

# Divergence and Curl (3A)

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- Divergence
- Curl
- Green's Theorem

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# 2-D Vector Field

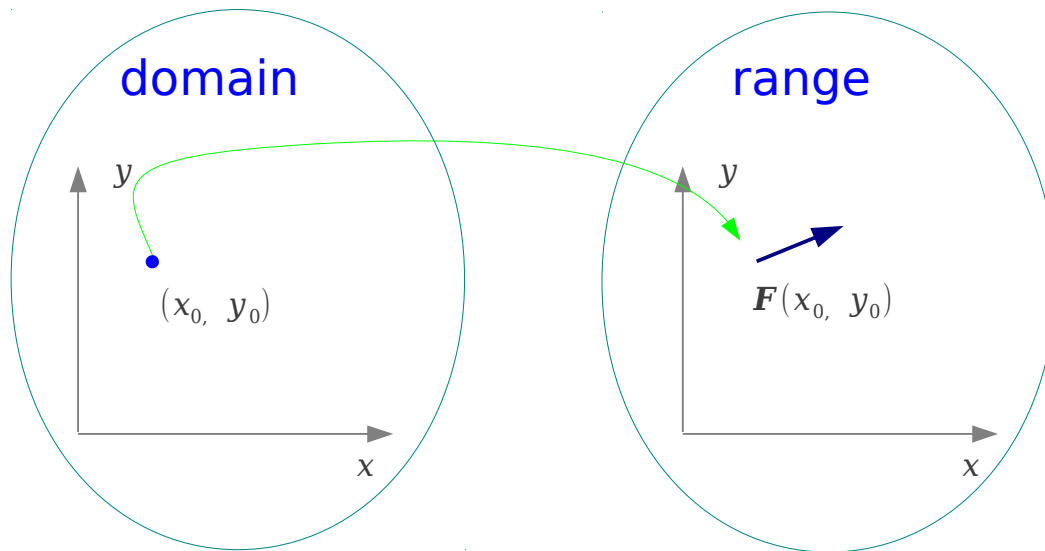
a given point in a 2-d space



A vector

$$(x_0, y_0)$$

$$\langle M(x_0, y_0), N(x_0, y_0) \rangle$$



2 functions

$$(x_0, y_0) \longrightarrow M(x_0, y_0)$$

$$(x_0, y_0) \longrightarrow N(x_0, y_0)$$

$$(x_0, y_0) \longrightarrow \mathbf{F}(x_0, y_0) = M(x_0, y_0)\mathbf{i} + N(x_0, y_0)\mathbf{j}$$

# 3-D Vector Field

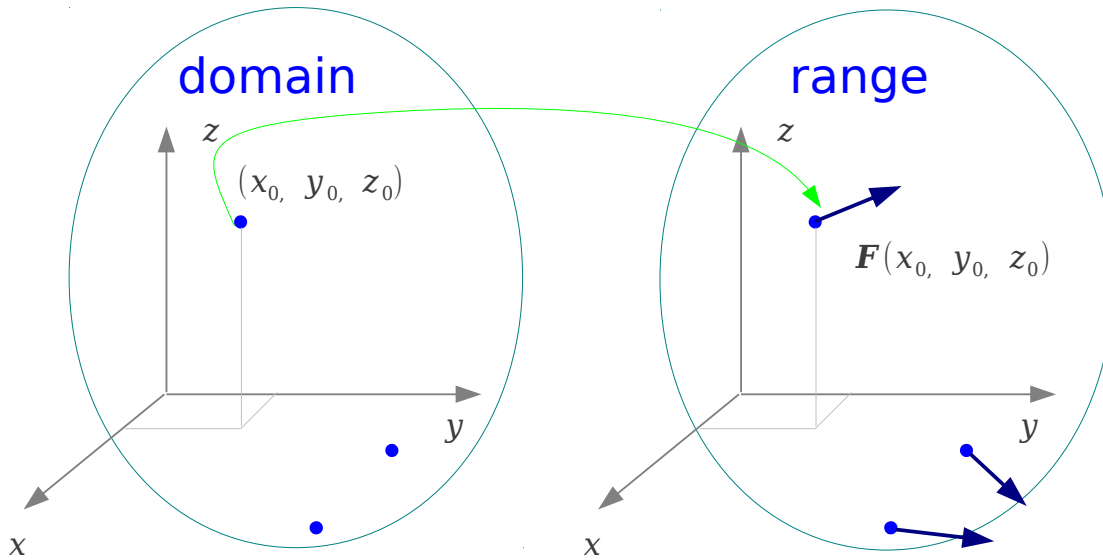
A given point in a 3-d space

$$(x_0, y_0, z_0)$$



A vector

$$\langle M(x_0, y_0, z_0), N(x_0, y_0, z_0), P(x_0, y_0, z_0) \rangle$$



3 functions

$$(x_0, y_0, z_0) \longrightarrow M(x_0, y_0, z_0)$$

$$(x_0, y_0, z_0) \longrightarrow N(x_0, y_0, z_0)$$

$$(x_0, y_0, z_0) \longrightarrow P(x_0, y_0, z_0)$$

$$(x_0, y_0, z_0) \longrightarrow \mathbf{F}(x_0, y_0, z_0) = M(x_0, y_0, z_0)\mathbf{i} + N(x_0, y_0, z_0)\mathbf{j} + P(x_0, y_0, z_0)\mathbf{k}$$

## 2-Divergence (1 - 5)

Flux across rectangle boundary

$$\approx \left( \frac{\partial M}{\partial x} \Delta x \right) \Delta y + \left( \frac{\partial N}{\partial y} \Delta y \right) \Delta x = \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta x \Delta y$$

Flux density

$$= \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)$$

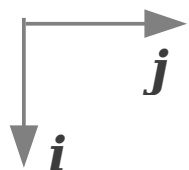
Divergence of  $\mathbf{F}$

Flux Density

# 2-D Divergence (1)

Velocity Fields  
of fluid flows

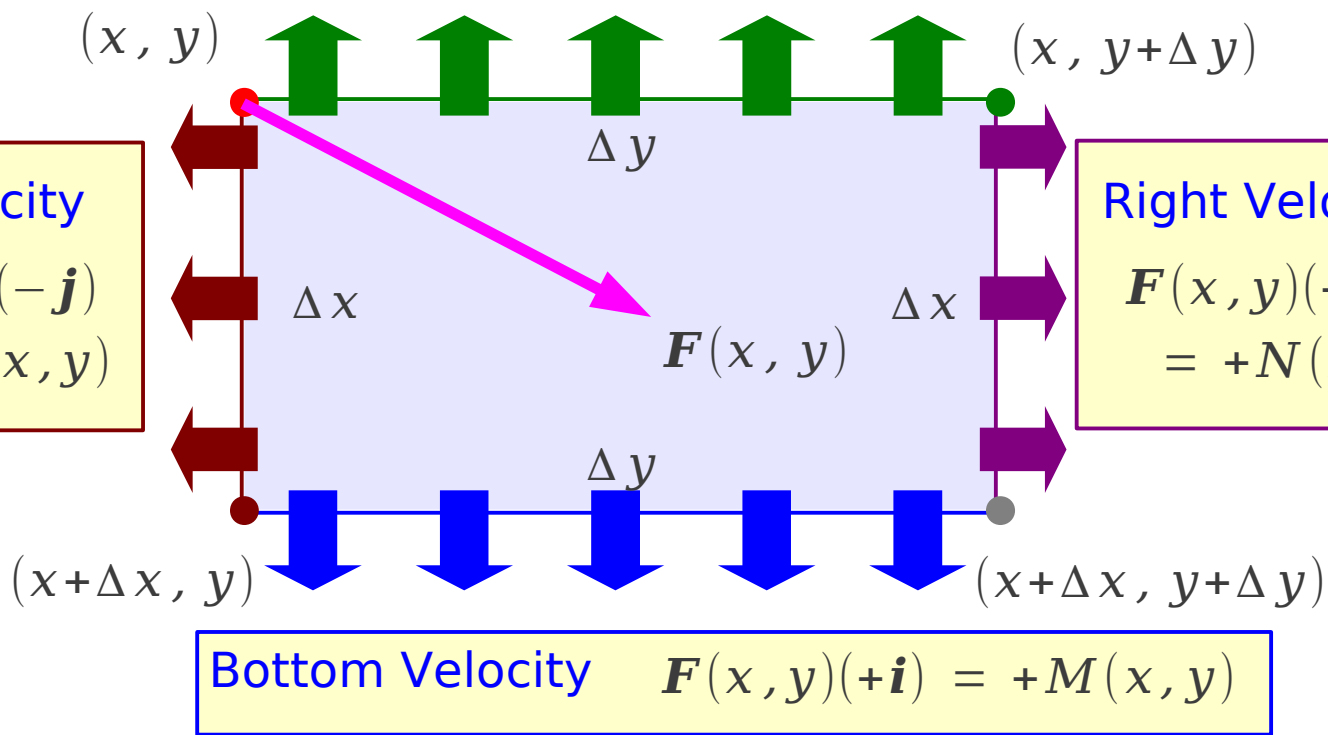
$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$



