thesis

```
lemma Diff_wens_constrains: "F ∈ (wens F act A - A) co wens F act A"
by (simp add: wens_def gfp_def wp_def awp_def constrains_def, blast)
```

— Proved instantly, yet remarkably fragile. If *Un_subset_iff* is declared as an iff-rule, then it's almost impossible to prove. One proof is via *meson* after expanding all definitions, but it's slow!

Assertion (7): 4.18 in the thesis. NOTE that many of these results hold for an arbitrary action. We often do not require `act ∈ Acts F`

```
lemma stable_wens: "F ∈ stable A ==> F ∈ stable (wens F act A)"
apply (simp add: stable_def)
apply (drule constrains_Un [OF Diff_wens_constrains [of F act A]])
apply (simp add: Un_Int_distrib2 Compl_partition2)
apply (erule constrains_weaken, blast)
apply (simp add: Un_subset_iff wens_weakening)
done
```

Assertion 4.20 in the thesis.

```
lemma wens_Int_eq_lemma:
      "[|T-B ⊆ awp F T; act ∈ Acts F|]
       ==> T ∩ wens F act B ⊆ wens F act (T∩B)"
apply (rule subset_wens)
apply (rule_tac P="λx. ?f x ⊆ ?b" in ssubst [OF wens_unfold])
apply (simp add: wp_def awp_def, blast)
done
```

Assertion (8): 4.21 in the thesis. Here we indeed require `act ∈ Acts F`

```
lemma wens_Int_eq:
      "[|T-B ⊆ awp F T; act ∈ Acts F|]
       ==> T ∩ wens F act B = T ∩ wens F act (T∩B)"
apply (rule equalityI)
 apply (simp_all add: Int_lower1 Int_subset_iff)
 apply (rule wens_Int_eq_lemma, assumption+)
apply (rule subset_trans [OF _ wens_mono [of "T∩B" B]], auto)
done
```

17.2 Defining the Weakest Ensures Set

```
consts
  wens_set :: "['a program, 'a set] => 'a set set"

inductive "wens_set F B"
 intros

  Basis: "B ∈ wens_set F B"
```

```
  Wens:  "[|X ∈ wens_set F B; act ∈ Acts F|] ==> wens F act X ∈ wens_set
F B"

  Union: "W ≠ {} ==> ∀U ∈ W. U ∈ wens_set F B ==> ⋃W ∈ wens_set F B"

lemma wens_set_imp_co: "A ∈ wens_set F B ==> F ∈ (A-B) co A"
apply (erule wens_set.induct)
  apply (simp add: constrains_def)
 apply (drule_tac act1=act and A1=X
        in constrains_Un [OF Diff_wens_constrains])
 apply (erule constrains_weaken, blast)
 apply (simp add: Un_subset_iff wens_weakening)
apply (rule constrains_weaken)
apply (rule_tac I=W and A="λv. v-B" and A'="λv. v" in constrains_UN, blast+)
done

lemma wens_set_imp_leadsTo: "A ∈ wens_set F B ==> F ∈ A leadsTo B"
apply (erule wens_set.induct)
  apply (rule leadsTo_refl)
 apply (blast intro: wens_ensures leadsTo_Trans)
apply (blast intro: leadsTo_Union)
done

lemma leadsTo_imp_wens_set: "F ∈ A leadsTo B ==> ∃C ∈ wens_set F B. A ⊆
C"
apply (erule leadsTo_induct_pre)
  apply (blast dest!: ensures_imp_wens intro: wens_set.Basis wens_set.Wens)

 apply (clarify, drule ensures_weaken_R, assumption)
 apply (blast dest!: ensures_imp_wens intro: wens_set.Wens)
apply (case_tac "S={}")
 apply (simp, blast intro: wens_set.Basis)
apply (clarsimp dest!: bchoice simp: ball_conj_distrib Bex_def)
apply (rule_tac x = "⋃{Z. ∃U∈S. Z = f U}" in exI)
apply (blast intro: wens_set.Union)
done
```

Assertion (9): 4.27 in the thesis.

```
lemma leadsTo_iff_wens_set: "(F ∈ A leadsTo B) = (∃C ∈ wens_set F B. A
⊆ C)"
by (blast intro: leadsTo_imp_wens_set leadsTo_weaken_L wens_set_imp_leadsTo)
```

This is the result that requires the definition of `wens_set` to require `W` to be non-empty in the Unio case, for otherwise we should always have `{} ∈ wens_set F B`.

```
lemma wens_set_imp_subset: "A ∈ wens_set F B ==> B ⊆ A"
apply (erule wens_set.induct)
  apply (blast intro: wens_weakening [THEN subsetD])+
done
```

17.3 Properties Involving Program Union

Assertion (4.30) of thesis, reoriented

```
lemma awp_Join_eq: "awp (F⊔G) B = awp F B ∩ awp G B"
by (simp add: awp_def wp_def, blast)

lemma wens_subset: "wens F act B - B ⊆ wp act B ∩ awp F (B ∪ wens F act
B)"
by (subst wens_unfold, fast)
```

Assertion (4.31)

```
lemma subset_wens_Join:
      "[|A = T ∩ wens F act B;  T-B ⊆ awp F T; A-B ⊆ awp G (A ∪ B)|]
       ==> A ⊆ wens (F⊔G) act B"
apply (subgoal_tac "(T ∩ wens F act B) - B ⊆
                    wp act B ∩ awp F (B ∪ wens F act B) ∩ awp F T")
 apply (rule subset_wens)
 apply (simp add: awp_Join_eq awp_Int_eq Int_subset_iff Un_commute)
 apply (simp add: awp_def wp_def, blast)
apply (insert wens_subset [of F act B], blast)
done
```

Assertion (4.32)

```
lemma wens_Join_subset: "wens (F⊔G) act B ⊆ wens F act B"
apply (simp add: wens_def)
apply (rule gfp_mono)
apply (auto simp add: awp_Join_eq)
done
```

Lemma, because the inductive step is just too messy.

```
lemma wens_Union_inductive_step:
  assumes awpF: "T-B ⊆ awp F T"
      and awpG: "!!X. X ∈ wens_set F B ==> (T∩X) - B ⊆ awp G (T∩X)"
  shows "[|X ∈ wens_set F B; act ∈ Acts F; Y ⊆ X; T∩X = T∩Y|]
         ==> wens (F⊔G) act Y ⊆ wens F act X ∧
             T ∩ wens F act X = T ∩ wens (F⊔G) act Y"
apply (subgoal_tac "wens (F⊔G) act Y ⊆ wens F act X")
 prefer 2
 apply (blast dest: wens_mono intro: wens_Join_subset [THEN subsetD], simp)
apply (rule equalityI)
 prefer 2 apply blast
apply (simp add: Int_lower1 Int_subset_iff)
apply (frule wens_set_imp_subset)
apply (subgoal_tac "T-X ⊆ awp F T")
 prefer 2 apply (blast intro: awpF [THEN subsetD])
apply (rule_tac B = "wens (F⊔G) act (T∩X)" in subset_trans)
 prefer 2 apply (blast intro!: wens_mono)
apply (subst wens_Int_eq, assumption+)
apply (rule subset_wens_Join [of _ T], simp, blast)
apply (subgoal_tac "T ∩ wens F act (T∩X) ∪ T∩X = T ∩ wens F act X")
 prefer 2
 apply (subst wens_Int_eq [symmetric], assumption+)
 apply (blast intro: wens_weakening [THEN subsetD], simp)
apply (blast intro: awpG [THEN subsetD] wens_set.Wens)
done
```

```
theorem wens_Union:
  assumes awpF: "T-B ⊆ awp F T"
      and awpG: "!!X. X ∈ wens_set F B ==> (T∩X) - B ⊆ awp G (T∩X)"
      and major: "X ∈ wens_set F B"
  shows "∃Y ∈ wens_set (F⊔G) B. Y ⊆ X & T∩X = T∩Y"
apply (rule wens_set.induct [OF major])
```

Basis: trivial

```
  apply (blast intro: wens_set.Basis)
```

Inductive step

```
 apply clarify
 apply (rule_tac x = "wens (F⊔G) act Y" in rev_bexI)
  apply (force intro: wens_set.Wens)
 apply (simp add: wens_Union_inductive_step [OF awpF awpG])
```

Union: by Axiom of Choice

```
apply (simp add: ball_conj_distrib Bex_def)
apply (clarify dest!: bchoice)
apply (rule_tac x = "⋃{Z. ∃U∈W. Z = f U}" in exI)
apply (blast intro: wens_set.Union)
done

theorem leadsTo_Join:
  assumes leadsTo: "F ∈ A leadsTo B"
      and awpF: "T-B ⊆ awp F T"
      and awpG: "!!X. X ∈ wens_set F B ==> (T∩X) - B ⊆ awp G (T∩X)"
  shows "F⊔G ∈ T∩A leadsTo B"
apply (rule leadsTo [THEN leadsTo_imp_wens_set, THEN bexE])
apply (rule wens_Union [THEN bexE])
   apply (rule awpF)
  apply (erule awpG, assumption)
apply (blast intro: wens_set_imp_leadsTo [THEN leadsTo_weaken_L])
done
```

17.4 The Set *wens_set F B* for a Single-Assignment Program

Thesis Section 4.3.3

We start by proving laws about single-assignment programs

```
lemma awp_single_eq [simp]:
     "awp (mk_program (init, {act}, allowed)) B = B ∩ wp act B"
by (force simp add: awp_def wp_def)

lemma wp_Un_subset: "wp act A ∪ wp act B ⊆ wp act (A ∪ B)"
by (force simp add: wp_def)

lemma wp_Un_eq: "single_valued act ==> wp act (A ∪ B) = wp act A ∪ wp act
B"
apply (rule equalityI)
 apply (force simp add: wp_def single_valued_def)
apply (rule wp_Un_subset)
done
```

```
lemma wp_UN_subset: "(⋃i∈I. wp act (A i)) ⊆ wp act (⋃i∈I. A i)"
by (force simp add: wp_def)

lemma wp_UN_eq:
     "[|single_valued act; I≠{}|]
      ==> wp act (⋃i∈I. A i) = (⋃i∈I. wp act (A i))"
apply (rule equalityI)
 prefer 2 apply (rule wp_UN_subset)
 apply (simp add: wp_def Image_INT_eq)
done

lemma wens_single_eq:
     "wens (mk_program (init, {act}, allowed)) act B = B ∪ wp act B"
by (simp add: wens_def gfp_def wp_def, blast)
```

Next, we express the wens_set for single-assignment programs

```
constdefs
  wens_single_finite :: "[('a*'a) set, 'a set, nat] => 'a set"
    "wens_single_finite act B k == ⋃i ∈ atMost k. ((wp act)^i) B"

  wens_single :: "[('a*'a) set, 'a set] => 'a set"
    "wens_single act B == ⋃i. ((wp act)^i) B"

lemma wens_single_Un_eq:
      "single_valued act
       ==> wens_single act B ∪ wp act (wens_single act B) = wens_single act
B"
apply (rule equalityI)
 apply (simp_all add: Un_upper1 Un_subset_iff)
apply (simp add: wens_single_def wp_UN_eq, clarify)
apply (rule_tac a="Suc(i)" in UN_I, auto)
done

lemma atMost_nat_nonempty: "atMost (k::nat) ≠ {}"
by force

lemma wens_single_finite_0 [simp]: "wens_single_finite act B 0 = B"
by (simp add: wens_single_finite_def)

lemma wens_single_finite_Suc:
      "single_valued act
       ==> wens_single_finite act B (Suc k) =
           wens_single_finite act B k ∪ wp act (wens_single_finite act B k)"
apply (simp add: wens_single_finite_def image_def
                 wp_UN_eq [OF _ atMost_nat_nonempty])
apply (force elim!: le_SucE)
done

lemma wens_single_finite_Suc_eq_wens:
     "single_valued act
       ==> wens_single_finite act B (Suc k) =
           wens (mk_program (init, {act}, allowed)) act
                (wens_single_finite act B k)"
```

```
by (simp add: wens_single_finite_Suc wens_single_eq)

lemma def_wens_single_finite_Suc_eq_wens:
     "[|F = mk_program (init, {act}, allowed); single_valued act|]
      ==> wens_single_finite act B (Suc k) =
          wens F act (wens_single_finite act B k)"
by (simp add: wens_single_finite_Suc_eq_wens)

lemma wens_single_finite_Un_eq:
      "single_valued act
       ==> wens_single_finite act B k ∪ wp act (wens_single_finite act B k)
           ∈ range (wens_single_finite act B)"
by (simp add: wens_single_finite_Suc [symmetric])

lemma wens_single_eq_Union:
      "wens_single act B = ⋃range (wens_single_finite act B)"
by (simp add: wens_single_finite_def wens_single_def, blast)

lemma wens_single_finite_eq_Union:
     "wens_single_finite act B n = (⋃k∈atMost n. wens_single_finite act B
k)"
apply (auto simp add: wens_single_finite_def)
apply (blast intro: le_trans)
done

lemma wens_single_finite_mono:
     "m ≤ n ==> wens_single_finite act B m ⊆ wens_single_finite act B n"
by (force simp add:  wens_single_finite_eq_Union [of act B n])

lemma wens_single_finite_subset_wens_single:
      "wens_single_finite act B k ⊆ wens_single act B"
by (simp add: wens_single_eq_Union, blast)

lemma subset_wens_single_finite:
      "[|W ⊆ wens_single_finite act B ` (atMost k); single_valued act; W≠{}|]
       ==> ∃m. ⋃W = wens_single_finite act B m"
apply (induct k)
 apply (rule_tac x=0 in exI, simp, blast)
apply (auto simp add: atMost_Suc)
apply (case_tac "wens_single_finite act B (Suc k) ∈ W")
 prefer 2 apply blast
apply (drule_tac x="Suc k" in spec)
apply (erule notE, rule equalityI)
 prefer 2 apply blast
apply (subst wens_single_finite_eq_Union)
apply (simp add: atMost_Suc, blast)
done
```

lemma for Union case

```
lemma Union_eq_wens_single:
      "⟦∀k. ¬ W ⊆ wens_single_finite act B ` {..k};
        W ⊆ insert (wens_single act B)
            (range (wens_single_finite act B))⟧
       ⟹ ⋃W = wens_single act B"
```

```
apply (case_tac "wens_single act B ∈ W")
 apply (blast dest: wens_single_finite_subset_wens_single [THEN subsetD])

apply (simp add: wens_single_eq_Union)
apply (rule equalityI, blast)
apply (simp add: UN_subset_iff, clarify)
apply (subgoal_tac "∃y∈W. ∃n. y = wens_single_finite act B n & i≤n")
 apply (blast intro: wens_single_finite_mono [THEN subsetD])
apply (drule_tac x=i in spec)
apply (force simp add: atMost_def)
done

lemma wens_set_subset_single:
      "single_valued act
       ==> wens_set (mk_program (init, {act}, allowed)) B ⊆
           insert (wens_single act B) (range (wens_single_finite act B))"
apply (rule subsetI)
apply (erule wens_set.induct)
```

Basis

```
  apply (force simp add: wens_single_finite_def)
```

Wens inductive step

```
 apply (case_tac "acta = Id", simp)
 apply (simp add: wens_single_eq)
 apply (elim disjE)
 apply (simp add: wens_single_Un_eq)
 apply (force simp add: wens_single_finite_Un_eq)
```

Union inductive step

```
apply (case_tac "∃k. W ⊆ wens_single_finite act B ` (atMost k)")
 apply (blast dest!: subset_wens_single_finite, simp)
apply (rule disjI1 [OF Union_eq_wens_single], blast+)
done

lemma wens_single_finite_in_wens_set:
      "single_valued act ⟹
         wens_single_finite act B k
         ∈ wens_set (mk_program (init, {act}, allowed)) B"
apply (induct_tac k)
 apply (simp add: wens_single_finite_def wens_set.Basis)
apply (simp add: wens_set.Wens
                 wens_single_finite_Suc_eq_wens [of act B _ init allowed])

done

lemma single_subset_wens_set:
      "single_valued act
       ==> insert (wens_single act B) (range (wens_single_finite act B)) ⊆

           wens_set (mk_program (init, {act}, allowed)) B"
apply (simp add: wens_single_eq_Union UN_eq)
apply (blast intro: wens_set.Union wens_single_finite_in_wens_set)
done
```

Theorem (4.29)

```
theorem wens_set_single_eq:
     "[|F = mk_program (init, {act}, allowed); single_valued act|]
      ==> wens_set F B =
          insert (wens_single act B) (range (wens_single_finite act B))"
apply (rule equalityI)
 apply (simp add: wens_set_subset_single)
apply (erule ssubst, erule single_subset_wens_set)
done
```

Generalizing Misra's Fixed Point Union Theorem (4.41)

```
lemma fp_leadsTo_Join:
    "[|T-B ⊆ awp F T; T-B ⊆ FP G; F ∈ A leadsTo B|] ==> F⊔G ∈ T∩A leadsTo
B"
apply (rule leadsTo_Join, assumption, blast)
apply (simp add: FP_def awp_iff_constrains stable_def constrains_def, blast)

done

end
```

18 Progress Sets

```
theory ProgressSets imports Transformers begin
```

18.1 Complete Lattices and the Operator *cl*

```
constdefs
  lattice :: "'a set set => bool"
   — Meier calls them closure sets, but they are just complete lattices
   "lattice L ==
         (∀M. M ⊆ L --> ⋂M ∈ L) & (∀M. M ⊆ L --> ⋃M ∈ L)"

  cl :: "['a set set, 'a set] => 'a set"
   — short for "closure"
   "cl L r == ⋂{x. x∈L & r ⊆ x}"

lemma UNIV_in_lattice: "lattice L ==> UNIV ∈ L"
by (force simp add: lattice_def)

lemma empty_in_lattice: "lattice L ==> {} ∈ L"
by (force simp add: lattice_def)

lemma Union_in_lattice: "[|M ⊆ L; lattice L|] ==> ⋃M ∈ L"
by (simp add: lattice_def)

lemma Inter_in_lattice: "[|M ⊆ L; lattice L|] ==> ⋂M ∈ L"
by (simp add: lattice_def)

lemma UN_in_lattice:
     "[|lattice L; !!i. i∈I ==> r i ∈ L|] ==> (⋃i∈I. r i) ∈ L"
apply (simp add: Set.UN_eq)
```

```
apply (blast intro: Union_in_lattice)
done

lemma INT_in_lattice:
     "[|lattice L; !!i. i∈I ==> r i ∈ L|] ==> (⋂i∈I. r i)  ∈ L"
apply (simp add: INT_eq)
apply (blast intro: Inter_in_lattice)
done

lemma Un_in_lattice: "[|x∈L; y∈L; lattice L|] ==> x∪y ∈ L"
apply (simp only: Un_eq_Union)
apply (blast intro: Union_in_lattice)
done

lemma Int_in_lattice: "[|x∈L; y∈L; lattice L|] ==> x∩y ∈ L"
apply (simp only: Int_eq_Inter)
apply (blast intro: Inter_in_lattice)
done

lemma lattice_stable: "lattice {X. F ∈ stable X}"
by (simp add: lattice_def stable_def constrains_def, blast)
```

The next three results state that `cl L r` is the minimal element of `L` that includes `r`.

```
lemma cl_in_lattice: "lattice L ==> cl L r ∈ L"
apply (simp add: lattice_def cl_def)
apply (erule conjE)
apply (drule spec, erule mp, blast)
done

lemma cl_least: "[|c∈L; r⊆c|] ==> cl L r ⊆ c"
by (force simp add: cl_def)
```

The next three lemmas constitute assertion (4.61)

```
lemma cl_mono: "r ⊆ r' ==> cl L r ⊆ cl L r'"
by (simp add: cl_def, blast)

lemma subset_cl: "r ⊆ cl L r"
by (simp add: cl_def, blast)
```

A reformulation of `?r ⊆ cl ?L ?r`

```
lemma clI: "x ∈ r ==> x ∈ cl L r"
by (simp add: cl_def, blast)
```

A reformulation of ⟦`?c ∈ ?L; ?r ⊆ ?c`⟧ ⟹ `cl ?L ?r ⊆ ?c`

```
lemma clD: "[|c ∈ cl L r; B ∈ L; r ⊆ B|] ==> c ∈ B"
by (force simp add: cl_def)

lemma cl_UN_subset: "(⋃i∈I. cl L (r i)) ⊆ cl L (⋃i∈I. r i)"
by (simp add: cl_def, blast)

lemma cl_Un: "lattice L ==> cl L (r∪s) = cl L r ∪ cl L s"
apply (rule equalityI)
```

```
 prefer 2
  apply (simp add: cl_def, blast)
apply (rule cl_least)
 apply (blast intro: Un_in_lattice cl_in_lattice)
apply (blast intro: subset_cl [THEN subsetD])
done

lemma cl_UN: "lattice L ==> cl L (⋃i∈I. r i) = (⋃i∈I. cl L (r i))"
apply (rule equalityI)
 prefer 2 apply (simp add: cl_def, blast)
apply (rule cl_least)
 apply (blast intro: UN_in_lattice cl_in_lattice)
apply (blast intro: subset_cl [THEN subsetD])
done

lemma cl_Int_subset: "cl L (r∩s) ⊆ cl L r ∩ cl L s"
by (simp add: cl_def, blast)

lemma cl_idem [simp]: "cl L (cl L r) = cl L r"
by (simp add: cl_def, blast)

lemma cl_ident: "r∈L ==> cl L r = r"
by (force simp add: cl_def)

lemma cl_empty [simp]: "lattice L ==> cl L {} = {}"
by (simp add: cl_ident empty_in_lattice)

lemma cl_UNIV [simp]: "lattice L ==> cl L UNIV = UNIV"
by (simp add: cl_ident UNIV_in_lattice)
```

Assertion (4.62)

```
lemma cl_ident_iff: "lattice L ==> (cl L r = r) = (r∈L)"
apply (rule iffI)
 apply (erule subst)
 apply (erule cl_in_lattice)
apply (erule cl_ident)
done

lemma cl_subset_in_lattice: "[|cl L r ⊆ r; lattice L|] ==> r∈L"
by (simp add: cl_ident_iff [symmetric] equalityI subset_cl)
```

18.2 Progress Sets and the Main Lemma

A progress set satisfies certain closure conditions and is a simple way of including the set *wens_set F B*.

```
constdefs
  closed :: "['a program, 'a set, 'a set,  'a set set] => bool"
   "closed F T B L == ∀M. ∀act ∈ Acts F. B⊆M & T∩M ∈ L -->
                              T ∩ (B ∪ wp act M) ∈ L"

  progress_set :: "['a program, 'a set, 'a set] => 'a set set set"
   "progress_set F T B ==
      {L. lattice L & B ∈ L & T ∈ L & closed F T B L}"
```

```
lemma closedD:
   "[|closed F T B L; act ∈ Acts F; B⊆M; T∩M ∈ L|]
    ==> T ∩ (B ∪ wp act M) ∈ L"
by (simp add: closed_def)
```

Note: the formalization below replaces Meier's `q` by `B` and `m` by `X`.

Part of the proof of the claim at the bottom of page 97. It's proved separately because the argument requires a generalization over all `act ∈ Acts F`.

```
lemma lattice_awp_lemma:
  assumes TXC:  "T∩X ∈ C" — induction hypothesis in theorem below
      and BsubX:  "B ⊆ X"   — holds in inductive step
      and latt: "lattice C"
      and TC:   "T ∈ C"
      and BC:   "B ∈ C"
      and clos: "closed F T B C"
    shows "T ∩ (B ∪ awp F (X ∪ cl C (T∩r))) ∈ C"
apply (simp del: INT_simps add: awp_def INT_extend_simps)
apply (rule INT_in_lattice [OF latt])
apply (erule closedD [OF clos])
apply (simp add: subset_trans [OF BsubX Un_upper1])
apply (subgoal_tac "T ∩ (X ∪ cl C (T∩r)) = (T∩X) ∪ cl C (T∩r)")
 prefer 2 apply (blast intro: TC clD)
apply (erule ssubst)
apply (blast intro: Un_in_lattice latt cl_in_lattice TXC)
done
```

Remainder of the proof of the claim at the bottom of page 97.

```
lemma lattice_lemma:
  assumes TXC:  "T∩X ∈ C" — induction hypothesis in theorem below
      and BsubX:  "B ⊆ X"   — holds in inductive step
      and act:  "act ∈ Acts F"
      and latt: "lattice C"
      and TC:   "T ∈ C"
      and BC:   "B ∈ C"
      and clos: "closed F T B C"
    shows "T ∩ (wp act X ∩ awp F (X ∪ cl C (T∩r)) ∪ X) ∈ C"
apply (subgoal_tac "T ∩ (B ∪ wp act X) ∈ C")
 prefer 2 apply (simp add: closedD [OF clos] act BsubX TXC)
apply (drule Int_in_lattice
             [OF _ lattice_awp_lemma [OF TXC BsubX latt TC BC clos, of r]
                  latt])
apply (subgoal_tac
         "T ∩ (B ∪ wp act X) ∩ (T ∩ (B ∪ awp F (X ∪ cl C (T∩r)))) =
          T ∩ (B ∪ wp act X ∩ awp F (X ∪ cl C (T∩r)))")
 prefer 2 apply blast
apply simp
apply (drule Un_in_lattice [OF _ TXC latt])
apply (subgoal_tac
         "T ∩ (B ∪ wp act X ∩ awp F (X ∪ cl C (T∩r))) ∪ T∩X =
          T ∩ (wp act X ∩ awp F (X ∪ cl C (T∩r)) ∪ X)")
 apply simp
apply (blast intro: BsubX [THEN subsetD])
```

done

Induction step for the main lemma

```
lemma progress_induction_step:
  assumes TXC:  "T∩X ∈ C" — induction hypothesis in theorem below
      and act:  "act ∈ Acts F"
      and Xwens: "X ∈ wens_set F B"
      and latt: "lattice C"
      and  TC:  "T ∈ C"
      and  BC:  "B ∈ C"
      and clos: "closed F T B C"
      and Fstable: "F ∈ stable T"
  shows "T ∩ wens F act X ∈ C"
proof -
  from Xwens have BsubX: "B ⊆ X"
    by (rule wens_set_imp_subset)
  let ?r = "wens F act X"
  have "?r ⊆ (wp act X ∩ awp F (X∪?r)) ∪ X"
    by (simp add: wens_unfold [symmetric])
  then have "T∩?r ⊆ T ∩ ((wp act X ∩ awp F (X∪?r)) ∪ X)"
    by blast
  then have "T∩?r ⊆ T ∩ ((wp act X ∩ awp F (T ∩ (X∪?r))) ∪ X)"
    by (simp add: awp_Int_eq Fstable stable_imp_awp_ident, blast)
  then have "T∩?r ⊆ T ∩ ((wp act X ∩ awp F (X ∪ cl C (T∩?r))) ∪ X)"
    by (blast intro: awp_mono [THEN [2] rev_subsetD] subset_cl [THEN subsetD])
  then have "cl C (T∩?r) ⊆
             cl C (T ∩ ((wp act X ∩ awp F (X ∪ cl C (T∩?r))) ∪ X))"
    by (rule cl_mono)
  then have "cl C (T∩?r) ⊆
             T ∩ ((wp act X ∩ awp F (X ∪ cl C (T∩?r))) ∪ X)"
    by (simp add: cl_ident lattice_lemma [OF TXC BsubX act latt TC BC clos])
  then have "cl C (T∩?r) ⊆ (wp act X ∩ awp F (X ∪ cl C (T∩?r))) ∪ X"
    by blast
  then have "cl C (T∩?r) ⊆ ?r"
    by (blast intro!: subset_wens)
  then have cl_subset: "cl C (T∩?r) ⊆ T∩?r"
    by (simp add: Int_subset_iff cl_ident TC
                  subset_trans [OF cl_mono [OF Int_lower1]])
  show ?thesis
    by (rule cl_subset_in_lattice [OF cl_subset latt])
qed
```

Proved on page 96 of Meier's thesis. The special case when `T = UNIV` states that every progress set for the program `F` and set `B` includes the set `wens_set F B`.

```
lemma progress_set_lemma:
     "[|C ∈ progress_set F T B; r ∈ wens_set F B; F ∈ stable T|] ==> T∩r
∈ C"
apply (simp add: progress_set_def, clarify)
apply (erule wens_set.induct)
```

