Complex Integration (2A)

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Contour Integrals

$f(z)$ defined at points of a smooth curve C

a smooth curve C is defined by

$$x = x(t) \qquad a \le t \le b$$
$$y = y(t)$$

The contour integral of **f** along C

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy) = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$= \int_a^b [u\,x'(t) - v\,y'(t)]\,dt + i\int_C [v\,x'(t) + u\,y'(t)]\,dt$$

$$= \int_a^b (u+iv)(x'(t)+i\,y'(t))\,dt$$

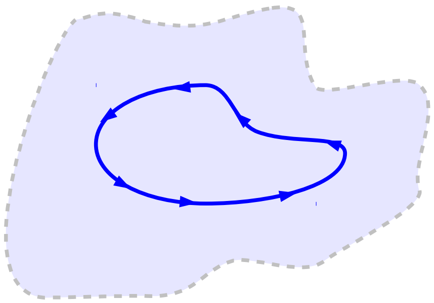
$$z(t) = x(t) + i\,y(t)$$
$$z'(t) = x'(t) + i\,y'(t)$$
$$a \le t \le b$$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

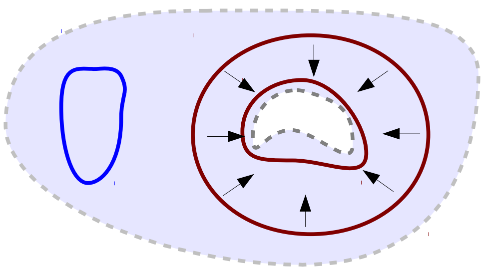
Connected Region

Connected Domains

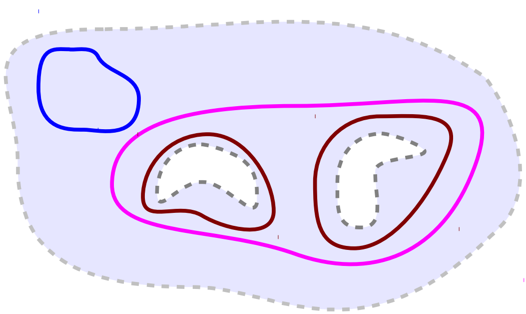
Simply Connected



Doubly Connected

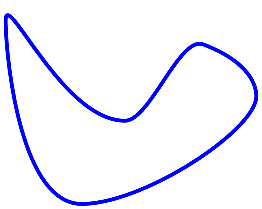


Triply Connected

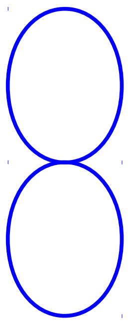


Closed Paths

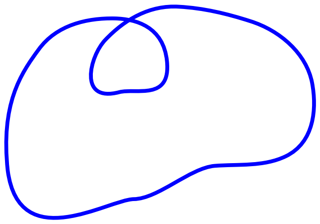
simple closed path



not simple closed path



not simple closed path



Contour Integration Evaluation

(1) Indefinite Integration of Analytic Functions

$f(z)$: **analytic** in a **simply connected domain** D $\quad f(z) = F'(z)$

➡ There exists an **indefinite integral** in D : an analytic function $F(z)$

➡ $$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$

for every path in D between z_0 and z_1

(2) Integration by the Use of the Path

C : a piecewise smooth path represented by $z = z(t) \quad (a \le t \le b)$

$f(z)$ a continuous function on C

➡ $$\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt$$

Contour Integration Evaluation – f(z) = 1/z

(1) Indefinite Integration of Analytic Functions

$z_1 = z_0$ ➡ $\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) = 0$

But $f(z) = \dfrac{1}{z}$ not analytic at $z = 0$ ➡ cannot use this method

(2) Integration by the Use of the Path