Top Velocity  $\mathbf{F}(x, y)(-\mathbf{i}) = -M(x, y)$

Left Velocity  
 $\mathbf{F}(x, y)(-\mathbf{j}) = -N(x, y)$

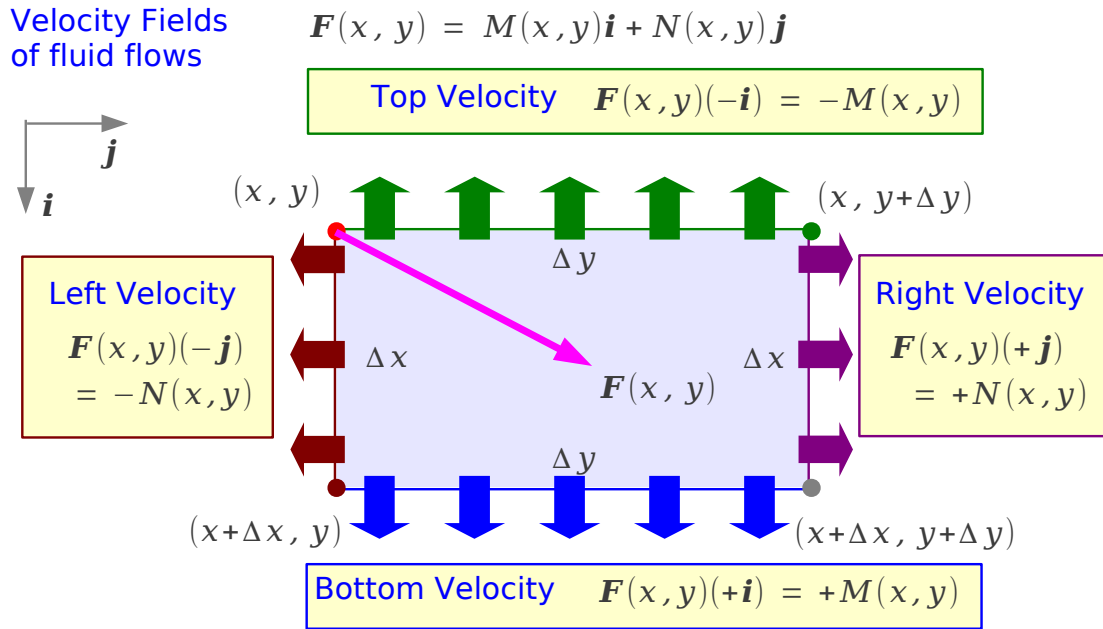
Right Velocity  
 $\mathbf{F}(x, y)(+\mathbf{j}) = +N(x, y)$



**Flow rate of outward bound fluid**

# 2-D Divergence (2)

Velocity Fields  
of fluid flows



**Flow rate of outward bound fluid**

The rate at which fluid leave the rectangle

Across top  $F(x, y) \cdot (-\mathbf{i})\Delta y = -M(x, y)\Delta y$

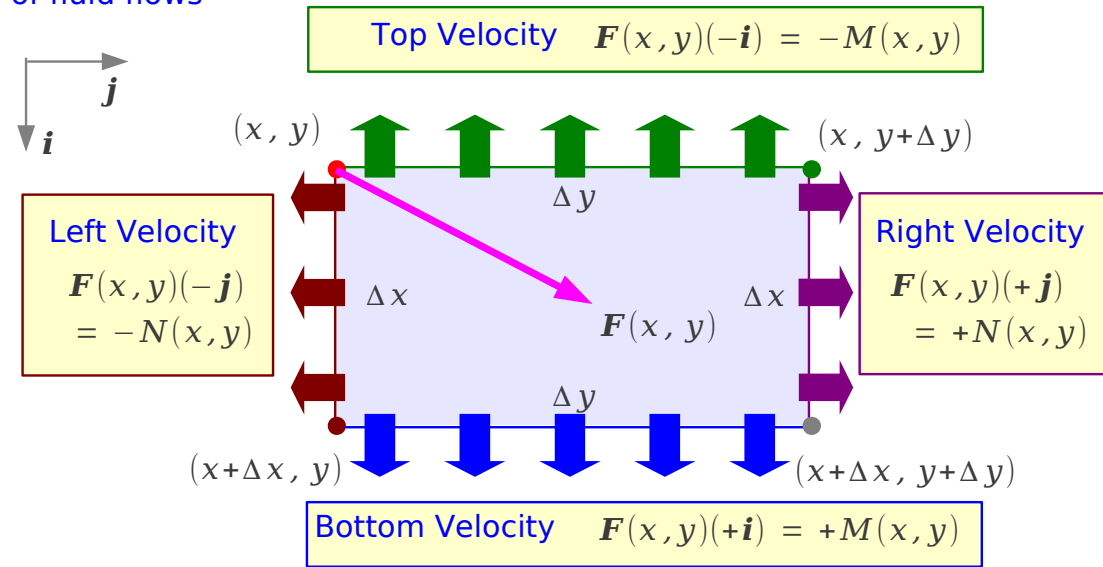
Across bottom  $F(x+\Delta x, y) \cdot (+\mathbf{i})\Delta y = M(x+\Delta x, y)\Delta y$

Across left  $F(x, y) \cdot (-\mathbf{j})\Delta x = -N(x, y)\Delta x$

Across right  $F(x, y+\Delta y) \cdot (+\mathbf{j})\Delta x = N(x, y+\Delta y)\Delta x$

# 2-D Divergence (3)

Velocity Fields of fluid flows



$$\begin{aligned}
 \mathbf{F}(x, y) \cdot (-\mathbf{i})\Delta y &= -M(x, y)\Delta y \\
 \mathbf{F}(x+\Delta x, y) \cdot (+\mathbf{i})\Delta y &= M(x+\Delta x, y)\Delta y \\
 \mathbf{F}(x, y) \cdot (-\mathbf{j})\Delta x &= -N(x, y)\Delta x \\
 \mathbf{F}(x, y+\Delta y) \cdot (+\mathbf{j})\Delta x &= N(x, y+\Delta y)\Delta x
 \end{aligned}$$

Flow rate of outward bound fluid

The rate at which fluid leave the rectangle

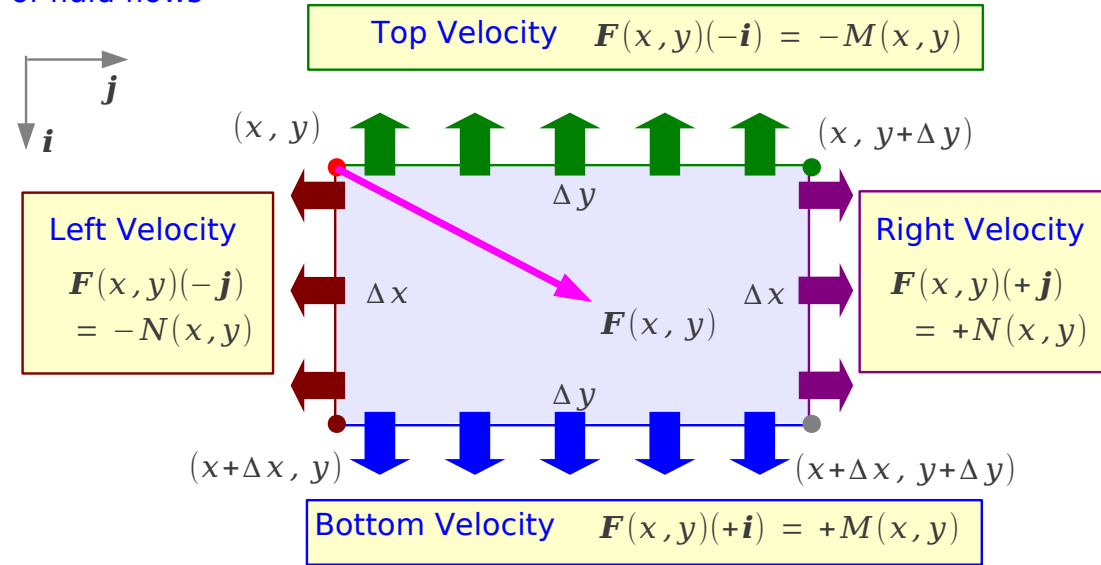
$$\text{Across top + bottom} \quad \{M(x+\Delta x, y) - M(x, y)\}\Delta y = \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y$$

$$\text{Across left + right} \quad \{N(x, y+\Delta y) - N(x, y)\}\Delta x = \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$$



# 2-D Divergence (4)

Velocity Fields of fluid flows



$$\{M(x+\Delta x, y) - M(x, y)\} \Delta y = \left( \frac{\partial M}{\partial x} \Delta x \right) \Delta y$$

$$\{N(x, y+\Delta y) - N(x, y)\} \Delta x = \left( \frac{\partial N}{\partial y} \Delta y \right) \Delta x$$

Flow rate of outward bound fluid

Flux across rectangle boundary

$$\approx \left( \frac{\partial M}{\partial x} \Delta x \right) \Delta y + \left( \frac{\partial N}{\partial y} \Delta y \right) \Delta x = \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta x \Delta y$$

