

Vectors (1A)

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Determinant

Determinant of order 2

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ & b_2 & b_3 \\ & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} & a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \quad \begin{bmatrix} & & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & & c_3 \end{bmatrix}$$

Determinant

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ & b_2 & b_3 \\ & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} & a_2 & \\ & b_1 & b_3 \\ & c_1 & c_3 \end{bmatrix} \quad \begin{bmatrix} & & a_3 \\ & b_1 & b_2 \\ & c_1 & c_2 \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = + a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Cross Product (1)

Determinant of order 3

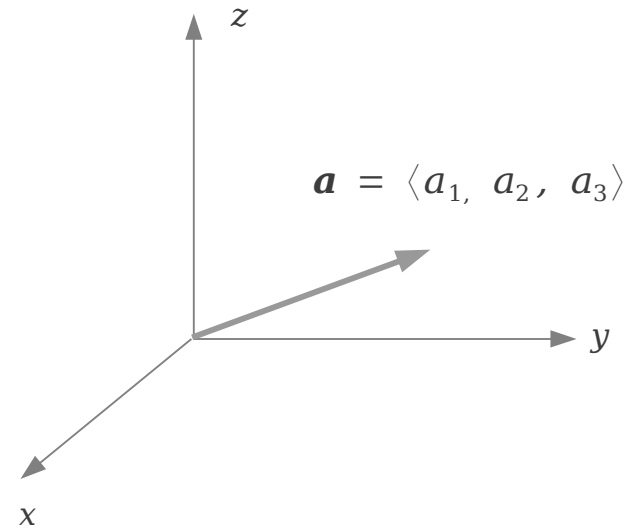
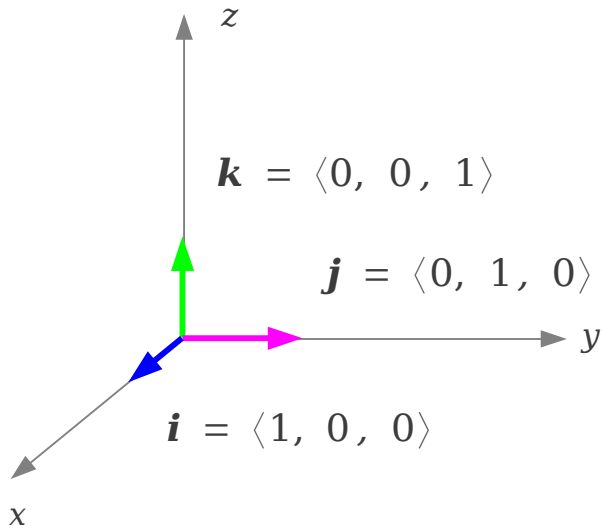
$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = \langle a_1, a_2, a_3 \rangle$$

$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} = \langle b_1, b_2, b_3 \rangle$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Cross Product (2)



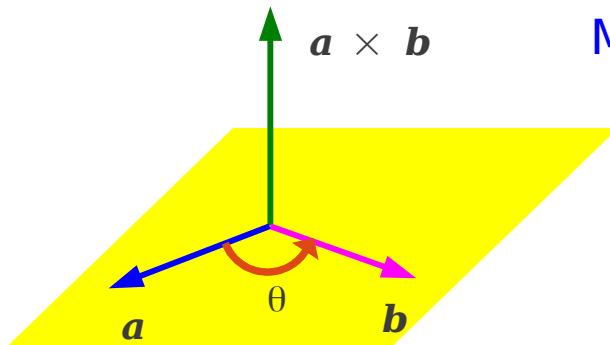
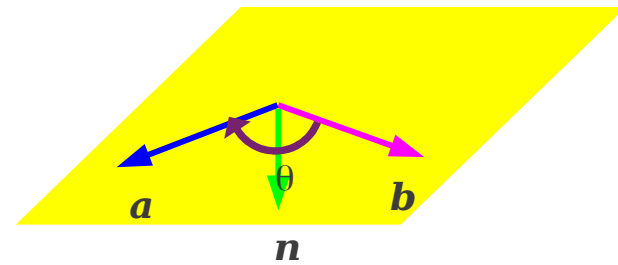
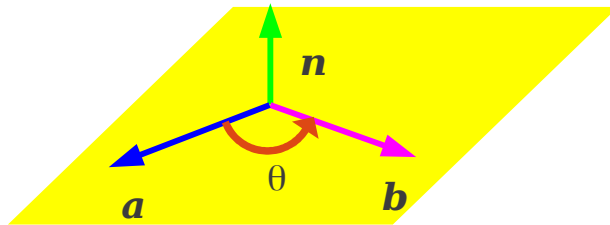
$$\begin{aligned}
 \mathbf{i} \times \mathbf{j} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k} \quad \leftarrow \text{normal to } \mathbf{i} \ \& \ \mathbf{j} \quad \rightarrow \quad \mathbf{j} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{k} \\
 \mathbf{j} \times \mathbf{k} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i} \quad \leftarrow \text{normal to } \mathbf{j} \ \& \ \mathbf{k} \quad \rightarrow \quad \mathbf{k} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i} \\
 \mathbf{k} \times \mathbf{i} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j} \quad \leftarrow \text{normal to } \mathbf{k} \ \& \ \mathbf{i} \quad \rightarrow \quad \mathbf{i} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\mathbf{j}
 \end{aligned}$$

Right Hand Rule

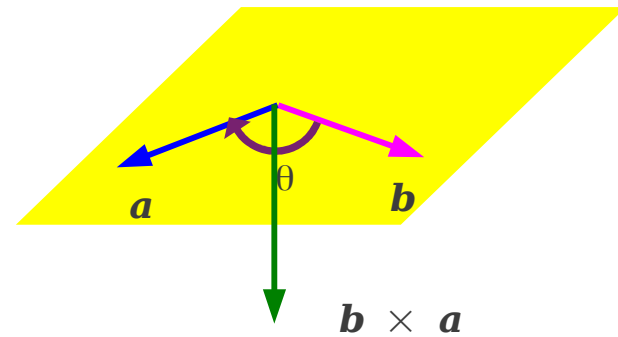
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\mathbf{b} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Normal direction \mathbf{n}



Magnitude = $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$



Line Equations (1)