Base

```
  apply (simp add: Int_in_lattice)
```

The difficult `wens` case

```
 apply (simp add: progress_induction_step)
```

Disjunctive case

```
apply (subgoal_tac "(⋃U∈W. T ∩ U) ∈ C")
 apply (simp add: Int_Union)
apply (blast intro: UN_in_lattice)
done
```

18.3 The Progress Set Union Theorem

```
lemma closed_mono:
  assumes BB': "B ⊆ B'"
      and TBwp: "T ∩ (B ∪ wp act M) ∈ C"
      and B'C: "B' ∈ C"
      and TC: "T ∈ C"
      and latt: "lattice C"
  shows "T ∩ (B' ∪ wp act M) ∈ C"
proof -
  from TBwp have "(T∩B) ∪ (T ∩ wp act M) ∈ C"
    by (simp add: Int_Un_distrib)
  then have TBBC: "(T∩B') ∪ ((T∩B) ∪ (T ∩ wp act M)) ∈ C"
    by (blast intro: Int_in_lattice Un_in_lattice TC B'C latt)
  show ?thesis
    by (rule eqelem_imp_iff [THEN iffD1, OF _ TBBC],
        blast intro: BB' [THEN subsetD])
qed


lemma progress_set_mono:
    assumes BB': "B ⊆ B'"
    shows
     "[| B' ∈ C;  C ∈ progress_set F T B|]
      ==> C ∈ progress_set F T B'"
by (simp add: progress_set_def closed_def closed_mono [OF BB']
                subset_trans [OF BB'])

theorem progress_set_Union:
  assumes leadsTo: "F ∈ A leadsTo B'"
      and prog: "C ∈ progress_set F T B"
      and Fstable: "F ∈ stable T"
      and BB': "B ⊆ B'"
      and B'C: "B' ∈ C"
      and Gco: "!!X. X∈C ==> G ∈ X-B co X"
  shows "F⊔G ∈ T∩A leadsTo B'"
apply (insert prog Fstable)
apply (rule leadsTo_Join [OF leadsTo])
  apply (force simp add: progress_set_def awp_iff_stable [symmetric])
apply (simp add: awp_iff_constrains)
apply (drule progress_set_mono [OF BB' B'C])
apply (blast intro: progress_set_lemma Gco constrains_weaken_L
                  BB' [THEN subsetD])
done
```

18.4 Some Progress Sets

lemma *UNIV_in_progress_set: "UNIV ∈ progress_set F T B"*
by *(simp add: progress_set_def lattice_def closed_def)*

18.4.1 Lattices and Relations

From Meier's thesis, section 4.5.3

constdefs
*relcl :: "'a set set => ('a * 'a) set"*
— Derived relation from a lattice
"relcl L == {(x,y). y ∈ cl L {x}}"

*latticeof :: "('a * 'a) set => 'a set set"*
— Derived lattice from a relation: the set of upwards-closed sets
"latticeof r == {X. ∀s t. s ∈ X & (s,t) ∈ r --> t ∈ X}"

lemma *relcl_refl: "(a,a) ∈ relcl L"*
by *(simp add: relcl_def subset_cl [THEN subsetD])*

lemma *relcl_trans:*
"[| (a,b) ∈ relcl L; (b,c) ∈ relcl L; lattice L |] ==> (a,c) ∈ relcl L"
apply *(simp add: relcl_def)*
apply *(blast intro: clD cl_in_lattice)*
done

lemma *refl_relcl: "lattice L ==> refl UNIV (relcl L)"*
by *(simp add: reflI relcl_def subset_cl [THEN subsetD])*

lemma *trans_relcl: "lattice L ==> trans (relcl L)"*
by *(blast intro: relcl_trans transI)*

lemma *lattice_latticeof: "lattice (latticeof r)"*
by *(auto simp add: lattice_def latticeof_def)*

lemma *lattice_singletonI:*
"[|lattice L; !!s. s ∈ X ==> {s} ∈ L|] ==> X ∈ L"
apply *(cut_tac UN_singleton [of X])*
apply *(erule subst)*
apply *(simp only: UN_in_lattice)*
done

Equation (4.71) of Meier's thesis. He gives no proof.

lemma *cl_latticeof:*
"[|refl UNIV r; trans r|]
==> cl (latticeof r) X = {t. ∃s. s∈X & (s,t) ∈ r}"
apply *(rule equalityI)*
apply *(rule cl_least)*
apply *(simp (no_asm_use) add: latticeof_def trans_def, blast)*
apply *(simp add: latticeof_def refl_def, blast)*
apply *(simp add: latticeof_def, clarify)*
apply *(unfold cl_def, blast)*

done

Related to (4.71).

```
lemma cl_eq_Collect_relcl:
     "lattice L ==> cl L X = {t. ∃s. s∈X & (s,t) ∈ relcl L}"
apply (cut_tac UN_singleton [of X])
apply (erule subst)
apply (force simp only: relcl_def cl_UN)
done
```

Meier's theorem of section 4.5.3

```
theorem latticeof_relcl_eq: "lattice L ==> latticeof (relcl L) = L"
apply (rule equalityI)
 prefer 2 apply (force simp add: latticeof_def relcl_def cl_def, clarify)

apply (rename_tac X)
apply (rule cl_subset_in_lattice)
 prefer 2 apply assumption
apply (drule cl_ident_iff [OF lattice_latticeof, THEN iffD2])
apply (drule equalityD1)
apply (rule subset_trans)
 prefer 2 apply assumption
apply (thin_tac "?U ⊆ X")
apply (cut_tac A=X in UN_singleton)
apply (erule subst)
apply (simp only: cl_UN lattice_latticeof
                  cl_latticeof [OF refl_relcl trans_relcl])
apply (simp add: relcl_def)
done

theorem relcl_latticeof_eq:
     "[|refl UNIV r; trans r|] ==> relcl (latticeof r) = r"
by (simp add: relcl_def cl_latticeof, blast)
```

18.4.2 Decoupling Theorems

```
constdefs
  decoupled :: "['a program, 'a program] => bool"
   "decoupled F G ==
        ∀act ∈ Acts F. ∀B. G ∈ stable B --> G ∈ stable (wp act B)"
```

Rao's Decoupling Theorem

```
lemma stableco: "F ∈ stable A ==> F ∈ A-B co A"
by (simp add: stable_def constrains_def, blast)

theorem decoupling:
  assumes leadsTo: "F ∈ A leadsTo B"
      and Gstable: "G ∈ stable B"
      and dec:     "decoupled F G"
  shows "F⊔G ∈ A leadsTo B"
proof -
  have prog: "{X. G ∈ stable X} ∈ progress_set F UNIV B"
    by (simp add: progress_set_def lattice_stable Gstable closed_def
```

```
                stable_Un [OF Gstable] dec [unfolded decoupled_def])
  have "F⊔G ∈ (UNIV∩A) leadsTo B"
    by (rule progress_set_Union [OF leadsTo prog],
        simp_all add: Gstable stableco)
  thus ?thesis by simp
qed
```

Rao's Weak Decoupling Theorem

```
theorem weak_decoupling:
  assumes leadsTo: "F ∈ A leadsTo B"
      and stable: "F⊔G ∈ stable B"
      and dec:     "decoupled F (F⊔G)"
  shows "F⊔G ∈ A leadsTo B"
proof -
  have prog: "{X. F⊔G ∈ stable X} ∈ progress_set F UNIV B"
    by (simp del: Join_stable
             add: progress_set_def lattice_stable stable closed_def
                  stable_Un [OF stable] dec [unfolded decoupled_def])
  have "F⊔G ∈ (UNIV∩A) leadsTo B"
    by (rule progress_set_Union [OF leadsTo prog],
        simp_all del: Join_stable add: stable,
        simp add: stableco)
  thus ?thesis by simp
qed
```

The "Decoupling via `G'` Union Theorem"

```
theorem decoupling_via_aux:
  assumes leadsTo: "F ∈ A leadsTo B"
      and prog: "{X. G' ∈ stable X} ∈ progress_set F UNIV B"
      and GG':  "G ≤ G'"
                — Beware! This is the converse of the refinement relation!
  shows "F⊔G ∈ A leadsTo B"
proof -
  from prog have stable: "G' ∈ stable B"
    by (simp add: progress_set_def)
  have "F⊔G ∈ (UNIV∩A) leadsTo B"
    by (rule progress_set_Union [OF leadsTo prog],
        simp_all add: stable stableco component_stable [OF GG'])
  thus ?thesis by simp
qed
```

18.5 Composition Theorems Based on Monotonicity and Commutativity

18.5.1 Commutativity of `cl L` and assignment.

```
constdefs
  commutes :: "['a program, 'a set, 'a set,  'a set set] => bool"
   "commutes F T B L ==
       ∀M. ∀act ∈ Acts F. B ⊆ M -->
           cl L (T ∩ wp act M) ⊆ T ∩ (B ∪ wp act (cl L (T∩M)))"
```

From Meier's thesis, section 4.5.6

```
lemma commutativity1_lemma:
```

```
  assumes commutes: "commutes F T B L"
      and lattice:  "lattice L"
      and BL: "B ∈ L"
      and TL: "T ∈ L"
  shows "closed F T B L"
apply (simp add: closed_def, clarify)
apply (rule ProgressSets.cl_subset_in_lattice [OF _ lattice])
apply (simp add: Int_Un_distrib cl_Un [OF lattice] Un_subset_iff
                 cl_ident Int_in_lattice [OF TL BL lattice] Un_upper1)
apply (subgoal_tac "cl L (T ∩ wp act M) ⊆ T ∩ (B ∪ wp act (cl L (T ∩ M)))")

 prefer 2
 apply (cut_tac commutes, simp add: commutes_def)
apply (erule subset_trans)
apply (simp add: cl_ident)
apply (blast intro: rev_subsetD [OF _ wp_mono])
done
```

Version packaged with ⟦*?F ∈ ?A leadsTo ?B'; ?C ∈ progress_set ?F ?T ?B; ?F ∈ stable ?T; ?B ⊆ ?B'; ?B' ∈ ?C;* ⋀*X. X ∈ ?C* ⟹ *?G ∈ X - ?B co X*⟧ ⟹ *?F ⊔ ?G ∈ ?T ∩ ?A leadsTo ?B'*

```
lemma commutativity1:
  assumes leadsTo: "F ∈ A leadsTo B"
      and lattice:  "lattice L"
      and BL: "B ∈ L"
      and TL: "T ∈ L"
      and Fstable: "F ∈ stable T"
      and Gco: "!!X. X∈L ==> G ∈ X-B co X"
      and commutes: "commutes F T B L"
  shows "F⊔G ∈ T∩A leadsTo B"
by (rule progress_set_Union [OF leadsTo _ Fstable subset_refl BL Gco],
    simp add: progress_set_def commutativity1_lemma commutes lattice BL TL)
```

Possibly move to Relation.thy, after *single_valued*

```
constdefs
  funof :: "[('a*'b)set, 'a] => 'b"
   "funof r == (λx. THE y. (x,y) ∈ r)"

lemma funof_eq: "[|single_valued r; (x,y) ∈ r|] ==> funof r x = y"
by (simp add: funof_def single_valued_def, blast)

lemma funof_Pair_in:
     "[|single_valued r; x ∈ Domain r|] ==> (x, funof r x) ∈ r"
by (force simp add: funof_eq)

lemma funof_in:
     "[|r``{x} ⊆ A; single_valued r; x ∈ Domain r|] ==> funof r x ∈ A"
by (force simp add: funof_eq)

lemma funof_imp_wp: "[|funof act t ∈ A; single_valued act|] ==> t ∈ wp act
A"
by (force simp add: in_wp_iff funof_eq)
```

18.5.2 Commutativity of Functions and Relation

Thesis, page 109

From Meier's thesis, section 4.5.6

```
lemma commutativity2_lemma:
  assumes dcommutes:
        "∀act ∈ Acts F.
         ∀s ∈ T. ∀t. (s,t) ∈ relcl L -->
                     s ∈ B | t ∈ B | (funof act s, funof act t) ∈ relcl
L"
      and determ: "!!act. act ∈ Acts F ==> single_valued act"
      and total: "!!act. act ∈ Acts F ==> Domain act = UNIV"
      and lattice:  "lattice L"
      and BL: "B ∈ L"
      and TL: "T ∈ L"
      and Fstable: "F ∈ stable T"
  shows  "commutes F T B L"
apply (simp add: commutes_def del: Int_subset_iff, clarify)
apply (rename_tac t)
apply (subgoal_tac "∃s. (s,t) ∈ relcl L & s ∈ T ∩ wp act M")
 prefer 2
 apply (force simp add: cl_eq_Collect_relcl [OF lattice], simp, clarify)
apply (subgoal_tac "∀u∈L. s ∈ u --> t ∈ u")
 prefer 2
 apply (intro ballI impI)
 apply (subst cl_ident [symmetric], assumption)
 apply (simp add: relcl_def)
 apply (blast intro: cl_mono [THEN [2] rev_subsetD])
apply (subgoal_tac "funof act s ∈ T∩M")
 prefer 2
 apply (cut_tac Fstable)
 apply (force intro!: funof_in
              simp add: wp_def stable_def constrains_def determ total)
apply (subgoal_tac "s ∈ B | t ∈ B | (funof act s, funof act t) ∈ relcl L")
 prefer 2
 apply (rule dcommutes [rule_format], assumption+)
apply (subgoal_tac "t ∈ B | funof act t ∈ cl L (T∩M)")
 prefer 2
 apply (simp add: relcl_def)
 apply (blast intro: BL cl_mono [THEN [2] rev_subsetD])
apply (subgoal_tac "t ∈ B | t ∈ wp act (cl L (T∩M))")
 prefer 2
 apply (blast intro: funof_imp_wp determ)
apply (blast intro: TL cl_mono [THEN [2] rev_subsetD])
done
```

Version packaged with ⟦?F ∈ ?A leadsTo ?B'; ?C ∈ progress_set ?F ?T ?B; ?F ∈ stable ?T; ?B ⊆ ?B'; ?B' ∈ ?C; ⋀X. X ∈ ?C ⟹ ?G ∈ X - ?B co X⟧ ⟹ ?F ⊔ ?G ∈ ?T ∩ ?A leadsTo ?B'

```
lemma commutativity2:
  assumes leadsTo: "F ∈ A leadsTo B"
      and dcommutes:
        "∀act ∈ Acts F.
```

```
        ∀s ∈ T. ∀t. (s,t) ∈ relcl L -->
                    s ∈ B | t ∈ B | (funof act s, funof act t) ∈ relcl
L"
      and determ: "!!act. act ∈ Acts F ==> single_valued act"
      and total: "!!act. act ∈ Acts F ==> Domain act = UNIV"
      and lattice:  "lattice L"
      and BL: "B ∈ L"
      and TL: "T ∈ L"
      and Fstable: "F ∈ stable T"
      and Gco: "!!X. X∈L ==> G ∈ X-B co X"
  shows "F⊔G ∈ T∩A leadsTo B"
apply (rule commutativity1 [OF leadsTo lattice])
apply (simp_all add: Gco commutativity2_lemma dcommutes determ total
                     lattice BL TL Fstable)
done
```

18.6 Monotonicity

From Meier's thesis, section 4.5.7, page 110

end

19 Comprehensive UNITY Theory

```
theory UNITY_Main imports Detects PPROD Follows ProgressSets
uses "UNITY_tactics.ML" begin

method_setup safety = {*
    Method.ctxt_args (fn ctxt =>
        Method.METHOD (fn facts =>
            gen_constrains_tac (local_clasimpset_of ctxt) 1)) *}
    "for proving safety properties"

method_setup ensures_tac = {*
    fn args => fn ctxt =>
        Method.goal_args' (Scan.lift Args.name)
           (gen_ensures_tac (local_clasimpset_of ctxt))
           args ctxt *}
    "for proving progress properties"

end



theory Deadlock imports UNITY begin


lemma "[| F ∈ (A ∩ B) co A;  F ∈ (B ∩ A) co B |] ==> F ∈ stable (A ∩ B)"
by (unfold constrains_def stable_def, blast)



lemma Collect_le_Int_equals:
```

```
     "(⋂i ∈ atMost n. A(Suc i) ∩ A i) = (⋂i ∈ atMost (Suc n). A i)"
apply (induct_tac "n")
apply (auto simp add: atMost_Suc)
done


lemma UN_Int_Compl_subset:
     "(⋃i ∈ lessThan n. A i) ∩ (- A n) ⊆
      (⋃i ∈ lessThan n. (A i) ∩ (- A (Suc i)))"
apply (induct_tac "n", simp)
apply (simp add: lessThan_Suc, blast)
done



lemma INT_Un_Compl_subset:
     "(⋂i ∈ lessThan n. -A i ∪ A (Suc i))  ⊆
      (⋂i ∈ lessThan n. -A i) ∪ A n"
apply (induct_tac "n", simp)
apply (simp add: lessThan_Suc, blast)
done



lemma INT_le_equals_Int_lemma:
     "A 0 ∩ (-(A n) ∩ (⋂i ∈ lessThan n. -A i ∪ A (Suc i))) = {}"
by (blast intro: gr0I dest: INT_Un_Compl_subset [THEN subsetD])


lemma INT_le_equals_Int:
     "(⋂i ∈ atMost n. A i) =
      A 0 ∩ (⋂i ∈ lessThan n. -A i ∪ A(Suc i))"
apply (induct_tac "n", simp)
apply (simp add: Int_ac Int_Un_distrib Int_Un_distrib2
                 INT_le_equals_Int_lemma lessThan_Suc atMost_Suc)
done

lemma INT_le_Suc_equals_Int:
     "(⋂i ∈ atMost (Suc n). A i) =
      A 0 ∩ (⋂i ∈ atMost n. -A i ∪ A(Suc i))"
by (simp add: lessThan_Suc_atMost INT_le_equals_Int)



lemma
  assumes zeroprem: "F ∈ (A 0 ∩ A (Suc n)) co (A 0)"
      and allprem:
            "!!i. i ∈ atMost n ==> F ∈ (A(Suc i) ∩ A i) co (-A i ∪ A(Suc
i))"
  shows "F ∈ stable (⋂i ∈ atMost (Suc n). A i)"
apply (unfold stable_def)
apply (rule constrains_Int [THEN constrains_weaken])
   apply (rule zeroprem)
  apply (rule constrains_INT)
```

```
  apply (erule allprem)
 apply (simp add: Collect_le_Int_equals Int_assoc INT_absorb)
apply (simp add: INT_le_Suc_equals_Int)
done

end
```

```
theory Common imports UNITY_Main begin

consts
  ftime :: "nat=>nat"
  gtime :: "nat=>nat"

axioms
  fmono: "m ≤ n ==> ftime m ≤ ftime n"
  gmono: "m ≤ n ==> gtime m ≤ gtime n"

  fasc:  "m ≤ ftime n"
  gasc:  "m ≤ gtime n"

constdefs
  common :: "nat set"
    "common == {n. ftime n = n & gtime n = n}"

  maxfg :: "nat => nat set"
    "maxfg m == {t. t ≤ max (ftime m) (gtime m)}"



lemma common_stable:
     "[| ∀m. F ∈ {m} Co (maxfg m); n ∈ common |]
      ==> F ∈ Stable (atMost n)"
apply (drule_tac M = "{t. t ≤ n}" in Elimination_sing)
apply (simp add: atMost_def Stable_def common_def maxfg_def le_max_iff_disj)
apply (erule Constrains_weaken_R)
apply (blast intro: order_eq_refl fmono gmono le_trans)
done

lemma common_safety:
     "[| Init F ⊆ atMost n;
         ∀m. F ∈ {m} Co (maxfg m); n ∈ common |]
      ==> F ∈ Always (atMost n)"
by (simp add: AlwaysI common_stable)



lemma "SKIP ∈ {m} co (maxfg m)"
by (simp add: constrains_def maxfg_def le_max_iff_disj fasc)


lemma "mk_total_program
```

```
        (UNIV, {range(%t.(t,ftime t)), range(%t.(t,gtime t))}, UNIV)
      ∈ {m} co (maxfg m)"
apply (simp add: mk_total_program_def)
apply (simp add: constrains_def maxfg_def le_max_iff_disj fasc)
done


lemma "mk_total_program (UNIV, {range(%t.(t, max (ftime t) (gtime t)))},
UNIV)
      ∈ {m} co (maxfg m)"
apply (simp add: mk_total_program_def)
apply (simp add: constrains_def maxfg_def max_def gasc)
done


lemma  "mk_total_program
         (UNIV, { {(t, Suc t) | t. t < max (ftime t) (gtime t)} }, UNIV)
      ∈ {m} co (maxfg m)"
apply (simp add: mk_total_program_def)
apply (simp add: constrains_def maxfg_def max_def gasc)
done
```

```
declare atMost_Int_atLeast [simp]

lemma leadsTo_common_lemma:
     "[| ∀m. F ∈ {m} Co (maxfg m);
         ∀m ∈ lessThan n. F ∈ {m} LeadsTo (greaterThan m);
         n ∈ common |]
      ==> F ∈ (atMost n) LeadsTo common"
apply (rule LeadsTo_weaken_R)
apply (rule_tac f = id and l = n in GreaterThan_bounded_induct)
apply (simp_all (no_asm_simp))
apply (rule_tac [2] subset_refl)
apply (blast dest: PSP_Stable2 intro: common_stable LeadsTo_weaken_R)
done


lemma leadsTo_common:
     "[| ∀m. F ∈ {m} Co (maxfg m);
         ∀m ∈ -common. F ∈ {m} LeadsTo (greaterThan m);
         n ∈ common |]
      ==> F ∈ (atMost (LEAST n. n ∈ common)) LeadsTo common"
apply (rule leadsTo_common_lemma)
apply (simp_all (no_asm_simp))
apply (erule_tac [2] LeastI)
apply (blast dest!: not_less_Least)
done
```

```
end


theory Network imports UNITY begin


datatype pvar = Sent | Rcvd | Idle

datatype pname = Aproc | Bproc

types state = "pname * pvar => nat"

locale F_props =
  fixes F
  assumes rsA: "F ∈ stable {s. s(Bproc,Rcvd) ≤ s(Aproc,Sent)}"
      and rsB: "F ∈ stable {s. s(Aproc,Rcvd) ≤ s(Bproc,Sent)}"
    and sent_nondec: "F ∈ stable {s. m ≤ s(proc,Sent)}"
    and rcvd_nondec: "F ∈ stable {s. n ≤ s(proc,Rcvd)}"
    and rcvd_idle: "F ∈ {s. s(proc,Idle) = Suc 0 & s(proc,Rcvd) = m}
                        co {s. s(proc,Rcvd) = m --> s(proc,Idle) = Suc 0}"
    and sent_idle: "F ∈ {s. s(proc,Idle) = Suc 0 & s(proc,Sent) = n}
                        co {s. s(proc,Sent) = n}"


lemmas (in F_props)
        sent_nondec_A = sent_nondec [of _ Aproc]
    and sent_nondec_B = sent_nondec [of _ Bproc]
    and rcvd_nondec_A = rcvd_nondec [of _ Aproc]
    and rcvd_nondec_B = rcvd_nondec [of _ Bproc]
    and rcvd_idle_A = rcvd_idle [of Aproc]
    and rcvd_idle_B = rcvd_idle [of Bproc]
    and sent_idle_A = sent_idle [of Aproc]
    and sent_idle_B = sent_idle [of Bproc]

    and rs_AB = stable_Int [OF rsA rsB]
    and sent_nondec_AB = stable_Int [OF sent_nondec_A sent_nondec_B]
    and rcvd_nondec_AB = stable_Int [OF rcvd_nondec_A rcvd_nondec_B]
    and rcvd_idle_AB = constrains_Int [OF rcvd_idle_A rcvd_idle_B]
    and sent_idle_AB = constrains_Int [OF sent_idle_A sent_idle_B]
    and nondec_AB = stable_Int [OF sent_nondec_AB rcvd_nondec_AB]
    and idle_AB = constrains_Int [OF rcvd_idle_AB sent_idle_AB]
    and nondec_idle = constrains_Int [OF nondec_AB [unfolded stable_def]
                                         idle_AB]

lemma (in F_props)
  shows "F ∈ stable {s. s(Aproc,Idle) = Suc 0 & s(Bproc,Idle) = Suc 0 &
                        s(Aproc,Sent) = s(Bproc,Rcvd) &
                        s(Bproc,Sent) = s(Aproc,Rcvd) &
                        s(Aproc,Rcvd) = m & s(Bproc,Rcvd) = n}"
apply (unfold stable_def)
apply (rule constrainsI)
apply (drule constrains_Int [OF rs_AB [unfolded stable_def] nondec_idle,
```

```
                              THEN constrainsD], assumption)
apply simp_all
apply (blast del: le0, clarify)
apply (subgoal_tac "s' (Aproc, Rcvd) = s (Aproc, Rcvd)")
apply (subgoal_tac "s' (Bproc, Rcvd) = s (Bproc, Rcvd)")
apply simp
apply (blast intro: order_antisym le_trans eq_imp_le)+
done

end
```

20 The Token Ring

```
theory Token
imports "../WFair"

begin
```

From Misra, "A Logic for Concurrent Programming" (1994), sections 5.2 and 13.2.

