

# Fourier Integrals (3A)

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- Continuous Time Fourier Transform

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# CTFT of a Rect(t/T) function (1)

## Continuous Time Fourier Transform

Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

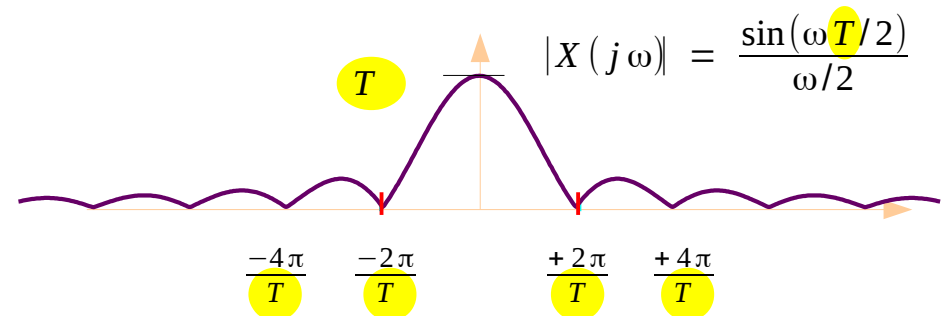
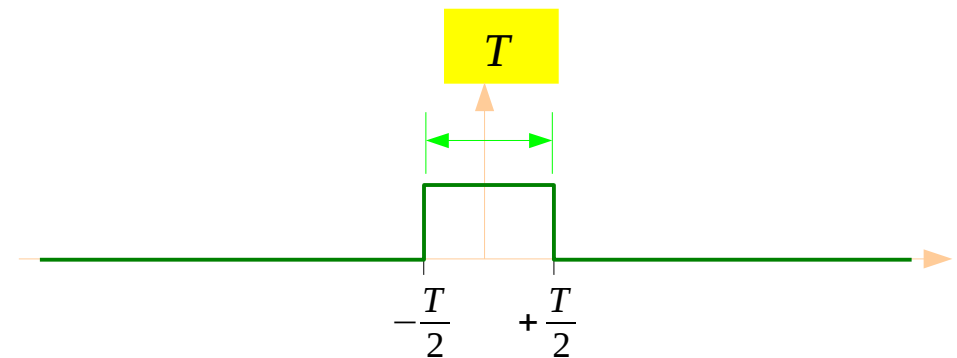
$$\begin{aligned} X(j\omega) &= \int_{-T/2}^{+T/2} e^{-j\omega t} dt \\ &= \left[ \frac{-1}{j\omega} e^{-j\omega t} \right]_{-T/2}^{+T/2} = -\frac{e^{-j\omega T/2} - e^{+j\omega T/2}}{j\omega} \\ &= \frac{\sin(\omega T/2)}{\omega/2} \end{aligned}$$

$$X(j0) = \lim_{\omega \rightarrow 0} \frac{\sin(\omega T/2)}{\omega/2} = \lim_{\omega \rightarrow 0} \frac{T \cos(\omega T/2)}{2 \cdot 1/2} = T$$

$$\sin(\omega T/2) = 0 \quad \rightarrow \quad \omega T/2 = \pi n$$

$$\rightarrow \quad \omega = \frac{2\pi}{T} n$$

$$\rightarrow \quad \omega = \pm \frac{2\pi}{T}, \pm \frac{4\pi}{T}, \pm \frac{6\pi}{T}, \dots$$



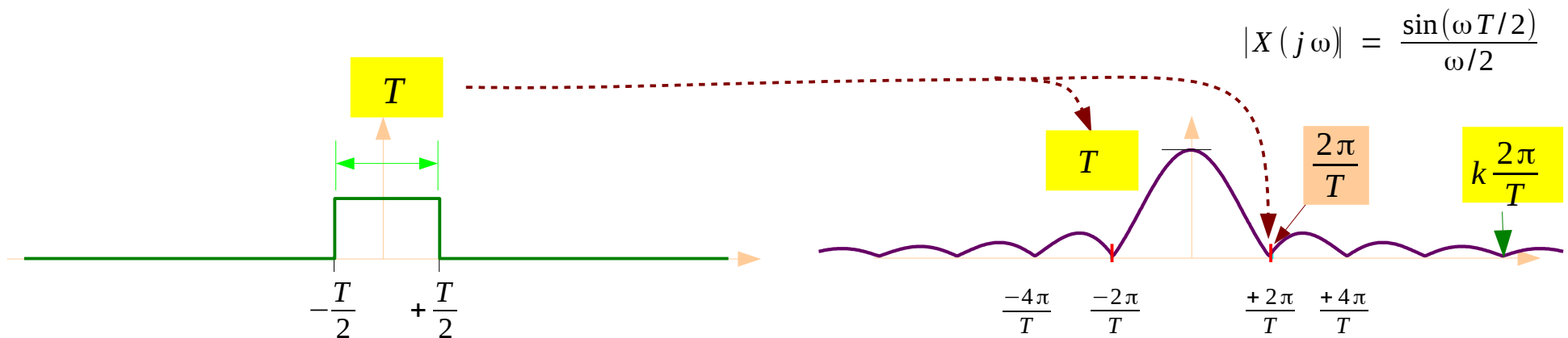
# CTFT of a Rect(t/T) function (2)

