Thm of interp. error (A. p. 134)

$f : \underset{\text{domain}}{\mathbb{R}} \to \underset{\text{range}}{\mathbb{R}}$ (set of real numbers)

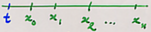
f diff. with $(n+1)$ continuous deriv.

on $I_t := \mathcal{H}(t, x_0, x_1, \ldots, x_n) :=$ smallest (1) interv. containing pts $(t, x_0, \ldots, x_n)$.

$x_0 < x_1 < \cdots < x_n$

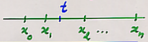
Case 1: $t < x_0$

$I_t = [t, x_n]$



Case 2: $t \in [x_0, x_n]$

$I_t = [x_0, x_n]$



Case 3: $x_n < t$

$I_t = [x_0, t]$

$$f_n(x) = \sum_{i=0}^{n} \ell_{i,n}(x) f(x_i) \quad (2)$$

Then $(\Rightarrow)$ $f(t) - f_n(t) \overset{(3)}{=} \dfrac{q_{n+1}(t)}{(n+1)!} f^{(n+1)}(\xi)$

$\xi \in I_t$, $q_{n+1}(x) \overset{(4)}{:=} (x - x_0)(x - x_1)\cdots(x - x_n)$

$= \prod_{j=0}^{n} (x - x_j) \in \mathcal{P}_{n+1}$ //

Pf: (Similar pf tech. used in other L10-2 error analyses)

Note: 1) Montesquieu

Similarity and difference betw (3) p. 10-1 (i.e, approx error) and Taylor series remainder (1) p. 2-3.

2) Consider $t = x_j$, $j = 0, 1, \dots, n$

RHS (3) p. 10-1 $= 0$ since $q_{n+1}(x_j) = 0$, $j = 0, \dots, n$

LHS (3) p. 10-1 $= 0$ since

$$f(x_j) - \underbrace{f_n(x_j)}_{\overset{\shortparallel}{f(x_j)}} = 0$$

why?²

$$f_n(x_j) = \sum_{i=0}^{n} \underbrace{l_{i,n}(x_j)}_{\overset{\shortparallel}{\delta_{ij}}} f(x_j) = f(x_j)$$

// (end note)

$E(x) := f(x) - f_n(x)$ interp. error

$$G(x) := E(x) - \frac{q_{n+1}(x)}{q_{n+1}(t)} E(t)$$

$x \in I_t$

$G(\cdot)$ has $(n+1)$ cont. deriv.