Vector Equation

Parameter

Direction Vector

$$\mathbf{r} = \mathbf{r}_2 + t\mathbf{a}$$

$$\mathbf{r}_2 = \langle x_2, y_2, z_2 \rangle$$

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

Parametric Equation

Component

$$x = x_2 + ta_1$$

$$y = y_2 + ta_2$$

$$z = z_2 + ta_3$$

$$ta_1 = x - x_2$$

$$ta_2 = y - y_2$$

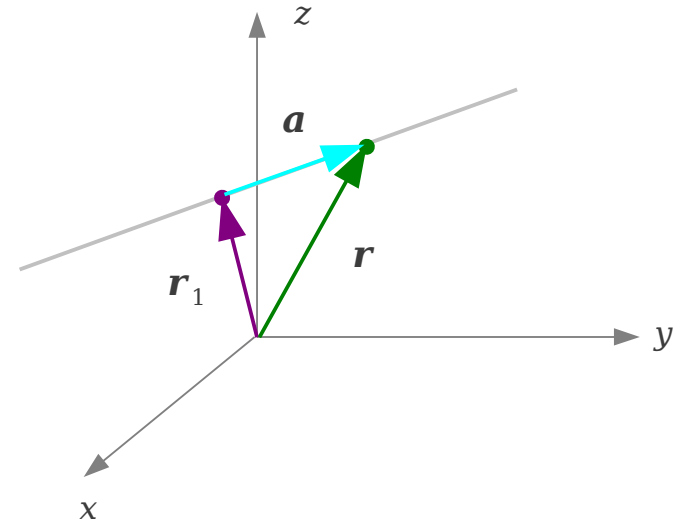
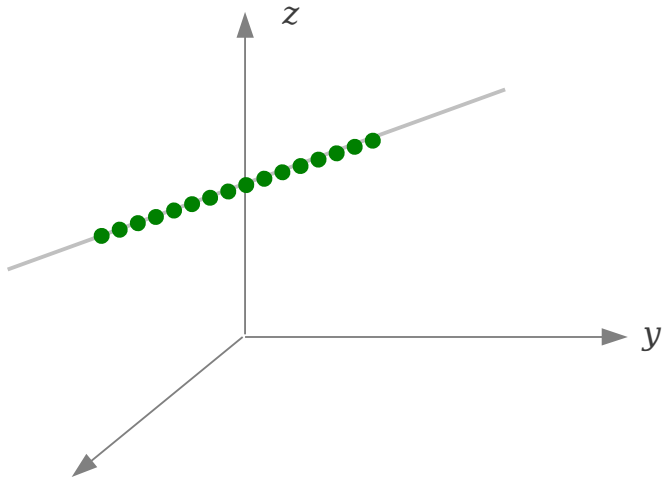
$$ta_3 = z - z_2$$

Symmetric Equation

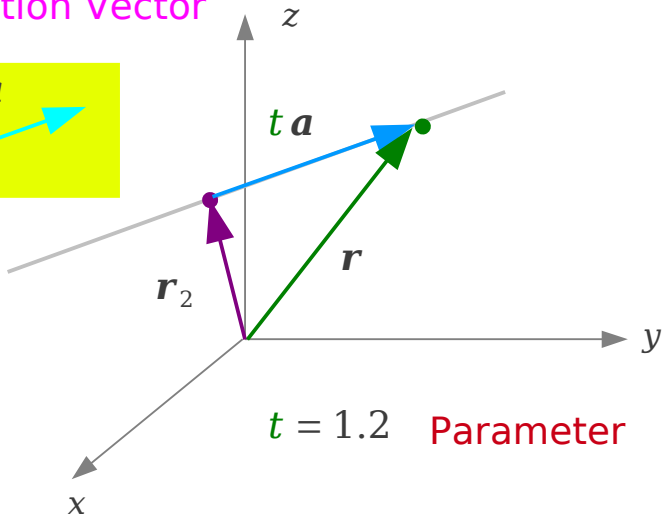
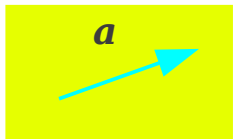
Elimination of parameter

$$t = \frac{x - x_2}{a_1} = \frac{y - y_2}{a_2} = \frac{z - z_2}{a_3}$$

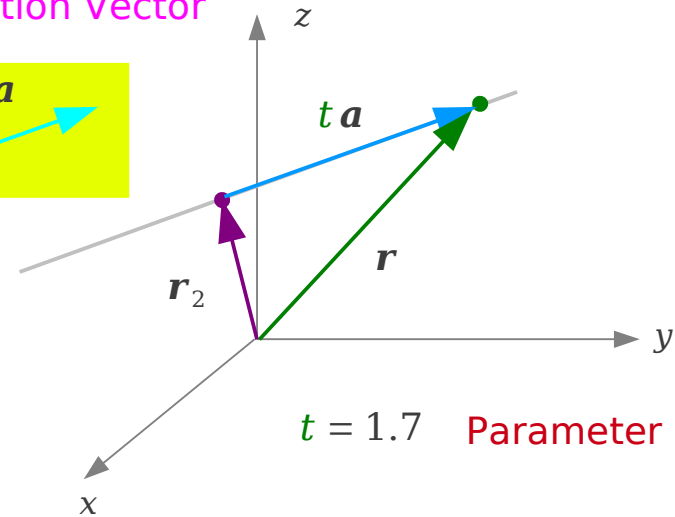
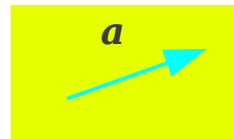
Line Equations (2)