20.1 Definitions

```
datatype pstate = Hungry | Eating | Thinking
```
— process states

```
record state =
  token :: "nat"
  proc  :: "nat => pstate"


constdefs
  HasTok :: "nat => state set"
    "HasTok i == {s. token s = i}"

  H :: "nat => state set"
    "H i == {s. proc s i = Hungry}"

  E :: "nat => state set"
    "E i == {s. proc s i = Eating}"

  T :: "nat => state set"
    "T i == {s. proc s i = Thinking}"


locale Token =
  fixes N and F and nodeOrder and "next"
  defines nodeOrder_def:
       "nodeOrder j == measure(%i. ((j+N)-i) mod N) ∩ {..<N} × {..<N}"
      and next_def:
       "next i == (Suc i) mod N"
  assumes N_positive [iff]: "0<N"
```

```
    and TR2:  "F ∈ (T i) co (T i ∪ H i)"
    and TR3:  "F ∈ (H i) co (H i ∪ E i)"
    and TR4:  "F ∈ (H i - HasTok i) co (H i)"
    and TR5:  "F ∈ (HasTok i) co (HasTok i ∪ -(E i))"
    and TR6:  "F ∈ (H i ∩ HasTok i) leadsTo (E i)"
    and TR7:  "F ∈ (HasTok i) leadsTo (HasTok (next i))"


lemma HasToK_partition: "[| s ∈ HasTok i; s ∈ HasTok j |] ==> i=j"
by (unfold HasTok_def, auto)

lemma not_E_eq: "(s ∉ E i) = (s ∈ H i | s ∈ T i)"
apply (simp add: H_def E_def T_def)
apply (case_tac "proc s i", auto)
done

lemma (in Token) token_stable: "F ∈ stable (-(E i) ∪ (HasTok i))"
apply (unfold stable_def)
apply (rule constrains_weaken)
apply (rule constrains_Un [OF constrains_Un [OF TR2 TR4] TR5])
apply (auto simp add: not_E_eq)
apply (simp_all add: H_def E_def T_def)
done
```

20.2 Progress under Weak Fairness

```
lemma (in Token) wf_nodeOrder: "wf(nodeOrder j)"
apply (unfold nodeOrder_def)
apply (rule wf_measure [THEN wf_subset], blast)
done

lemma (in Token) nodeOrder_eq:
     "[| i<N; j<N |] ==> ((next i, i) ∈ nodeOrder j) = (i ≠ j)"
apply (unfold nodeOrder_def next_def measure_def inv_image_def)
apply (auto simp add: mod_Suc mod_geq)
apply (auto split add: nat_diff_split simp add: linorder_neq_iff mod_geq)
done
```

From "A Logic for Concurrent Programming", but not used in Chapter 4. Note the use of `case_tac`. Reasoning about leadsTo takes practice!

```
lemma (in Token) TR7_nodeOrder:
     "[| i<N; j<N |] ==>
      F ∈ (HasTok i) leadsTo ({s. (token s, i) ∈ nodeOrder j} ∪ HasTok j)"
apply (case_tac "i=j")
apply (blast intro: subset_imp_leadsTo)
apply (rule TR7 [THEN leadsTo_weaken_R])
apply (auto simp add: HasTok_def nodeOrder_eq)
done
```

Chapter 4 variant, the one actually used below.

```
lemma (in Token) TR7_aux: "[| i<N; j<N; i≠j |]
      ==> F ∈ (HasTok i) leadsTo {s. (token s, i) ∈ nodeOrder j}"
apply (rule TR7 [THEN leadsTo_weaken_R])
apply (auto simp add: HasTok_def nodeOrder_eq)
```

```
done

lemma (in Token) token_lemma:
     "({s. token s < N} ∩ token -‘ {m}) = (if m<N then token -‘ {m} else {})"
by auto
```

Misra's TR9: the token reaches an arbitrary node

```
lemma  (in Token) leadsTo_j: "j<N ==> F ∈ {s. token s < N} leadsTo (HasTok
j)"
apply (rule leadsTo_weaken_R)
apply (rule_tac I = "-{j}" and f = token and B = "{}"
       in wf_nodeOrder [THEN bounded_induct])
apply (simp_all (no_asm_simp) add: token_lemma vimage_Diff HasTok_def)
 prefer 2 apply blast
apply clarify
apply (rule TR7_aux [THEN leadsTo_weaken])
apply (auto simp add: HasTok_def nodeOrder_def)
done
```

Misra's TR8: a hungry process eventually eats

```
lemma (in Token) token_progress:
     "j<N ==> F ∈ ({s. token s < N} ∩ H j) leadsTo (E j)"
apply (rule leadsTo_cancel1 [THEN leadsTo_Un_duplicate])
apply (rule_tac [2] TR6)
apply (rule psp [OF leadsTo_j TR3, THEN leadsTo_weaken], blast+)
done


end



theory Channel imports UNITY_Main begin

types state = "nat set"

consts
  F :: "state program"

constdefs
  minSet :: "nat set => nat option"
    "minSet A == if A={} then None else Some (LEAST x. x ∈ A)"

axioms

  UC1:  "F ∈ (minSet -‘ {Some x}) co (minSet -‘ (Some‘atLeast x))"



  UC2:  "F ∈ (minSet -‘ {Some x}) leadsTo {s. x ∉ s}"



lemma minSet_eq_SomeD: "minSet A = Some x ==> x ∈ A"
```

```
apply (unfold minSet_def)
apply (simp split: split_if_asm)
apply (fast intro: LeastI)
done

lemma minSet_empty [simp]: " minSet{} = None"
by (unfold minSet_def, simp)

lemma minSet_nonempty: "x ∈ A ==> minSet A = Some (LEAST x. x ∈ A)"
by (unfold minSet_def, auto)

lemma minSet_greaterThan:
     "F ∈ (minSet -‘ {Some x}) leadsTo (minSet -‘ (Some‘greaterThan x))"
apply (rule leadsTo_weaken)
apply (rule_tac x1=x in psp [OF UC2 UC1], safe)
apply (auto dest: minSet_eq_SomeD simp add: linorder_neq_iff)
done


lemma Channel_progress_lemma:
     "F ∈ (UNIV-{{}}) leadsTo (minSet -‘ (Some‘atLeast y))"
apply (rule leadsTo_weaken_R)
apply (rule_tac l = y and f = "the o minSet" and B = "{}"
       in greaterThan_bounded_induct, safe)
apply (simp_all (no_asm_simp))
apply (drule_tac [2] minSet_nonempty)
 prefer 2 apply simp
apply (rule minSet_greaterThan [THEN leadsTo_weaken], safe)
apply simp_all
apply (drule minSet_nonempty, simp)
done


lemma Channel_progress: "!!y::nat. F ∈ (UNIV-{{}}) leadsTo {s. y ∉ s}"
apply (rule Channel_progress_lemma [THEN leadsTo_weaken_R], clarify)
apply (frule minSet_nonempty)
apply (auto dest: Suc_le_lessD not_less_Least)
done

end



theory Lift
imports "../UNITY_Main"

begin

record state =
  floor :: "int"          — current position of the lift
  "open" :: "bool"        — whether the door is opened at floor
  stop  :: "bool"         — whether the lift is stopped at floor
  req   :: "int set"      — for each floor, whether the lift is requested
  up    :: "bool"         — current direction of movement
```

```
  move  :: "bool"
```
— whether moving takes precedence over opening

consts

```
  Min :: "int"
  Max :: "int"
```
— least and greatest floors
— least and greatest floors

axioms

```
  Min_le_Max [iff]: "Min ≤ Max"
```

constdefs

— Abbreviations: the "always" part

```
  above :: "state set"
    "above == {s. ∃i. floor s < i & i ≤ Max & i ∈ req s}"

  below :: "state set"
    "below == {s. ∃i. Min ≤ i & i < floor s & i ∈ req s}"

  queueing :: "state set"
    "queueing == above ∪ below"

  goingup :: "state set"
    "goingup   == above ∩ ({s. up s}  ∪ -below)"

  goingdown :: "state set"
    "goingdown == below ∩ ({s. ~ up s} ∪ -above)"

  ready :: "state set"
    "ready == {s. stop s & ~ open s & move s}"
```

— Further abbreviations

```
  moving :: "state set"
    "moving ==  {s. ~ stop s & ~ open s}"

  stopped :: "state set"
    "stopped == {s. stop s  & ~ open s & ~ move s}"

  opened :: "state set"
    "opened ==  {s. stop s  &  open s  &  move s}"

  closed :: "state set"
```
— but this is the same as ready!!
```
    "closed ==  {s. stop s  & ~ open s &  move s}"

  atFloor :: "int => state set"
    "atFloor n ==  {s. floor s = n}"

  Req :: "int => state set"
    "Req n ==  {s. n ∈ req s}"
```

— The program

```
request_act :: "(state*state) set"
  "request_act == {(s,s'). s' = s (|stop:=True, move:=False|)
                              & ~ stop s & floor s ∈ req s}"

open_act :: "(state*state) set"
  "open_act ==
       {(s,s'). s' = s (|open :=True,
                         req  := req s - {floor s},
                         move := True|)
                      & stop s & ~ open s & floor s ∈ req s
                      & ~(move s & s ∈ queueing)}"

close_act :: "(state*state) set"
  "close_act == {(s,s'). s' = s (|open := False|) & open s}"

req_up :: "(state*state) set"
  "req_up ==
       {(s,s'). s' = s (|stop  :=False,
                         floor := floor s + 1,
                         up    := True|)
                      & s ∈ (ready ∩ goingup)}"

req_down :: "(state*state) set"
  "req_down ==
       {(s,s'). s' = s (|stop  :=False,
                         floor := floor s - 1,
                         up    := False|)
                      & s ∈ (ready ∩ goingdown)}"

move_up :: "(state*state) set"
  "move_up ==
       {(s,s'). s' = s (|floor := floor s + 1|)
                      & ~ stop s & up s & floor s ∉ req s}"

move_down :: "(state*state) set"
  "move_down ==
       {(s,s'). s' = s (|floor := floor s - 1|)
                      & ~ stop s & ~ up s & floor s ∉ req s}"

button_press  :: "(state*state) set"
```

— This action is omitted from prior treatments, which therefore are unrealistic: nobody asks the lift to do anything! But adding this action invalidates many of the existing progress arguments: various "ensures" properties fail. Maybe it should be constrained to only allow button presses in the current direction of travel, like in a real lift.

```
   "button_press ==
       {(s,s'). ∃n. s' = s (|req := insert n (req s)|)
                     & Min ≤ n & n ≤ Max}"


Lift :: "state program"
```

— for the moment, we OMIT `button_press`

```
  "Lift == mk_total_program
```

```
              ({s. floor s = Min & ~ up s & move s & stop s &
                       ~ open s & req s = {}},
               {request_act, open_act, close_act,
                req_up, req_down, move_up, move_down},
               UNIV)"
```

— Invariants

```
  bounded :: "state set"
    "bounded == {s. Min ≤ floor s & floor s ≤ Max}"

  open_stop :: "state set"
    "open_stop == {s. open s --> stop s}"

  open_move :: "state set"
    "open_move == {s. open s --> move s}"

  stop_floor :: "state set"
    "stop_floor == {s. stop s & ~ move s --> floor s ∈ req s}"

  moving_up :: "state set"
    "moving_up == {s. ~ stop s & up s -->
                  (∃f. floor s ≤ f & f ≤ Max & f ∈ req s)}"

  moving_down :: "state set"
    "moving_down == {s. ~ stop s & ~ up s -->
                    (∃f. Min ≤ f & f ≤ floor s & f ∈ req s)}"

  metric :: "[int,state] => int"
    "metric ==
       %n s. if floor s < n then (if up s then n - floor s
                                  else (floor s - Min) + (n-Min))
             else
             if n < floor s then (if up s then (Max - floor s) + (Max-n)
                                  else floor s - n)
             else 0"
```

locale *Floor* =
  fixes *n*
  assumes *Min_le_n [iff]*: "Min ≤ n"
      and *n_le_Max [iff]*: "n ≤ Max"

lemma *not_mem_distinct*: "[| x ∉ A; y ∈ A |] ==> x ≠ y"
by *blast*

declare *Lift_def [THEN def_prg_Init, simp]*

declare *request_act_def [THEN def_act_simp, simp]*
declare *open_act_def [THEN def_act_simp, simp]*
declare *close_act_def [THEN def_act_simp, simp]*
declare *req_up_def [THEN def_act_simp, simp]*
declare *req_down_def [THEN def_act_simp, simp]*

```
declare move_up_def [THEN def_act_simp, simp]
declare move_down_def [THEN def_act_simp, simp]
declare button_press_def [THEN def_act_simp, simp]


declare above_def [THEN def_set_simp, simp]
declare below_def [THEN def_set_simp, simp]
declare queueing_def [THEN def_set_simp, simp]
declare goingup_def [THEN def_set_simp, simp]
declare goingdown_def [THEN def_set_simp, simp]
declare ready_def [THEN def_set_simp, simp]


declare bounded_def [simp]
        open_stop_def [simp]
        open_move_def [simp]
        stop_floor_def [simp]
        moving_up_def [simp]
        moving_down_def [simp]

lemma open_stop: "Lift ∈ Always open_stop"
apply (rule AlwaysI, force)
apply (unfold Lift_def, safety)
done

lemma stop_floor: "Lift ∈ Always stop_floor"
apply (rule AlwaysI, force)
apply (unfold Lift_def, safety)
done


lemma open_move: "Lift ∈ Always open_move"
apply (cut_tac open_stop)
apply (rule AlwaysI, force)
apply (unfold Lift_def, safety)
done

lemma moving_up: "Lift ∈ Always moving_up"
apply (rule AlwaysI, force)
apply (unfold Lift_def, safety)
apply (auto dest: order_le_imp_less_or_eq simp add: add1_zle_eq)
done

lemma moving_down: "Lift ∈ Always moving_down"
apply (rule AlwaysI, force)
apply (unfold Lift_def, safety)
apply (blast dest: order_le_imp_less_or_eq)
done

lemma bounded: "Lift ∈ Always bounded"
apply (cut_tac moving_up moving_down)
apply (rule AlwaysI, force)
apply (unfold Lift_def, safety, auto)
apply (drule not_mem_distinct, assumption, arith)+
```

done

20.3 Progress

declare *moving_def [THEN def_set_simp, simp]*
declare *stopped_def [THEN def_set_simp, simp]*
declare *opened_def [THEN def_set_simp, simp]*
declare *closed_def [THEN def_set_simp, simp]*
declare *atFloor_def [THEN def_set_simp, simp]*
declare *Req_def [THEN def_set_simp, simp]*

The HUG'93 paper mistakenly omits the Req n from these!

lemma *E_thm01: "Lift ∈ (stopped ∩ atFloor n) LeadsTo (opened ∩ atFloor n)"*
apply *(cut_tac stop_floor)*
apply *(unfold Lift_def, ensures_tac "open_act")*
done

lemma *E_thm02: "Lift ∈ (Req n ∩ stopped - atFloor n) LeadsTo*
(Req n ∩ opened - atFloor n)"
apply *(cut_tac stop_floor)*
apply *(unfold Lift_def, ensures_tac "open_act")*
done

lemma *E_thm03: "Lift ∈ (Req n ∩ opened - atFloor n) LeadsTo*
(Req n ∩ closed - (atFloor n - queueing))"
apply *(unfold Lift_def, ensures_tac "close_act")*
done

lemma *E_thm04: "Lift ∈ (Req n ∩ closed ∩ (atFloor n - queueing))*
LeadsTo (opened ∩ atFloor n)"
apply *(unfold Lift_def, ensures_tac "open_act")*
done

lemmas *linorder_leI = linorder_not_less [THEN iffD1]*

lemmas *(in Floor) le_MinD = Min_le_n [THEN order_antisym]*
and *Max_leD = n_le_Max [THEN [2] order_antisym]*

declare *(in Floor) le_MinD [dest!]*
and *linorder_leI [THEN le_MinD, dest!]*
and *Max_leD [dest!]*
and *linorder_leI [THEN Max_leD, dest!]*

lemma *(in Floor) E_thm05c:*
"Lift ∈ (Req n ∩ closed - (atFloor n - queueing))

```
            LeadsTo ((closed ∩ goingup ∩ Req n)  ∪
                     (closed ∩ goingdown ∩ Req n))"
by (auto intro!: subset_imp_LeadsTo simp add: linorder_neq_iff)


lemma (in Floor) lift_2: "Lift ∈ (Req n ∩ closed - (atFloor n - queueing))

            LeadsTo (moving ∩ Req n)"
apply (rule LeadsTo_Trans [OF E_thm05c LeadsTo_Un])
apply (unfold Lift_def)
apply (ensures_tac [2] "req_down")
apply (ensures_tac "req_up", auto)
done



declare split_if_asm [split]



lemma (in Floor) E_thm12a:
     "0 < N ==>
      Lift ∈ (moving ∩ Req n ∩ {s. metric n s = N} ∩
              {s. floor s ∉ req s} ∩ {s. up s})
             LeadsTo
               (moving ∩ Req n ∩ {s. metric n s < N})"
apply (cut_tac moving_up)
apply (unfold Lift_def, ensures_tac "move_up", safe)

apply (erule linorder_leI [THEN order_antisym, symmetric])
apply (auto simp add: metric_def)
done



lemma (in Floor) E_thm12b: "0 < N ==>
      Lift ∈ (moving ∩ Req n ∩ {s. metric n s = N} ∩
              {s. floor s ∉ req s} - {s. up s})
             LeadsTo (moving ∩ Req n ∩ {s. metric n s < N})"
apply (cut_tac moving_down)
apply (unfold Lift_def, ensures_tac "move_down", safe)

apply (erule linorder_leI [THEN order_antisym, symmetric])
apply (auto simp add: metric_def)
done


lemma (in Floor) lift_4:
     "0<N ==> Lift ∈ (moving ∩ Req n ∩ {s. metric n s = N} ∩
                           {s. floor s ∉ req s}) LeadsTo
                          (moving ∩ Req n ∩ {s. metric n s < N})"
apply (rule LeadsTo_Trans [OF subset_imp_LeadsTo
                              LeadsTo_Un [OF E_thm12a E_thm12b]], auto)
```

done

lemma (**in** *Floor*) *E_thm16a*: "0<N
  ==> Lift ∈ (closed ∩ Req n ∩ {s. metric n s = N} ∩ goingup) LeadsTo
             (moving ∩ Req n ∩ {s. metric n s < N})"
apply (*cut_tac bounded*)
apply (*unfold Lift_def, ensures_tac "req_up"*)
apply (*auto simp add: metric_def*)
done

lemma (**in** *Floor*) *E_thm16b*: "0<N ==>
      Lift ∈ (closed ∩ Req n ∩ {s. metric n s = N} ∩ goingdown) LeadsTo

                  (moving ∩ Req n ∩ {s. metric n s < N})"
apply (*cut_tac bounded*)
apply (*unfold Lift_def, ensures_tac "req_down"*)
apply (*auto simp add: metric_def*)
done

lemma (**in** *Floor*) *E_thm16c*:
     "0<N ==> Req n ∩ {s. metric n s = N} ⊆ goingup ∪ goingdown"
by (*force simp add: metric_def*)

lemma (**in** *Floor*) *lift_5*:
     "0<N ==> Lift ∈ (closed ∩ Req n ∩ {s. metric n s = N}) LeadsTo
                   (moving ∩ Req n ∩ {s. metric n s < N})"
apply (*rule LeadsTo_Trans [OF subset_imp_LeadsTo
                                LeadsTo_Un [OF E_thm16a E_thm16b]]*)
apply (*drule E_thm16c, auto*)
done

lemma (**in** *Floor*) *metric_eq_0D [dest]*:
     "[| metric n s = 0; Min ≤ floor s; floor s ≤ Max |] ==> floor s =
n"
by (*force simp add: metric_def*)

lemma (**in** *Floor*) *E_thm11*: "Lift ∈ (moving ∩ Req n ∩ {s. metric n s = 0})
LeadsTo

```
                    (stopped ∩ atFloor n)"
apply (cut_tac bounded)
apply (unfold Lift_def, ensures_tac "request_act", auto)
done


lemma (in Floor) E_thm13:
  "Lift ∈ (moving ∩ Req n ∩ {s. metric n s = N} ∩ {s. floor s ∈ req s})

  LeadsTo (stopped ∩ Req n ∩ {s. metric n s = N} ∩ {s. floor s ∈ req s})"
apply (unfold Lift_def, ensures_tac "request_act")
apply (auto simp add: metric_def)
done


lemma (in Floor) E_thm14: "0 < N ==>
      Lift ∈
        (stopped ∩ Req n ∩ {s. metric n s = N} ∩ {s. floor s ∈ req s})
        LeadsTo (opened ∩ Req n ∩ {s. metric n s = N})"
apply (unfold Lift_def, ensures_tac "open_act")
apply (auto simp add: metric_def)
done


lemma (in Floor) E_thm15: "Lift ∈ (opened ∩ Req n ∩ {s. metric n s = N})

            LeadsTo (closed ∩ Req n ∩ {s. metric n s = N})"
apply (unfold Lift_def, ensures_tac "close_act")
apply (auto simp add: metric_def)
done




lemma (in Floor) lift_3_Req: "0 < N ==>
      Lift ∈
        (moving ∩ Req n ∩ {s. metric n s = N} ∩ {s. floor s ∈ req s})
        LeadsTo (moving ∩ Req n ∩ {s. metric n s < N})"
apply (blast intro!: E_thm13 E_thm14 E_thm15 lift_5 intro: LeadsTo_Trans)
done



lemma (in Floor) Always_nonneg: "Lift ∈ Always {s. 0 ≤ metric n s}"
apply (rule bounded [THEN Always_weaken])
apply (auto simp add: metric_def)
done

lemmas (in Floor) R_thm11 = Always_LeadsTo_weaken [OF Always_nonneg E_thm11]

lemma (in Floor) lift_3:
     "Lift ∈ (moving ∩ Req n) LeadsTo (stopped ∩ atFloor n)"
apply (rule Always_nonneg [THEN integ_0_le_induct])
apply (case_tac "0 < z")
```

```
prefer 2 apply (force intro: R_thm11 order_antisym simp add: linorder_not_less)
apply (rule LeadsTo_weaken_R [OF asm_rl Un_upper1])
apply (rule LeadsTo_Trans [OF subset_imp_LeadsTo
                              LeadsTo_Un [OF lift_4 lift_3_Req]], auto)
done


lemma (in Floor) lift_1: "Lift ∈ (Req n) LeadsTo (opened ∩ atFloor n)"
apply (rule LeadsTo_Trans)
 prefer 2
 apply (rule LeadsTo_Un [OF E_thm04 LeadsTo_Un_post])
 apply (rule E_thm01 [THEN [2] LeadsTo_Trans_Un])
 apply (rule lift_3 [THEN [2] LeadsTo_Trans_Un])
 apply (rule lift_2 [THEN [2] LeadsTo_Trans_Un])
 apply (rule LeadsTo_Trans_Un [OF E_thm02 E_thm03])
apply (rule open_move [THEN Always_LeadsToI])
apply (rule Always_LeadsToI [OF open_stop subset_imp_LeadsTo], clarify)

apply (case_tac "open x", auto)
done


end



theory Mutex imports UNITY_Main begin

record state =
  p :: bool
  m :: int
  n :: int
  u :: bool
  v :: bool

types command = "(state*state) set"

constdefs



  U0 :: command
    "U0 == {(s,s'). s' = s (|u:=True, m:=1|) & m s = 0}"

  U1 :: command
    "U1 == {(s,s'). s' = s (|p:= v s, m:=2|) & m s = 1}"

  U2 :: command
    "U2 == {(s,s'). s' = s (|m:=3|) & ~ p s & m s = 2}"

  U3 :: command
    "U3 == {(s,s'). s' = s (|u:=False, m:=4|) & m s = 3}"
```

```
U4 :: command
  "U4 == {(s,s'). s' = s (|p:=True, m:=0|) & m s = 4}"



V0 :: command
  "V0 == {(s,s'). s' = s (|v:=True, n:=1|) & n s = 0}"

V1 :: command
  "V1 == {(s,s'). s' = s (|p:= ~ u s, n:=2|) & n s = 1}"

V2 :: command
  "V2 == {(s,s'). s' = s (|n:=3|) & p s & n s = 2}"

V3 :: command
  "V3 == {(s,s'). s' = s (|v:=False, n:=4|) & n s = 3}"

V4 :: command
  "V4 == {(s,s'). s' = s (|p:=False, n:=0|) & n s = 4}"

Mutex :: "state program"
  "Mutex == mk_total_program
              ({s. ~ u s & ~ v s & m s = 0 & n s = 0},
               {U0, U1, U2, U3, U4, V0, V1, V2, V3, V4},
               UNIV)"



IU :: "state set"
  "IU == {s. (u s = (1 ≤ m s & m s ≤ 3)) & (m s = 3 --> ~ p s)}"

IV :: "state set"
  "IV == {s. (v s = (1 ≤ n s & n s ≤ 3)) & (n s = 3 --> p s)}"



bad_IU :: "state set"
  "bad_IU == {s. (u s = (1 ≤ m s & m s ≤ 3)) &
                 (3 ≤ m s & m s ≤ 4 --> ~ p s)}"
```

declare *Mutex_def [THEN def_prg_Init, simp]*

declare *U0_def [THEN def_act_simp, simp]*
declare *U1_def [THEN def_act_simp, simp]*
declare *U2_def [THEN def_act_simp, simp]*
declare *U3_def [THEN def_act_simp, simp]*
declare *U4_def [THEN def_act_simp, simp]*
declare *V0_def [THEN def_act_simp, simp]*
declare *V1_def [THEN def_act_simp, simp]*
declare *V2_def [THEN def_act_simp, simp]*
declare *V3_def [THEN def_act_simp, simp]*
declare *V4_def [THEN def_act_simp, simp]*

```
declare IU_def [THEN def_set_simp, simp]
declare IV_def [THEN def_set_simp, simp]
declare bad_IU_def [THEN def_set_simp, simp]

lemma IU: "Mutex ∈ Always IU"
apply (rule AlwaysI, force)
apply (unfold Mutex_def, safety)
done


lemma IV: "Mutex ∈ Always IV"
apply (rule AlwaysI, force)
apply (unfold Mutex_def, safety)
done


lemma mutual_exclusion: "Mutex ∈ Always {s. ~ (m s = 3 & n s = 3)}"
apply (rule Always_weaken)
apply (rule Always_Int_I [OF IU IV], auto)
done



lemma "Mutex ∈ Always bad_IU"
apply (rule AlwaysI, force)
apply (unfold Mutex_def, safety, auto)

oops


lemma eq_123: "((1::int) ≤ i & i ≤ 3) = (i = 1 | i = 2 | i = 3)"
by arith




lemma U_F0: "Mutex ∈ {s. m s=2} Unless {s. m s=3}"
by (unfold Unless_def Mutex_def, safety)

lemma U_F1: "Mutex ∈ {s. m s=1} LeadsTo {s. p s = v s & m s = 2}"
by (unfold Mutex_def, ensures_tac U1)

lemma U_F2: "Mutex ∈ {s. ~ p s & m s = 2} LeadsTo {s. m s = 3}"
apply (cut_tac IU)
apply (unfold Mutex_def, ensures_tac U2)
done

lemma U_F3: "Mutex ∈ {s. m s = 3} LeadsTo {s. p s}"
apply (rule_tac B = "{s. m s = 4}" in LeadsTo_Trans)
 apply (unfold Mutex_def)
 apply (ensures_tac U3)
apply (ensures_tac U4)
done
```

```
lemma U_lemma2: "Mutex ∈ {s. m s = 2} LeadsTo {s. p s}"
apply (rule LeadsTo_Diff [OF LeadsTo_weaken_L
                             Int_lower2 [THEN subset_imp_LeadsTo]])
apply (rule LeadsTo_Trans [OF U_F2 U_F3], auto)
done

lemma U_lemma1: "Mutex ∈ {s. m s = 1} LeadsTo {s. p s}"
by (rule LeadsTo_Trans [OF U_F1 [THEN LeadsTo_weaken_R] U_lemma2], blast)

lemma U_lemma123: "Mutex ∈ {s. 1 ≤ m s & m s ≤ 3} LeadsTo {s. p s}"
by (simp add: eq_123 Collect_disj_eq LeadsTo_Un_distrib U_lemma1 U_lemma2
U_F3)


lemma u_Leadsto_p: "Mutex ∈ {s. u s} LeadsTo {s. p s}"
by (rule Always_LeadsTo_weaken [OF IU U_lemma123], auto)




lemma V_F0: "Mutex ∈ {s. n s=2} Unless {s. n s=3}"
by (unfold Unless_def Mutex_def, safety)

lemma V_F1: "Mutex ∈ {s. n s=1} LeadsTo {s. p s = (~ u s) & n s = 2}"
by (unfold Mutex_def, ensures_tac "V1")

lemma V_F2: "Mutex ∈ {s. p s & n s = 2} LeadsTo {s. n s = 3}"
apply (cut_tac IV)
apply (unfold Mutex_def, ensures_tac "V2")
done

lemma V_F3: "Mutex ∈ {s. n s = 3} LeadsTo {s. ~ p s}"
apply (rule_tac B = "{s. n s = 4}" in LeadsTo_Trans)
 apply (unfold Mutex_def)
 apply (ensures_tac V3)
apply (ensures_tac V4)
done

lemma V_lemma2: "Mutex ∈ {s. n s = 2} LeadsTo {s. ~ p s}"
apply (rule LeadsTo_Diff [OF LeadsTo_weaken_L
                             Int_lower2 [THEN subset_imp_LeadsTo]])
apply (rule LeadsTo_Trans [OF V_F2 V_F3], auto)
done

lemma V_lemma1: "Mutex ∈ {s. n s = 1} LeadsTo {s. ~ p s}"
by (rule LeadsTo_Trans [OF V_F1 [THEN LeadsTo_weaken_R] V_lemma2], blast)

lemma V_lemma123: "Mutex ∈ {s. 1 ≤ n s & n s ≤ 3} LeadsTo {s. ~ p s}"
by (simp add: eq_123 Collect_disj_eq LeadsTo_Un_distrib V_lemma1 V_lemma2
V_F3)
```

```
lemma v_Leadsto_not_p: "Mutex ∈ {s. v s} LeadsTo {s. ~ p s}"
by (rule Always_LeadsTo_weaken [OF IV V_lemma123], auto)
```

```
lemma m1_Leadsto_3: "Mutex ∈ {s. m s = 1} LeadsTo {s. m s = 3}"
apply (rule LeadsTo_cancel2 [THEN LeadsTo_Un_duplicate])
apply (rule_tac [2] U_F2)
apply (simp add: Collect_conj_eq)
apply (subst Un_commute)
apply (rule LeadsTo_cancel2 [THEN LeadsTo_Un_duplicate])
 apply (rule_tac [2] PSP_Unless [OF v_Leadsto_not_p U_F0])
apply (rule U_F1 [THEN LeadsTo_weaken_R], auto)
done

lemma n1_Leadsto_3: "Mutex ∈ {s. n s = 1} LeadsTo {s. n s = 3}"
apply (rule LeadsTo_cancel2 [THEN LeadsTo_Un_duplicate])
apply (rule_tac [2] V_F2)
apply (simp add: Collect_conj_eq)
apply (subst Un_commute)
apply (rule LeadsTo_cancel2 [THEN LeadsTo_Un_duplicate])
 apply (rule_tac [2] PSP_Unless [OF u_Leadsto_p V_F0])
apply (rule V_F1 [THEN LeadsTo_weaken_R], auto)
done

end
```

```
theory Reach imports UNITY_Main begin

typedecl vertex

types    state = "vertex=>bool"

consts
  init ::  "vertex"

  edges :: "(vertex*vertex) set"

constdefs

  asgt  :: "[vertex,vertex] => (state*state) set"
    "asgt u v == {(s,s'). s' = s(v:= s u | s v)}"

  Rprg :: "state program"
    "Rprg == mk_total_program ({%v. v=init}, ⋃(u,v)∈edges. {asgt u v}, UNIV)"

  reach_invariant :: "state set"
    "reach_invariant == {s. (∀v. s v --> (init, v) ∈ edges^*) & s init}"
```

```
  fixedpoint :: "state set"
    "fixedpoint == {s. ∀(u,v)∈edges. s u --> s v}"

  metric :: "state => nat"
    "metric s == card {v. ~ s v}"
```