C : the unit circle ➡ $z(t) = \cos t + i\sin t = e^{it}$ $(0 \le t \le 2\pi)$

$z'(t) = -\sin t + i\cos t = ie^{it}$

$$\int_C f(z)\,dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$$

Contour Integration Evaluation – f(z) = z^m

(1) Indefinite Integration of Analytic Functions

$z_1 = z_0$ ➡ $\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) = 0$

But $f(z) = z^m$ not analytic at $z = 0$ for $m < 0$ ➡ cannot use this method

(2) Integration by the Use of the Path

C : the unit circle ➡ $z(t) = \cos t + i \sin t = e^{it}$ $(0 \le t \le 2\pi)$

$z'(t) = -\sin t + i \cos t = i e^{it}$

$$\int_C f(z)\,dz = \int_0^{2\pi} e^{mit} i e^{it}\,dt = \int_0^{2\pi} i e^{i(m+1)t}\,dt = i\left[\int_0^{2\pi} \cos((m+1)t)\,dt + i \int_0^{2\pi} \sin((m+1)t)\,dt\right]$$

$$\int_C z^m\,dz = \begin{cases} 2\pi i & (m = -1) \\ 0 & (m \neq -1) \end{cases}$$

Cauchy's Theorem

$f(z)$: **analytic** in a **simply connected domain** D

$f'(z)$: **continuous** in a **simply connected domain** D

⇒ for every **simple closed contour** C in D

$$\oint_C f(z)\,dz = 0$$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy) = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Cauchy-Goursat Theorem

$f(z)$: **analytic** in a **simply connected domain** D

⇒ for every **simple closed contour** C in D

$$\oint_C f(z)\,dz = 0$$

$f'(z)$: ~~**continuous**~~ in a simply connected domain D

- simple closed curve
- a continuously turning tangent
- except possibly at a finite number of points
- allow a finite number of corners (not smooth)

Cauchy's Integral Formula (1)

$f(z)$: **analytic** on and inside simple close curve C

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}dz$$

the value of $f(z)$ at a point $z = a$ inside C

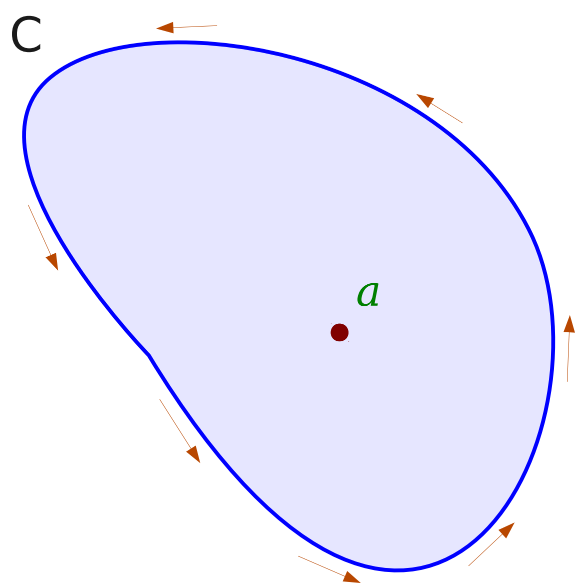
$$f(z) = \frac{1}{2\pi i}\oint \frac{f(w)}{w-z}dw$$

Cauchy's Integral Formula (2)

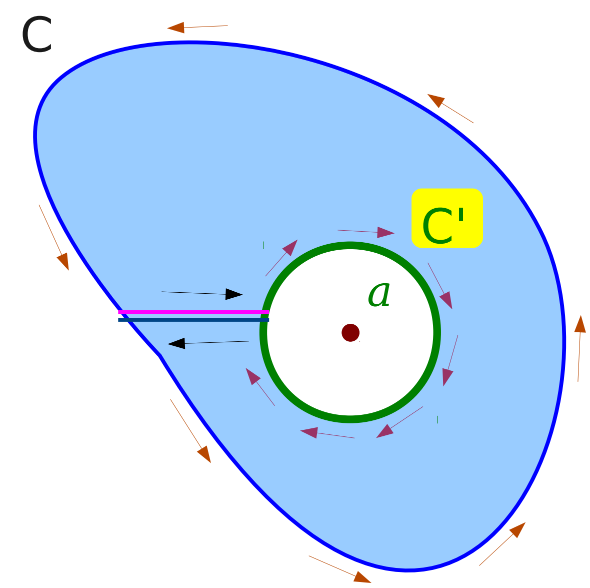
$f(z)$: **analytic** on and inside simple close curve C