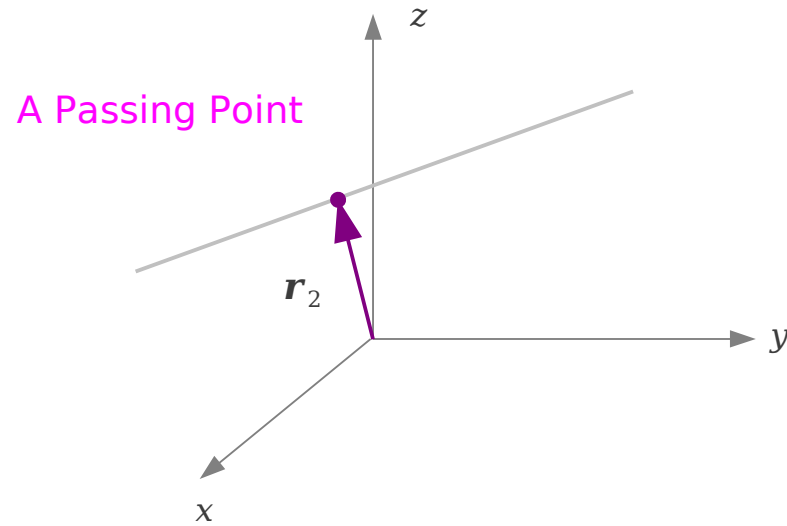
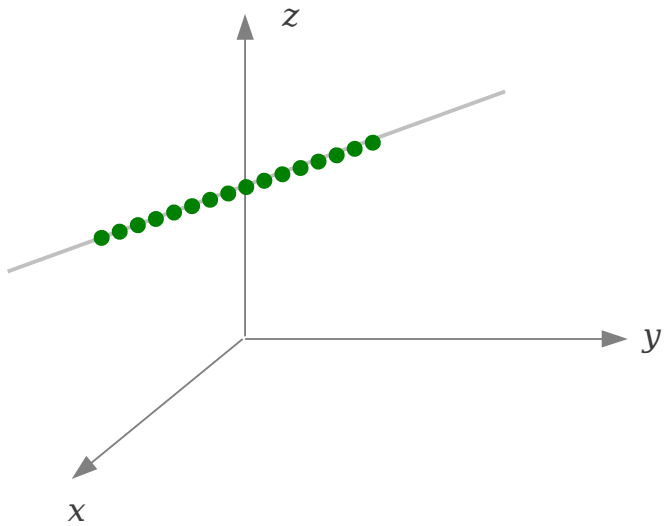
Direction Vector



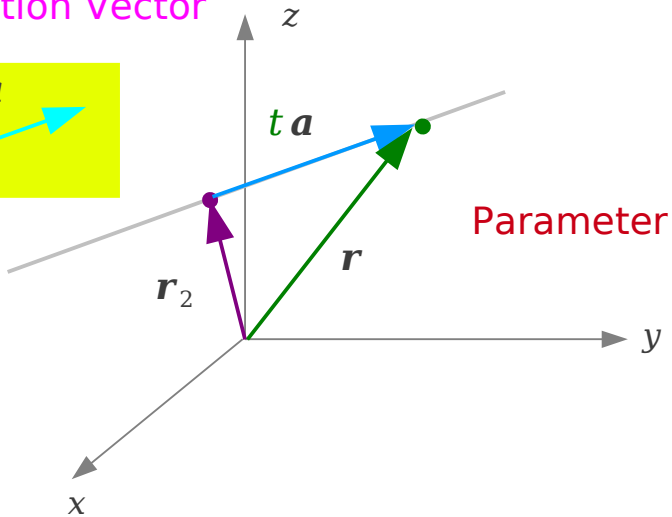
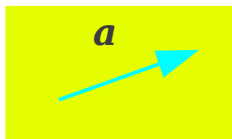
Direction Vector



Line Equations (3)



Direction Vector



$$\mathbf{r} = \mathbf{r}_2 + t\mathbf{a}$$

$$\mathbf{r}_2 = \langle x_2, y_2, z_2 \rangle$$

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

Plane Equations (1)

Vector equation

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

Normal Vector

$$\mathbf{r} = \langle x, y, z \rangle$$

$$\mathbf{r}_1 = \langle x_1, y_1, z_1 \rangle$$

$$\mathbf{n} = \langle a, b, c \rangle$$

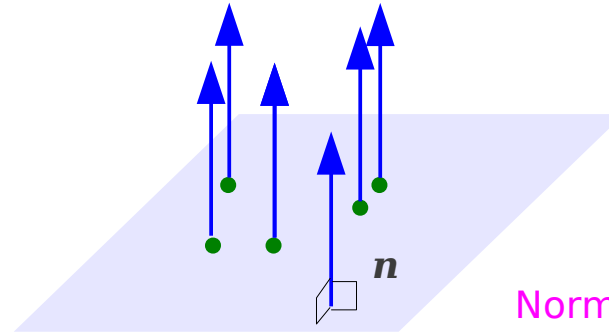
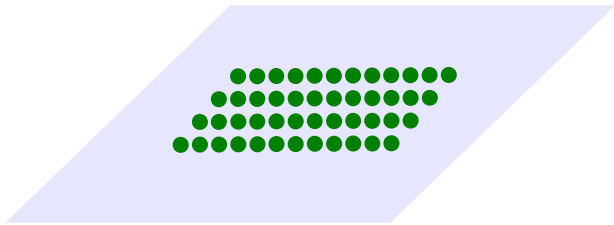
$$\mathbf{r} - \mathbf{r}_1 = \langle x - x_1, y - y_1, z - z_1 \rangle$$

$$\mathbf{n} = \langle a, b, c \rangle$$

Cartesian equation

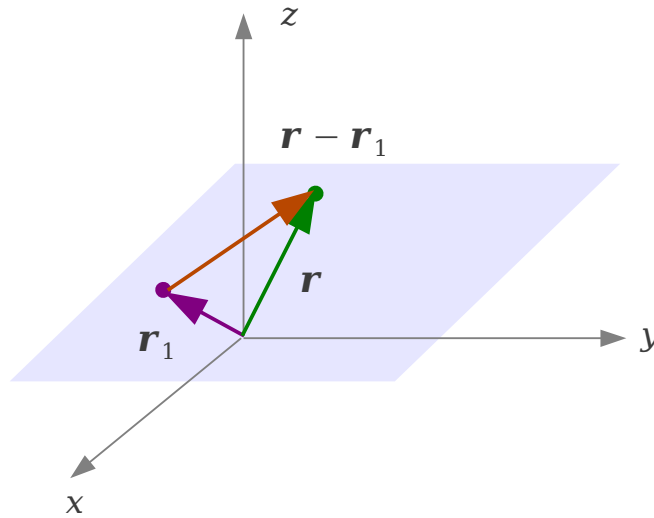
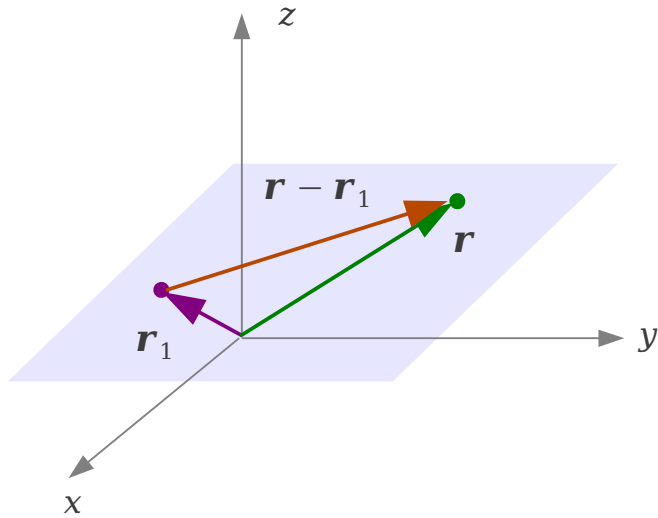
$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Plane Equations (2)