Flux density

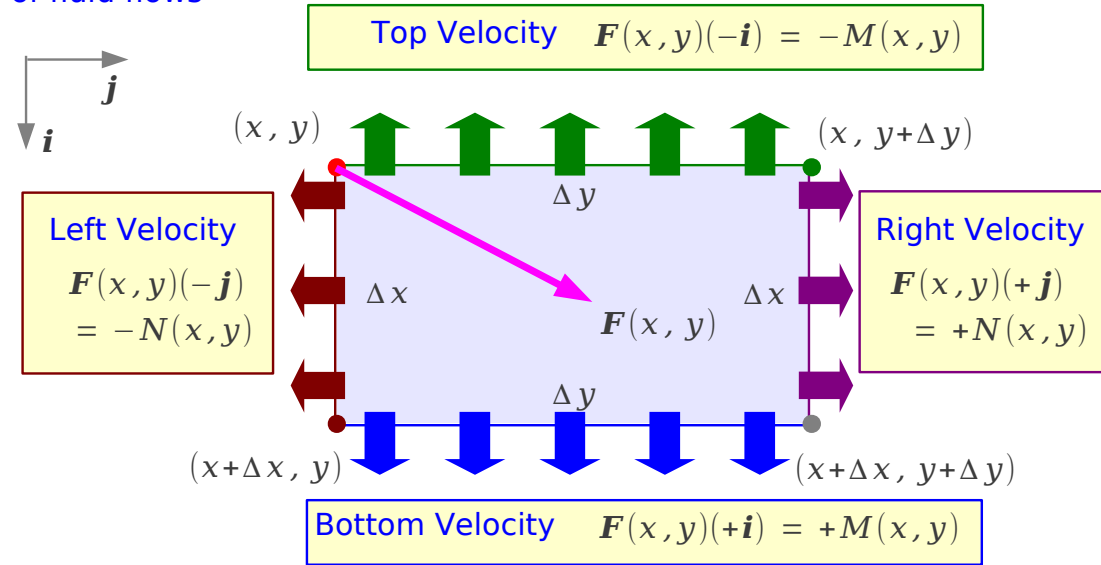
$$= \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)$$

Divergence of  $\mathbf{F}$

Flux Density

# 2-D Divergence (5)

Velocity Fields of fluid flows



$$\{M(x+\Delta x, y) - M(x, y)\} \Delta y = \left( \frac{\partial M}{\partial x} \Delta x \right) \Delta y$$

$$\{N(x, y+\Delta y) - N(x, y)\} \Delta x = \left( \frac{\partial N}{\partial y} \Delta y \right) \Delta x$$

Flow rate of outward bound fluid

Flux across rectangle boundary

$$\approx \left( \frac{\partial M}{\partial x} \Delta x \right) \Delta y + \left( \frac{\partial N}{\partial y} \Delta y \right) \Delta x = \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta x \Delta y$$

Flux density

$$= \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)$$

Divergence of  $\mathbf{F}$

Flux Density

# 2-D Divergence (a - d)

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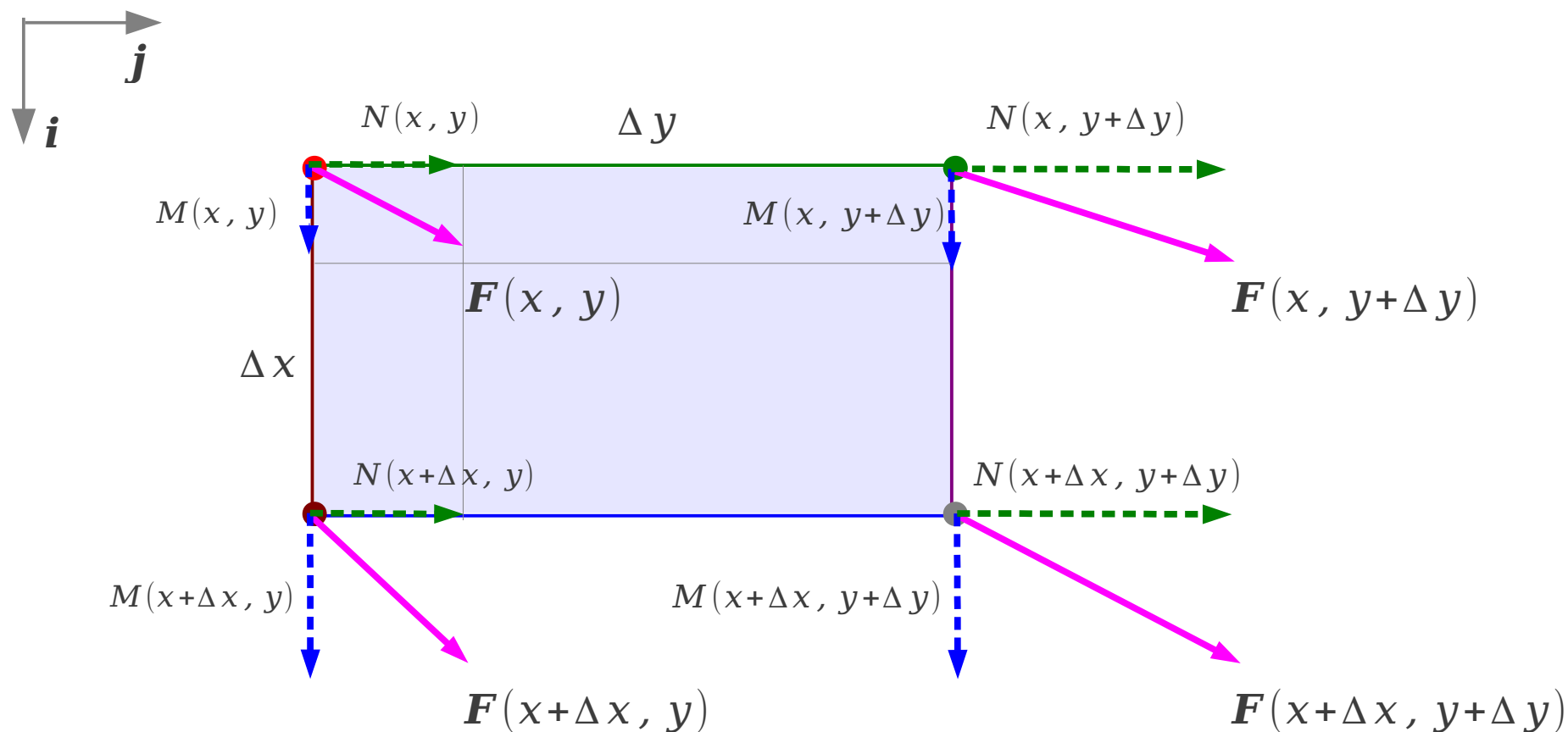
2-D Divergence and Del Operator

3-D Divergence and Del Operator

$$\nabla \cdot \mathbf{F}$$

# 2-D Divergence (a)

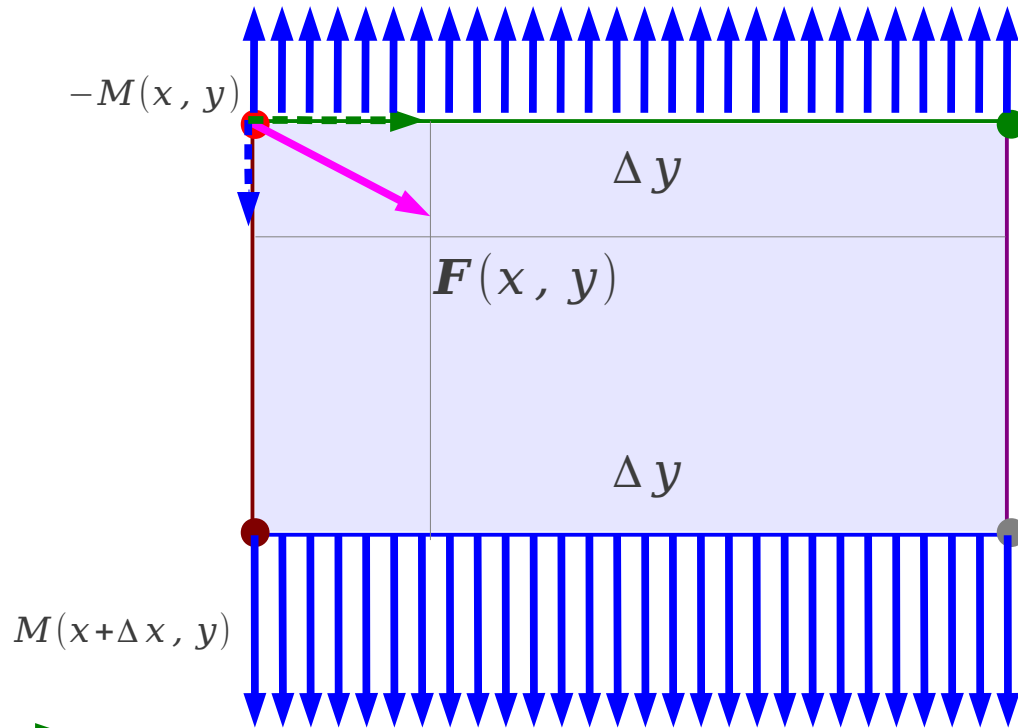
$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$