*We assume that the set of vertices is finite

axioms
```
  finite_graph:  "finite (UNIV :: vertex set)"
```

lemma *ifE [elim!]:*
```
    "[| if P then Q else R;
        [| P;   Q |] ==> S;
        [| ~ P; R |] ==> S |] ==> S"
```
by *(simp split: split_if_asm)*

declare *Rprg_def [THEN def_prg_Init, simp]*

declare *asgt_def [THEN def_act_simp,simp]*

All vertex sets are finite

declare *finite_subset [OF subset_UNIV finite_graph, iff]*

declare *reach_invariant_def [THEN def_set_simp, simp]*

lemma *reach_invariant: "Rprg ∈ Always reach_invariant"*
apply *(rule AlwaysI, force)*
apply *(unfold Rprg_def, safety)*
apply *(blast intro: rtrancl_trans)*
done

lemma *fixedpoint_invariant_correct:*
```
     "fixedpoint ∩ reach_invariant = { %v. (init, v) ∈ edges^* }"
```
apply *(unfold fixedpoint_def)*
apply *(rule equalityI)*
apply *(auto intro!: ext)*
apply *(erule rtrancl_induct, auto)*
done

lemma *lemma1:*
```
     "FP Rprg ⊆ fixedpoint"
```
apply *(simp add: FP_def fixedpoint_def Rprg_def mk_total_program_def)*
apply *(auto simp add: stable_def constrains_def)*
apply *(drule bspec, assumption)*

```
apply (simp add: Image_singleton image_iff)
apply (drule fun_cong, auto)
done

lemma lemma2:
     "fixedpoint ⊆ FP Rprg"
apply (simp add: FP_def fixedpoint_def Rprg_def mk_total_program_def)
apply (auto intro!: ext simp add: stable_def constrains_def)
done

lemma FP_fixedpoint: "FP Rprg = fixedpoint"
by (rule equalityI [OF lemma1 lemma2])



lemma Compl_fixedpoint: "- fixedpoint = (⋃(u,v)∈edges. {s. s u & ~ s v})"
apply (simp add: FP_fixedpoint [symmetric] Rprg_def mk_total_program_def)
apply (auto simp add: Compl_FP UN_UN_flatten)
 apply (rule fun_upd_idem, force)
apply (force intro!: rev_bexI simp add: fun_upd_idem_iff)
done

lemma Diff_fixedpoint:
     "A - fixedpoint = (⋃(u,v)∈edges. A ∩ {s. s u & ~ s v})"
by (simp add: Diff_eq Compl_fixedpoint, blast)



lemma Suc_metric: "~ s x ==> Suc (metric (s(x:=True))) = metric s"
apply (unfold metric_def)
apply (subgoal_tac "{v. ~ (s (x:=True)) v} = {v. ~ s v} - {x}")
 prefer 2 apply force
apply (simp add: card_Suc_Diff1)
done

lemma metric_less [intro!]: "~ s x ==> metric (s(x:=True)) < metric s"
by (erule Suc_metric [THEN subst], blast)

lemma metric_le: "metric (s(y:=s x | s y)) ≤ metric s"
apply (case_tac "s x --> s y")
apply (auto intro: less_imp_le simp add: fun_upd_idem)
done

lemma LeadsTo_Diff_fixedpoint:
     "Rprg ∈ ((metric-‘{m}) - fixedpoint) LeadsTo (metric-‘(lessThan m))"
apply (simp (no_asm) add: Diff_fixedpoint Rprg_def)
apply (rule LeadsTo_UN, auto)
apply (ensures_tac "asgt a b")
 prefer 2 apply blast
apply (simp (no_asm_use) add: linorder_not_less)
apply (drule metric_le [THEN order_antisym])
apply (auto elim: less_not_refl3 [THEN [2] rev_notE])
```

done

lemma *LeadsTo_Un_fixedpoint:*
"Rprg ∈ (metric-‘{m}) LeadsTo (metric-‘(lessThan m) ∪ fixedpoint)"
apply *(rule LeadsTo_Diff [OF LeadsTo_Diff_fixedpoint [THEN LeadsTo_weaken_R]*
subset_imp_LeadsTo], auto)
done

lemma *LeadsTo_fixedpoint: "Rprg ∈ UNIV LeadsTo fixedpoint"*
apply *(rule LessThan_induct, auto)*
apply *(rule LeadsTo_Un_fixedpoint)*
done

lemma *LeadsTo_correct: "Rprg ∈ UNIV LeadsTo { %v. (init, v) ∈ edges^* }"*
apply *(subst fixedpoint_invariant_correct [symmetric])*
apply *(rule Always_LeadsTo_weaken [OF reach_invariant LeadsTo_fixedpoint],*
auto)
done

end

theory *Reachability* **imports** *Detects Reach* **begin**

types *edge = "(vertex*vertex)"*

record *state =*
reach :: "vertex => bool"
nmsg :: "edge => nat"

consts *REACHABLE :: "edge set"*
root :: "vertex"
E :: "edge set"
V :: "vertex set"

inductive *"REACHABLE"*
intros
base: "v ∈ V ==> ((v,v) ∈ REACHABLE)"
step: "((u,v) ∈ REACHABLE) & (v,w) ∈ E ==> ((u,w) ∈ REACHABLE)"

constdefs
reachable :: "vertex => state set"
"reachable p == {s. reach s p}"

nmsg_eq :: "nat => edge => state set"
"nmsg_eq k == %e. {s. nmsg s e = k}"

nmsg_gt :: "nat => edge => state set"
"nmsg_gt k == %e. {s. k < nmsg s e}"

nmsg_gte :: "nat => edge => state set"

```
  "nmsg_gte k == %e. {s. k ≤ nmsg s e}"

  nmsg_lte  :: "nat => edge => state set"
  "nmsg_lte k  == %e. {s. nmsg s e ≤ k}"

  final :: "state set"
  "final == (⋂v∈V. reachable v <==> {s. (root, v) ∈ REACHABLE}) ∩
            (INTER E (nmsg_eq 0))"
```

axioms

```
    Graph1: "root ∈ V"

    Graph2: "(v,w) ∈ E ==> (v ∈ V) & (w ∈ V)"

    MA1:  "F ∈ Always (reachable root)"

    MA2:  "v ∈ V ==> F ∈ Always (- reachable v ∪ {s. ((root,v) ∈ REACHABLE)})"

    MA3:  "[|v ∈ V;w ∈ V|] ==> F ∈ Always (-(nmsg_gt 0 (v,w)) ∪ (reachable
v))"

    MA4:  "(v,w) ∈ E ==>
           F ∈ Always (-(reachable v) ∪ (nmsg_gt 0 (v,w)) ∪ (reachable w))"

    MA5:  "[|v ∈ V; w ∈ V|]
           ==> F ∈ Always (nmsg_gte 0 (v,w) ∩ nmsg_lte (Suc 0) (v,w))"

    MA6:  "[|v ∈ V|] ==> F ∈ Stable (reachable v)"

    MA6b: "[|v ∈ V;w ∈ W|] ==> F ∈ Stable (reachable v ∩ nmsg_lte k (v,w))"

    MA7:  "[|v ∈ V;w ∈ V|] ==> F ∈ UNIV LeadsTo nmsg_eq 0 (v,w)"
```

lemmas *E_imp_in_V_L = Graph2 [THEN conjunct1, standard]*
lemmas *E_imp_in_V_R = Graph2 [THEN conjunct2, standard]*

lemma *lemma2:*
"(v,w) ∈ E ==> F ∈ reachable v LeadsTo nmsg_eq 0 (v,w) ∩ reachable v"
apply *(rule MA7 [THEN PSP_Stable, THEN LeadsTo_weaken_L])*
apply *(rule_tac [3] MA6)*
apply *(auto simp add: E_imp_in_V_L E_imp_in_V_R)*
done

lemma *Induction_base: "(v,w) ∈ E ==> F ∈ reachable v LeadsTo reachable w"*
apply *(rule MA4 [THEN Always_LeadsTo_weaken])*
apply *(rule_tac [2] lemma2)*
apply *(auto simp add: nmsg_eq_def nmsg_gt_def)*
done

lemma *REACHABLE_LeadsTo_reachable:*
"(v,w) ∈ REACHABLE ==> F ∈ reachable v LeadsTo reachable w"
apply *(erule REACHABLE.induct)*

```
apply (rule subset_imp_LeadsTo, blast)
apply (blast intro: LeadsTo_Trans Induction_base)
done

lemma Detects_part1: "F ∈ {s. (root,v) ∈ REACHABLE} LeadsTo reachable v"
apply (rule single_LeadsTo_I)
apply (simp split add: split_if_asm)
apply (rule MA1 [THEN Always_LeadsToI])
apply (erule REACHABLE_LeadsTo_reachable [THEN LeadsTo_weaken_L], auto)
done


lemma Reachability_Detected:
     "v ∈ V ==> F ∈ (reachable v) Detects {s. (root,v) ∈ REACHABLE}"
apply (unfold Detects_def, auto)
 prefer 2 apply (blast intro: MA2 [THEN Always_weaken])
apply (rule Detects_part1 [THEN LeadsTo_weaken_L], blast)
done


lemma LeadsTo_Reachability:
     "v ∈ V ==> F ∈ UNIV LeadsTo (reachable v <==> {s. (root,v) ∈ REACHABLE})"
by (erule Reachability_Detected [THEN Detects_Imp_LeadstoEQ])




lemma Eq_lemma1:
     "(reachable v <==> {s. (root,v) ∈ REACHABLE}) =
      {s. ((s ∈ reachable v) = ((root,v) ∈ REACHABLE))}"
by (unfold Equality_def, blast)


lemma Eq_lemma2:
     "(reachable v <==> (if (root,v) ∈ REACHABLE then UNIV else {})) =
      {s. ((s ∈ reachable v) = ((root,v) ∈ REACHABLE))}"
by (unfold Equality_def, auto)




lemma final_lemma1:
     "(⋂v ∈ V. ⋂w ∈ V. {s. ((s ∈ reachable v) = ((root,v) ∈ REACHABLE))
&
                                s ∈ nmsg_eq 0 (v,w)})
      ⊆ final"
apply (unfold final_def Equality_def, auto)
apply (frule E_imp_in_V_R)
apply (frule E_imp_in_V_L, blast)
done
```

```
lemma final_lemma2:
 "E≠{}
  ==> (⋂v ∈ V. ⋂e ∈ E. {s. ((s ∈ reachable v) = ((root,v) ∈ REACHABLE))}

                          ∩ nmsg_eq 0 e)    ⊆  final"
apply (unfold final_def Equality_def)
apply (auto split add: split_if_asm)
apply (frule E_imp_in_V_L, blast)
done

lemma final_lemma3:
     "E≠{}
      ==> (⋂v ∈ V. ⋂e ∈ E.
           (reachable v <==> {s. (root,v) ∈ REACHABLE}) ∩ nmsg_eq 0 e)
          ⊆ final"
apply (frule final_lemma2)
apply (simp (no_asm_use) add: Eq_lemma2)
done


lemma final_lemma4:
     "E≠{}
      ==> (⋂v ∈ V. ⋂e ∈ E.
           {s. ((s ∈ reachable v) = ((root,v) ∈ REACHABLE))} ∩ nmsg_eq 0
e)
          = final"
apply (rule subset_antisym)
apply (erule final_lemma2)
apply (unfold final_def Equality_def, blast)
done

lemma final_lemma5:
     "E≠{}
      ==> (⋂v ∈ V. ⋂e ∈ E.
           ((reachable v) <==> {s. (root,v) ∈ REACHABLE}) ∩ nmsg_eq 0 e)

          = final"
apply (frule final_lemma4)
apply (simp (no_asm_use) add: Eq_lemma2)
done


lemma final_lemma6:
     "(⋂v ∈ V. ⋂w ∈ V.
       (reachable v <==> {s. (root,v) ∈ REACHABLE}) ∩ nmsg_eq 0 (v,w))
      ⊆ final"
apply (simp (no_asm) add: Eq_lemma2 Int_def)
apply (rule final_lemma1)
done


lemma final_lemma7:
     "final =
```

```
      (⋂v ∈ V. ⋂w ∈ V.
       ((reachable v) <==> {s. (root,v) ∈ REACHABLE}) ∩
       (-{s. (v,w) ∈ E} ∪ (nmsg_eq 0 (v,w))))"
apply (unfold final_def)
apply (rule subset_antisym, blast)
apply (auto split add: split_if_asm)
apply (blast dest: E_imp_in_V_L E_imp_in_V_R)+
done
```

```
lemma not_REACHABLE_imp_Stable_not_reachable:
     "[| v ∈ V; (root,v) ∉ REACHABLE |] ==> F ∈ Stable (- reachable v)"
apply (drule MA2 [THEN AlwaysD], auto)
done

lemma Stable_reachable_EQ_R:
     "v ∈ V ==> F ∈ Stable (reachable v <==> {s. (root,v) ∈ REACHABLE})"
apply (simp (no_asm) add: Equality_def Eq_lemma2)
apply (blast intro: MA6 not_REACHABLE_imp_Stable_not_reachable)
done


lemma lemma4:
     "((nmsg_gte 0 (v,w) ∩ nmsg_lte (Suc 0) (v,w)) ∩
       (- nmsg_gt 0 (v,w) ∪ A))
      ⊆ A ∪ nmsg_eq 0 (v,w)"
apply (unfold nmsg_gte_def nmsg_lte_def nmsg_gt_def nmsg_eq_def, auto)
done


lemma lemma5:
     "reachable v ∩ nmsg_eq 0 (v,w) =
      ((nmsg_gte 0 (v,w) ∩ nmsg_lte (Suc 0) (v,w)) ∩
       (reachable v ∩ nmsg_lte 0 (v,w)))"
by (unfold nmsg_gte_def nmsg_lte_def nmsg_gt_def nmsg_eq_def, auto)

lemma lemma6:
     "- nmsg_gt 0 (v,w) ∪ reachable v ⊆ nmsg_eq 0 (v,w) ∪ reachable v"
apply (unfold nmsg_gte_def nmsg_lte_def nmsg_gt_def nmsg_eq_def, auto)
done

lemma Always_reachable_OR_nmsg_0:
     "[|v ∈ V; w ∈ V|] ==> F ∈ Always (reachable v ∪ nmsg_eq 0 (v,w))"
apply (rule Always_Int_I [OF MA5 MA3, THEN Always_weaken])
apply (rule_tac [5] lemma4, auto)
done

lemma Stable_reachable_AND_nmsg_0:
     "[|v ∈ V; w ∈ V|] ==> F ∈ Stable (reachable v ∩ nmsg_eq 0 (v,w))"
```

```
apply (subst lemma5)
apply (blast intro: MA5 Always_imp_Stable [THEN Stable_Int] MA6b)
done

lemma Stable_nmsg_0_OR_reachable:
     "[|v ∈ V; w ∈ V|] ==> F ∈ Stable (nmsg_eq 0 (v,w) ∪ reachable v)"
by (blast intro!: Always_weaken [THEN Always_imp_Stable] lemma6 MA3)

lemma not_REACHABLE_imp_Stable_not_reachable_AND_nmsg_0:
     "[| v ∈ V; w ∈ V; (root,v) ∉ REACHABLE |]
      ==> F ∈ Stable (- reachable v ∩ nmsg_eq 0 (v,w))"
apply (rule Stable_Int [OF MA2 [THEN Always_imp_Stable]
                           Stable_nmsg_0_OR_reachable,
            THEN Stable_eq])
   prefer 4 apply blast
apply auto
done

lemma Stable_reachable_EQ_R_AND_nmsg_0:
     "[| v ∈ V; w ∈ V |]
      ==> F ∈ Stable ((reachable v <==> {s. (root,v) ∈ REACHABLE}) ∩
                      nmsg_eq 0 (v,w))"
by (simp add: Equality_def Eq_lemma2  Stable_reachable_AND_nmsg_0
              not_REACHABLE_imp_Stable_not_reachable_AND_nmsg_0)




lemma UNIV_lemma: "UNIV ⊆ (⋂v ∈ V. UNIV)"
by blast

lemmas UNIV_LeadsTo_completion =
    LeadsTo_weaken_L [OF Finite_stable_completion UNIV_lemma]

lemma LeadsTo_final_E_empty: "E={} ==> F ∈ UNIV LeadsTo final"
apply (unfold final_def, simp)
apply (rule UNIV_LeadsTo_completion)
  apply safe
 apply (erule LeadsTo_Reachability [simplified])
apply (drule Stable_reachable_EQ_R, simp)
done


lemma Leadsto_reachability_AND_nmsg_0:
     "[| v ∈ V; w ∈ V |]
      ==> F ∈ UNIV LeadsTo
             ((reachable v <==> {s. (root,v): REACHABLE}) ∩ nmsg_eq 0 (v,w))"
apply (rule LeadsTo_Reachability [THEN LeadsTo_Trans], blast)
apply (subgoal_tac
         "F ∈ (reachable v <==> {s. (root,v) ∈ REACHABLE}) ∩
```

```
             UNIV LeadsTo (reachable v <==> {s. (root,v) ∈ REACHABLE}) ∩

             nmsg_eq 0 (v,w) ")
apply simp
apply (rule PSP_Stable2)
apply (rule MA7)
apply (rule_tac [3] Stable_reachable_EQ_R, auto)
done

lemma LeadsTo_final_E_NOT_empty: "E≠{} ==> F ∈ UNIV LeadsTo final"
apply (rule LeadsTo_weaken_L [OF LeadsTo_weaken_R UNIV_lemma])
apply (rule_tac [2] final_lemma6)
apply (rule Finite_stable_completion)
  apply blast
 apply (rule UNIV_LeadsTo_completion)
   apply (blast intro: Stable_INT Stable_reachable_EQ_R_AND_nmsg_0
                   Leadsto_reachability_AND_nmsg_0)+
done

lemma LeadsTo_final: "F ∈ UNIV LeadsTo final"
apply (case_tac "E={}")
 apply (rule_tac [2] LeadsTo_final_E_NOT_empty)
apply (rule LeadsTo_final_E_empty, auto)
done




lemma Stable_final_E_empty: "E={} ==> F ∈ Stable final"
apply (unfold final_def, simp)
apply (rule Stable_INT)
apply (drule Stable_reachable_EQ_R, simp)
done


lemma Stable_final_E_NOT_empty: "E≠{} ==> F ∈ Stable final"
apply (subst final_lemma7)
apply (rule Stable_INT)
apply (rule Stable_INT)
apply (simp (no_asm) add: Eq_lemma2)
apply safe
apply (rule Stable_eq)
apply (subgoal_tac [2]
     "({s. (s ∈ reachable v) = ((root,v) ∈ REACHABLE) } ∩ nmsg_eq 0 (v,w))
=
      ({s. (s ∈ reachable v) = ( (root,v) ∈ REACHABLE) } ∩ (- UNIV ∪ nmsg_eq
0 (v,w)))")
prefer 2 apply blast
prefer 2 apply blast
apply (rule Stable_reachable_EQ_R_AND_nmsg_0
            [simplified Eq_lemma2 Collect_const])
apply (blast, blast)
apply (rule Stable_eq)
```

```
 apply (rule Stable_reachable_EQ_R [simplified Eq_lemma2 Collect_const])
 apply simp
apply (subgoal_tac
       "({s. (s ∈ reachable v) = ((root,v) ∈ REACHABLE) }) =
        ({s. (s ∈ reachable v) = ( (root,v) ∈ REACHABLE) } Int
             (- {} ∪ nmsg_eq 0 (v,w)))")
apply blast+
done

lemma Stable_final: "F ∈ Stable final"
apply (case_tac "E={}")
 prefer 2 apply (blast intro: Stable_final_E_NOT_empty)
apply (blast intro: Stable_final_E_empty)
done

end
```

21 Analyzing the Needham-Schroeder Public-Key Protocol in UNITY

theory *NSP_Bad* **imports** *Public UNITY_Main* **begin**

This is the flawed version, vulnerable to Lowe's attack. From page 260 of Burrows, Abadi and Needham. A Logic of Authentication. Proc. Royal Soc. 426 (1989).

types *state = "event list"*

constdefs

```
  Fake :: "(state*state) set"
    "Fake == {(s,s').
              ∃B X. s' = Says Spy B X # s
                    & X ∈ synth (analz (spies s))}"



  NS1 :: "(state*state) set"
    "NS1 == {(s1,s').
             ∃A1 B NA.
                 s' = Says A1 B (Crypt (pubK B) {|Nonce NA, Agent A1|}) # s1
              & Nonce NA ∉ used s1}"


  NS2 :: "(state*state) set"
    "NS2 == {(s2,s').
             ∃A' A2 B NA NB.
                 s' = Says B A2 (Crypt (pubK A2) {|Nonce NA, Nonce NB|}) #
s2
```

```
              & Says A' B (Crypt (pubK B) {|Nonce NA, Agent A2|}) ∈ set s2
              & Nonce NB ∉ used s2}"


  NS3 :: "(state*state) set"
    "NS3 == {(s3,s').
             ∃A3 B' B NA NB.
                 s' = Says A3 B (Crypt (pubK B) (Nonce NB)) # s3
               & Says A3  B (Crypt (pubK B) {|Nonce NA, Agent A3|}) ∈ set
s3
               & Says B' A3 (Crypt (pubK A3) {|Nonce NA, Nonce NB|}) ∈ set
s3}"


constdefs
  Nprg :: "state program"

    "Nprg == mk_total_program({[]}, {Fake, NS1, NS2, NS3}, UNIV)"

declare spies_partsEs [elim]
declare analz_into_parts [dest]
declare Fake_parts_insert_in_Un  [dest]
```

For other theories, e.g. Mutex and Lift, using [iff] slows proofs down. Here, it facilitates re-use of the Auth proofs.

```
declare Fake_def [THEN def_act_simp, iff]
declare NS1_def [THEN def_act_simp, iff]
declare NS2_def [THEN def_act_simp, iff]
declare NS3_def [THEN def_act_simp, iff]

declare Nprg_def [THEN def_prg_Init, simp]
```

A "possibility property": there are traces that reach the end. Replace by LEADSTO proof!

```
lemma "A ≠ B ==>
       ∃NB. ∃s ∈ reachable Nprg. Says A B (Crypt (pubK B) (Nonce NB)) ∈
set s"
apply (intro exI bexI)
apply (rule_tac [2] act = "totalize_act NS3" in reachable.Acts)
apply (rule_tac [3] act = "totalize_act NS2" in reachable.Acts)
apply (rule_tac [4] act = "totalize_act NS1" in reachable.Acts)
apply (rule_tac [5] reachable.Init)
apply (simp_all (no_asm_simp) add: Nprg_def totalize_act_def)
apply auto
done
```

21.1 Inductive Proofs about ns_public

```
lemma ns_constrainsI:
     "(!!act s s'. [| act ∈ {Id, Fake, NS1, NS2, NS3};
                      (s,s') ∈ act;  s ∈ A |] ==> s' ∈ A')
      ==> Nprg ∈ A co A'"
apply (simp add: Nprg_def mk_total_program_def)
```

```
apply (rule constrainsI, auto)
done
```

This ML code does the inductions directly.

```
ML{*
val ns_constrainsI = thm "ns_constrainsI";

fun ns_constrains_tac(cs,ss) i =
   SELECT_GOAL
      (EVERY [REPEAT (etac Always_ConstrainsI 1),
              REPEAT (resolve_tac [StableI, stableI,
                                   constrains_imp_Constrains] 1),
              rtac ns_constrainsI 1,
              full_simp_tac ss 1,
              REPEAT (FIRSTGOAL (etac disjE)),
              ALLGOALS (clarify_tac (cs delrules [impI,impCE])),
              REPEAT (FIRSTGOAL analz_mono_contra_tac),
              ALLGOALS (asm_simp_tac ss)]) i;

(*Tactic for proving secrecy theorems*)
fun ns_induct_tac(cs,ss) =
  (SELECT_GOAL o EVERY)
     [rtac AlwaysI 1,
      force_tac (cs,ss) 1,
      (*"reachable" gets in here*)
      rtac (Always_reachable RS Always_ConstrainsI RS StableI) 1,
      ns_constrains_tac(cs,ss) 1];
*}

method_setup ns_induct = {*
    Method.ctxt_args (fn ctxt =>
        Method.METHOD (fn facts =>
            ns_induct_tac (local_clasimpset_of ctxt) 1)) *}
    "for inductive reasoning about the Needham-Schroeder protocol"
```

Converts invariants into statements about reachable states

```
lemmas Always_Collect_reachableD =
     Always_includes_reachable [THEN subsetD, THEN CollectD]
```

Spy never sees another agent's private key! (unless it's bad at start)

```
lemma Spy_see_priK:
     "Nprg ∈ Always {s. (Key (priK A) ∈ parts (spies s)) = (A ∈ bad)}"
apply ns_induct
apply blast
done
declare Spy_see_priK [THEN Always_Collect_reachableD, simp]

lemma Spy_analz_priK:
     "Nprg ∈ Always {s. (Key (priK A) ∈ analz (spies s)) = (A ∈ bad)}"
by (rule Always_reachable [THEN Always_weaken], auto)
declare Spy_analz_priK [THEN Always_Collect_reachableD, simp]
```

21.2 Authenticity properties obtained from NS2

It is impossible to re-use a nonce in both NS1 and NS2 provided the nonce is secret. (Honest users generate fresh nonces.)

```
lemma no_nonce_NS1_NS2:
 "Nprg
  ∈ Always {s. Crypt (pubK C) {|NA', Nonce NA|} ∈ parts (spies s) -->
               Crypt (pubK B) {|Nonce NA, Agent A|} ∈ parts (spies s) -->
               Nonce NA ∈ analz (spies s)}"
apply ns_induct
apply (blast intro: analz_insertI)+
done
```

Adding it to the claset slows down proofs...

```
lemmas no_nonce_NS1_NS2_reachable =
       no_nonce_NS1_NS2 [THEN Always_Collect_reachableD, rule_format]
```

Unicity for NS1: nonce NA identifies agents A and B

```
lemma unique_NA_lemma:
     "Nprg
  ∈ Always {s. Nonce NA ∉ analz (spies s) -->
               Crypt(pubK B) {|Nonce NA, Agent A|} ∈ parts(spies s) -->
               Crypt(pubK B') {|Nonce NA, Agent A'|} ∈ parts(spies s) -->
               A=A' & B=B'}"
apply ns_induct
apply auto
```

Fake, NS1 are non-trivial

```
done
```

Unicity for NS1: nonce NA identifies agents A and B

```
lemma unique_NA:
     "[| Crypt(pubK B)  {|Nonce NA, Agent A|} ∈ parts(spies s);
         Crypt(pubK B') {|Nonce NA, Agent A'|} ∈ parts(spies s);
         Nonce NA ∉ analz (spies s);
         s ∈ reachable Nprg |]
      ==> A=A' & B=B'"
by (blast dest: unique_NA_lemma [THEN Always_Collect_reachableD])
```

Secrecy: Spy does not see the nonce sent in msg NS1 if A and B are secure

```
lemma Spy_not_see_NA:
     "[| A ∉ bad;  B ∉ bad |]
  ==> Nprg ∈ Always
              {s. Says A B (Crypt(pubK B) {|Nonce NA, Agent A|}) ∈ set s
                  --> Nonce NA ∉ analz (spies s)}"
apply ns_induct
```

NS3

```
prefer 4 apply (blast intro: no_nonce_NS1_NS2_reachable)
```

NS2

```
prefer 3 apply (blast dest: unique_NA)
```

NS1

prefer *2* **apply** *blast*

Fake

apply *spy_analz*
done

Authentication for A: if she receives message 2 and has used NA to start a run, then B has sent message 2.