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}dz$$

the value of $f(z)$ at a point $z = a$ inside C



$$\oint_C f(z)\,dz = 0$$



$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} + \oint_{cw\ C'} \frac{f(z)\,dz}{z-a} = 0$$

Cauchy's Integral Formula (3)

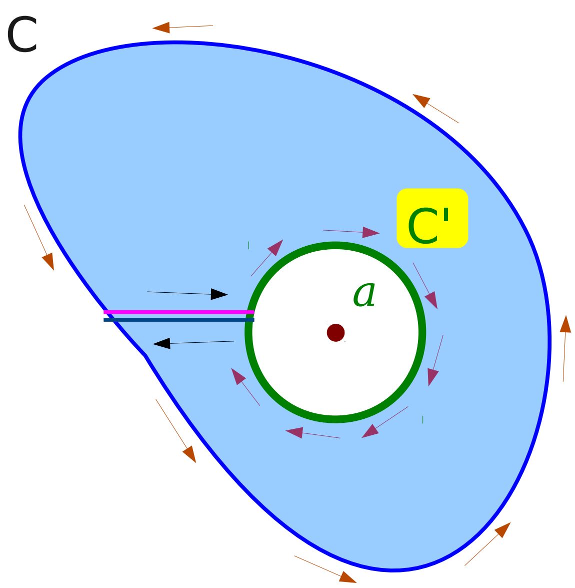
$f(z)$: **analytic** on and inside simple close curve C

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}dz$$

the value of $f(z)$ at a point $z = a$ inside C

$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} + \oint_{cw\ C'} \frac{f(z)\,dz}{z-a} = 0$$

$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \oint_{ccw\ C'} \frac{f(z)\,dz}{z-a}$$



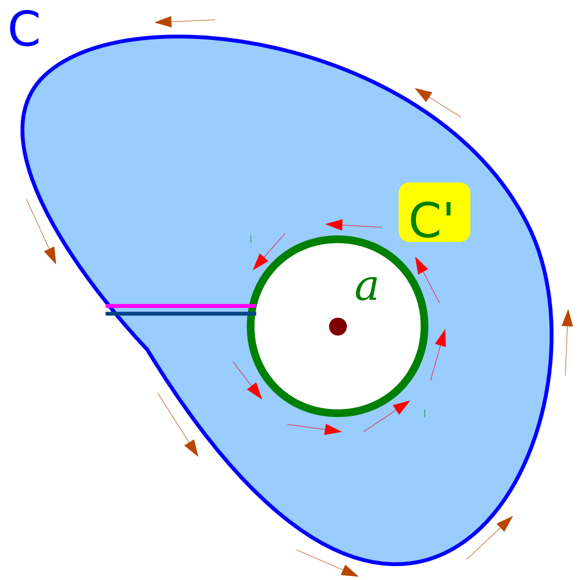
Cauchy's Integral Formula (4)

$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \oint_{ccw\ C'} \frac{f(z)\,dz}{z-a}$$