# 2-D Divergence (Top, Bottom) (b)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

$$\mathbf{F}(x, y) \cdot (-\mathbf{i})\Delta y = -M(x, y)\Delta y$$



$$\frac{\{M(x+\Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x}$$

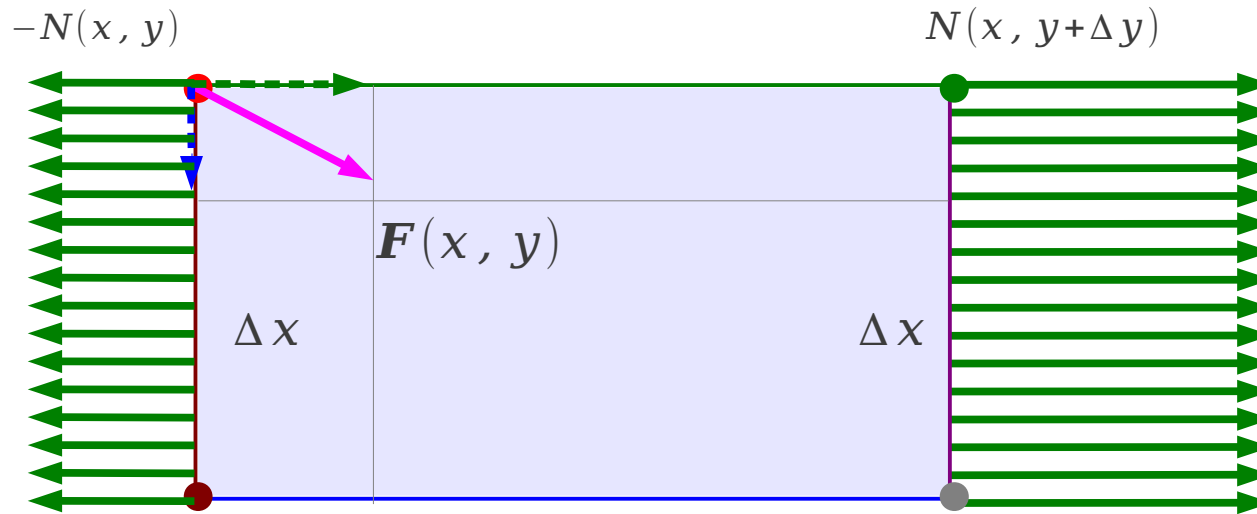
$$\{M(x+\Delta x, y) - M(x, y)\}\Delta y = \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y$$

$$\mathbf{F}(x+\Delta x, y) \cdot (+\mathbf{i})\Delta y = M(x+\Delta x, y)\Delta y$$

# 2-D Divergence (left, right) (c)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

$$\mathbf{F}(x, y) \cdot (-\mathbf{j})\Delta x = -N(x, y)\Delta x$$



$$\mathbf{F}(x, y + \Delta y) \cdot (+\mathbf{j})\Delta x = N(x, y + \Delta y)\Delta x$$

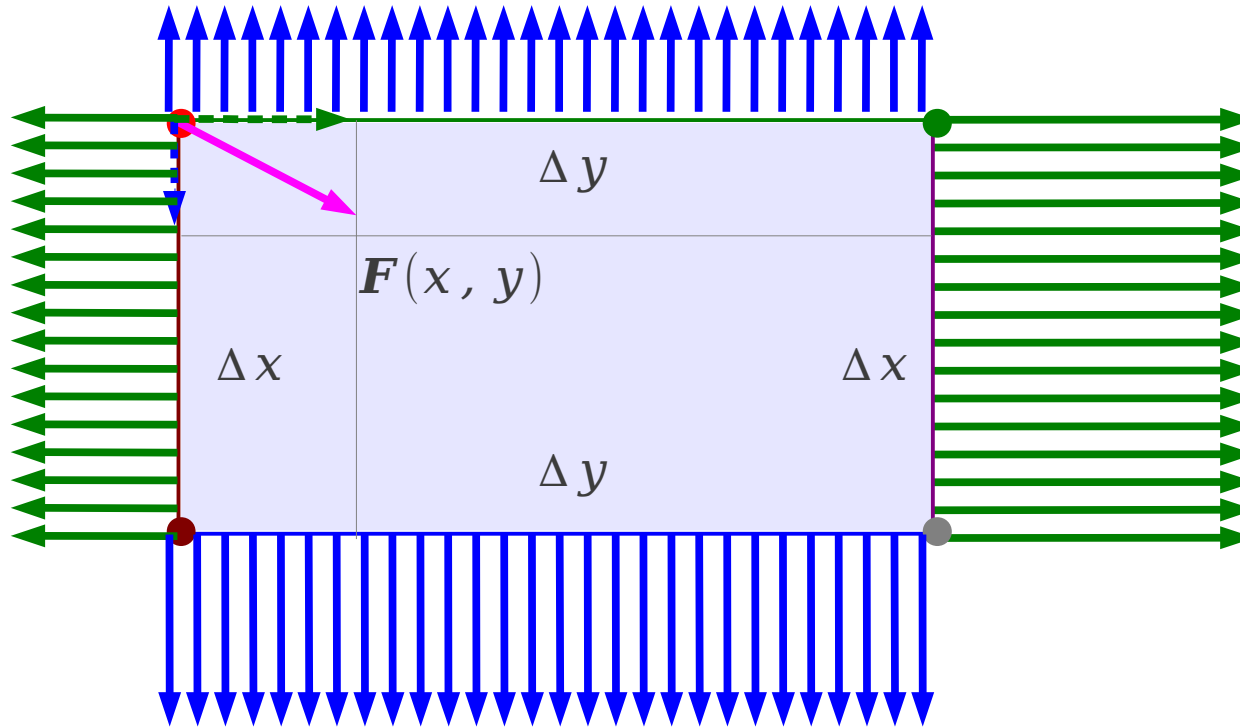
$$\frac{N(x, y + \Delta y) - N(x, y)}{\Delta y} \approx \frac{\partial N}{\partial y}$$

$$\{N(x, y + \Delta y) - N(x, y)\}\Delta x = \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$$

# 2-D Divergence (d)

Flux across rectangle boundary

$$\approx \left( \frac{\partial M}{\partial x} \Delta x \right) \Delta y + \left( \frac{\partial N}{\partial y} \Delta y \right) \Delta x = \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta x \Delta y$$

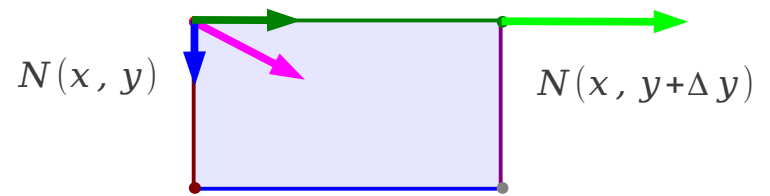
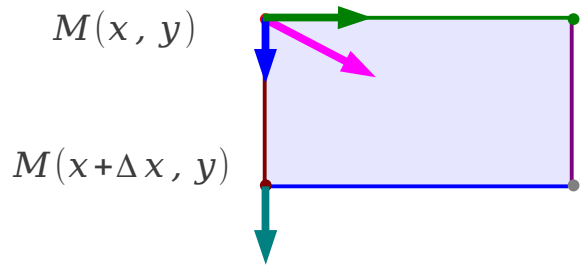


Flux density =  $\left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)$

Divergence of  $\mathbf{F}$

Flux Density

# Inward & Outward Bound



$$\frac{\{M(x + \Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x} > 0 \quad \text{Outward}$$

$$\frac{\{N(x, y + \Delta y) - N(x, y)\}}{\Delta y} \approx \frac{\partial N}{\partial y} > 0 \quad \text{Outward}$$

$$\frac{\{M(x + \Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x} < 0 \quad \text{Inward}$$

$$\frac{\{N(x, y + \Delta y) - N(x, y)\}}{\Delta y} \approx \frac{\partial N}{\partial y} < 0 \quad \text{Inward}$$

$$\begin{aligned} & \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \right) \cdot (M \mathbf{i}) \\ &= \nabla \cdot (M \mathbf{i}) \end{aligned}$$

$$\begin{aligned} & \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \right) \cdot (N \mathbf{j}) \\ &= \nabla \cdot (N \mathbf{j}) \end{aligned}$$