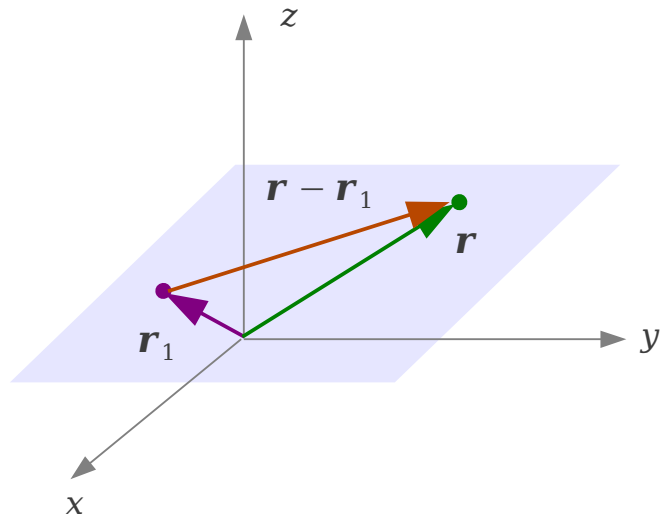
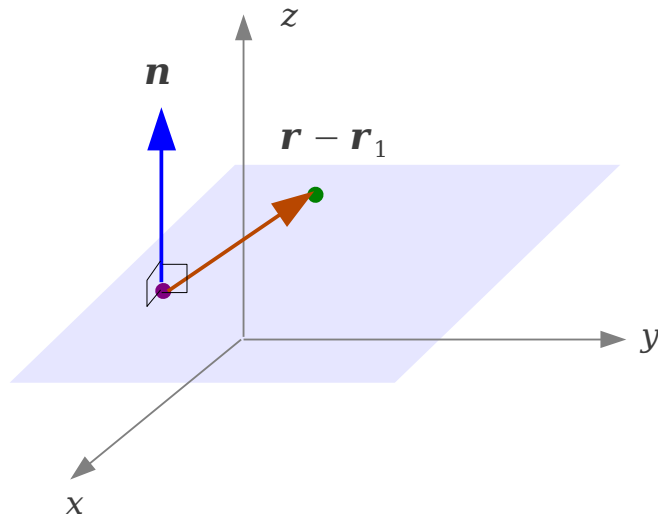
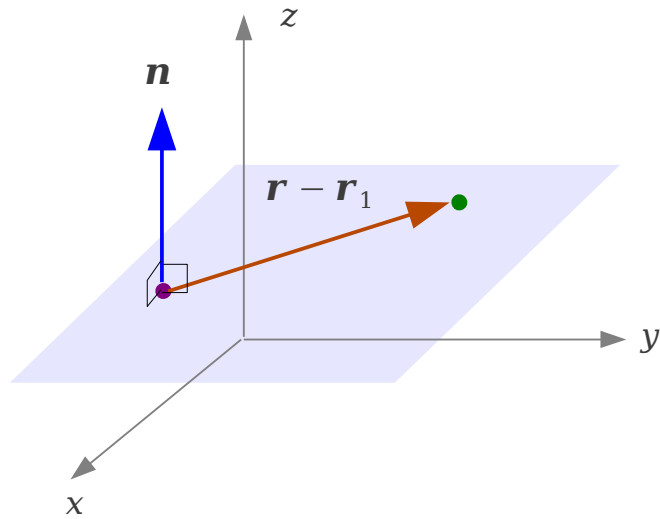


Normal Vector

No Parameter



Plane Equations (3)



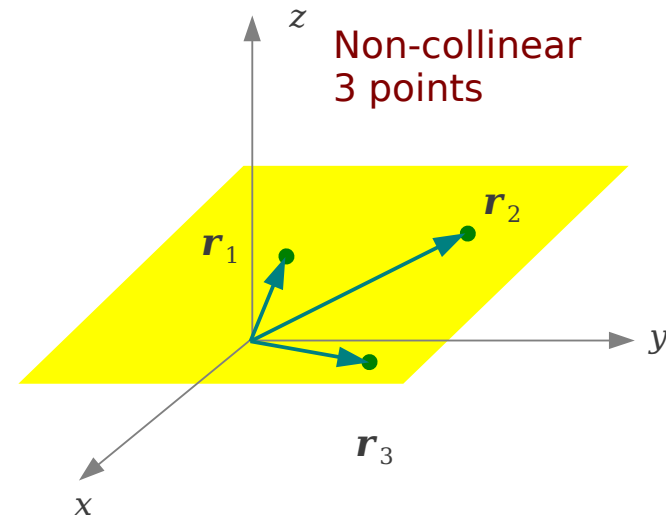
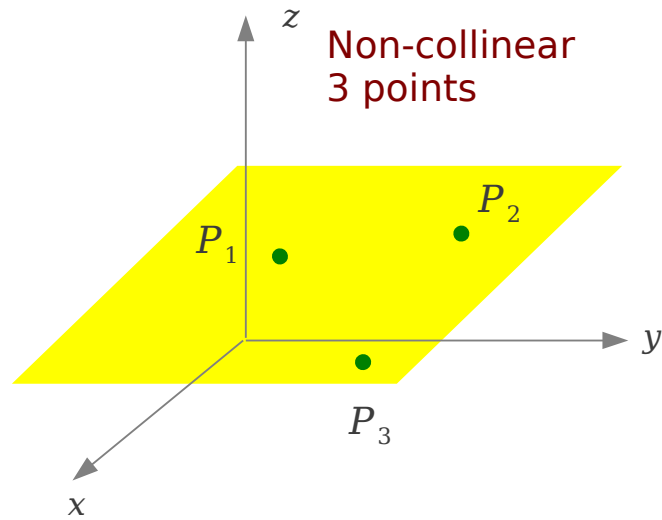
$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