$$= 2\pi i\ f(a)$$

along C' $\quad z - a = \rho e^{i\theta}$

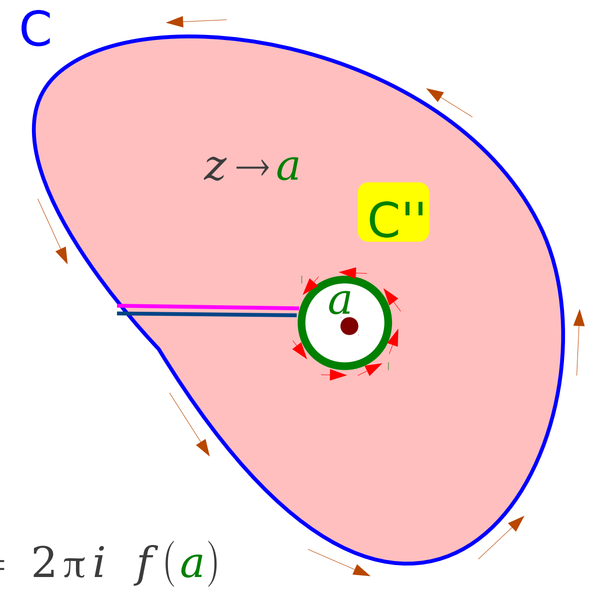
as $\quad z \to a \Rightarrow \rho \to 0$



$$z = a - \rho e^{i\theta}$$

$$dz = i\rho e^{i\theta} d\theta$$

$$\frac{dz}{z-a} = \frac{i\rho e^{i\theta} d\theta}{\rho e^{i\theta}}$$



$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \int_0^{2\pi} f(z)\, i\, d\theta = 2\pi i\ f(a)$$

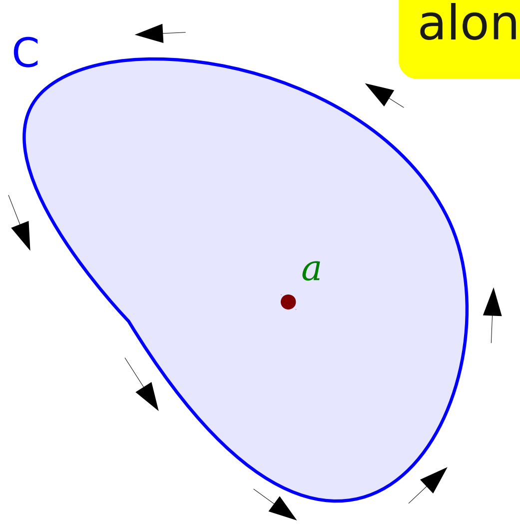
Cauchy's Integral Formula (5)

$$\frac{dz}{(z-a)^2} = \frac{i\rho e^{i\theta} d\theta}{(\rho e^{i\theta})^2}$$

$$\oint_{ccw\ C} \frac{f(z)\,dz}{(z-a)^2} = \int_0^{2\pi} \frac{f(z) i}{\rho e^{i\theta}} d\theta$$

$$= \int_0^{2\pi} \frac{f(z)}{\rho} i e^{-i\theta} d\theta = \left[-\frac{f(z)}{\rho} e^{-i\theta}\right]_0^{2\pi}$$

$$= -\frac{f(z)}{\rho}(e^{-i2\pi} - e^{-i0}) = 0$$



along C' $z - a = \rho e^{i\theta}$

$z = a - \rho e^{i\theta}$

$dz = i\rho e^{i\theta} d\theta$

$$dz = i\rho e^{i\theta} d\theta$$

$$\oint_{ccw\ C} f(z)\,dz = \int_0^{2\pi} f(z) i\rho e^{i\theta} d\theta$$

$$= \left[f(z)\rho e^{i\theta}\right]_0^{2\pi}$$

$$= f(z)\rho(e^{-i2\pi} - e^{-i0}) = 0$$

$$(z-a)\,dz = \rho e^{i\theta} i\rho e^{i\theta} d\theta$$

$$\oint_{ccw\ C} (z-a) f(z)\,dz = \int_0^{2\pi} f(z) i(\rho e^{i\theta})^2 d\theta$$

$$= \int_0^{2\pi} f(z)\rho^2 i e^{i2\theta} d\theta = \left[f(z)\frac{\rho}{2} e^{i2\theta}\right]_0^{2\pi}$$

$$= f(z)\frac{\rho}{2}(e^{-i4\pi} - e^{-i0}) = 0$$

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”