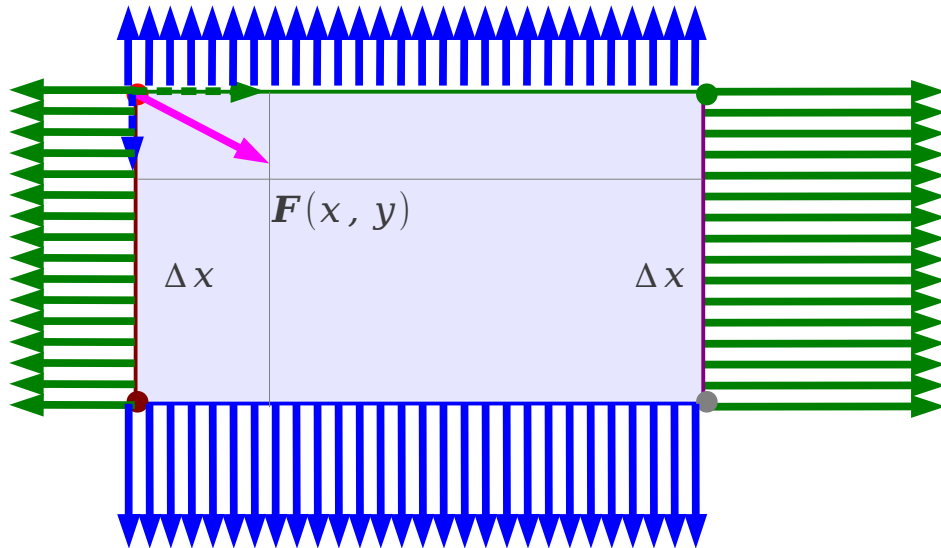
# 2-D Divergence and Del Operator

2-D Vector Field

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

Flux across rectangle boundary

$$\approx \left( \frac{\partial M}{\partial x} \Delta x \right) \Delta y + \left( \frac{\partial N}{\partial y} \Delta y \right) \Delta x = \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta x \Delta y$$



$$\text{Flux density} = \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)$$

$$= \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \right) \cdot (M \mathbf{i} + N \mathbf{j})$$

$$= \nabla \cdot \mathbf{F}$$

Divergence of  $\mathbf{F}$

# 3-D Divergence and Del Operator

2-D Vector Field

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

$$\begin{aligned}\text{Divergence of } \mathbf{F} &= \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \\ &= \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \right) \cdot (M\mathbf{i} + N\mathbf{j}) = \nabla \cdot \mathbf{F}\end{aligned}$$

3-D Vector Field

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

$$\begin{aligned}\text{Divergence of } \mathbf{F} &= \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) \\ &= \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \cdot (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) = \nabla \cdot \mathbf{F}\end{aligned}$$



## 2-Curl (1 - 5)

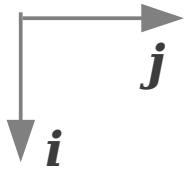
Circulation around rectangle boundary

$$\approx \left( \frac{\partial N}{\partial x} \Delta x \right) \Delta y - \left( \frac{\partial M}{\partial y} \Delta y \right) \Delta x = \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \Delta x \Delta y$$

Circulation density =  $\left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$     **k-component**    **Circulation Density**  
**Curl of  $\mathbf{F}$**

# 2-D Curl (1)

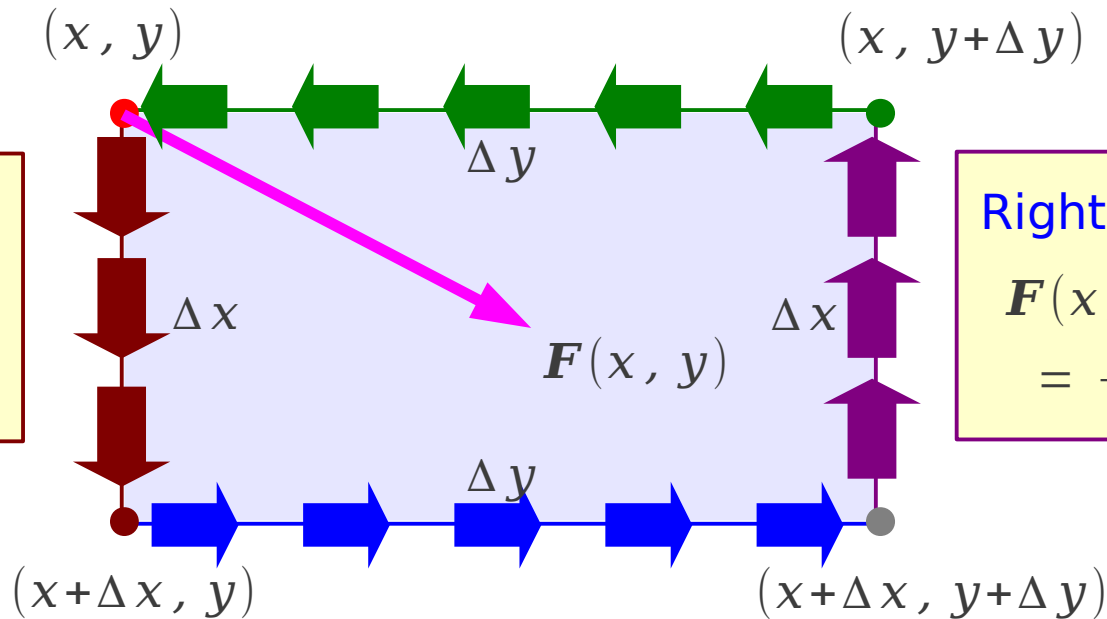
Velocity Fields of fluid flows



$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

Top Velocity  $\mathbf{F}(x, y)(-\mathbf{j}) = -N(x, y)$

Left Velocity  $\mathbf{F}(x, y)(+\mathbf{i}) = +M(x, y)$



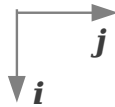
Right Velocity  $\mathbf{F}(x, y)(-\mathbf{i}) = -M(x, y)$

Bottom Velocity  $\mathbf{F}(x, y)(+\mathbf{j}) = +N(x, y)$

**Flow rate of counter clock wise circulating fluid**

# 2-D Curl (2)

Velocity Fields  
of fluid flows

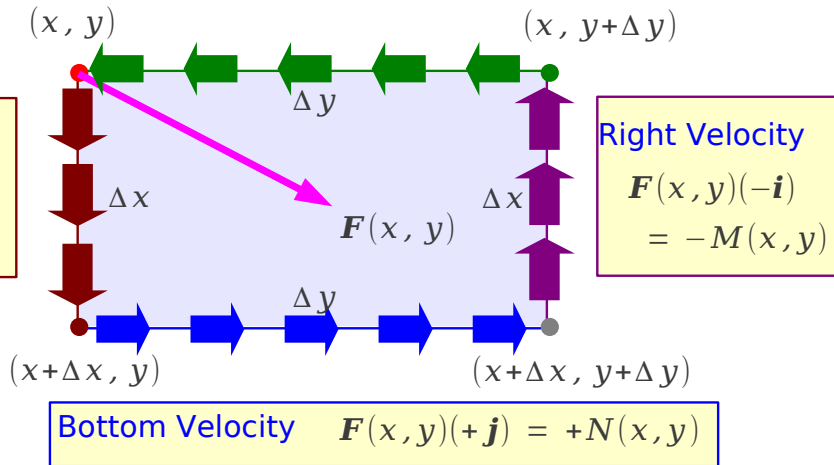


$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

$$\text{Top Velocity } \mathbf{F}(x, y)(-\mathbf{j}) = -N(x, y)$$

Left Velocity

$$\mathbf{F}(x, y)(+\mathbf{i}) = +M(x, y)$$



Right Velocity

$$\mathbf{F}(x, y)(-\mathbf{i}) = -M(x, y)$$

$$\text{Bottom Velocity } \mathbf{F}(x, y)(+\mathbf{j}) = +N(x, y)$$

**Flow rate of counter clock wise circulating fluid**

The flow rate of counter clock wise circulation

Across top  $\mathbf{F}(x, y) \cdot (-\mathbf{j})\Delta y = -N(x, y)\Delta y$

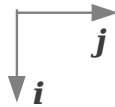
Across bottom  $\mathbf{F}(x+\Delta x, y) \cdot (+\mathbf{j})\Delta y = N(x+\Delta x, y)\Delta y$

Across left  $\mathbf{F}(x, y) \cdot (+\mathbf{i})\Delta x = M(x, y)\Delta x$

Across right  $\mathbf{F}(x, y+\Delta y) \cdot (-\mathbf{i})\Delta x = -M(x, y+\Delta y)\Delta x$

# 2-D Curl (3)

Velocity Fields of fluid flows

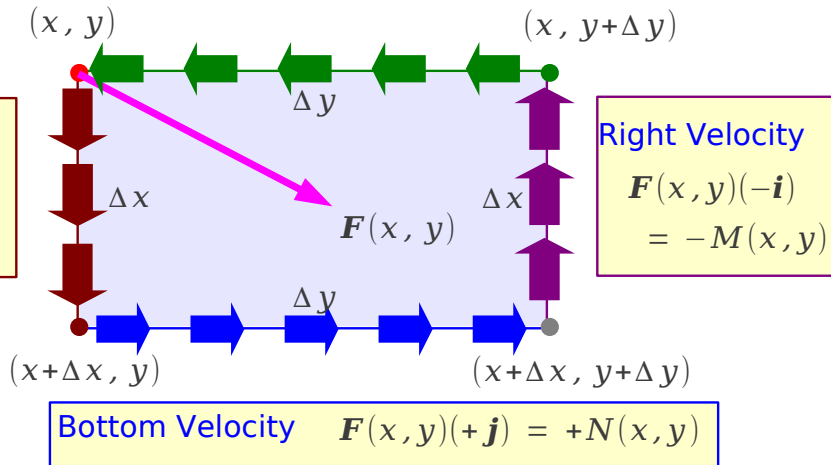


$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

$$\text{Top Velocity } \mathbf{F}(x, y)(-\mathbf{j}) = -N(x, y)$$

Left Velocity

$$\mathbf{F}(x, y)(+\mathbf{i}) = +M(x, y)$$



Right Velocity

$$\mathbf{F}(x, y)(-\mathbf{i}) = -M(x, y)$$

$$\text{Bottom Velocity } \mathbf{F}(x, y)(+\mathbf{j}) = +N(x, y)$$

$$\begin{aligned} \mathbf{F}(x, y) \cdot (-\mathbf{j})\Delta y &= -N(x, y)\Delta y \\ \mathbf{F}(x+\Delta x, y) \cdot (+\mathbf{j})\Delta y &= N(x+\Delta x, y)\Delta y \\ \mathbf{F}(x, y) \cdot (+\mathbf{i})\Delta x &= M(x, y)\Delta x \\ \mathbf{F}(x, y+\Delta y) \cdot (-\mathbf{i})\Delta x &= -M(x, y+\Delta y)\Delta x \end{aligned}$$