$$\mathbf{r} = \langle x, y, z \rangle$$

$$\mathbf{r}_1 = \langle x_1, y_1, z_1 \rangle$$

$$\mathbf{n} = \langle a, b, c \rangle$$

Normal Vector & 3 Points



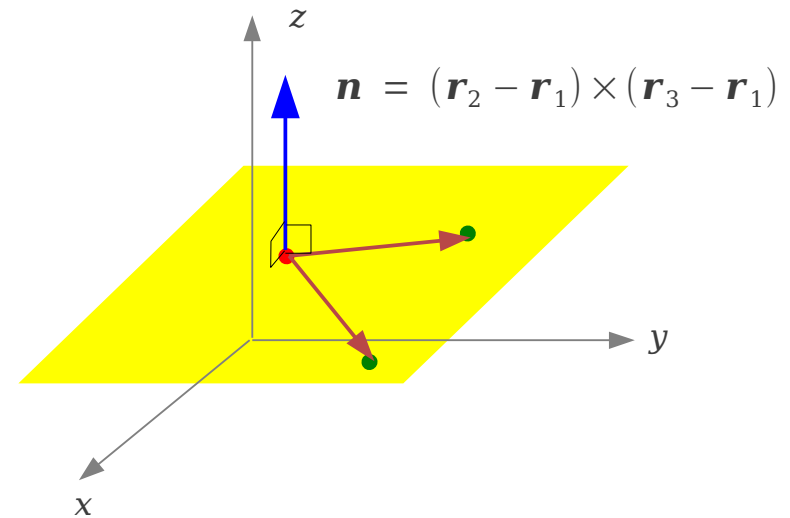
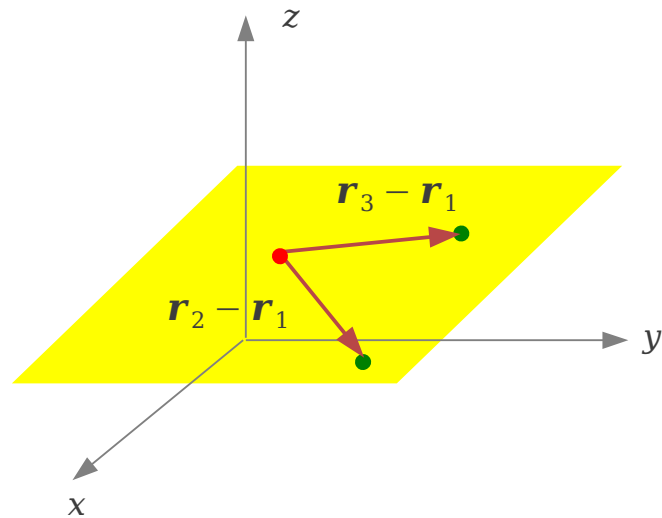
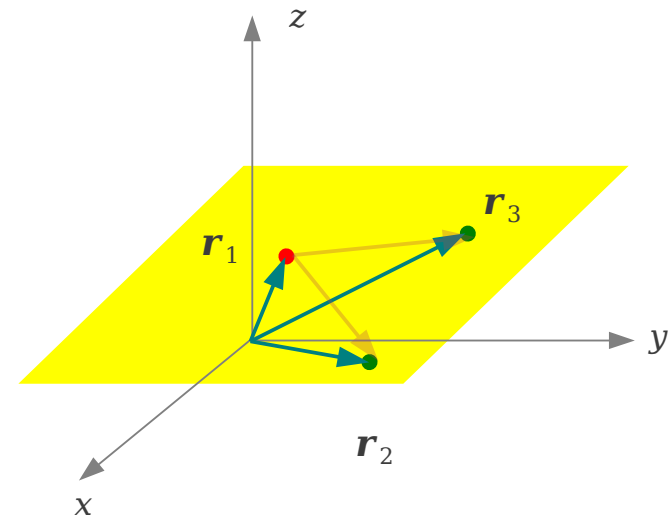
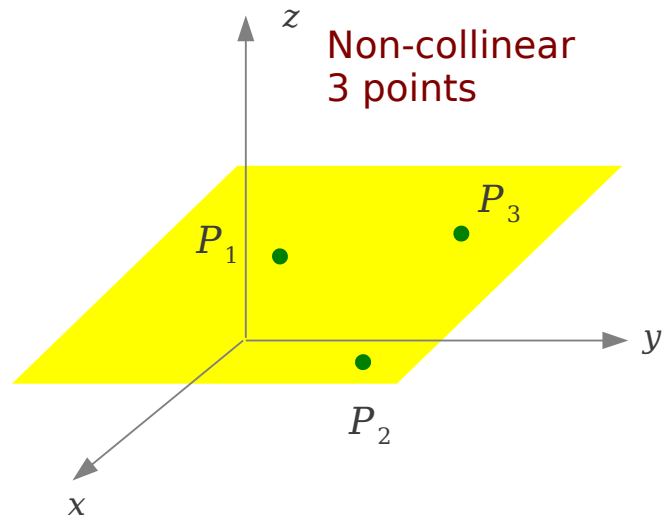
Graph of a plane