lemma *A_trusts_NS2:*
```
 "[| A ∉ bad;  B ∉ bad |]
  ==> Nprg ∈ Always
              {s. Says A B (Crypt(pubK B) {|Nonce NA, Agent A|}) ∈ set s &
                  Crypt(pubK A) {|Nonce NA, Nonce NB|} ∈ parts (knows Spy
s)
         --> Says B A (Crypt(pubK A) {|Nonce NA, Nonce NB|}) ∈ set s}"
```

apply *(insert Spy_not_see_NA [of A B NA], simp, ns_induct)*
apply *(auto dest: unique_NA)*
done

If the encrypted message appears then it originated with Alice in NS1

lemma *B_trusts_NS1:*
```
     "Nprg ∈ Always
              {s. Nonce NA ∉ analz (spies s) -->
                  Crypt (pubK B) {|Nonce NA, Agent A|} ∈ parts (spies s)
         --> Says A B (Crypt (pubK B) {|Nonce NA, Agent A|}) ∈ set s}"
```
apply *ns_induct*
apply *blast*
done

21.3 Authenticity properties obtained from NS2

Unicity for NS2: nonce NB identifies nonce NA and agent A. Proof closely follows that of *unique_NA*.

lemma *unique_NB_lemma:*
```
 "Nprg
  ∈ Always {s. Nonce NB ∉ analz (spies s)  -->
               Crypt (pubK A) {|Nonce NA, Nonce NB|} ∈ parts (spies s) -->
               Crypt(pubK A'){|Nonce NA', Nonce NB|} ∈ parts(spies s) -->
               A=A' & NA=NA'}"
```
apply *ns_induct*
apply *auto*

Fake, NS2 are non-trivial

done

lemma *unique_NB:*
```
     "[| Crypt(pubK A) {|Nonce NA, Nonce NB|} ∈ parts(spies s);
         Crypt(pubK A'){|Nonce NA', Nonce NB|} ∈ parts(spies s);
         Nonce NB ∉ analz (spies s);
         s ∈ reachable Nprg |]
```

```
      ==> A=A' & NA=NA'"
apply (blast dest: unique_NB_lemma [THEN Always_Collect_reachableD])
done
```

NB remains secret PROVIDED Alice never responds with round 3

```
lemma Spy_not_see_NB:
     "[| A ∉ bad;  B ∉ bad |]
  ==> Nprg ∈ Always
              {s. Says B A (Crypt (pubK A) {|Nonce NA, Nonce NB|}) ∈ set s
&
                  (ALL C. Says A C (Crypt (pubK C) (Nonce NB)) ∉ set s)
                  --> Nonce NB ∉ analz (spies s)}"
apply ns_induct
apply (simp_all (no_asm_simp) add: all_conj_distrib)
```

NS3: because NB determines A

```
prefer 4 apply (blast dest: unique_NB)
```

NS2: by freshness and unicity of NB

```
prefer 3 apply (blast intro: no_nonce_NS1_NS2_reachable)
```

NS1: by freshness

```
prefer 2 apply blast
```

Fake

```
apply spy_analz
done
```

Authentication for B: if he receives message 3 and has used NB in message 2, then A has sent message 3–to somebody....

```
lemma B_trusts_NS3:
     "[| A ∉ bad;  B ∉ bad |]
  ==> Nprg ∈ Always
              {s. Crypt (pubK B) (Nonce NB) ∈ parts (spies s) &
                  Says B A  (Crypt (pubK A) {|Nonce NA, Nonce NB|}) ∈ set
s
                  --> (∃C. Says A C (Crypt (pubK C) (Nonce NB)) ∈ set s)}"

apply (insert Spy_not_see_NB [of A B NA NB], simp, ns_induct)
apply (simp_all (no_asm_simp) add: ex_disj_distrib)
apply auto
```

NS3: because NB determines A. This use of `unique_NB` is robust.

```
apply (blast intro: unique_NB [THEN conjunct1])
done
```

Can we strengthen the secrecy theorem? NO

```
lemma "[| A ∉ bad;  B ∉ bad |]
  ==> Nprg ∈ Always
              {s. Says B A (Crypt (pubK A) {|Nonce NA, Nonce NB|}) ∈ set s
                  --> Nonce NB ∉ analz (spies s)}"
apply ns_induct
```

apply *auto*

Fake

apply *spy_analz*

NS2: by freshness and unicity of NB

apply *(blast intro: no_nonce_NS1_NS2_reachable)*

NS3: unicity of NB identifies A and NA, but not B

apply *(frule_tac A'=A* **in** *Says_imp_spies [THEN parts.Inj, THEN unique_NB])*
apply *(erule Says_imp_spies [THEN parts.Inj], auto)*
apply *(rename_tac s B' C)*

This is the attack!

```
1. ⋀s B' C.
      ⟦A ∉ bad; B ∉ bad; s ∈ reachable Nprg;
       Says A C (Crypt (pubK C) {|Nonce NA, Agent A|}) ∈ set s;
       Says B' A (Crypt (pubK A) {|Nonce NA, Nonce NB|}) ∈ set s;
       C ∈ bad;
       Says B A (Crypt (pubK A) {|Nonce NA, Nonce NB|}) ∈ set s;
       Nonce NB ∉ analz (knows Spy s)⟧
      ⟹ False
```

oops

end

theory *Handshake* **imports** *UNITY_Main* **begin**

record *state =*
  BB :: bool
  NF :: nat
  NG :: nat

constdefs

```
  cmdF :: "(state*state) set"
    "cmdF == {(s,s'). s' = s (|NF:= Suc(NF s), BB:=False|) & BB s}"

  F :: "state program"
    "F == mk_total_program ({s. NF s = 0 & BB s}, {cmdF}, UNIV)"


  cmdG :: "(state*state) set"
    "cmdG == {(s,s'). s' = s (|NG:= Suc(NG s), BB:=True|) & ~ BB s}"

  G :: "state program"
    "G == mk_total_program ({s. NG s = 0 & BB s}, {cmdG}, UNIV)"
```

```
  invFG :: "state set"
    "invFG == {s. NG s <= NF s & NF s <= Suc (NG s) & (BB s = (NF s = NG s))}"


declare F_def [THEN def_prg_Init, simp]
        G_def [THEN def_prg_Init, simp]

        cmdF_def [THEN def_act_simp, simp]
        cmdG_def [THEN def_act_simp, simp]

        invFG_def [THEN def_set_simp, simp]


lemma invFG: "(F Join G) : Always invFG"
apply (rule AlwaysI)
apply force
apply (rule constrains_imp_Constrains [THEN StableI])
apply auto
 apply (unfold F_def, safety)
apply (unfold G_def, safety)
done

lemma lemma2_1: "(F Join G) : ({s. NF s = k} - {s. BB s}) LeadsTo
                              ({s. NF s = k} Int {s. BB s})"
apply (rule stable_Join_ensures1[THEN leadsTo_Basis, THEN leadsTo_imp_LeadsTo])
 apply (unfold F_def, safety)
apply (unfold G_def, ensures_tac "cmdG")
done

lemma lemma2_2: "(F Join G) : ({s. NF s = k} Int {s. BB s}) LeadsTo
                              {s. k < NF s}"
apply (rule stable_Join_ensures2[THEN leadsTo_Basis, THEN leadsTo_imp_LeadsTo])
 apply (unfold F_def, ensures_tac "cmdF")
apply (unfold G_def, safety)
done

lemma progress: "(F Join G) : UNIV LeadsTo {s. m < NF s}"
apply (rule LeadsTo_weaken_R)
apply (rule_tac f = "NF" and l = "Suc m" and B = "{}"
       in GreaterThan_bounded_induct)

apply (auto intro!: lemma2_1 lemma2_2
            intro: LeadsTo_Trans LeadsTo_Diff simp add: vimage_def)
done

end
```

22 A Family of Similar Counters: Original Version

theory *Counter* **imports** *UNITY_Main* **begin**

datatype *name = C | c nat*
types *state = "name=>int"*

consts
  sum :: "[nat,state]=>int"
  sumj :: "[nat, nat, state]=>int"

primrec
  "sum 0 s = 0"
  "sum (Suc i) s = s (c i) + sum i s"

primrec
  "sumj 0 i s = 0"
  "sumj (Suc n) i s = (if n=i then sum n s else s (c n) + sumj n i s)"

types *command = "(state*state)set"*

constdefs
  a :: "nat=>command"
 "a i == {(s, s'). s'=s(c i:= s (c i) + 1, C:= s C + 1)}"

  Component :: "nat => state program"
  "Component i ==
    mk_total_program({s. s C = 0 & s (c i) = 0}, {a i},
                    ⋃G ∈ preserves (%s. s (c i)). Acts G)"

declare *Component_def [THEN def_prg_Init, simp]*
declare *a_def [THEN def_act_simp, simp]*

lemma *sum_upd_gt [rule_format]: "∀n. I<n --> sum I (s(c n := x)) = sum I s"*
by *(induct_tac "I", auto)*

lemma *sum_upd_eq: "sum I (s(c I := x)) = sum I s"*
apply *(induct_tac "I")*
apply *(auto simp add: sum_upd_gt [unfolded fun_upd_def])*
done

lemma *sum_upd_C: "sum I (s(C := x)) = sum I s"*
by *(induct_tac "I", auto)*

lemma *sumj_upd_ci: "sumj I i (s(c i := x)) = sumj I i s"*
apply *(induct_tac "I")*
apply *(auto simp add: sum_upd_eq [unfolded fun_upd_def])*

done

lemma *sumj_upd_C: "sumj I i (s(C := x)) = sumj I i s"*
apply *(induct_tac "I")*
apply *(auto simp add: sum_upd_C [unfolded fun_upd_def])*
done

lemma *sumj_sum_gt [rule_format]: "∀i. I<i--> (sumj I i s = sum I s)"*
by *(induct_tac "I", auto)*

lemma *sumj_sum_eq: "(sumj I I s = sum I s)"*
apply *(induct_tac "I", auto)*
apply *(simp (no_asm) add: sumj_sum_gt)*
done

lemma *sum_sumj [rule_format]: "∀i. i<I-->(sum I s = s (c i) + sumj I i s)"*
apply *(induct_tac "I")*
apply *(auto simp add: linorder_neq_iff sumj_sum_eq)*
done

lemma *p2: "Component i ∈ stable {s. s C = s (c i) + k}"*
by *(simp add: Component_def, safety)*

lemma *p3: "Component i ∈ stable {s. ∀v. v≠c i & v≠C --> s v = k v}"*
by *(simp add: Component_def, safety)*

lemma *p2_p3_lemma1:*
"(∀k. Component i ∈ stable ({s. s C = s (c i) + sumj I i k}
∩ {s. ∀v. v≠c i & v≠C --> s v = k v}))
= (Component i ∈ stable {s. s C = s (c i) + sumj I i s})"
apply *(simp add: Component_def mk_total_program_def)*
apply *(auto simp add: constrains_def stable_def sumj_upd_C sumj_upd_ci)*
done

lemma *p2_p3_lemma2:*
"∀k. Component i ∈ stable ({s. s C = s (c i) + sumj I i k} Int
{s. ∀v. v≠c i & v≠C --> s v = k v})"
by *(blast intro: stable_Int [OF p2 p3])*

lemma *p2_p3: "Component i ∈ stable {s. s C = s (c i) + sumj I i s}"*
by *(auto intro!: p2_p3_lemma2 simp add: p2_p3_lemma1 [symmetric])*

lemma *sum_0' [rule_format]: "(∀i. i < I --> s (c i) = 0) --> sum I s = 0"*
by *(induct_tac "I", auto)*

lemma *safety:*
"0<I ==> (⨆i ∈ {i. i<I}. Component i) ∈ invariant {s. s C = sum I s}"

apply *(simp (no_asm) add: invariant_def JN_stable sum_sumj)*
apply *(force intro: p2_p3 sum_0')*
done

end

23 A Family of Similar Counters: Version with Compatibility

theory *Counterc* **imports** *UNITY_Main* **begin**

typedecl *state*
arities *state :: type*

consts
C :: "state=>int"
c :: "state=>nat=>int"

consts
sum :: "[nat,state]=>int"
sumj :: "[nat, nat, state]=>int"

primrec
"sum 0 s = 0"
"sum (Suc i) s = (c s) i + sum i s"

primrec
"sumj 0 i s = 0"
"sumj (Suc n) i s = (if n=i then sum n s else (c s) n + sumj n i s)"

types *command = "(state*state)set"*

constdefs
a :: "nat=>command"
"a i == {(s, s'). (c s') i = (c s) i + 1 & (C s') = (C s) + 1}"

Component :: "nat => state program"
"Component i == mk_total_program({s. C s = 0 & (c s) i = 0},
{a i},
⋃G ∈ preserves (%s. (c s) i). Acts G)"

declare *Component_def [THEN def_prg_Init, simp]*
declare *Component_def [THEN def_prg_AllowedActs, simp]*
declare *a_def [THEN def_act_simp, simp]*

lemma *sum_sumj_eq1 [rule_format]: "∀i. I<i--> (sum I s = sumj I i s)"*
by *(induct_tac "I", auto)*

lemma *sum_sumj_eq2 [rule_format]: "i<I --> sum I s = c s i + sumj I i s"*
apply *(induct_tac "I")*

```
apply (auto simp add: linorder_neq_iff sum_sumj_eq1)
done

lemma sum_ext [rule_format]:
     "(∀i. i<I --> c s' i = c s i) --> (sum I s' = sum I s)"
by (induct_tac "I", auto)

lemma sumj_ext [rule_format]:
     "(∀j. j<I & j≠i --> c s' j =  c s j) --> (sumj I i s' = sumj I i s)"
apply (induct_tac "I", safe)
apply (auto intro!: sum_ext)
done


lemma sum0 [rule_format]: "(∀i. i<I --> c s i = 0) -->  sum I s = 0"
by (induct_tac "I", auto)




lemma Component_ok_iff:
     "(Component i ok G) =
      (G ∈ preserves (%s. c s i) & Component i ∈ Allowed G)"
apply (auto simp add: ok_iff_Allowed Component_def [THEN def_total_prg_Allowed])
done
declare Component_ok_iff [iff]
declare OK_iff_ok [iff]
declare preserves_def [simp]


lemma p2: "Component i ∈ stable {s. C s = (c s) i + k}"
by (simp add: Component_def, safety)

lemma p3:
     "[| OK I Component; i∈I |]
      ==> Component i ∈ stable {s. ∀j∈I. j≠i --> c s j = c k j}"
apply simp
apply (unfold Component_def mk_total_program_def)
apply (simp (no_asm_use) add: stable_def constrains_def)
apply blast
done


lemma p2_p3_lemma1:
     "[| OK {i. i<I} Component; i<I |] ==>
      ∀k. Component i ∈ stable ({s. C s = c s i + sumj I i k} Int
                                {s. ∀j∈{i. i<I}. j≠i --> c s j = c k j})"
by (blast intro: stable_Int [OF p2 p3])

lemma p2_p3_lemma2:
     "(∀k. F ∈ stable ({s. C s = (c s) i + sumj I i k} Int
                       {s. ∀j∈{i. i<I}. j≠i --> c s j = c k j}))
      ==> (F ∈ stable {s. C s = c s i + sumj I i s})"
apply (simp add: constrains_def stable_def)
```

```
apply (force intro!: sumj_ext)
done


lemma p2_p3:
     "[| OK {i. i<I} Component; i<I |]
      ==> Component i ∈ stable {s. C s = c s i + sumj I i s}"
by (blast intro: p2_p3_lemma1 [THEN p2_p3_lemma2])



lemma safety:
     "[| 0<I; OK {i. i<I} Component |]
      ==> (⨆i∈{i. i<I}. (Component i)) ∈ invariant {s. C s = sum I s}"
apply (simp (no_asm) add: invariant_def JN_stable sum_sumj_eq2)
apply (auto intro!: sum0 p2_p3)
done

end



theory PriorityAux
imports "../UNITY_Main"
begin

typedecl vertex
arities vertex :: type

constdefs
  symcl :: "(vertex*vertex)set=>(vertex*vertex)set"
  "symcl r == r ∪ (r^-1)"
    — symmetric closure: removes the orientation of a relation

  neighbors :: "[vertex, (vertex*vertex)set]=>vertex set"
  "neighbors i r == ((r ∪ r^-1)``{i}) - {i}"
    — Neighbors of a vertex i

  R :: "[vertex, (vertex*vertex)set]=>vertex set"
  "R i r == r``{i}"

  A :: "[vertex, (vertex*vertex)set]=>vertex set"
  "A i r == (r^-1)``{i}"

  reach :: "[vertex, (vertex*vertex)set]=> vertex set"
  "reach i r == (r^+)``{i}"
    — reachable and above vertices: the original notation was R* and A*

  above :: "[vertex, (vertex*vertex)set]=> vertex set"
  "above i r == ((r^-1)^+)``{i}"

  reverse :: "[vertex, (vertex*vertex) set]=>(vertex*vertex)set"
  "reverse i r == (r - {(x,y). x=i | y=i} ∩ r) ∪ ({(x,y). x=i|y=i} ∩ r)^-1"
```

```
  derive1 :: "[vertex, (vertex*vertex)set, (vertex*vertex)set]=>bool"
```
— The original definition
```
  "derive1 i r q == symcl r = symcl q &
                    (∀k k'. k≠i & k'≠i -->((k,k'):r) = ((k,k'):q)) &
                    A i r = {} & R i q = {}"

  derive :: "[vertex, (vertex*vertex)set, (vertex*vertex)set]=>bool"
```
— Our alternative definition
```
  "derive i r q == A i r = {} & (q = reverse i r)"
```

axioms
```
  finite_vertex_univ:  "finite (UNIV :: vertex set)"
```
— we assume that the universe of vertices is finite

declare *derive_def [simp] derive1_def [simp] symcl_def [simp]*
A_def [simp] R_def [simp]
above_def [simp] reach_def [simp]
reverse_def [simp] neighbors_def [simp]

All vertex sets are finite

declare *finite_subset [OF subset_UNIV finite_vertex_univ, iff]*

and relatons over vertex are finite too

lemmas *finite_UNIV_Prod =*
finite_Prod_UNIV [OF finite_vertex_univ finite_vertex_univ]

declare *finite_subset [OF subset_UNIV finite_UNIV_Prod, iff]*

lemma *image0_trancl_iff_image0_r: "((r^+)``{i} = {}) = (r``{i} = {})"*
apply *auto*
apply *(erule trancl_induct, auto)*
done

lemma *image0_r_iff_image0_trancl: "(r``{i}={}) = (ALL x. ((i,x):r^+) = False)"*
apply *auto*
apply *(drule image0_trancl_iff_image0_r [THEN ssubst], auto)*
done

lemma *acyclic_eq_wf: "!!r::(vertex*vertex)set. acyclic r = wf r"*
by *(auto simp add: wf_iff_acyclic_if_finite)*

lemma *derive_derive1_eq: "derive i r q = derive1 i r q"*
by *auto*

lemma *lemma1_a:*
"[| x ∈ reach i q; derive1 k r q |] ==> x≠k --> x ∈ reach i r"

```
apply (unfold reach_def)
apply (erule ImageE)
apply (erule trancl_induct)
 apply (case_tac "i=k", simp_all)
  apply (blast intro: r_into_trancl, blast, clarify)
apply (drule_tac x = y in spec)
apply (drule_tac x = z in spec)
apply (blast dest: r_into_trancl intro: trancl_trans)
done

lemma reach_lemma: "derive k r q ==> reach i q ⊆ (reach i r ∪ {k})"
apply clarify
apply (drule lemma1_a)
apply (auto simp add: derive_derive1_eq
            simp del: reach_def derive_def derive1_def)
done


lemma reach_above_lemma:
      "(∀i. reach i q ⊆ (reach i r ∪ {k})) =
       (∀x. x≠k --> (∀i. i ∉ above x r --> i ∉ above x q))"
by (auto simp add: trancl_converse)


lemma maximal_converse_image0:
     "(z, i):r^+ ==> (∀y. (y, z):r --> (y,i) ∉ r^+) = ((r^-1)``{z}={})"
apply auto
apply (frule_tac r = r in trancl_into_trancl2, auto)
done

lemma above_lemma_a:
     "acyclic r ==> A i r≠{}-->(∃j ∈ above i r. A j r = {})"
apply (simp add: acyclic_eq_wf wf_eq_minimal)
apply (drule_tac x = " ((r^-1) ^+) ``{i}" in spec)
apply auto
apply (simp add: maximal_converse_image0 trancl_converse)
done

lemma above_lemma_b:
     "acyclic r ==> above i r≠{}-->(∃j ∈ above i r. above j r = {})"
apply (drule above_lemma_a)
apply (auto simp add: image0_trancl_iff_image0_r)
done

end
```

24 The priority system

theory *Priority* **imports** *PriorityAux* **begin**

From Charpentier and Chandy, Examples of Program Composition Illustrating the Use of Universal Properties In J. Rolim (editor), Parallel and Distributed Processing, Spriner LNCS 1586 (1999), pages 1215-1227.

```
types state = "(vertex*vertex)set"
types command = "vertex=>(state*state)set"

consts
  init :: "(vertex*vertex)set"
```
— the initial state

Following the definitions given in section 4.4

```
constdefs
  highest :: "[vertex, (vertex*vertex)set]=>bool"
  "highest i r == A i r = {}"
```
— i has highest priority in r

```
  lowest :: "[vertex, (vertex*vertex)set]=>bool"
  "lowest i r == R i r = {}"
```
— i has lowest priority in r

```
  act :: command
  "act i == {(s, s'). s'=reverse i s & highest i s}"

  Component :: "vertex=>state program"
  "Component i == mk_total_program({init}, {act i}, UNIV)"
```
— All components start with the same initial state

Some Abbreviations

```
constdefs
  Highest :: "vertex=>state set"
  "Highest i == {s. highest i s}"

  Lowest :: "vertex=>state set"
  "Lowest i == {s. lowest i s}"

  Acyclic :: "state set"
  "Acyclic == {s. acyclic s}"


  Maximal :: "state set"
```
— Every "above" set has a maximal vertex

```
  "Maximal == ⋂i. {s. ~highest i s-->(∃j ∈ above i  s. highest j s)}"

  Maximal' :: "state set"
```
— Maximal vertex: equivalent definition

```
  "Maximal' == ⋂i. Highest i Un (⋃j. {s. j ∈ above i s} Int Highest j)"


  Safety :: "state set"
  "Safety == ⋂i. {s. highest i s --> (∀j ∈ neighbors i s. ~highest j s)}"




  system :: "state program"
  "system == JN i. Component i"
```

```
declare highest_def [simp] lowest_def [simp]
declare Highest_def [THEN def_set_simp, simp]
    and Lowest_def  [THEN def_set_simp, simp]

declare Component_def [THEN def_prg_Init, simp]
declare act_def [THEN def_act_simp, simp]
```

24.1 Component correctness proofs

neighbors is stable

```
lemma Component_neighbors_stable: "Component i ∈ stable {s. neighbors k
s = n}"
by (simp add: Component_def, safety, auto)
```

property 4

```
lemma Component_waits_priority: "Component i: {s. ((i,j):s) = b} Int (- Highest
i) co {s. ((i,j):s)=b}"
by (simp add: Component_def, safety)
```

property 5: charpentier and Chandy mistakenly express it as 'transient Highest i'. Consider the case where i has neighbors

```
lemma Component_yields_priority:
 "Component i: {s. neighbors i s ≠ {}} Int Highest i
              ensures - Highest i"
apply (simp add: Component_def)
apply (ensures_tac "act i", blast+)
done
```

or better

```
lemma Component_yields_priority': "Component i ∈ Highest i ensures Lowest
i"
apply (simp add: Component_def)
apply (ensures_tac "act i", blast+)
done
```

property 6: Component doesn't introduce cycle

```
lemma Component_well_behaves: "Component i ∈ Highest i co Highest i Un Lowest
i"
by (simp add: Component_def, safety, fast)
```

property 7: local axiom

```
lemma locality: "Component i ∈ stable {s. ∀j k. j≠i & k≠i--> ((j,k):s)
= b j k}"
by (simp add: Component_def, safety)
```

24.2 System properties

property 8: strictly universal

```
lemma Safety: "system ∈ stable Safety"
apply (unfold Safety_def)
```

```
apply (rule stable_INT)
apply (simp add: system_def, safety, fast)
done
```

property 13: universal

```
lemma p13: "system ∈ {s. s = q} co {s. s=q} Un {s. ∃i. derive i q s}"
by (simp add: system_def Component_def mk_total_program_def totalize_JN, safety,
blast)
```

property 14: the 'above set' of a Component that hasn't got priority doesn't increase

```
lemma above_not_increase:
     "system ∈ -Highest i Int {s. j∉above i s} co {s. j∉above i s}"
apply (insert reach_lemma [of concl: j])
apply (simp add: system_def Component_def mk_total_program_def totalize_JN,

       safety)
apply (simp add: trancl_converse, blast)
done

lemma above_not_increase':
     "system ∈ -Highest i Int {s. above i s = x} co {s. above i s <= x}"
apply (insert above_not_increase [of i])
apply (simp add: trancl_converse constrains_def, blast)
done
```

p15: universal property: all Components well behave

```
lemma system_well_behaves [rule_format]:
     "∀i. system ∈ Highest i co Highest i Un Lowest i"
apply clarify
apply (simp add: system_def Component_def mk_total_program_def totalize_JN,

       safety, auto)
done


lemma Acyclic_eq: "Acyclic = (⋂i. {s. i∉above i s})"
by (auto simp add: Acyclic_def acyclic_def trancl_converse)


lemmas system_co =
      constrains_Un [OF above_not_increase [rule_format] system_well_behaves]


lemma Acyclic_stable: "system ∈ stable Acyclic"
apply (simp add: stable_def Acyclic_eq)
apply (auto intro!: constrains_INT system_co [THEN constrains_weaken]
            simp add: image0_r_iff_image0_trancl trancl_converse)
done


lemma Acyclic_subset_Maximal: "Acyclic <= Maximal"
apply (unfold Acyclic_def Maximal_def, clarify)
```

```
apply (drule above_lemma_b, auto)
done
```

property 17: original one is an invariant

```
lemma Acyclic_Maximal_stable: "system ∈ stable (Acyclic Int Maximal)"
by (simp add: Acyclic_subset_Maximal [THEN Int_absorb2] Acyclic_stable)
```

property 5: existential property

```
lemma Highest_leadsTo_Lowest: "system ∈ Highest i leadsTo Lowest i"
apply (simp add: system_def Component_def mk_total_program_def totalize_JN)
apply (ensures_tac "act i", auto)
done
```

a lowest i can never be in any abover set

```
lemma Lowest_above_subset: "Lowest i <= (⋂k. {s. i∉above k s})"
by (auto simp add: image0_r_iff_image0_trancl trancl_converse)
```

property 18: a simpler proof than the original, one which uses psp

```
lemma Highest_escapes_above: "system ∈ Highest i leadsTo (⋂k. {s. i∉above
k s})"
apply (rule leadsTo_weaken_R)
apply (rule_tac [2] Lowest_above_subset)
apply (rule Highest_leadsTo_Lowest)
done

lemma Highest_escapes_above':
     "system ∈ Highest j Int {s. j ∈ above i s} leadsTo {s. j∉above i s}"
by (blast intro: leadsTo_weaken [OF Highest_escapes_above Int_lower1 INT_lower])
```

24.3 The main result: above set decreases

The original proof of the following formula was wrong

```
lemma Highest_iff_above0: "Highest i = {s. above i s ={}}"
by (auto simp add: image0_trancl_iff_image0_r)

lemmas above_decreases_lemma =
     psp [THEN leadsTo_weaken, OF Highest_escapes_above' above_not_increase']



lemma above_decreases:
     "system ∈ (⋃j. {s. above i s = x} Int {s. j ∈ above i s} Int Highest
j)
               leadsTo {s. above i s < x}"
apply (rule leadsTo_UN)
apply (rule single_leadsTo_I, clarify)
apply (rule_tac x2 = "above i x" in above_decreases_lemma)
apply (simp_all (no_asm_use) add: Highest_iff_above0)
apply blast+
done
```

```
lemma Maximal_eq_Maximal': "Maximal = Maximal'"
by (unfold Maximal_def Maximal'_def Highest_def, blast)

lemma Acyclic_subset:
   "x≠{} ==>
    Acyclic Int {s. above i s = x} <=
    (⋃j. {s. above i s = x} Int {s. j ∈ above i s} Int Highest j)"
apply (rule_tac B = "Maximal' Int {s. above i s = x}" in subset_trans)
apply (simp (no_asm) add: Maximal_eq_Maximal' [symmetric])
apply (blast intro: Acyclic_subset_Maximal [THEN subsetD])
apply (simp (no_asm) del: above_def add: Maximal'_def Highest_iff_above0)
apply blast
done

lemmas above_decreases' = leadsTo_weaken_L [OF above_decreases Acyclic_subset]
lemmas above_decreases_psp = psp_stable [OF above_decreases' Acyclic_stable]

lemma above_decreases_psp':
"x≠{}==> system ∈ Acyclic Int {s. above i s = x} leadsTo
                  Acyclic Int {s. above i s < x}"
by (erule above_decreases_psp [THEN leadsTo_weaken], blast, auto)


lemmas finite_psubset_induct = wf_finite_psubset [THEN leadsTo_wf_induct]


lemma Progress: "system ∈ Acyclic leadsTo Highest i"
apply (rule_tac f = "%s. above i s" in finite_psubset_induct)
apply (simp del: above_def
            add: Highest_iff_above0 vimage_def finite_psubset_def, clarify)
apply (case_tac "m={}")
apply (rule Int_lower2 [THEN [2] leadsTo_weaken_L])
apply (force simp add: leadsTo_refl)
apply (rule_tac A' = "Acyclic Int {x. above i x < m}" in leadsTo_weaken_R)
apply (blast intro: above_decreases_psp')+
done
```