Flow rate of counter clock wise circulating fluid

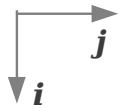
The flow rate of counter clock wise circulation

$$\text{Across top + bottom} \quad \{N(x+\Delta x, y) - N(x, y)\}\Delta y = \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y$$

$$\text{Across left + right} \quad -\{M(x, y+\Delta y) - M(x, y)\}\Delta x = -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

# 2-D Curl (4)

Velocity Fields of fluid flows

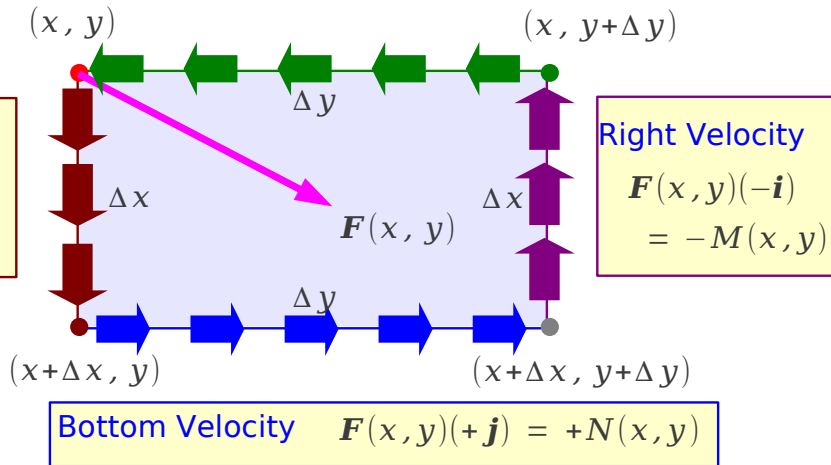


$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

$$\text{Top Velocity } \mathbf{F}(x, y)(-\mathbf{j}) = -N(x, y)$$

Left Velocity

$$\mathbf{F}(x, y)(+\mathbf{i}) = +M(x, y)$$



Right Velocity

$$\mathbf{F}(x, y)(-\mathbf{i}) = -M(x, y)$$

$$\text{Bottom Velocity } \mathbf{F}(x, y)(+\mathbf{j}) = +N(x, y)$$

$$\begin{aligned} & \{N(x+\Delta x, y) - N(x, y)\} \Delta y \\ &= \left( \frac{\partial N}{\partial x} \Delta x \right) \Delta y \\ & - \{M(x, y+\Delta y) - M(x, y)\} \Delta x \\ &= - \left( \frac{\partial M}{\partial y} \Delta y \right) \Delta x \end{aligned}$$

Flow rate of counter clock wise circulating fluid

Circulation around rectangle boundary

$$\approx \left( \frac{\partial N}{\partial x} \Delta x \right) \Delta y - \left( \frac{\partial M}{\partial y} \Delta y \right) \Delta x = \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \Delta x \Delta y$$

Circulation density

$$= \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

k-component  
Curl of  $\mathbf{F}$

Circulation Density



# 2-D Curl (a - d)

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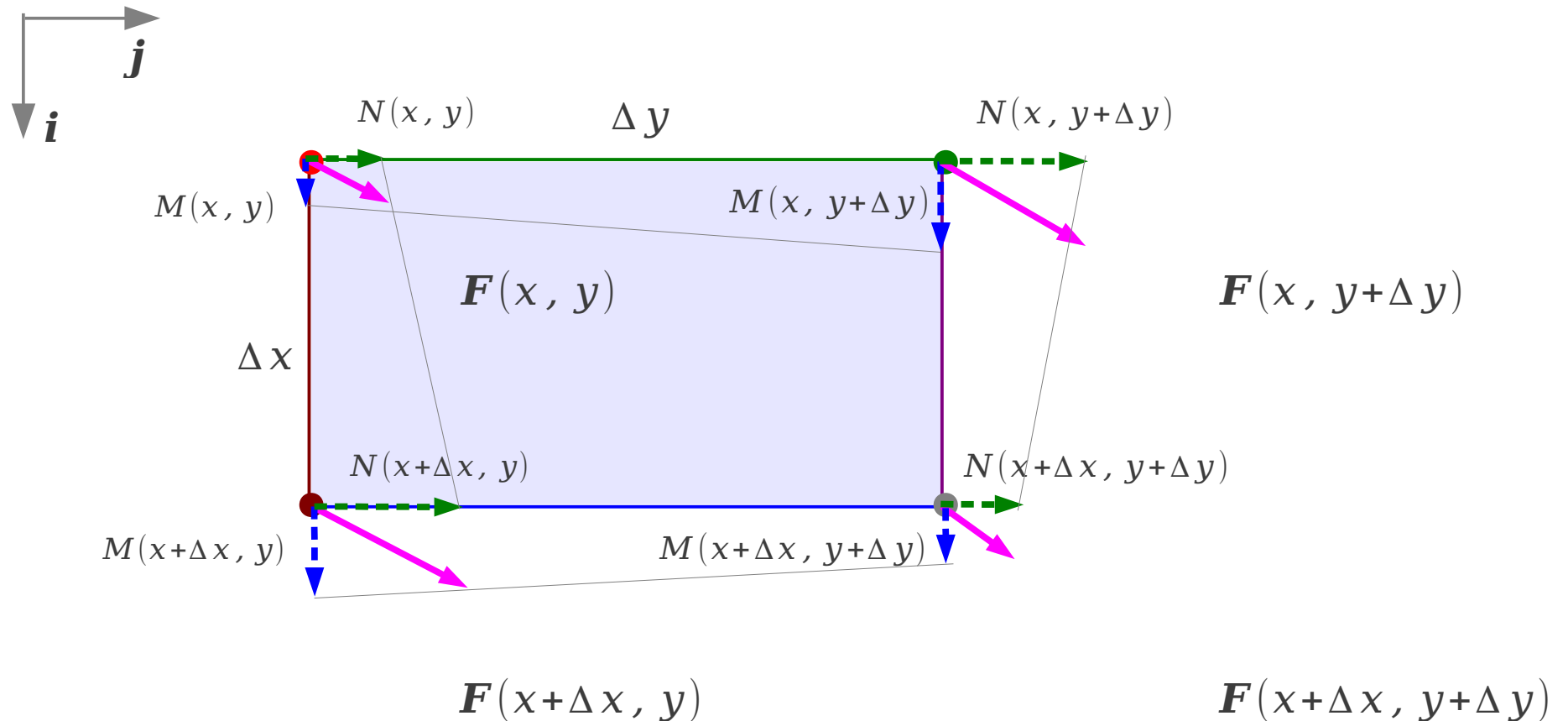
2-D Curl and Del Operator

3-D Curl and Del Operator

$$\nabla \times \mathbf{F}$$

# 2-D Curl (a)

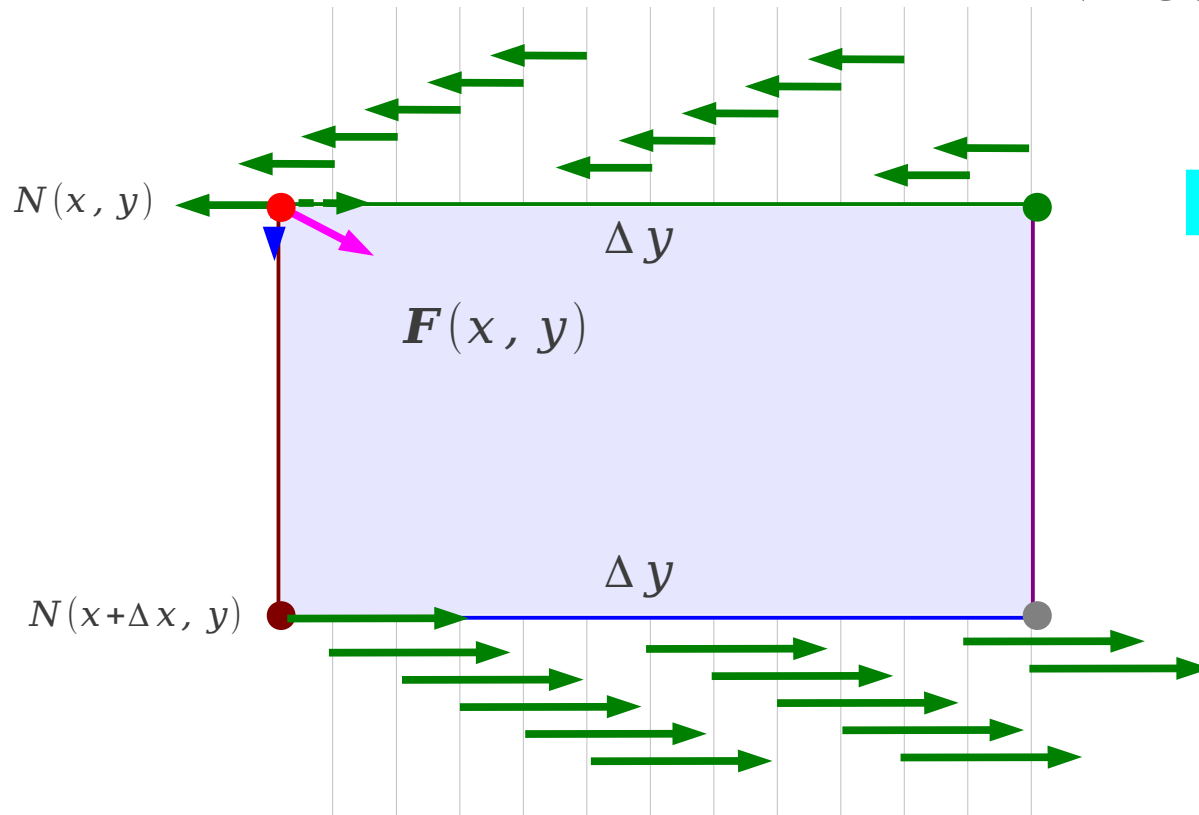
$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$