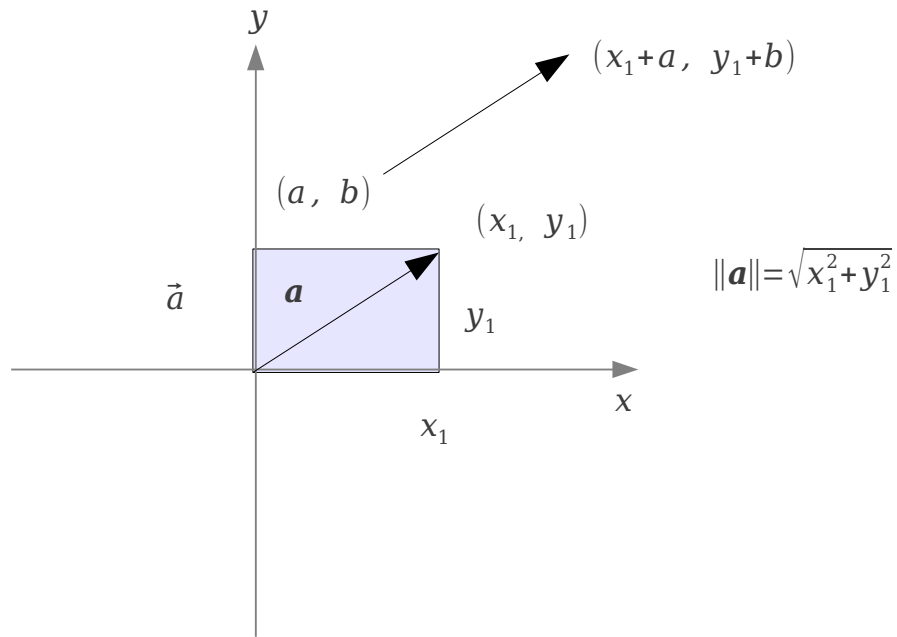
Line intersection of two planes

Point of intersection of a line and plane

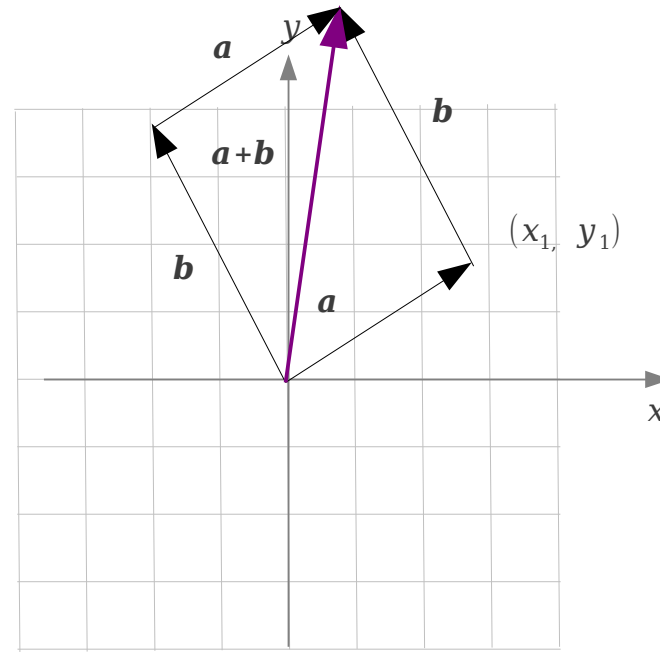
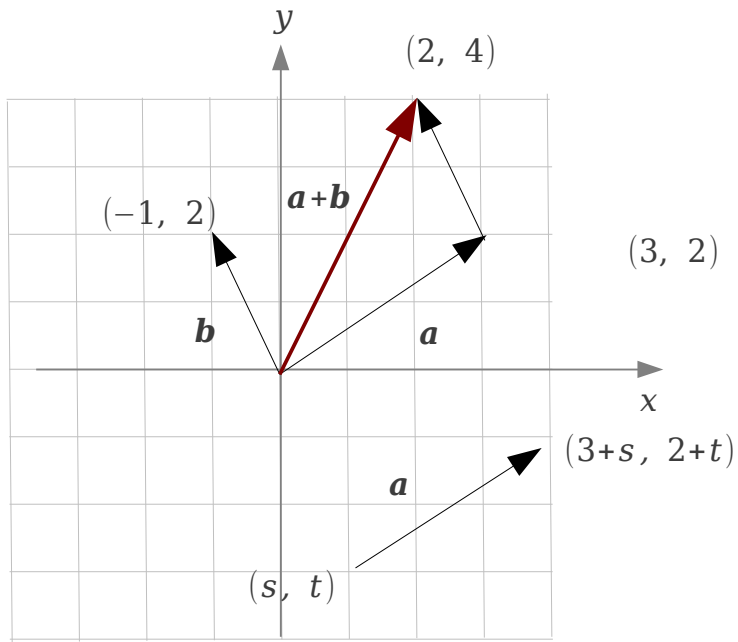
Normal Vector & 3 Points



Vectors

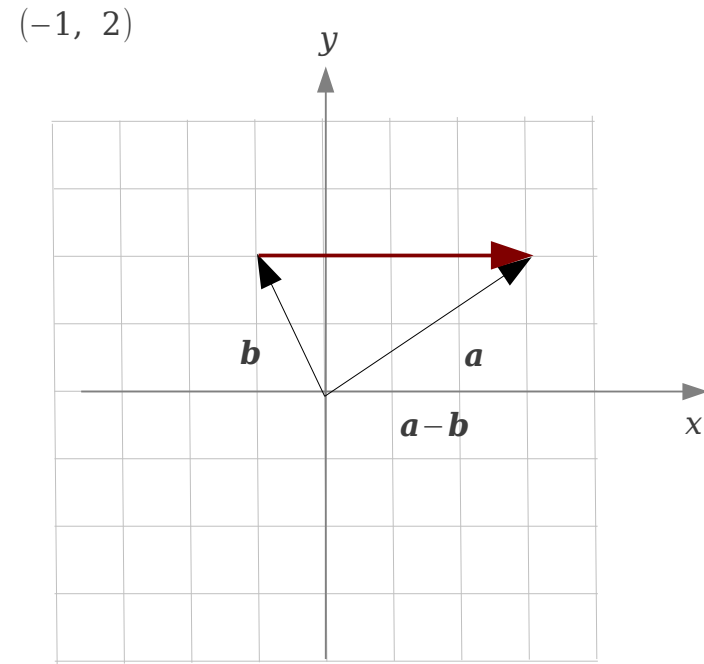
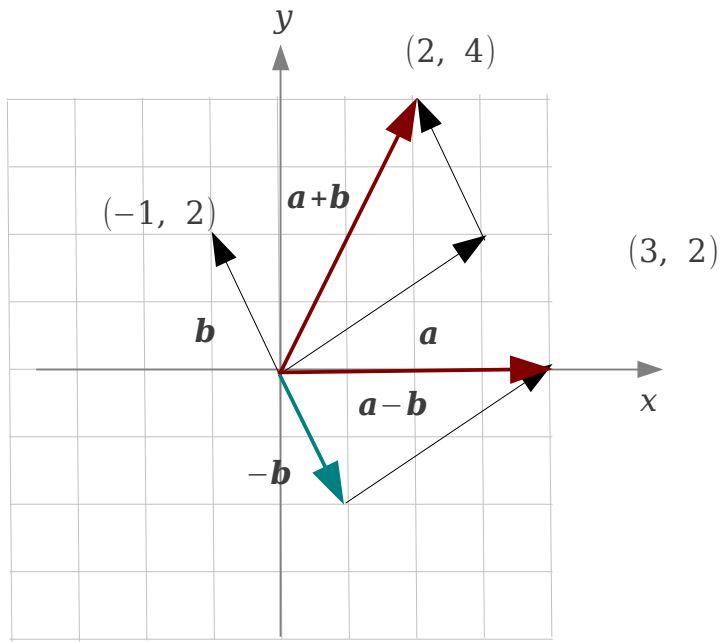