We have proved all (relevant) theorems given in the paper. We didn't assume any thing about the relation `r`. It is not necessary that `r` be a priority relation as assumed in the original proof. It suffices that we start from a state which is finite and acyclic.

end

```
theory TimerArray imports UNITY_Main begin

types 'a state = "nat * 'a"

constdefs
  count  :: "'a state => nat"
    "count s == fst s"

  decr  :: "('a state * 'a state) set"
```

```
    "decr == UN n uu. {((Suc n, uu), (n,uu))}"

  Timer :: "'a state program"
    "Timer == mk_total_program (UNIV, {decr}, UNIV)"


declare Timer_def [THEN def_prg_Init, simp]

declare count_def [simp] decr_def [simp]


lemma Timer_leadsTo_zero: "Timer : UNIV leadsTo {s. count s = 0}"
apply (rule_tac f = count in lessThan_induct, simp)
apply (case_tac "m")
 apply (force intro!: subset_imp_leadsTo)
apply (unfold Timer_def, ensures_tac "decr")
done

lemma Timer_preserves_snd [iff]: "Timer : preserves snd"
apply (rule preservesI)
apply (unfold Timer_def, safety)
done


declare PLam_stable [simp]

lemma TimerArray_leadsTo_zero:
     "finite I
      ==> (plam i: I. Timer) : UNIV leadsTo {(s,uu). ALL i:I. s i = 0}"
apply (erule_tac A'1 = "%i. lift_set i ({0} <*> UNIV)"
       in finite_stable_completion [THEN leadsTo_weaken])
apply auto

 prefer 2
 apply (simp add: Timer_def, safety)

apply (rule_tac f = "sub i o fst" in lessThan_induct)
apply (case_tac "m")

apply (auto intro: subset_imp_leadsTo
        simp add: insert_absorb
                  lift_set_Un_distrib [symmetric] lessThan_Suc [symmetric]

                Times_Un_distrib1 [symmetric] Times_Diff_distrib1 [symmetric])
apply (rename_tac "n")
apply (rule PLam_leadsTo_Basis)
apply (auto simp add: lessThan_Suc [symmetric])
apply (unfold Timer_def mk_total_program_def, safety)
apply (rule_tac act = decr in totalize_transientI, auto)
done

end
```

25 Common Declarations for Chandy and Charpentier's Allocator

theory *AllocBase* **imports** *UNITY_Main* **begin**

consts
  NbT :: *nat*
  Nclients :: *nat*

axioms
  NbT_pos: "0 < NbT"

consts *tokens* :: "nat list => nat"
primrec
  "tokens [] = 0"
  "tokens (x#xs) = x + tokens xs"

consts
  bag_of :: "'a list => 'a multiset"

primrec
  "bag_of [] = {#}"
  "bag_of (x#xs) = {#x#} + bag_of xs"

lemma *setsum_fun_mono* [*rule_format*]:
    "!!f :: nat=>nat.
     (ALL i. i<n --> f i <= g i) -->
     setsum f (lessThan n) <= setsum g (lessThan n)"
apply (*induct_tac* "n")
apply (*auto simp add*: *lessThan_Suc*)
apply (*drule_tac x = n* **in** *spec*, *arith*)
done

lemma *tokens_mono_prefix* [*rule_format*]:
    "ALL xs. xs <= ys --> tokens xs <= tokens ys"
apply (*induct_tac* "ys")
apply (*auto simp add*: *prefix_Cons*)
done

lemma *mono_tokens*: "mono tokens"
apply (*unfold mono_def*)
apply (*blast intro*: *tokens_mono_prefix*)
done

lemma *bag_of_append* [*simp*]: "bag_of (l@l') = bag_of l + bag_of l'"
apply (*induct_tac* "l", *simp*)
apply (*simp add*: *add_ac*)
done

lemma *mono_bag_of*: "mono (bag_of :: 'a list => ('a::order) multiset)"

```
apply (rule monoI)
apply (unfold prefix_def)
apply (erule genPrefix.induct, auto)
apply (simp add: union_le_mono)
apply (erule order_trans)
apply (rule union_upper1)
done


declare setsum_cong [cong]

lemma bag_of_sublist_lemma:
     "(∑ i∈ A Int lessThan k. {#if i<k then f i else g i#}) =
      (∑ i∈ A Int lessThan k. {#f i#})"
by (rule setsum_cong, auto)

lemma bag_of_sublist:
     "bag_of (sublist l A) =
      (∑ i∈ A Int lessThan (length l). {# l!i #})"
apply (rule_tac xs = l in rev_induct, simp)
apply (simp add: sublist_append Int_insert_right lessThan_Suc nth_append
                 bag_of_sublist_lemma add_ac)
done


lemma bag_of_sublist_Un_Int:
     "bag_of (sublist l (A Un B)) + bag_of (sublist l (A Int B)) =
      bag_of (sublist l A) + bag_of (sublist l B)"
apply (subgoal_tac "A Int B Int {..<length l} =
                    (A Int {..<length l}) Int (B Int {..<length l}) ")
apply (simp add: bag_of_sublist Int_Un_distrib2 setsum_Un_Int, blast)
done

lemma bag_of_sublist_Un_disjoint:
     "A Int B = {}
      ==> bag_of (sublist l (A Un B)) =
          bag_of (sublist l A) + bag_of (sublist l B)"
by (simp add: bag_of_sublist_Un_Int [symmetric])

lemma bag_of_sublist_UN_disjoint [rule_format]:
     "[| finite I; ALL i:I. ALL j:I. i~=j --> A i Int A j = {} |]
      ==> bag_of (sublist l (UNION I A)) =
          (∑ i∈I. bag_of (sublist l (A i)))"
apply (simp del: UN_simps
            add: UN_simps [symmetric] add: bag_of_sublist)
apply (subst setsum_UN_disjoint, auto)
done

ML
{*
val NbT_pos = thm "NbT_pos";
val setsum_fun_mono = thm "setsum_fun_mono";
val tokens_mono_prefix = thm "tokens_mono_prefix";
```

```
val mono_tokens = thm "mono_tokens";
val bag_of_append = thm "bag_of_append";
val mono_bag_of = thm "mono_bag_of";
val bag_of_sublist_lemma = thm "bag_of_sublist_lemma";
val bag_of_sublist = thm "bag_of_sublist";
val bag_of_sublist_Un_Int = thm "bag_of_sublist_Un_Int";
val bag_of_sublist_Un_disjoint = thm "bag_of_sublist_Un_disjoint";
val bag_of_sublist_UN_disjoint = thm "bag_of_sublist_UN_disjoint";
*}
```

end

theory *Alloc*
imports *AllocBase PPROD*
begin

record *clientState =*
  giv :: "nat list"
  ask :: "nat list"
  rel :: "nat list"

record *'a clientState_d =*
  clientState +
  dummy :: 'a

constdefs

  non_dummy :: "'a clientState_d => clientState"
    "non_dummy s == (|giv = giv s, ask = ask s, rel = rel s|)"

  client_map :: "'a clientState_d => clientState'a"*
    "client_map == funPair non_dummy dummy"

record *allocState =*
  allocGiv :: "nat => nat list"
  allocAsk :: "nat => nat list"
  allocRel :: "nat => nat list"

record *'a allocState_d =*
  allocState +
  dummy   :: 'a

record *'a systemState =*
  allocState +
  client :: "nat => clientState"
  dummy :: 'a

constdefs

```
system_safety :: "'a systemState program set"
  "system_safety ==
   Always {s. (SUM i: lessThan Nclients. (tokens o giv o sub i o client)s)
   <= NbT + (SUM i: lessThan Nclients. (tokens o rel o sub i o client)s)}"


system_progress :: "'a systemState program set"
  "system_progress == INT i : lessThan Nclients.
                      INT h.
                        {s. h <= (ask o sub i o client)s} LeadsTo
                        {s. h pfixLe (giv o sub i o client) s}"

system_spec :: "'a systemState program set"
  "system_spec == system_safety Int system_progress"




client_increasing :: "'a clientState_d program set"
  "client_increasing ==
       UNIV guarantees  Increasing ask Int Increasing rel"


client_bounded :: "'a clientState_d program set"
  "client_bounded ==
       UNIV guarantees  Always {s. ALL elt : set (ask s). elt <= NbT}"


client_progress :: "'a clientState_d program set"
  "client_progress ==
       Increasing giv  guarantees
       (INT h. {s. h <= giv s & h pfixGe ask s}
               LeadsTo {s. tokens h <= (tokens o rel) s})"


client_preserves :: "'a clientState_d program set"
  "client_preserves == preserves giv Int preserves clientState_d.dummy"


client_allowed_acts :: "'a clientState_d program set"
  "client_allowed_acts ==
     {F. AllowedActs F =
          insert Id (UNION (preserves (funPair rel ask)) Acts)}"

client_spec :: "'a clientState_d program set"
  "client_spec == client_increasing Int client_bounded Int client_progress
                  Int client_allowed_acts Int client_preserves"
```

```
alloc_increasing :: "'a allocState_d program set"
  "alloc_increasing ==
       UNIV  guarantees
       (INT i : lessThan Nclients. Increasing (sub i o allocGiv))"


alloc_safety :: "'a allocState_d program set"
  "alloc_safety ==
       (INT i : lessThan Nclients. Increasing (sub i o allocRel))
       guarantees
       Always {s. (SUM i: lessThan Nclients. (tokens o sub i o allocGiv)s)
       <= NbT + (SUM i: lessThan Nclients. (tokens o sub i o allocRel)s)}"


alloc_progress :: "'a allocState_d program set"
  "alloc_progress ==
       (INT i : lessThan Nclients. Increasing (sub i o allocAsk) Int
                                   Increasing (sub i o allocRel))
       Int
       Always {s. ALL i<Nclients.
                   ALL elt : set ((sub i o allocAsk) s). elt <= NbT}
       Int
       (INT i : lessThan Nclients.
        INT h. {s. h <= (sub i o allocGiv)s & h pfixGe (sub i o allocAsk)s}
               LeadsTo
               {s. tokens h <= (tokens o sub i o allocRel)s})
       guarantees
           (INT i : lessThan Nclients.
            INT h. {s. h <= (sub i o allocAsk) s}
                   LeadsTo
                   {s. h pfixLe (sub i o allocGiv) s})"



alloc_preserves :: "'a allocState_d program set"
  "alloc_preserves == preserves allocRel Int preserves allocAsk Int
                      preserves allocState_d.dummy"


alloc_allowed_acts :: "'a allocState_d program set"
  "alloc_allowed_acts ==
     {F. AllowedActs F =
          insert Id (UNION (preserves allocGiv) Acts)}"

alloc_spec :: "'a allocState_d program set"
  "alloc_spec == alloc_increasing Int alloc_safety Int alloc_progress Int
                 alloc_allowed_acts Int alloc_preserves"
```

```
network_ask :: "'a systemState program set"
  "network_ask == INT i : lessThan Nclients.
                  Increasing (ask o sub i o client)  guarantees
                  ((sub i o allocAsk) Fols (ask o sub i o client))"


network_giv :: "'a systemState program set"
  "network_giv == INT i : lessThan Nclients.
                  Increasing (sub i o allocGiv)
                  guarantees
                  ((giv o sub i o client) Fols (sub i o allocGiv))"


network_rel :: "'a systemState program set"
  "network_rel == INT i : lessThan Nclients.
                  Increasing (rel o sub i o client)
                  guarantees
                  ((sub i o allocRel) Fols (rel o sub i o client))"


network_preserves :: "'a systemState program set"
  "network_preserves ==
     preserves allocGiv  Int
     (INT i : lessThan Nclients. preserves (rel o sub i o client)  Int
                                 preserves (ask o sub i o client))"


network_allowed_acts :: "'a systemState program set"
  "network_allowed_acts ==
     {F. AllowedActs F =
         insert Id
          (UNION (preserves allocRel Int
                  (INT i: lessThan Nclients. preserves(giv o sub i o client)))
                Acts)}"

network_spec :: "'a systemState program set"
  "network_spec == network_ask Int network_giv Int
                   network_rel Int network_allowed_acts Int
                   network_preserves"



sysOfAlloc :: "((nat => clientState) * 'a) allocState_d => 'a systemState"
  "sysOfAlloc == %s. let (cl,xtr) = allocState_d.dummy s
                     in (| allocGiv = allocGiv s,
                           allocAsk = allocAsk s,
                           allocRel = allocRel s,
                           client   = cl,
                           dummy    = xtr|)"


sysOfClient :: "(nat => clientState) * 'a allocState_d => 'a systemState"
  "sysOfClient == %(cl,al). (| allocGiv = allocGiv al,
```

```
                                allocAsk = allocAsk al,
                                allocRel = allocRel al,
                                client   = cl,
                                systemState.dummy = allocState_d.dummy al|)"

consts
    Alloc   :: "'a allocState_d program"
    Client  :: "'a clientState_d program"
    Network :: "'a systemState program"
    System  :: "'a systemState program"

axioms
    Alloc:   "Alloc   : alloc_spec"
    Client:  "Client  : client_spec"
    Network: "Network : network_spec"

defs
    System_def:
      "System == rename sysOfAlloc Alloc Join Network Join
                  (rename sysOfClient
                   (plam x: lessThan Nclients. rename client_map Client))"



ML {* use_legacy_bindings (the_context ()) *}

end
```

26 Implementation of a multiple-client allocator from a single-client allocator

```
theory AllocImpl imports AllocBase Follows PPROD begin




record 'b merge =
  In   :: "nat => 'b list"
  Out  :: "'b list"
  iOut :: "nat list"

record ('a,'b) merge_d =
  "'b merge" +
  dummy :: 'a

constdefs
  non_dummy :: "('a,'b) merge_d => 'b merge"
    "non_dummy s == (|In = In s, Out = Out s, iOut = iOut s|)"

record 'b distr =
```

```
  In  :: "'b list"
  iIn :: "nat list"
  Out :: "nat => 'b list"

record ('a,'b) distr_d =
  "'b distr" +
  dummy :: 'a

record allocState =
  giv :: "nat list"
  ask :: "nat list"
  rel :: "nat list"

record 'a allocState_d =
  allocState +
  dummy    :: 'a

record 'a systemState =
  allocState +
  mergeRel :: "nat merge"
  mergeAsk :: "nat merge"
  distr    :: "nat distr"
  dummy    :: 'a
```

constdefs

```
  merge_increasing :: "('a,'b) merge_d program set"
    "merge_increasing ==
         UNIV guarantees (Increasing merge.Out) Int (Increasing merge.iOut)"


  merge_eqOut :: "('a,'b) merge_d program set"
    "merge_eqOut ==
         UNIV guarantees
         Always {s. length (merge.Out s) = length (merge.iOut s)}"


  merge_bounded :: "('a,'b) merge_d program set"
    "merge_bounded ==
         UNIV guarantees
         Always {s. ∀elt ∈ set (merge.iOut s). elt < Nclients}"


  merge_follows :: "('a,'b) merge_d program set"
    "merge_follows ==
         (⋂i ∈ lessThan Nclients. Increasing (sub i o merge.In))
         guarantees
         (⋂i ∈ lessThan Nclients.
          (%s. sublist (merge.Out s)
                       {k. k < size(merge.iOut s) & merge.iOut s! k = i})
```

```
        Fols (sub i o merge.In))"


merge_preserves :: "('a,'b) merge_d program set"
  "merge_preserves == preserves merge.In Int preserves merge_d.dummy"


merge_allowed_acts :: "('a,'b) merge_d program set"
  "merge_allowed_acts ==
     {F. AllowedActs F =
          insert Id (UNION (preserves (funPair merge.Out merge.iOut)) Acts)}"

merge_spec :: "('a,'b) merge_d program set"
  "merge_spec == merge_increasing Int merge_eqOut Int merge_bounded Int
                 merge_follows Int merge_allowed_acts Int merge_preserves"




distr_follows :: "('a,'b) distr_d program set"
  "distr_follows ==
       Increasing distr.In Int Increasing distr.iIn Int
       Always {s. ∀elt ∈ set (distr.iIn s). elt < Nclients}
       guarantees
       (⋂i ∈ lessThan Nclients.
        (sub i o distr.Out) Fols
        (%s. sublist (distr.In s)
                     {k. k < size(distr.iIn s) & distr.iIn s ! k = i}))"

distr_allowed_acts :: "('a,'b) distr_d program set"
  "distr_allowed_acts ==
     {D. AllowedActs D = insert Id (UNION (preserves distr.Out) Acts)}"

distr_spec :: "('a,'b) distr_d program set"
  "distr_spec == distr_follows Int distr_allowed_acts"




alloc_increasing :: "'a allocState_d program set"
  "alloc_increasing == UNIV  guarantees  Increasing giv"


alloc_safety :: "'a allocState_d program set"
  "alloc_safety ==
       Increasing rel
       guarantees  Always {s. tokens (giv s) ≤ NbT + tokens (rel s)}"


alloc_progress :: "'a allocState_d program set"
  "alloc_progress ==
       Increasing ask Int Increasing rel Int
       Always {s. ∀elt ∈ set (ask s). elt ≤ NbT}
       Int
```

```
        (⋂h. {s. h ≤ giv s & h pfixGe (ask s)}
               LeadsTo
               {s. tokens h ≤ tokens (rel s)})
       guarantees  (⋂h. {s. h ≤ ask s} LeadsTo {s. h pfixLe giv s})"


  alloc_preserves :: "'a allocState_d program set"
    "alloc_preserves == preserves rel Int
                        preserves ask Int
                        preserves allocState_d.dummy"



  alloc_allowed_acts :: "'a allocState_d program set"
    "alloc_allowed_acts ==
       {F. AllowedActs F = insert Id (UNION (preserves giv) Acts)}"

  alloc_spec :: "'a allocState_d program set"
    "alloc_spec == alloc_increasing Int alloc_safety Int alloc_progress Int
                   alloc_allowed_acts Int alloc_preserves"

locale Merge =
  fixes M :: "('a,'b::order) merge_d program"
  assumes
    Merge_spec:  "M  ∈ merge_spec"

locale Distrib =
  fixes D :: "('a,'b::order) distr_d program"
  assumes
    Distrib_spec:  "D ∈ distr_spec"




declare subset_preserves_o [THEN subsetD, intro]
declare funPair_o_distrib [simp]
declare Always_INT_distrib [simp]
declare o_apply [simp del]
```

26.1 Theorems for Merge

```
lemma (in Merge) Merge_Allowed:
     "Allowed M = (preserves merge.Out) Int (preserves merge.iOut)"
apply (cut_tac Merge_spec)
apply (auto simp add: merge_spec_def merge_allowed_acts_def Allowed_def
                      safety_prop_Acts_iff)
done

lemma (in Merge) M_ok_iff [iff]:
     "M ok G = (G ∈ preserves merge.Out & G ∈ preserves merge.iOut &
                    M ∈ Allowed G)"
by (auto simp add: Merge_Allowed ok_iff_Allowed)
```

```
lemma (in Merge) Merge_Always_Out_eq_iOut:
     "[| G ∈ preserves merge.Out; G ∈ preserves merge.iOut; M ∈ Allowed G
|]
      ==> M Join G ∈ Always {s. length (merge.Out s) = length (merge.iOut
s)}"
apply (cut_tac Merge_spec)
apply (force dest: guaranteesD simp add: merge_spec_def merge_eqOut_def)
done

lemma (in Merge) Merge_Bounded:
     "[| G ∈ preserves merge.iOut; G ∈ preserves merge.Out; M ∈ Allowed G
|]
      ==> M Join G ∈ Always {s. ∀elt ∈ set (merge.iOut s). elt < Nclients}"
apply (cut_tac Merge_spec)
apply (force dest: guaranteesD simp add: merge_spec_def merge_bounded_def)
done

lemma (in Merge) Merge_Bag_Follows_lemma:
     "[| G ∈ preserves merge.iOut; G ∈ preserves merge.Out; M ∈ Allowed G
|]
  ==> M Join G ∈ Always
         {s. (∑i ∈ lessThan Nclients. bag_of (sublist (merge.Out s)
                                  {k. k < length (iOut s) & iOut s ! k = i}))
=
              (bag_of o merge.Out) s}"
apply (rule Always_Compl_Un_eq [THEN iffD1])
apply (blast intro: Always_Int_I [OF Merge_Always_Out_eq_iOut Merge_Bounded])
apply (rule UNIV_AlwaysI, clarify)
apply (subst bag_of_sublist_UN_disjoint [symmetric])
  apply (simp)
 apply blast
apply (simp add: set_conv_nth)
apply (subgoal_tac
       "(⋃i ∈ lessThan Nclients. {k. k < length (iOut x) & iOut x ! k = i})
=
       lessThan (length (iOut x))")
 apply (simp (no_asm_simp) add: o_def)
apply blast
done

lemma (in Merge) Merge_Bag_Follows:
     "M ∈ (⋂i ∈ lessThan Nclients. Increasing (sub i o merge.In))
          guarantees
              (bag_of o merge.Out) Fols
              (%s. ∑i ∈ lessThan Nclients. (bag_of o sub i o merge.In) s)"
apply (rule Merge_Bag_Follows_lemma [THEN Always_Follows1, THEN guaranteesI],
auto)
apply (rule Follows_setsum)
apply (cut_tac Merge_spec)
apply (auto simp add: merge_spec_def merge_follows_def o_def)
apply (drule guaranteesD)
  prefer 3
  apply (best intro: mono_bag_of [THEN mono_Follows_apply, THEN subsetD],
```

```
auto)
done
```

26.2 Theorems for Distributor

```
lemma (in Distrib) Distr_Increasing_Out:
     "D ∈ Increasing distr.In Int Increasing distr.iIn Int
          Always {s. ∀elt ∈ set (distr.iIn s). elt < Nclients}
          guarantees
          (⋂i ∈ lessThan Nclients. Increasing (sub i o distr.Out))"
apply (cut_tac Distrib_spec)
apply (simp add: distr_spec_def distr_follows_def)
apply clarify
apply (blast intro: guaranteesI Follows_Increasing1 dest: guaranteesD)
done

lemma (in Distrib) Distr_Bag_Follows_lemma:
     "[| G ∈ preserves distr.Out;
         D Join G ∈ Always {s. ∀elt ∈ set (distr.iIn s). elt < Nclients}
|]
  ==> D Join G ∈ Always
          {s. (∑i ∈ lessThan Nclients. bag_of (sublist (distr.In s)
                                    {k. k < length (iIn s) & iIn s ! k = i}))
=
              bag_of (sublist (distr.In s) (lessThan (length (iIn s))))}"
apply (erule Always_Compl_Un_eq [THEN iffD1])
apply (rule UNIV_AlwaysI, clarify)
apply (subst bag_of_sublist_UN_disjoint [symmetric])
  apply (simp (no_asm))
 apply blast
apply (simp add: set_conv_nth)
apply (subgoal_tac
       "(⋃i ∈ lessThan Nclients. {k. k < length (iIn x) & iIn x ! k = i})
=
        lessThan (length (iIn x))")
 apply (simp (no_asm_simp))
apply blast
done

lemma (in Distrib) D_ok_iff [iff]:
     "D ok G = (G ∈ preserves distr.Out & D ∈ Allowed G)"
apply (cut_tac Distrib_spec)
apply (auto simp add: distr_spec_def distr_allowed_acts_def Allowed_def
                      safety_prop_Acts_iff ok_iff_Allowed)
done

lemma (in Distrib) Distr_Bag_Follows:
 "D ∈ Increasing distr.In Int Increasing distr.iIn Int
      Always {s. ∀elt ∈ set (distr.iIn s). elt < Nclients}
      guarantees
       (⋂i ∈ lessThan Nclients.
        (%s. ∑i ∈ lessThan Nclients. (bag_of o sub i o distr.Out) s)
        Fols
        (%s. bag_of (sublist (distr.In s) (lessThan (length(distr.iIn s))))))"
```

```
apply (rule guaranteesI, clarify)
apply (rule Distr_Bag_Follows_lemma [THEN Always_Follows2], auto)
apply (rule Follows_setsum)
apply (cut_tac Distrib_spec)
apply (auto simp add: distr_spec_def distr_follows_def o_def)
apply (drule guaranteesD)
  prefer 3
  apply (best intro: mono_bag_of [THEN mono_Follows_apply, THEN subsetD],
auto)
done
```

26.3 Theorems for Allocator

```
lemma alloc_refinement_lemma:
     "!!f::nat=>nat. (⋂i ∈ lessThan n. {s. f i ≤ g i s})
      ⊆ {s. (SUM x: lessThan n. f x) ≤ (SUM x: lessThan n. g x s)}"
apply (induct_tac "n")
apply (auto simp add: lessThan_Suc)
done

lemma alloc_refinement:
"(⋂i ∈ lessThan Nclients. Increasing (sub i o allocAsk) Int
                          Increasing (sub i o allocRel))
  Int
  Always {s. ∀i. i<Nclients -->
             (∀elt ∈ set ((sub i o allocAsk) s). elt ≤ NbT)}
  Int
  (⋂i ∈ lessThan Nclients.
   ⋂h. {s. h ≤ (sub i o allocGiv)s & h pfixGe (sub i o allocAsk)s}
        LeadsTo {s. tokens h ≤ (tokens o sub i o allocRel)s})
  ⊆
 (⋂i ∈ lessThan Nclients. Increasing (sub i o allocAsk) Int
                          Increasing (sub i o allocRel))
  Int
  Always {s. ∀i. i<Nclients -->
             (∀elt ∈ set ((sub i o allocAsk) s). elt ≤ NbT)}
  Int
  (⋂hf. (⋂i ∈ lessThan Nclients.
         {s. hf i ≤ (sub i o allocGiv)s & hf i pfixGe (sub i o allocAsk)s})
  LeadsTo {s. (∑i ∈ lessThan Nclients. tokens (hf i)) ≤
             (∑i ∈ lessThan Nclients. (tokens o sub i o allocRel)s)})"
apply (auto simp add: ball_conj_distrib)
apply (rename_tac F hf)
apply (rule LeadsTo_weaken_R [OF Finite_stable_completion alloc_refinement_lemma],
blast, blast)
apply (subgoal_tac "F ∈ Increasing (tokens o (sub i o allocRel))")
 apply (simp add: Increasing_def o_assoc)
apply (blast intro: mono_tokens [THEN mono_Increasing_o, THEN subsetD])
done

end
```

27 Distributed Resource Management System: the Client

theory *Client* **imports** *Rename AllocBase* **begin**

types
tokbag = nat — tokbags could be multisets...or any ordered type?