# 2-D Curl (top, bottom) (b)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

$$\mathbf{F}(x, y) \cdot (-\mathbf{j})\Delta y = -N(x, y)\Delta y$$

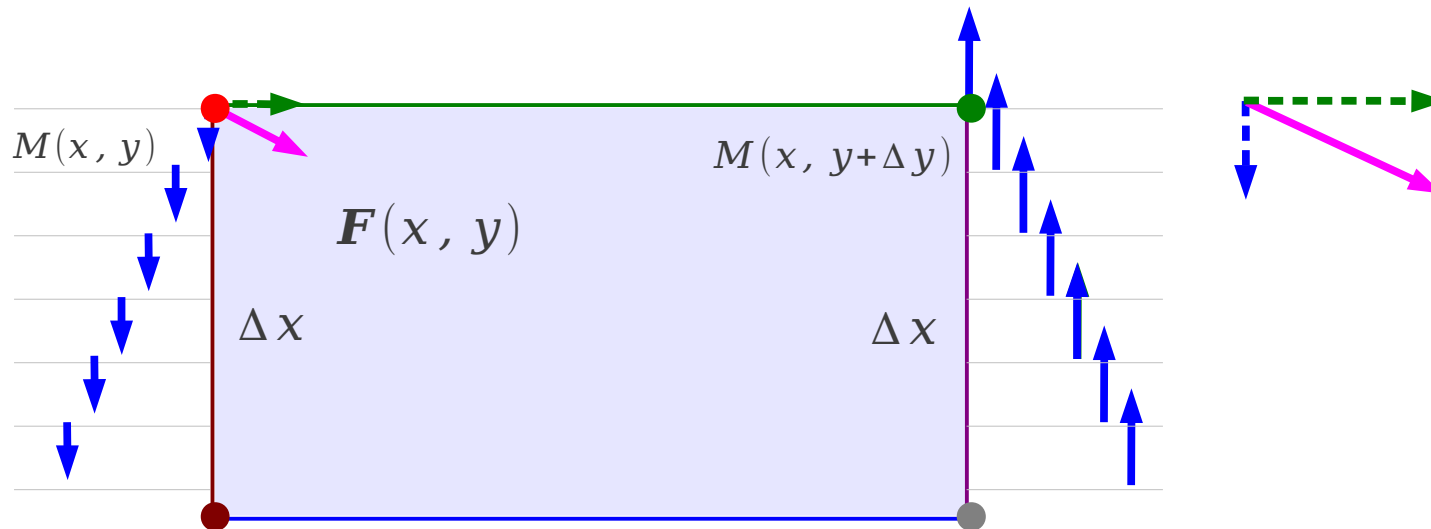


$$\mathbf{F}(x + \Delta x, y) \cdot (+\mathbf{j})\Delta y = N(x + \Delta x, y)\Delta y$$

## 2-D Curl (left, right) (c)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

$$\mathbf{F}(x, y) \cdot (+\mathbf{i})\Delta x = M(x, y)\Delta x \quad \mathbf{F}(x, y+\Delta y) \cdot (-\mathbf{i})\Delta x = -M(x, y+\Delta y)\Delta x$$



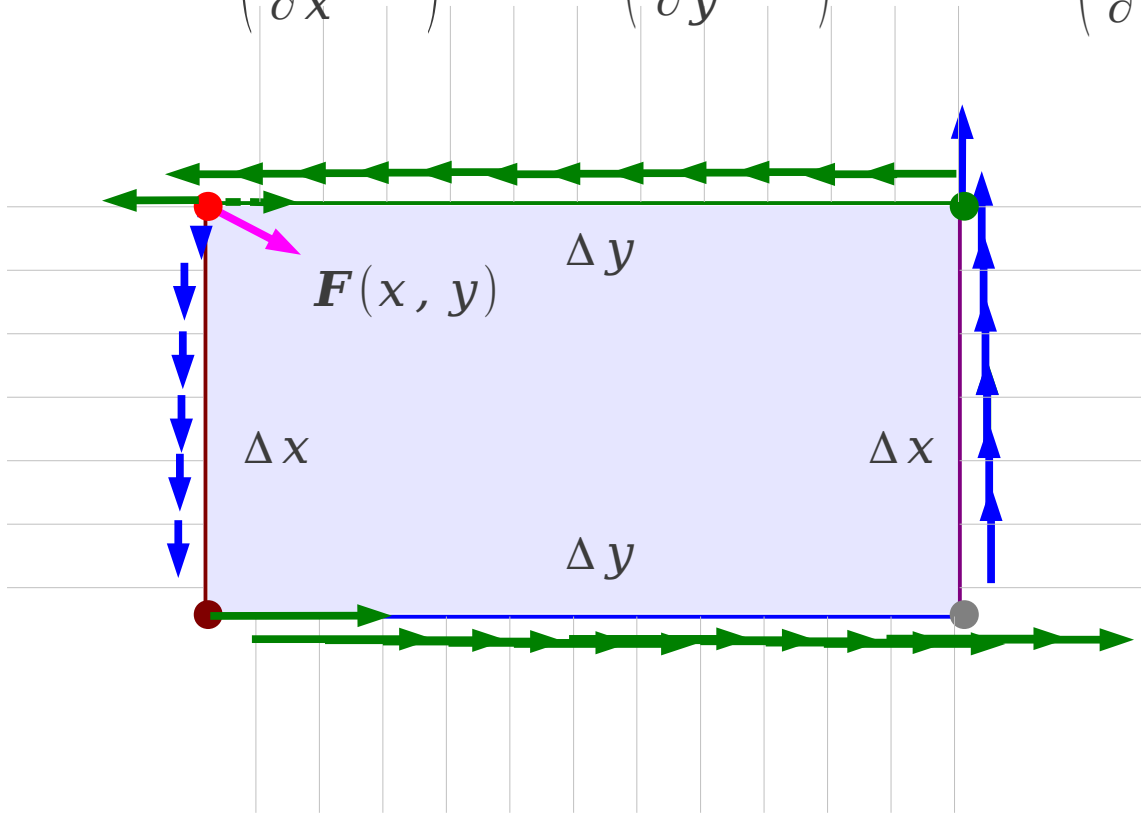
$$\frac{-\{M(x, y+\Delta y) - M(x, y)\}}{\Delta y} = -\left(\frac{\partial M}{\partial y}\right)$$

$$-\{M(x, y+\Delta y) - M(x, y)\}\Delta x = -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

# 2-D Curl (d)

Circulation around rectangle boundary

$$\approx \left( \frac{\partial N}{\partial x} \Delta x \right) \Delta y - \left( \frac{\partial M}{\partial y} \Delta y \right) \Delta x = \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \Delta x \Delta y$$

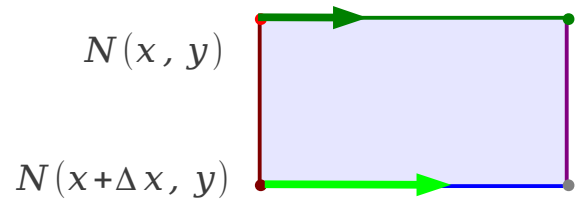


Circulation density =  $\left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$

k-component  
Curl of  $\mathbf{F}$

Circulation Density

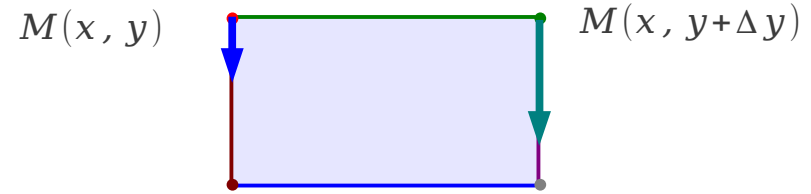
# Clock-Wise & Counter-Clock-Wise



$$\frac{\{N(x+\Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} > 0 \quad \text{CCW}$$

$$\frac{\{N(x+\Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} < 0 \quad \text{CW}$$

$$\left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + 0 \mathbf{k} \right) \times (N \mathbf{j}) = \nabla \times (N \mathbf{j})$$



$$\frac{\{M(x, y+\Delta y) - M(x, y)\}}{\Delta y} \approx \frac{\partial M}{\partial y} > 0$$

$$\frac{\{M(x, y+\Delta y) - M(x, y)\}}{\Delta y} \approx \frac{\partial M}{\partial y} < 0$$

$$\left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + 0 \mathbf{k} \right) \times (M \mathbf{i}) = \nabla \times (M \mathbf{i})$$