Vector Addition



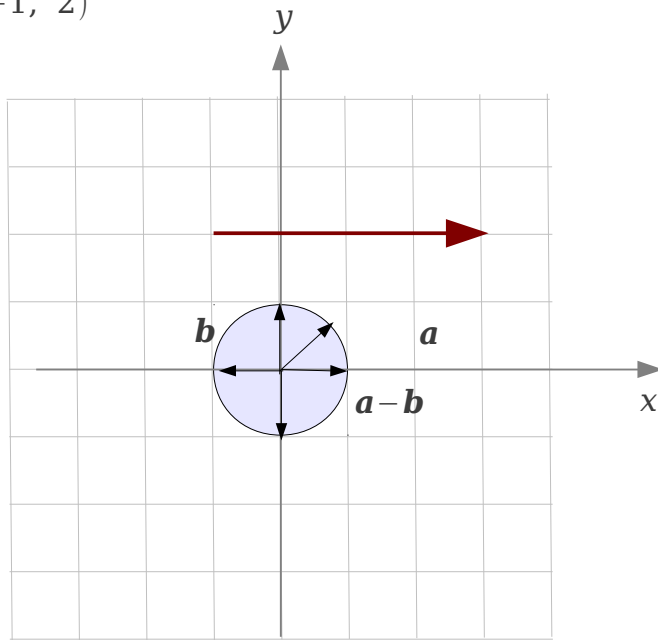
$$\|a\| = \sqrt{x_1^2 + y_1^2}$$

Vector Subtraction

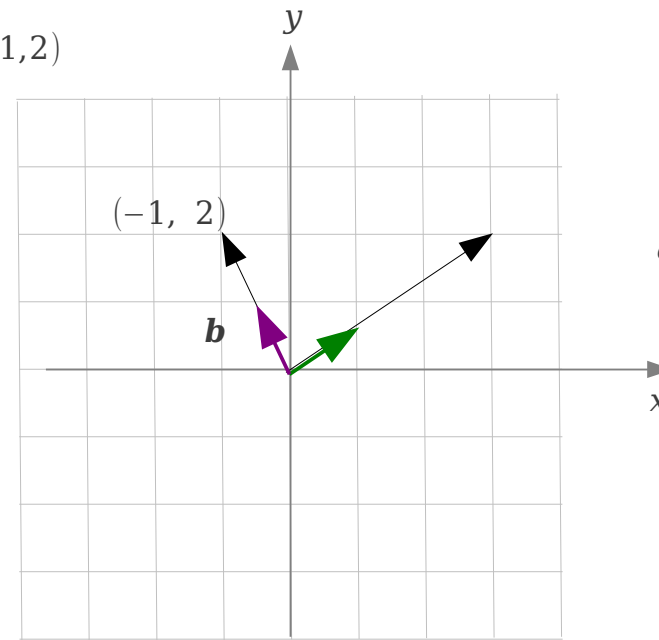


Unit Vector

$(-1, 2)$



$\mathbf{b}=(-1,2)$



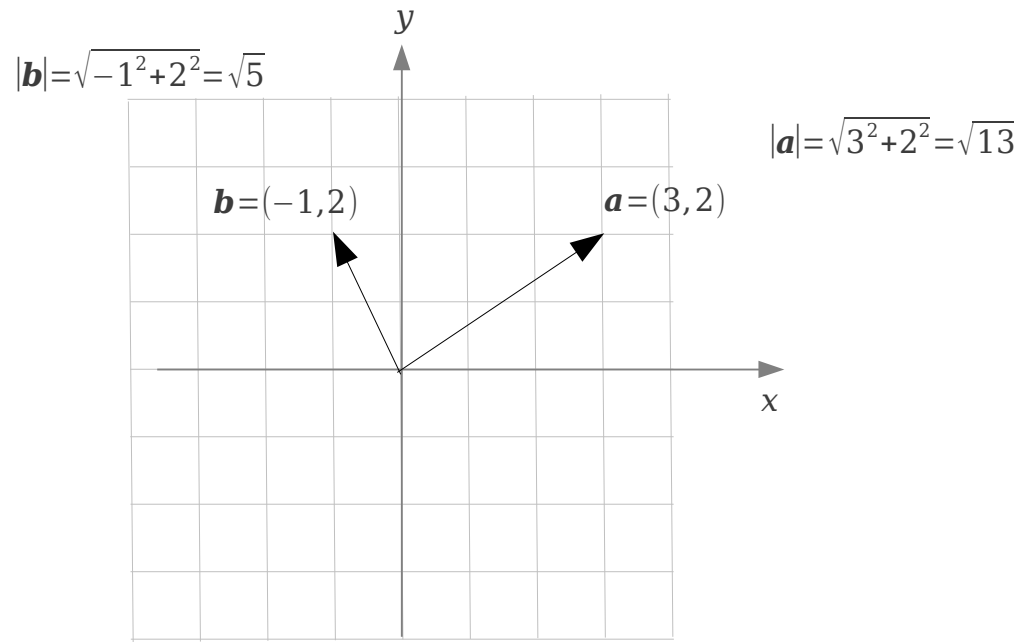
$$|\mathbf{a}| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$\mathbf{a} = (3, 2)$$