record *state =*
giv :: "tokbag list" — input history: tokens granted
ask :: "tokbag list" — output history: tokens requested
rel :: "tokbag list" — output history: tokens released
tok :: tokbag — current token request

record *'a state_d =*
state +
dummy :: 'a — new variables

constdefs

```
rel_act :: "('a state_d * 'a state_d) set"
  "rel_act == {(s,s').
               ∃nrel. nrel = size (rel s) &
                      s' = s (| rel := rel s @ [giv s!nrel] |) &
                      nrel < size (giv s) &
                      ask s!nrel ≤ giv s!nrel}"
```

```
tok_act :: "('a state_d * 'a state_d) set"
   "tok_act == {(s,s'). s'=s | s' = s (|tok := Suc (tok s mod NbT) |)}"

ask_act :: "('a state_d * 'a state_d) set"
  "ask_act == {(s,s'). s'=s |
                       (s' = s (|ask := ask s @ [tok s]|))}"

Client :: "'a state_d program"
  "Client ==
     mk_total_program
          ({s. tok s ∈ atMost NbT &
               giv s = [] & ask s = [] & rel s = []},
           {rel_act, tok_act, ask_act},
           ⋃G ∈ preserves rel Int preserves ask Int preserves tok.
                 Acts G)"

non_dummy :: "'a state_d => state"
```

```
    "non_dummy s == (|giv = giv s, ask = ask s, rel = rel s, tok = tok s|)"


  client_map :: "'a state_d => state*'a"
    "client_map == funPair non_dummy dummy"


declare Client_def [THEN def_prg_Init, simp]
declare Client_def [THEN def_prg_AllowedActs, simp]
declare rel_act_def [THEN def_act_simp, simp]
declare tok_act_def [THEN def_act_simp, simp]
declare ask_act_def [THEN def_act_simp, simp]

lemma Client_ok_iff [iff]:
     "(Client ok G) =
      (G ∈ preserves rel & G ∈ preserves ask & G ∈ preserves tok &
       Client ∈ Allowed G)"
by (auto simp add: ok_iff_Allowed Client_def [THEN def_total_prg_Allowed])
```

Safety property 1: ask, rel are increasing

```
lemma increasing_ask_rel:
     "Client ∈ UNIV guarantees Increasing ask Int Increasing rel"
apply (auto intro!: increasing_imp_Increasing simp add: guar_def preserves_subset_increasing
[THEN subsetD])
apply (auto simp add: Client_def increasing_def)
apply (safety, auto)+
done

declare nth_append [simp] append_one_prefix [simp]
```

Safety property 2: the client never requests too many tokens. With no Substitution Axiom, we must prove the two invariants simultaneously.

```
lemma ask_bounded_lemma:
     "Client ok G
      ==> Client Join G ∈
              Always ({s. tok s ≤ NbT}  Int
                      {s. ∀elt ∈ set (ask s). elt ≤ NbT})"
apply auto
apply (rule invariantI [THEN stable_Join_Always2], force)
 prefer 2
 apply (fast elim!: preserves_subset_stable [THEN subsetD] intro!: stable_Int)

apply (simp add: Client_def, safety)
apply (cut_tac m = "tok s" in NbT_pos [THEN mod_less_divisor], auto)
done
```

export version, with no mention of tok in the postcondition, but unfortunately tok must be declared local.

```
lemma ask_bounded:
     "Client ∈ UNIV guarantees Always {s. ∀elt ∈ set (ask s). elt ≤ NbT}"
apply (rule guaranteesI)
apply (erule ask_bounded_lemma [THEN Always_weaken])
apply (rule Int_lower2)
```

done

** Towards proving the liveness property **

lemma *stable_rel_le_giv: "Client ∈ stable {s. rel s ≤ giv s}"*
by *(simp add: Client_def, safety, auto)*

lemma *Join_Stable_rel_le_giv:*
"[| Client Join G ∈ Increasing giv; G ∈ preserves rel |]
==> Client Join G ∈ Stable {s. rel s ≤ giv s}"
by *(rule stable_rel_le_giv [THEN Increasing_preserves_Stable], auto)*

lemma *Join_Always_rel_le_giv:*
"[| Client Join G ∈ Increasing giv; G ∈ preserves rel |]
==> Client Join G ∈ Always {s. rel s ≤ giv s}"
by *(force intro: AlwaysI Join_Stable_rel_le_giv)*

lemma *transient_lemma:*
"Client ∈ transient {s. rel s = k & k<h & h ≤ giv s & h pfixGe ask s}"
apply *(simp add: Client_def mk_total_program_def)*
apply *(rule_tac act = rel_act* **in** *totalize_transientI)*
apply *(auto simp add: Domain_def Client_def)*
apply *(blast intro: less_le_trans prefix_length_le strict_prefix_length_less)*
apply *(auto simp add: prefix_def genPrefix_iff_nth Ge_def)*
apply *(blast intro: strict_prefix_length_less)*
done

lemma *induct_lemma:*
"[| Client Join G ∈ Increasing giv; Client ok G |]
==> Client Join G ∈ {s. rel s = k & k<h & h ≤ giv s & h pfixGe ask s}
LeadsTo {s. k < rel s & rel s ≤ giv s &
h ≤ giv s & h pfixGe ask s}"
apply *(rule single_LeadsTo_I)*
apply *(frule increasing_ask_rel [THEN guaranteesD], auto)*
apply *(rule transient_lemma [THEN Join_transient_I1, THEN transient_imp_leadsTo,*
THEN leadsTo_imp_LeadsTo, THEN PSP_Stable, THEN LeadsTo_weaken])
apply *(rule Stable_Int [THEN Stable_Int, THEN Stable_Int])*
apply *(erule_tac f = giv* **and** *x = "giv s"* **in** *IncreasingD)*
apply *(erule_tac f = ask* **and** *x = "ask s"* **in** *IncreasingD)*
apply *(erule_tac f = rel* **and** *x = "rel s"* **in** *IncreasingD)*
apply *(erule Join_Stable_rel_le_giv, blast)*
apply *(blast intro: order_less_imp_le order_trans)*
apply *(blast intro: sym order_less_le [THEN iffD2] order_trans*
prefix_imp_pfixGe pfixGe_trans)
done

lemma *rel_progress_lemma:*
"[| Client Join G ∈ Increasing giv; Client ok G |]
==> Client Join G ∈ {s. rel s < h & h ≤ giv s & h pfixGe ask s}
LeadsTo {s. h ≤ rel s}"
apply *(rule_tac f = "%s. size h - size (rel s) "* **in** *LessThan_induct)*
apply *(auto simp add: vimage_def)*
apply *(rule single_LeadsTo_I)*

```
apply (rule induct_lemma [THEN LeadsTo_weaken], auto)
 apply (blast intro: order_less_le [THEN iffD2] dest: common_prefix_linear)
apply (drule strict_prefix_length_less)+
apply arith
done


lemma client_progress_lemma:
     "[| Client Join G ∈ Increasing giv;  Client ok G |]
      ==> Client Join G ∈ {s. h ≤ giv s & h pfixGe ask s}
                          LeadsTo {s. h ≤ rel s}"
apply (rule Join_Always_rel_le_giv [THEN Always_LeadsToI], simp_all)
apply (rule LeadsTo_Un [THEN LeadsTo_weaken_L])
  apply (blast intro: rel_progress_lemma)
 apply (rule subset_refl [THEN subset_imp_LeadsTo])
apply (blast intro: order_less_le [THEN iffD2] dest: common_prefix_linear)
done
```

Progress property: all tokens that are given will be released

```
lemma client_progress:
     "Client ∈
       Increasing giv  guarantees
       (INT h. {s. h ≤ giv s & h pfixGe ask s} LeadsTo {s. h ≤ rel s})"
apply (rule guaranteesI, clarify)
apply (blast intro: client_progress_lemma)
done
```

This shows that the Client won't alter other variables in any state that it is combined with

```
lemma client_preserves_dummy: "Client ∈ preserves dummy"
by (simp add: Client_def preserves_def, clarify, safety, auto)
```

* Obsolete lemmas from first version of the Client *

```
lemma stable_size_rel_le_giv:
     "Client ∈ stable {s. size (rel s) ≤ size (giv s)}"
by (simp add: Client_def, safety, auto)
```

clients return the right number of tokens

```
lemma ok_guar_rel_prefix_giv:
     "Client ∈ Increasing giv  guarantees  Always {s. rel s ≤ giv s}"
apply (rule guaranteesI)
apply (rule AlwaysI, force)
apply (blast intro: Increasing_preserves_Stable stable_rel_le_giv)
done


end
```

28 Projections of State Sets

theory *Project* **imports** *Extend* **begin**

constdefs
```
  projecting :: "['c program => 'c set, 'a*'b => 'c,
                  'a program, 'c program set, 'a program set] => bool"
    "projecting C h F X' X ==
       ∀G. extend h F⊔G ∈ X' --> F⊔project h (C G) G ∈ X"

  extending :: "['c program => 'c set, 'a*'b => 'c, 'a program,
                 'c program set, 'a program set] => bool"
    "extending C h F Y' Y ==
       ∀G. extend h F  ok G --> F⊔project h (C G) G ∈ Y
              --> extend h F⊔G ∈ Y'"

  subset_closed :: "'a set set => bool"
    "subset_closed U == ∀A ∈ U. Pow A ⊆ U"
```

lemma (**in** *Extend*) *project_extend_constrains_I:*
```
     "F ∈ A co B ==> project h C (extend h F) ∈ A co B"
```
apply *(auto simp add: extend_act_def project_act_def constrains_def)*
done

28.1 Safety

lemma (**in** *Extend*) *project_unless [rule_format]:*
```
     "[| G ∈ stable C;  project h C G ∈ A unless B |]
      ==> G ∈ (C ∩ extend_set h A) unless (extend_set h B)"
```
apply *(simp add: unless_def project_constrains)*
apply *(blast dest: stable_constrains_Int intro: constrains_weaken)*
done

lemma (**in** *Extend*) *Join_project_constrains:*
```
     "(F⊔project h C G ∈ A co B)  =
        (extend h F⊔G ∈ (C ∩ extend_set h A) co (extend_set h B) &
         F ∈ A co B)"
```
apply *(simp (no_asm) add: project_constrains)*
apply *(blast intro: extend_constrains [THEN iffD2, THEN constrains_weaken]*

dest: constrains_imp_subset)
done

lemma (**in** *Extend*) *Join_project_stable:*
```
     "extend h F⊔G ∈ stable C
      ==> (F⊔project h C G ∈ stable A)  =
          (extend h F⊔G ∈ stable (C ∩ extend_set h A) &
           F ∈ stable A)"
```
apply *(unfold stable_def)*
apply *(simp only: Join_project_constrains)*
apply *(blast intro: constrains_weaken dest: constrains_Int)*
done

lemma (**in** *Extend*) *project_constrains_I:*
```
     "extend h F⊔G ∈ extend_set h A co extend_set h B
      ==> F⊔project h C G ∈ A co B"
apply (simp add: project_constrains extend_constrains)
apply (blast intro: constrains_weaken dest: constrains_imp_subset)
done
```

lemma (**in** *Extend*) *project_increasing_I:*
```
     "extend h F⊔G ∈ increasing (func o f)
      ==> F⊔project h C G ∈ increasing func"
apply (unfold increasing_def stable_def)
apply (simp del: Join_constrains
            add: project_constrains_I extend_set_eq_Collect)
done
```

lemma (**in** *Extend*) *Join_project_increasing:*
```
     "(F⊔project h UNIV G ∈ increasing func)  =
      (extend h F⊔G ∈ increasing (func o f))"
apply (rule iffI)
apply (erule_tac [2] project_increasing_I)
apply (simp del: Join_stable
            add: increasing_def Join_project_stable)
apply (auto simp add: extend_set_eq_Collect extend_stable [THEN iffD1])
done
```

lemma (**in** *Extend*) *project_constrains_D:*
```
     "F⊔project h UNIV G ∈ A co B
      ==> extend h F⊔G ∈ extend_set h A co extend_set h B"
by (simp add: project_constrains extend_constrains)
```

28.2 "projecting" and union/intersection (no converses)

lemma *projecting_Int:*
```
     "[| projecting C h F XA' XA;  projecting C h F XB' XB |]
      ==> projecting C h F (XA' ∩ XB') (XA ∩ XB)"
by (unfold projecting_def, blast)
```

lemma *projecting_Un:*
```
     "[| projecting C h F XA' XA;  projecting C h F XB' XB |]
      ==> projecting C h F (XA' ∪ XB') (XA ∪ XB)"
by (unfold projecting_def, blast)
```

lemma *projecting_INT:*
```
     "[| !!i. i ∈ I ==> projecting C h F (X' i) (X i) |]
      ==> projecting C h F (⋂i ∈ I. X' i) (⋂i ∈ I. X i)"
by (unfold projecting_def, blast)
```

lemma *projecting_UN:*
```
     "[| !!i. i ∈ I ==> projecting C h F (X' i) (X i) |]
      ==> projecting C h F (⋃i ∈ I. X' i) (⋃i ∈ I. X i)"
by (unfold projecting_def, blast)
```

lemma *projecting_weaken:*

```
    "[| projecting C h F X' X;  U'<=X';  X ⊆ U |] ==> projecting C h F U'
U"
by (unfold projecting_def, auto)

lemma projecting_weaken_L:
    "[| projecting C h F X' X;  U'<=X' |] ==> projecting C h F U' X"
by (unfold projecting_def, auto)

lemma extending_Int:
    "[| extending C h F YA' YA;  extending C h F YB' YB |]
     ==> extending C h F (YA' ∩ YB') (YA ∩ YB)"
by (unfold extending_def, blast)

lemma extending_Un:
    "[| extending C h F YA' YA;  extending C h F YB' YB |]
     ==> extending C h F (YA' ∪ YB') (YA ∪ YB)"
by (unfold extending_def, blast)

lemma extending_INT:
    "[| !!i. i ∈ I ==> extending C h F (Y' i) (Y i) |]
     ==> extending C h F (⋂i ∈ I. Y' i) (⋂i ∈ I. Y i)"
by (unfold extending_def, blast)

lemma extending_UN:
    "[| !!i. i ∈ I ==> extending C h F (Y' i) (Y i) |]
     ==> extending C h F (⋃i ∈ I. Y' i) (⋃i ∈ I. Y i)"
by (unfold extending_def, blast)

lemma extending_weaken:
    "[| extending C h F Y' Y;  Y'<=V';  V ⊆ Y |] ==> extending C h F V' V"
by (unfold extending_def, auto)

lemma extending_weaken_L:
    "[| extending C h F Y' Y;  Y'<=V' |] ==> extending C h F V' Y"
by (unfold extending_def, auto)

lemma projecting_UNIV: "projecting C h F X' UNIV"
by (simp add: projecting_def)

lemma (in Extend) projecting_constrains:
    "projecting C h F (extend_set h A co extend_set h B) (A co B)"
apply (unfold projecting_def)
apply (blast intro: project_constrains_I)
done

lemma (in Extend) projecting_stable:
    "projecting C h F (stable (extend_set h A)) (stable A)"
apply (unfold stable_def)
apply (rule projecting_constrains)
done

lemma (in Extend) projecting_increasing:
    "projecting C h F (increasing (func o f)) (increasing func)"
apply (unfold projecting_def)
```

```
apply (blast intro: project_increasing_I)
done

lemma (in Extend) extending_UNIV: "extending C h F UNIV Y"
apply (simp (no_asm) add: extending_def)
done

lemma (in Extend) extending_constrains:
     "extending (%G. UNIV) h F (extend_set h A co extend_set h B) (A co B)"
apply (unfold extending_def)
apply (blast intro: project_constrains_D)
done

lemma (in Extend) extending_stable:
     "extending (%G. UNIV) h F (stable (extend_set h A)) (stable A)"
apply (unfold stable_def)
apply (rule extending_constrains)
done

lemma (in Extend) extending_increasing:
     "extending (%G. UNIV) h F (increasing (func o f)) (increasing func)"
by (force simp only: extending_def Join_project_increasing)
```

28.3 Reachability and project

```
lemma (in Extend) reachable_imp_reachable_project:
     "[| reachable (extend h F⊔G) ⊆ C;
         z ∈ reachable (extend h F⊔G) |]
      ==> f z ∈ reachable (F⊔project h C G)"
apply (erule reachable.induct)
apply (force intro!: reachable.Init simp add: split_extended_all, auto)
 apply (rule_tac act = x in reachable.Acts)
 apply auto
 apply (erule extend_act_D)
apply (rule_tac act1 = "Restrict C act"
       in project_act_I [THEN [3] reachable.Acts], auto)
done

lemma (in Extend) project_Constrains_D:
     "F⊔project h (reachable (extend h F⊔G)) G ∈ A Co B
      ==> extend h F⊔G ∈ (extend_set h A) Co (extend_set h B)"
apply (unfold Constrains_def)
apply (simp del: Join_constrains
            add: Join_project_constrains, clarify)
apply (erule constrains_weaken)
apply (auto intro: reachable_imp_reachable_project)
done

lemma (in Extend) project_Stable_D:
     "F⊔project h (reachable (extend h F⊔G)) G ∈ Stable A
      ==> extend h F⊔G ∈ Stable (extend_set h A)"
apply (unfold Stable_def)
apply (simp (no_asm_simp) add: project_Constrains_D)
done
```

```
lemma (in Extend) project_Always_D:
     "F⊔project h (reachable (extend h F⊔G)) G ∈ Always A
      ==> extend h F⊔G ∈ Always (extend_set h A)"
apply (unfold Always_def)
apply (force intro: reachable.Init simp add: project_Stable_D split_extended_all)
done

lemma (in Extend) project_Increasing_D:
     "F⊔project h (reachable (extend h F⊔G)) G ∈ Increasing func
      ==> extend h F⊔G ∈ Increasing (func o f)"
apply (unfold Increasing_def, auto)
apply (subst extend_set_eq_Collect [symmetric])
apply (simp (no_asm_simp) add: project_Stable_D)
done
```

28.4 Converse results for weak safety: benefits of the argument C

```
lemma (in Extend) reachable_project_imp_reachable:
     "[| C ⊆ reachable(extend h F⊔G);
         x ∈ reachable (F⊔project h C G) |]
      ==> ∃y. h(x,y) ∈ reachable (extend h F⊔G)"
apply (erule reachable.induct)
apply  (force intro: reachable.Init)
apply (auto simp add: project_act_def)
apply (force del: Id_in_Acts intro: reachable.Acts extend_act_D)+
done

lemma (in Extend) project_set_reachable_extend_eq:
     "project_set h (reachable (extend h F⊔G)) =
      reachable (F⊔project h (reachable (extend h F⊔G)) G)"
by (auto dest: subset_refl [THEN reachable_imp_reachable_project]
               subset_refl [THEN reachable_project_imp_reachable])


lemma (in Extend) reachable_extend_Join_subset:
     "reachable (extend h F⊔G) ⊆ C
      ==> reachable (extend h F⊔G) ⊆
          extend_set h (reachable (F⊔project h C G))"
apply (auto dest: reachable_imp_reachable_project)
done

lemma (in Extend) project_Constrains_I:
     "extend h F⊔G ∈ (extend_set h A) Co (extend_set h B)
      ==> F⊔project h (reachable (extend h F⊔G)) G ∈ A Co B"
apply (unfold Constrains_def)
apply (simp del: Join_constrains
            add: Join_project_constrains extend_set_Int_distrib)
apply (rule conjI)
 prefer 2
 apply (force elim: constrains_weaken_L
              dest!: extend_constrains_project_set
                     subset_refl [THEN reachable_project_imp_reachable])
```

```
apply (blast intro: constrains_weaken_L)
done

lemma (in Extend) project_Stable_I:
     "extend h F⊔G ∈ Stable (extend_set h A)
      ==> F⊔project h (reachable (extend h F⊔G)) G ∈ Stable A"
apply (unfold Stable_def)
apply (simp (no_asm_simp) add: project_Constrains_I)
done

lemma (in Extend) project_Always_I:
     "extend h F⊔G ∈ Always (extend_set h A)
      ==> F⊔project h (reachable (extend h F⊔G)) G ∈ Always A"
apply (unfold Always_def)
apply (auto simp add: project_Stable_I)
apply (unfold extend_set_def, blast)
done

lemma (in Extend) project_Increasing_I:
    "extend h F⊔G ∈ Increasing (func o f)
     ==> F⊔project h (reachable (extend h F⊔G)) G ∈ Increasing func"
apply (unfold Increasing_def, auto)
apply (simp (no_asm_simp) add: extend_set_eq_Collect project_Stable_I)
done

lemma (in Extend) project_Constrains:
     "(F⊔project h (reachable (extend h F⊔G)) G ∈ A Co B)  =
      (extend h F⊔G ∈ (extend_set h A) Co (extend_set h B))"
apply (blast intro: project_Constrains_I project_Constrains_D)
done

lemma (in Extend) project_Stable:
     "(F⊔project h (reachable (extend h F⊔G)) G ∈ Stable A)  =
      (extend h F⊔G ∈ Stable (extend_set h A))"
apply (unfold Stable_def)
apply (rule project_Constrains)
done

lemma (in Extend) project_Increasing:
   "(F⊔project h (reachable (extend h F⊔G)) G ∈ Increasing func)  =
    (extend h F⊔G ∈ Increasing (func o f))"
apply (simp (no_asm_simp) add: Increasing_def project_Stable extend_set_eq_Collect)
done
```

28.5 A lot of redundant theorems: all are proved to facilitate reasoning about guarantees.

```
lemma (in Extend) projecting_Constrains:
     "projecting (%G. reachable (extend h F⊔G)) h F
                 (extend_set h A Co extend_set h B) (A Co B)"

apply (unfold projecting_def)
apply (blast intro: project_Constrains_I)
done
```

```
lemma (in Extend) projecting_Stable:
     "projecting (%G. reachable (extend h F⊔G)) h F
                 (Stable (extend_set h A)) (Stable A)"
apply (unfold Stable_def)
apply (rule projecting_Constrains)
done

lemma (in Extend) projecting_Always:
     "projecting (%G. reachable (extend h F⊔G)) h F
                 (Always (extend_set h A)) (Always A)"
apply (unfold projecting_def)
apply (blast intro: project_Always_I)
done

lemma (in Extend) projecting_Increasing:
     "projecting (%G. reachable (extend h F⊔G)) h F
                 (Increasing (func o f)) (Increasing func)"
apply (unfold projecting_def)
apply (blast intro: project_Increasing_I)
done

lemma (in Extend) extending_Constrains:
     "extending (%G. reachable (extend h F⊔G)) h F
                  (extend_set h A Co extend_set h B) (A Co B)"
apply (unfold extending_def)
apply (blast intro: project_Constrains_D)
done

lemma (in Extend) extending_Stable:
     "extending (%G. reachable (extend h F⊔G)) h F
                  (Stable (extend_set h A)) (Stable A)"
apply (unfold extending_def)
apply (blast intro: project_Stable_D)
done

lemma (in Extend) extending_Always:
     "extending (%G. reachable (extend h F⊔G)) h F
                  (Always (extend_set h A)) (Always A)"
apply (unfold extending_def)
apply (blast intro: project_Always_D)
done

lemma (in Extend) extending_Increasing:
     "extending (%G. reachable (extend h F⊔G)) h F
                  (Increasing (func o f)) (Increasing func)"
apply (unfold extending_def)
apply (blast intro: project_Increasing_D)
done
```

28.6 leadsETo in the precondition (??)

28.6.1 transient

```
lemma (in Extend) transient_extend_set_imp_project_transient:
     "[| G ∈ transient (C ∩ extend_set h A);  G ∈ stable C |]
      ==> project h C G ∈ transient (project_set h C ∩ A)"
apply (auto simp add: transient_def Domain_project_act)
apply (subgoal_tac "act `` (C ∩ extend_set h A) ⊆ - extend_set h A")
 prefer 2
 apply (simp add: stable_def constrains_def, blast)

apply (erule_tac V = "?AA ⊆ -C ∪ ?BB" in thin_rl)
apply (drule bspec, assumption)
apply (simp add: extend_set_def project_act_def, blast)
done


lemma (in Extend) project_extend_transient_D:
     "project h C (extend h F) ∈ transient (project_set h C ∩ D)
      ==> F ∈ transient (project_set h C ∩ D)"
apply (simp add: transient_def Domain_project_act, safe)
apply blast+
done
```

28.6.2 ensures – a primitive combining progress with safety

```
lemma (in Extend) ensures_extend_set_imp_project_ensures:
     "[| extend h F ∈ stable C;  G ∈ stable C;
         extend h F⊔G ∈ A ensures B;  A-B = C ∩ extend_set h D |]
      ==> F⊔project h C G
            ∈ (project_set h C ∩ project_set h A) ensures (project_set h B)"
apply (simp add: ensures_def project_constrains Join_transient extend_transient,
       clarify)
apply (intro conjI)

apply (blast intro: extend_stable_project_set
                  [THEN stableD, THEN constrains_Int, THEN constrains_weaken]

             dest!: extend_constrains_project_set equalityD1)

apply (erule stableD [THEN constrains_Int, THEN constrains_weaken])
    apply assumption
   apply (simp (no_asm_use) add: extend_set_def)
   apply blast
 apply (simp add: extend_set_Int_distrib extend_set_Un_distrib)
 apply (blast intro!: extend_set_project_set [THEN subsetD], blast)

apply auto
 prefer 2
 apply (force dest!: equalityD1
              intro: transient_extend_set_imp_project_transient
                        [THEN transient_strengthen])
apply (simp (no_asm_use) add: Int_Diff)
apply (force dest!: equalityD1
```

```
            intro: transient_extend_set_imp_project_transient
              [THEN project_extend_transient_D, THEN transient_strengthen])
done
```

Transferring a transient property upwards

```
lemma (in Extend) project_transient_extend_set:
     "project h C G ∈ transient (project_set h C ∩ A - B)
      ==> G ∈ transient (C ∩ extend_set h A - extend_set h B)"
apply (simp add: transient_def project_set_def extend_set_def project_act_def)
apply (elim disjE bexE)
 apply (rule_tac x=Id in bexI)
  apply (blast intro!: rev_bexI )+
done

lemma (in Extend) project_unless2 [rule_format]:
     "[| G ∈ stable C;  project h C G ∈ (project_set h C ∩ A) unless B |]

      ==> G ∈ (C ∩ extend_set h A) unless (extend_set h B)"
by (auto dest: stable_constrains_Int intro: constrains_weaken
         simp add: unless_def project_constrains Diff_eq Int_assoc
                   Int_extend_set_lemma)


lemma (in Extend) extend_unless:
   "[|extend h F ∈ stable C; F ∈ A unless B|]
    ==> extend h F ∈ C ∩ extend_set h A unless extend_set h B"
apply (simp add: unless_def stable_def)
apply (drule constrains_Int)
apply (erule extend_constrains [THEN iffD2])
apply (erule constrains_weaken, blast)
apply blast
done


lemma (in Extend) Join_project_ensures [rule_format]:
     "[| extend h F⊔G ∈ stable C;
         F⊔project h C G ∈ A ensures B |]
      ==> extend h F⊔G ∈ (C ∩ extend_set h A) ensures (extend_set h B)"
apply (auto simp add: ensures_eq extend_unless project_unless)
apply (blast intro:  extend_transient [THEN iffD2] transient_strengthen)
apply (blast intro: project_transient_extend_set transient_strengthen)
done
```

Lemma useful for both STRONG and WEAK progress, but the transient condition's very strong

```
lemma (in Extend) PLD_lemma:
     "[| extend h F⊔G ∈ stable C;
         F⊔project h C G ∈ (project_set h C ∩ A) leadsTo B |]
      ==> extend h F⊔G ∈
          C ∩ extend_set h (project_set h C ∩ A) leadsTo (extend_set h B)"
apply (erule leadsTo_induct)
  apply (blast intro: leadsTo_Basis Join_project_ensures)
 apply (blast intro: psp_stable2 [THEN leadsTo_weaken_L] leadsTo_Trans)
apply (simp del: UN_simps add: Int_UN_distrib leadsTo_UN extend_set_Union)
```

```
done

lemma (in Extend) project_leadsTo_D_lemma:
     "[| extend h F⊔G ∈ stable C;
         F⊔project h C G ∈ (project_set h C ∩ A) leadsTo B |]
      ==> extend h F⊔G ∈ (C ∩ extend_set h A) leadsTo (extend_set h B)"
apply (rule PLD_lemma [THEN leadsTo_weaken])
apply (auto simp add: split_extended_all)
done

lemma (in Extend) Join_project_LeadsTo:
     "[| C = (reachable (extend h F⊔G));
         F⊔project h C G ∈ A LeadsTo B |]
      ==> extend h F⊔G ∈ (extend_set h A) LeadsTo (extend_set h B)"
by (simp del: Join_stable    add: LeadsTo_def project_leadsTo_D_lemma
                                  project_set_reachable_extend_eq)
```