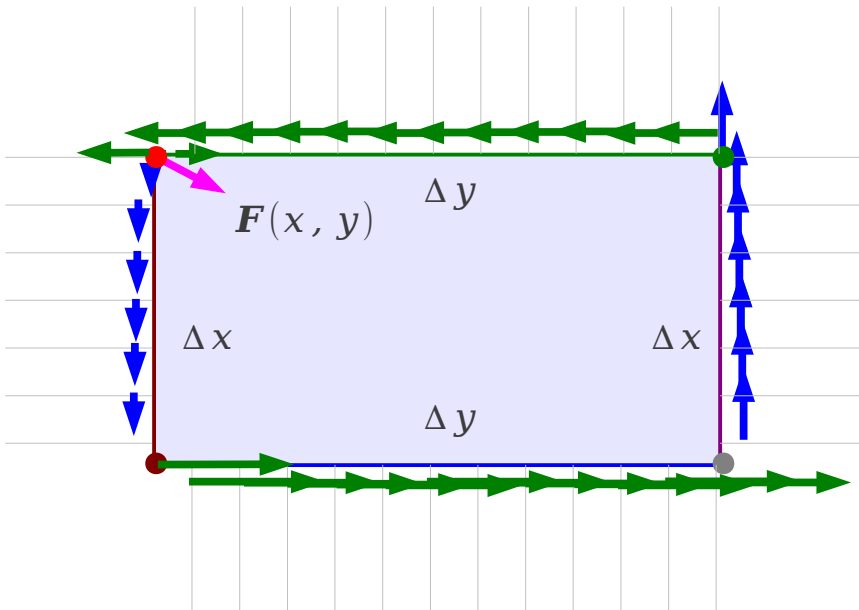
# 2-D Curl and Del Operator

## 2-D Vector Field

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

## Circulation around rectangle boundary

$$\approx \left( \frac{\partial N}{\partial x} \Delta x \right) \Delta y - \left( \frac{\partial M}{\partial y} \Delta y \right) \Delta x = \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \Delta x \Delta y$$



$$\begin{aligned} \text{Circulation density} &= \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + 0\mathbf{k} \right) \times (M\mathbf{i} + N\mathbf{j} + 0\mathbf{k}) \end{aligned}$$

$$\text{Curl of } \mathbf{F} = \nabla \times \mathbf{F}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix}$$

# 3-D Curl and Del Operator

2-D Vector Field

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

$$\begin{aligned} \text{Curl of } \mathbf{F} &= \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} && \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix} \\ &= \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + 0 \mathbf{k} \right) \times (M\mathbf{i} + N\mathbf{j} + 0\mathbf{k}) = \nabla \times \mathbf{F} \end{aligned}$$

3-D Vector Field

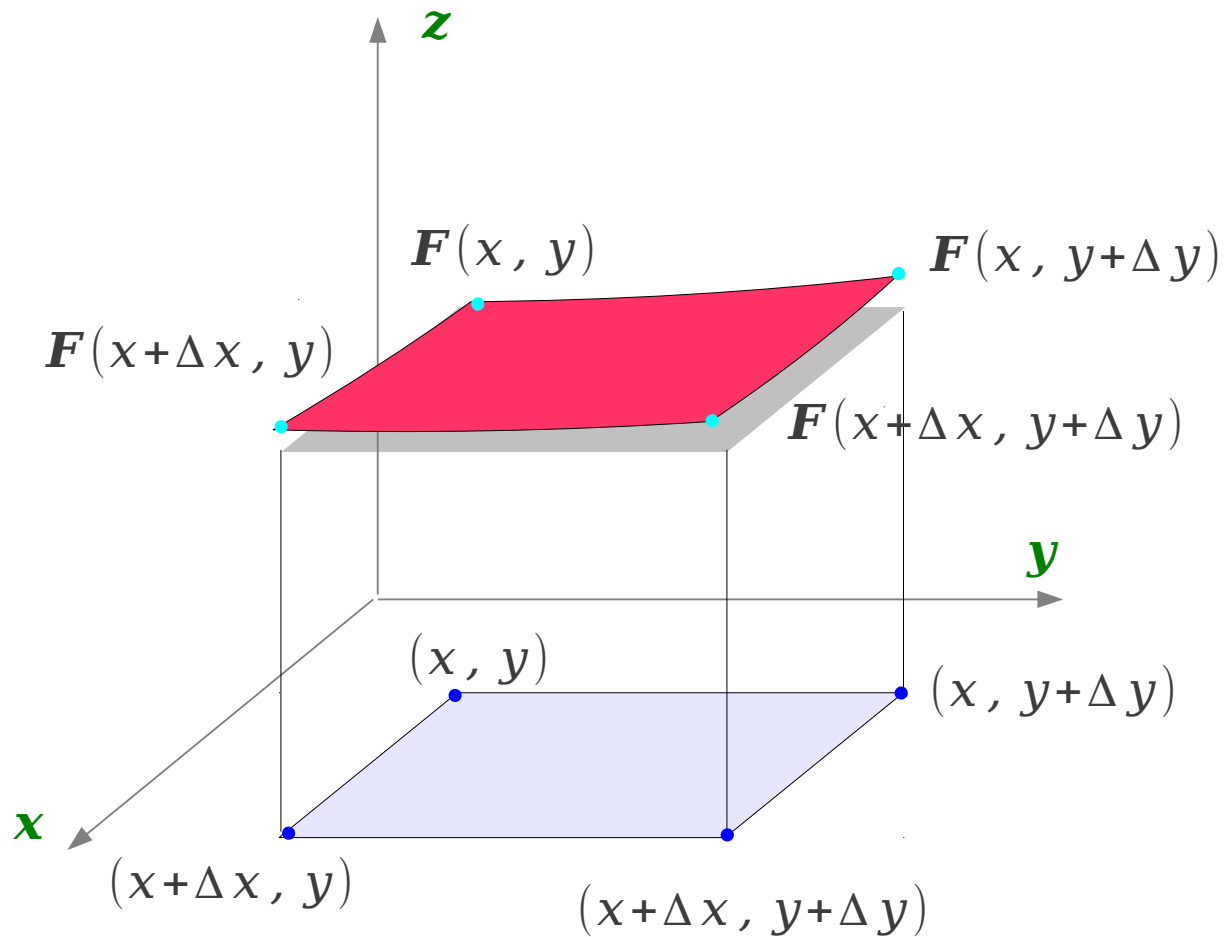
$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

$$\begin{aligned} \text{Curl of } \mathbf{F} &= \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} - \left( \frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j} + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} && \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\ &= \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \times (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) = \nabla \times \mathbf{F} \end{aligned}$$



*i*

# 2-D Divergence



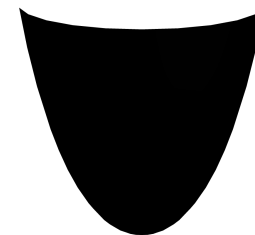
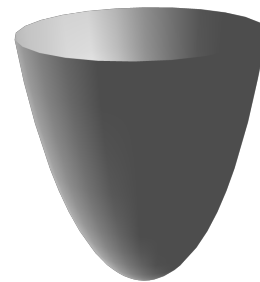
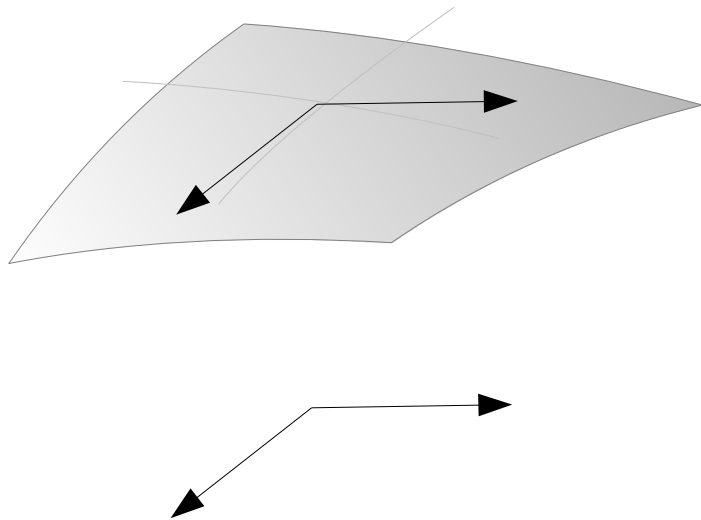
# Chain Rule

Function of two variable

$$y = f(u, v)$$

$$u = g(x, y)$$

$$v = h(x, y)$$



## References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [4] D.G. Zill, “Advanced Engineering Mathematics”