$$= \left(\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right)$$

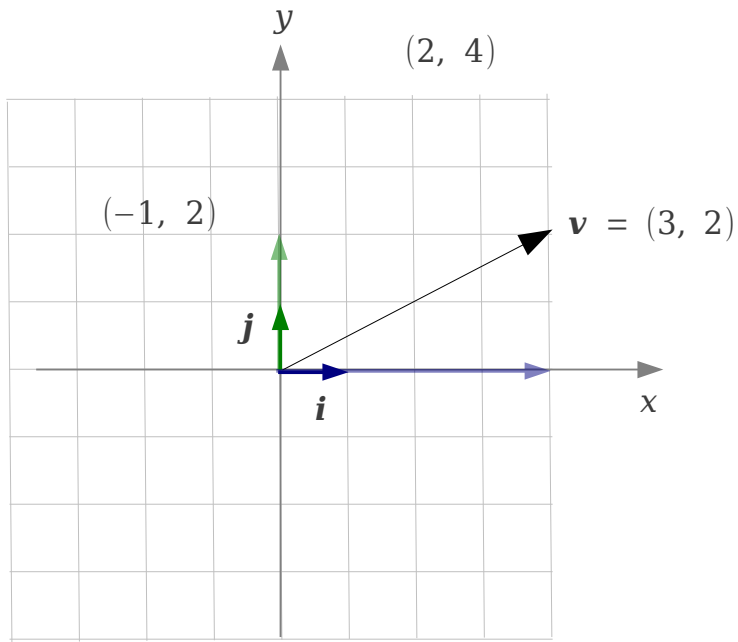
$$= \sqrt{\frac{3^2}{13} + \frac{2^2}{13}}$$

Vector Magnitude



$$\mathbf{a} \cdot \mathbf{b} = 3 \cdot -1 + 2 \cdot 2 = -3 + 4 = 1$$

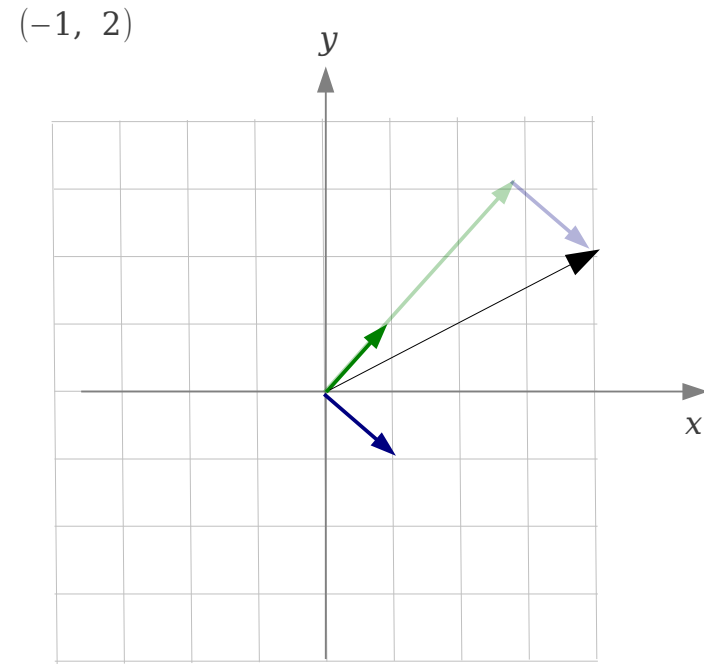
Basis



$$\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$$

$$a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{0} \implies a = b = 0$$

basis



$$a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \mathbf{0} \implies a = b = 0$$

basis

Vector Triple Product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

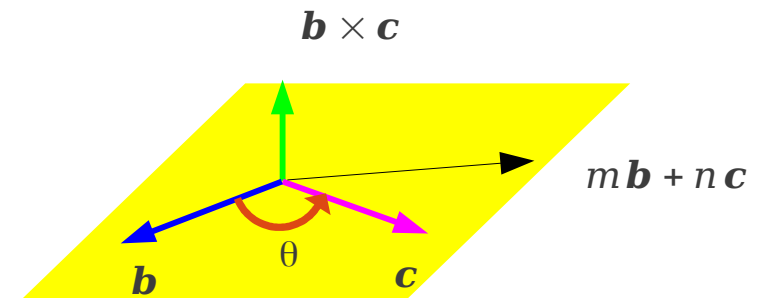
⇒ Perpendicular to $\mathbf{b} \times \mathbf{c}$

⇒ Any vector perpendicular to $\mathbf{b} \times \mathbf{c}$
lies in the plane perpendicular to $\mathbf{b} \times \mathbf{c}$

⇒ lies in the plane of \mathbf{b} and \mathbf{c}

Perpendicular to the plane of

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \Rightarrow m\mathbf{b} + n\mathbf{c}$$
$$(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$



Scalar Triple Product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

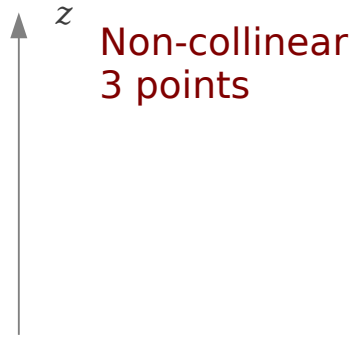
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

$$\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

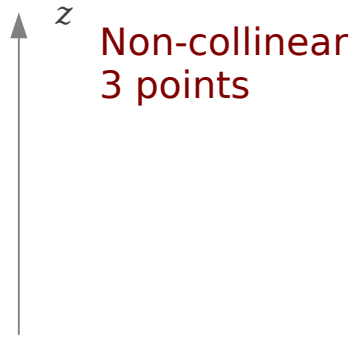
$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot \left(\mathbf{i} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right) \\ &= \left(a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{aligned}$$

Normal Vector & 3 Points



$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Normal Vector & 3 Points



References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [4] D.G. Zill, “Advanced Engineering Mathematics”