28.7 Towards the theorem *project_Ensures_D*

```
lemma (in Extend) project_ensures_D_lemma:
     "[| G ∈ stable ((C ∩ extend_set h A) - (extend_set h B));
         F⊔project h C G ∈ (project_set h C ∩ A) ensures B;
         extend h F⊔G ∈ stable C |]
      ==> extend h F⊔G ∈ (C ∩ extend_set h A) ensures (extend_set h B)"

apply (auto intro!: project_unless2 [unfolded unless_def]
            intro: project_extend_constrains_I
            simp add: ensures_def)


 prefer 2
 apply (blast intro: project_transient_extend_set)

apply (force elim!: extend_transient [THEN iffD2, THEN transient_strengthen]
             simp add: split_extended_all)
done

lemma (in Extend) project_ensures_D:
     "[| F⊔project h UNIV G ∈ A ensures B;
         G ∈ stable (extend_set h A - extend_set h B) |]
      ==> extend h F⊔G ∈ (extend_set h A) ensures (extend_set h B)"
apply (rule project_ensures_D_lemma [of _ UNIV, THEN revcut_rl], auto)
done

lemma (in Extend) project_Ensures_D:
     "[| F⊔project h (reachable (extend h F⊔G)) G ∈ A Ensures B;
         G ∈ stable (reachable (extend h F⊔G) ∩ extend_set h A -
                     extend_set h B) |]
      ==> extend h F⊔G ∈ (extend_set h A) Ensures (extend_set h B)"
apply (unfold Ensures_def)
apply (rule project_ensures_D_lemma [THEN revcut_rl])
apply (auto simp add: project_set_reachable_extend_eq [symmetric])
done
```

28.8 Guarantees

lemma (**in** *Extend*) *project_act_Restrict_subset_project_act:*
"project_act h (Restrict C act) ⊆ project_act h act"
apply *(auto simp add: project_act_def)*
done

lemma (**in** *Extend*) *subset_closed_ok_extend_imp_ok_project:*
"[| extend h F ok G; subset_closed (AllowedActs F) |]
==> F ok project h C G"
apply *(auto simp add: ok_def)*
apply *(rename_tac act)*
apply *(drule subsetD, blast)*
apply *(rule_tac x = "Restrict C (extend_act h act)"* **in** *rev_image_eqI)*
apply *simp +*
apply *(cut_tac project_act_Restrict_subset_project_act)*
apply *(auto simp add: subset_closed_def)*
done

lemma (**in** *Extend*) *project_guarantees_raw:*
assumes *xguary: "F ∈ X guarantees Y"*
and *closed: "subset_closed (AllowedActs F)"*
and *project: "!!G. extend h F⊔G ∈ X'*
==> F⊔project h (C G) G ∈ X"
and *extend: "!!G. [| F⊔project h (C G) G ∈ Y |]*
==> extend h F⊔G ∈ Y'"
shows *"extend h F ∈ X' guarantees Y'"*
apply *(rule xguary [THEN guaranteesD, THEN extend, THEN guaranteesI])*
apply *(blast intro: closed subset_closed_ok_extend_imp_ok_project)*
apply *(erule project)*
done

lemma (**in** *Extend*) *project_guarantees:*
"[| F ∈ X guarantees Y; subset_closed (AllowedActs F);
projecting C h F X' X; extending C h F Y' Y |]
==> extend h F ∈ X' guarantees Y'"
apply *(rule guaranteesI)*
apply *(auto simp add: guaranteesD projecting_def extending_def*
subset_closed_ok_extend_imp_ok_project)
done

28.9 guarantees corollaries

28.9.1 Some could be deleted: the required versions are easy to prove

lemma (**in** *Extend*) *extend_guar_increasing:*
"[| F ∈ UNIV guarantees increasing func;
subset_closed (AllowedActs F) |]
==> extend h F ∈ X' guarantees increasing (func o f)"

```
apply (erule project_guarantees)
apply (rule_tac [3] extending_increasing)
apply (rule_tac [2] projecting_UNIV, auto)
done

lemma (in Extend) extend_guar_Increasing:
     "[| F ∈ UNIV guarantees Increasing func;
         subset_closed (AllowedActs F) |]
      ==> extend h F ∈ X' guarantees Increasing (func o f)"
apply (erule project_guarantees)
apply (rule_tac [3] extending_Increasing)
apply (rule_tac [2] projecting_UNIV, auto)
done

lemma (in Extend) extend_guar_Always:
     "[| F ∈ Always A guarantees Always B;
         subset_closed (AllowedActs F) |]
      ==> extend h F
            ∈ Always(extend_set h A) guarantees Always(extend_set h B)"
apply (erule project_guarantees)
apply (rule_tac [3] extending_Always)
apply (rule_tac [2] projecting_Always, auto)
done
```

28.9.2 Guarantees with a leadsTo postcondition

```
lemma (in Extend) project_leadsTo_D:
     "F⊔project h UNIV G ∈ A leadsTo B
      ==> extend h F⊔G ∈ (extend_set h A) leadsTo (extend_set h B)"
apply (rule_tac C1 = UNIV in project_leadsTo_D_lemma [THEN leadsTo_weaken],
auto)
done

lemma (in Extend) project_LeadsTo_D:
     "F⊔project h (reachable (extend h F⊔G)) G ∈ A LeadsTo B
       ==> extend h F⊔G ∈ (extend_set h A) LeadsTo (extend_set h B)"
apply (rule refl [THEN Join_project_LeadsTo], auto)
done

lemma (in Extend) extending_leadsTo:
     "extending (%G. UNIV) h F
                (extend_set h A leadsTo extend_set h B) (A leadsTo B)"
apply (unfold extending_def)
apply (blast intro: project_leadsTo_D)
done

lemma (in Extend) extending_LeadsTo:
     "extending (%G. reachable (extend h F⊔G)) h F
                (extend_set h A LeadsTo extend_set h B) (A LeadsTo B)"
apply (unfold extending_def)
apply (blast intro: project_LeadsTo_D)
done
```

ML

```
{*
val projecting_Int = thm "projecting_Int";
val projecting_Un = thm "projecting_Un";
val projecting_INT = thm "projecting_INT";
val projecting_UN = thm "projecting_UN";
val projecting_weaken = thm "projecting_weaken";
val projecting_weaken_L = thm "projecting_weaken_L";
val extending_Int = thm "extending_Int";
val extending_Un = thm "extending_Un";
val extending_INT = thm "extending_INT";
val extending_UN = thm "extending_UN";
val extending_weaken = thm "extending_weaken";
val extending_weaken_L = thm "extending_weaken_L";
val projecting_UNIV = thm "projecting_UNIV";
*}
```

end

29 Progress Under Allowable Sets

theory *ELT* **imports** *Project* **begin**

consts

```
  elt :: "['a set set, 'a program] => ('a set * 'a set) set"
```

inductive *"elt CC F"*
intros

```
   Basis:  "[| F : A ensures B;  A-B : (insert {} CC) |] ==> (A,B) : elt CC
F"

   Trans:  "[| (A,B) : elt CC F;  (B,C) : elt CC F |] ==> (A,C) : elt CC F"

   Union:  "ALL A: S. (A,B) : elt CC F ==> (Union S, B) : elt CC F"
```

constdefs

```
  givenBy :: "['a => 'b] => 'a set set"
    "givenBy f == range (%B. f-` B)"


  leadsETo :: "['a set, 'a set set, 'a set] => 'a program set"
                                          ("(3_/ leadsTo[_]/ _)" [80,0,80] 80)
    "leadsETo A CC B == {F. (A,B) : elt CC F}"

  LeadsETo :: "['a set, 'a set set, 'a set] => 'a program set"
                                          ("(3_/ LeadsTo[_]/ _)" [80,0,80] 80)
```

```
    "LeadsETo A CC B ==
      {F. F : (reachable F Int A) leadsTo[(%C. reachable F Int C) ` CC] B}"
```

```
lemma givenBy_id [simp]: "givenBy id = UNIV"
by (unfold givenBy_def, auto)

lemma givenBy_eq_all: "(givenBy v) = {A. ALL x:A. ALL y. v x = v y --> y:
A}"
apply (unfold givenBy_def, safe)
apply (rule_tac [2] x = "v ` ?u" in image_eqI, auto)
done

lemma givenByI: "(!!x y. [| x:A;  v x = v y |] ==> y: A) ==> A: givenBy v"
by (subst givenBy_eq_all, blast)

lemma givenByD: "[| A: givenBy v;  x:A;  v x = v y |] ==> y: A"
by (unfold givenBy_def, auto)

lemma empty_mem_givenBy [iff]: "{} : givenBy v"
by (blast intro!: givenByI)

lemma givenBy_imp_eq_Collect: "A: givenBy v ==> EX P. A = {s. P(v s)}"
apply (rule_tac x = "%n. EX s. v s = n & s : A" in exI)
apply (simp (no_asm_use) add: givenBy_eq_all)
apply blast
done

lemma Collect_mem_givenBy: "{s. P(v s)} : givenBy v"
by (unfold givenBy_def, best)

lemma givenBy_eq_Collect: "givenBy v = {A. EX P. A = {s. P(v s)}}"
by (blast intro: Collect_mem_givenBy givenBy_imp_eq_Collect)


lemma preserves_givenBy_imp_stable:
     "[| F : preserves v;  D : givenBy v |] ==> F : stable D"
by (force simp add: preserves_subset_stable [THEN subsetD] givenBy_eq_Collect)

lemma givenBy_o_subset: "givenBy (w o v) <= givenBy v"
apply (simp (no_asm) add: givenBy_eq_Collect)
apply best
done

lemma givenBy_DiffI:
     "[| A : givenBy v;  B : givenBy v |] ==> A-B : givenBy v"
apply (simp (no_asm_use) add: givenBy_eq_Collect)
apply safe
apply (rule_tac x = "%z. ?R z & ~ ?Q z" in exI)
apply (tactic "deepen_tac (set_cs addSIs [equalityI]) 0 1")
done
```

```
lemma leadsETo_Basis [intro]:
     "[| F: A ensures B;  A-B: insert {} CC |] ==> F : A leadsTo[CC] B"
apply (unfold leadsETo_def)
apply (blast intro: elt.Basis)
done

lemma leadsETo_Trans:
     "[| F : A leadsTo[CC] B;  F : B leadsTo[CC] C |] ==> F : A leadsTo[CC]
C"
apply (unfold leadsETo_def)
apply (blast intro: elt.Trans)
done



lemma leadsETo_Un_duplicate:
     "F : A leadsTo[CC] (A' Un A') ==> F : A leadsTo[CC] A'"
by (simp add: Un_ac)

lemma leadsETo_Un_duplicate2:
     "F : A leadsTo[CC] (A' Un C Un C) ==> F : A leadsTo[CC] (A' Un C)"
by (simp add: Un_ac)


lemma leadsETo_Union:
    "(!!A. A : S ==> F : A leadsTo[CC] B) ==> F : (Union S) leadsTo[CC] B"
apply (unfold leadsETo_def)
apply (blast intro: elt.Union)
done

lemma leadsETo_UN:
    "(!!i. i : I ==> F : (A i) leadsTo[CC] B)
     ==> F : (UN i:I. A i) leadsTo[CC] B"
apply (subst Union_image_eq [symmetric])
apply (blast intro: leadsETo_Union)
done


lemma leadsETo_induct:
  "[| F : za leadsTo[CC] zb;
      !!A B. [| F : A ensures B;  A-B : insert {} CC |] ==> P A B;
      !!A B C. [| F : A leadsTo[CC] B; P A B; F : B leadsTo[CC] C; P B C |]

               ==> P A C;
      !!B S. ALL A:S. F : A leadsTo[CC] B & P A B ==> P (Union S) B
   |] ==> P za zb"
apply (unfold leadsETo_def)
apply (drule CollectD)
apply (erule elt.induct, blast+)
done
```

lemma *leadsETo_mono: "CC' <= CC ==> (A leadsTo[CC'] B) <= (A leadsTo[CC] B)"*
apply *safe*
apply *(erule leadsETo_induct)*
prefer *3* **apply** *(blast intro: leadsETo_Union)*
prefer *2* **apply** *(blast intro: leadsETo_Trans)*
apply *(blast intro: leadsETo_Basis)*
done

lemma *leadsETo_Trans_Un:*
"[| F : A leadsTo[CC] B; F : B leadsTo[DD] C |]
==> F : A leadsTo[CC Un DD] C"
by *(blast intro: leadsETo_mono [THEN subsetD] leadsETo_Trans)*

lemma *leadsETo_Union_Int:*
"(!!A. A : S ==> F : (A Int C) leadsTo[CC] B)
==> F : (Union S Int C) leadsTo[CC] B"
apply *(unfold leadsETo_def)*
apply *(simp only: Int_Union_Union)*
apply *(blast intro: elt.Union)*
done

lemma *leadsETo_Un:*
"[| F : A leadsTo[CC] C; F : B leadsTo[CC] C |]
==> F : (A Un B) leadsTo[CC] C"
apply *(subst Un_eq_Union)*
apply *(blast intro: leadsETo_Union)*
done

lemma *single_leadsETo_I:*
"(!!x. x : A ==> F : {x} leadsTo[CC] B) ==> F : A leadsTo[CC] B"
by *(subst UN_singleton [symmetric], rule leadsETo_UN, blast)*

lemma *subset_imp_leadsETo: "A<=B ==> F : A leadsTo[CC] B"*
by *(simp add: subset_imp_ensures [THEN leadsETo_Basis]*
Diff_eq_empty_iff [THEN iffD2])

lemmas *empty_leadsETo = empty_subsetI [THEN subset_imp_leadsETo, simp]*

lemma *leadsETo_weaken_R:*
"[| F : A leadsTo[CC] A'; A'<=B' |] ==> F : A leadsTo[CC] B'"
by *(blast intro: subset_imp_leadsETo leadsETo_Trans)*

lemma *leadsETo_weaken_L [rule_format]:*
"[| F : A leadsTo[CC] A'; B<=A |] ==> F : B leadsTo[CC] A'"

```
by (blast intro: leadsETo_Trans subset_imp_leadsETo)


lemma leadsETo_Un_distrib:
     "F : (A Un B) leadsTo[CC] C  =
      (F : A leadsTo[CC] C & F : B leadsTo[CC] C)"
by (blast intro: leadsETo_Un leadsETo_weaken_L)

lemma leadsETo_UN_distrib:
     "F : (UN i:I. A i) leadsTo[CC] B  =
      (ALL i : I. F : (A i) leadsTo[CC] B)"
by (blast intro: leadsETo_UN leadsETo_weaken_L)

lemma leadsETo_Union_distrib:
     "F : (Union S) leadsTo[CC] B  =  (ALL A : S. F : A leadsTo[CC] B)"
by (blast intro: leadsETo_Union leadsETo_weaken_L)

lemma leadsETo_weaken:
     "[| F : A leadsTo[CC'] A'; B<=A; A'<=B';  CC' <= CC |]
      ==> F : B leadsTo[CC] B'"
apply (drule leadsETo_mono [THEN subsetD], assumption)
apply (blast del: subsetCE
             intro: leadsETo_weaken_R leadsETo_weaken_L leadsETo_Trans)
done

lemma leadsETo_givenBy:
     "[| F : A leadsTo[CC] A';  CC <= givenBy v |]
      ==> F : A leadsTo[givenBy v] A'"
by (blast intro: empty_mem_givenBy leadsETo_weaken)



lemma leadsETo_Diff:
     "[| F : (A-B) leadsTo[CC] C; F : B leadsTo[CC] C |]
      ==> F : A leadsTo[CC] C"
by (blast intro: leadsETo_Un leadsETo_weaken)



lemma leadsETo_Un_Un:
     "[| F : A leadsTo[CC] A';  F : B leadsTo[CC] B' |]
      ==> F : (A Un B) leadsTo[CC] (A' Un B')"
by (blast intro: leadsETo_Un leadsETo_weaken_R)




lemma leadsETo_cancel2:
     "[| F : A leadsTo[CC] (A' Un B); F : B leadsTo[CC] B' |]
      ==> F : A leadsTo[CC] (A' Un B')"
by (blast intro: leadsETo_Un_Un subset_imp_leadsETo leadsETo_Trans)

lemma leadsETo_cancel1:
     "[| F : A leadsTo[CC] (B Un A'); F : B leadsTo[CC] B' |]
```

```
    ==> F : A leadsTo[CC] (B' Un A')"
apply (simp add: Un_commute)
apply (blast intro!: leadsETo_cancel2)
done

lemma leadsETo_cancel_Diff1:
     "[| F : A leadsTo[CC] (B Un A'); F : (B-A') leadsTo[CC] B' |]
    ==> F : A leadsTo[CC] (B' Un A')"
apply (rule leadsETo_cancel1)
 prefer 2 apply assumption
apply simp_all
done




lemma e_psp_stable:
   "[| F : A leadsTo[CC] A';  F : stable B;  ALL C:CC. C Int B : CC |]
    ==> F : (A Int B) leadsTo[CC] (A' Int B)"
apply (unfold stable_def)
apply (erule leadsETo_induct)
prefer 3 apply (blast intro: leadsETo_Union_Int)
prefer 2 apply (blast intro: leadsETo_Trans)
apply (rule leadsETo_Basis)
prefer 2 apply (force simp add: Diff_Int_distrib2 [symmetric])
apply (simp add: ensures_def Diff_Int_distrib2 [symmetric]
                 Int_Un_distrib2 [symmetric])
apply (blast intro: transient_strengthen constrains_Int)
done

lemma e_psp_stable2:
     "[| F : A leadsTo[CC] A'; F : stable B;  ALL C:CC. C Int B : CC |]
      ==> F : (B Int A) leadsTo[CC] (B Int A')"
by (simp (no_asm_simp) add: e_psp_stable Int_ac)

lemma e_psp:
     "[| F : A leadsTo[CC] A'; F : B co B';
         ALL C:CC. C Int B Int B' : CC |]
      ==> F : (A Int B') leadsTo[CC] ((A' Int B) Un (B' - B))"
apply (erule leadsETo_induct)
prefer 3 apply (blast intro: leadsETo_Union_Int)

apply (rule_tac [2] leadsETo_Un_duplicate2)
apply (erule_tac [2] leadsETo_cancel_Diff1)
prefer 2
 apply (simp add: Int_Diff Diff_triv)
 apply (blast intro: leadsETo_weaken_L dest: constrains_imp_subset)

apply (rule leadsETo_Basis)
apply (blast intro: psp_ensures)
apply (subgoal_tac "A Int B' - (Ba Int B Un (B' - B)) = (A - Ba) Int B Int
B'")
apply auto
```

```
done

lemma e_psp2:
     "[| F : A leadsTo[CC] A'; F : B co B';
         ALL C:CC. C Int B Int B' : CC |]
      ==> F : (B' Int A) leadsTo[CC] ((B Int A') Un (B' - B))"
by (simp add: e_psp Int_ac)



lemma gen_leadsETo_imp_Join_leadsETo:
     "[| F: (A leadsTo[givenBy v] B);  G : preserves v;
         F⊔G : stable C |]
      ==> F⊔G : ((C Int A) leadsTo[(%D. C Int D) ` givenBy v] B)"
apply (erule leadsETo_induct)
  prefer 3
  apply (subst Int_Union)
  apply (blast intro: leadsETo_UN)
prefer 2
 apply (blast intro: e_psp_stable2 [THEN leadsETo_weaken_L] leadsETo_Trans)
apply (rule leadsETo_Basis)
apply (auto simp add: Diff_eq_empty_iff [THEN iffD2]
                      Int_Diff ensures_def givenBy_eq_Collect Join_transient)
prefer 3 apply (blast intro: transient_strengthen)
apply (drule_tac [2] P1 = P in preserves_subset_stable [THEN subsetD])
apply (drule_tac P1 = P in preserves_subset_stable [THEN subsetD])
apply (unfold stable_def)
apply (blast intro: constrains_Int [THEN constrains_weaken])+
done



lemma leadsETo_subset_leadsTo: "(A leadsTo[CC] B) <= (A leadsTo B)"
apply safe
apply (erule leadsETo_induct)
  prefer 3 apply (blast intro: leadsTo_Union)
 prefer 2 apply (blast intro: leadsTo_Trans, blast)
done

lemma leadsETo_UNIV_eq_leadsTo: "(A leadsTo[UNIV] B) = (A leadsTo B)"
apply safe
apply (erule leadsETo_subset_leadsTo [THEN subsetD])

apply (erule leadsTo_induct)
  prefer 3 apply (blast intro: leadsETo_Union)
 prefer 2 apply (blast intro: leadsETo_Trans, blast)
done
```

```
lemma LeadsETo_eq_leadsETo:
     "A LeadsTo[CC] B =
        {F. F : (reachable F Int A) leadsTo[(%C. reachable F Int C) ` CC]
        (reachable F Int B)}"
apply (unfold LeadsETo_def)
apply (blast dest: e_psp_stable2 intro: leadsETo_weaken)
done



lemma LeadsETo_Trans:
     "[| F : A LeadsTo[CC] B;  F : B LeadsTo[CC] C |]
      ==> F : A LeadsTo[CC] C"
apply (simp add: LeadsETo_eq_leadsETo)
apply (blast intro: leadsETo_Trans)
done

lemma LeadsETo_Union:
     "(!!A. A : S ==> F : A LeadsTo[CC] B) ==> F : (Union S) LeadsTo[CC] B"
apply (simp add: LeadsETo_def)
apply (subst Int_Union)
apply (blast intro: leadsETo_UN)
done

lemma LeadsETo_UN:
     "(!!i. i : I ==> F : (A i) LeadsTo[CC] B)
      ==> F : (UN i:I. A i) LeadsTo[CC] B"
apply (simp only: Union_image_eq [symmetric])
apply (blast intro: LeadsETo_Union)
done


lemma LeadsETo_Un:
     "[| F : A LeadsTo[CC] C; F : B LeadsTo[CC] C |]
      ==> F : (A Un B) LeadsTo[CC] C"
apply (subst Un_eq_Union)
apply (blast intro: LeadsETo_Union)
done


lemma single_LeadsETo_I:
     "(!!s. s : A ==> F : {s} LeadsTo[CC] B) ==> F : A LeadsTo[CC] B"
by (subst UN_singleton [symmetric], rule LeadsETo_UN, blast)

lemma subset_imp_LeadsETo:
     "A <= B ==> F : A LeadsTo[CC] B"
apply (simp (no_asm) add: LeadsETo_def)
apply (blast intro: subset_imp_leadsETo)
done

lemmas empty_LeadsETo = empty_subsetI [THEN subset_imp_LeadsETo, standard]

lemma LeadsETo_weaken_R [rule_format]:
     "[| F : A LeadsTo[CC] A';  A' <= B' |] ==> F : A LeadsTo[CC] B'"
```

```
apply (simp (no_asm_use) add: LeadsETo_def)
apply (blast intro: leadsETo_weaken_R)
done

lemma LeadsETo_weaken_L [rule_format]:
     "[| F : A LeadsTo[CC] A';  B <= A |] ==> F : B LeadsTo[CC] A'"
apply (simp (no_asm_use) add: LeadsETo_def)
apply (blast intro: leadsETo_weaken_L)
done

lemma LeadsETo_weaken:
     "[| F : A LeadsTo[CC'] A';
         B <= A;  A' <= B';  CC' <= CC |]
      ==> F : B LeadsTo[CC] B'"
apply (simp (no_asm_use) add: LeadsETo_def)
apply (blast intro: leadsETo_weaken)
done

lemma LeadsETo_subset_LeadsTo: "(A LeadsTo[CC] B) <= (A LeadsTo B)"
apply (unfold LeadsETo_def LeadsTo_def)
apply (blast intro: leadsETo_subset_leadsTo [THEN subsetD])
done


lemma reachable_ensures:
     "F : A ensures B ==> F : (reachable F Int A) ensures B"
apply (rule stable_ensures_Int [THEN ensures_weaken_R], auto)
done

lemma lel_lemma:
     "F : A leadsTo B ==> F : (reachable F Int A) leadsTo[Pow(reachable F)]
B"
apply (erule leadsTo_induct)
  apply (blast intro: reachable_ensures leadsETo_Basis)
 apply (blast dest: e_psp_stable2 intro: leadsETo_Trans leadsETo_weaken_L)
apply (subst Int_Union)
apply (blast intro: leadsETo_UN)
done

lemma LeadsETo_UNIV_eq_LeadsTo: "(A LeadsTo[UNIV] B) = (A LeadsTo B)"
apply safe
apply (erule LeadsETo_subset_LeadsTo [THEN subsetD])

apply (unfold LeadsETo_def LeadsTo_def)
apply (blast intro: lel_lemma [THEN leadsETo_weaken])
done



lemma (in Extend) givenBy_o_eq_extend_set:
     "givenBy (v o f) = extend_set h ` (givenBy v)"
apply (simp add: givenBy_eq_Collect)
apply (rule equalityI, best)
```

```
apply blast
done

lemma (in Extend) givenBy_eq_extend_set: "givenBy f = range (extend_set h)"
by (simp add: givenBy_eq_Collect, best)

lemma (in Extend) extend_set_givenBy_I:
     "D : givenBy v ==> extend_set h D : givenBy (v o f)"
apply (simp (no_asm_use) add: givenBy_eq_all, blast)
done

lemma (in Extend) leadsETo_imp_extend_leadsETo:
     "F : A leadsTo[CC] B
      ==> extend h F : (extend_set h A) leadsTo[extend_set h ` CC]
                       (extend_set h B)"
apply (erule leadsETo_induct)
  apply (force intro: leadsETo_Basis subset_imp_ensures
               simp add: extend_ensures extend_set_Diff_distrib [symmetric])
 apply (blast intro: leadsETo_Trans)
apply (simp add: leadsETo_UN extend_set_Union)
done



lemma (in Extend) Join_project_ensures_strong:
     "[| project h C G ~: transient (project_set h C Int (A-B)) |
           project_set h C Int (A - B) = {};
         extend h F⊔G : stable C;
         F⊔project h C G : (project_set h C Int A) ensures B |]
      ==> extend h F⊔G : (C Int extend_set h A) ensures (extend_set h B)"
apply (subst Int_extend_set_lemma [symmetric])
apply (rule Join_project_ensures)
apply (auto simp add: Int_Diff)
done






lemma (in Extend) pli_lemma:
     "[| extend h F⊔G : stable C;
         F⊔project h C G
           : project_set h C Int project_set h A leadsTo project_set h B |]

      ==> F⊔project h C G
            : project_set h C Int project_set h A leadsTo
              project_set h C Int project_set h B"
apply (rule psp_stable2 [THEN leadsTo_weaken_L])
apply (auto simp add: project_stable_project_set extend_stable_project_set)
done

lemma (in Extend) project_leadsETo_I_lemma:
```

```
     "[| extend h F⊔G : stable C;
         extend h F⊔G :
           (C Int A) leadsTo[(%D. C Int D)‘givenBy f]  B |]
  ==> F⊔project h C G
    : (project_set h C Int project_set h (C Int A)) leadsTo (project_set h
B)"
apply (erule leadsETo_induct)
  prefer 3
  apply (simp only: Int_UN_distrib project_set_Union)
  apply (blast intro: leadsTo_UN)
 prefer 2 apply (blast intro: leadsTo_Trans pli_lemma)
apply (simp add: givenBy_eq_extend_set)
apply (rule leadsTo_Basis)
apply (blast intro: ensures_extend_set_imp_project_ensures)
done

lemma (in Extend) project_leadsETo_I:
     "extend h F⊔G : (extend_set h A) leadsTo[givenBy f] (extend_set h B)
      ==> F⊔project h UNIV G : A leadsTo B"
apply (rule project_leadsETo_I_lemma [THEN leadsTo_weaken], auto)
done

lemma (in Extend) project_LeadsETo_I:
     "extend h F⊔G : (extend_set h A) LeadsTo[givenBy f] (extend_set h B)

      ==> F⊔project h (reachable (extend h F⊔G)) G
           : A LeadsTo B"
apply (simp (no_asm_use) add: LeadsTo_def LeadsETo_def)
apply (rule project_leadsETo_I_lemma [THEN leadsTo_weaken])
apply (auto simp add: project_set_reachable_extend_eq [symmetric])
done

lemma (in Extend) projecting_leadsTo:
     "projecting (%G. UNIV) h F
                 (extend_set h A leadsTo[givenBy f] extend_set h B)
                 (A leadsTo B)"
apply (unfold projecting_def)
apply (force dest: project_leadsETo_I)
done

lemma (in Extend) projecting_LeadsTo:
     "projecting (%G. reachable (extend h F⊔G)) h F
                 (extend_set h A LeadsTo[givenBy f] extend_set h B)
                 (A LeadsTo B)"
apply (unfold projecting_def)
apply (force dest: project_LeadsETo_I)
done

end
```