## Continuous Time Fourier Transform

Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

$$X(j\omega) = \int_{-T/2}^{+T/2} e^{-j\omega t} dt = \frac{\sin(\omega T/2)}{\omega/2}$$



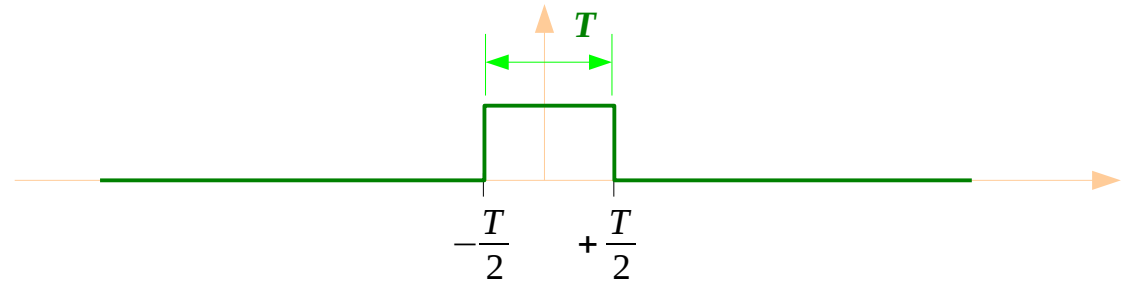
# CTFT and CTFS

## Continuous Time Fourier Transform

## Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

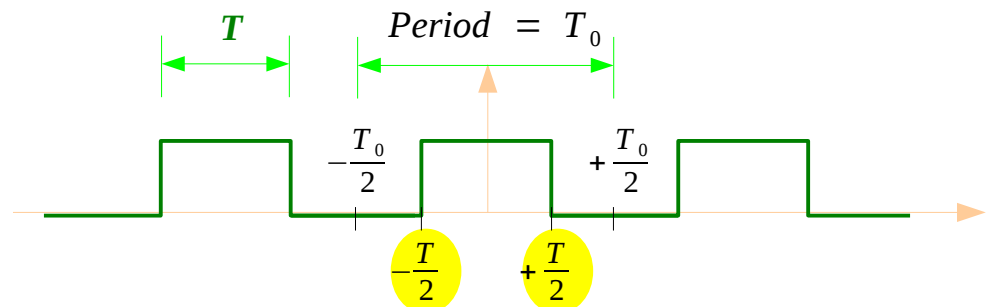


## Continuous Time Fourier Series

## Periodic Continuous Time Signal

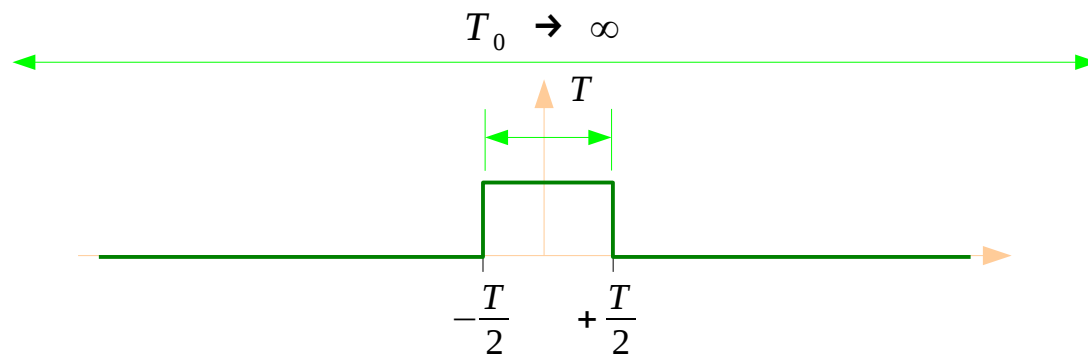
$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt \quad \longleftrightarrow \quad x(t) = \sum_{k=-\infty}^{\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0} \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

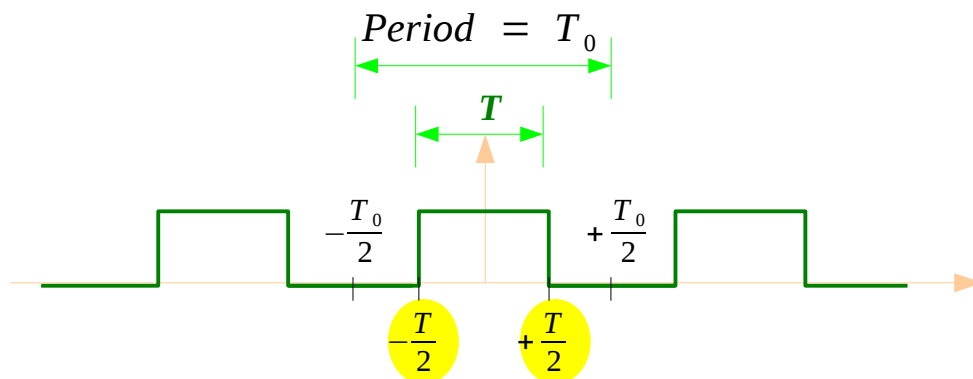


# CTFT ← CTFS

## Aperiodic Continuous Time Signal Continuous Time Fourier Transform



## Periodic Continuous Time Signal Continuous Time Fourier Series



$$x(t)$$

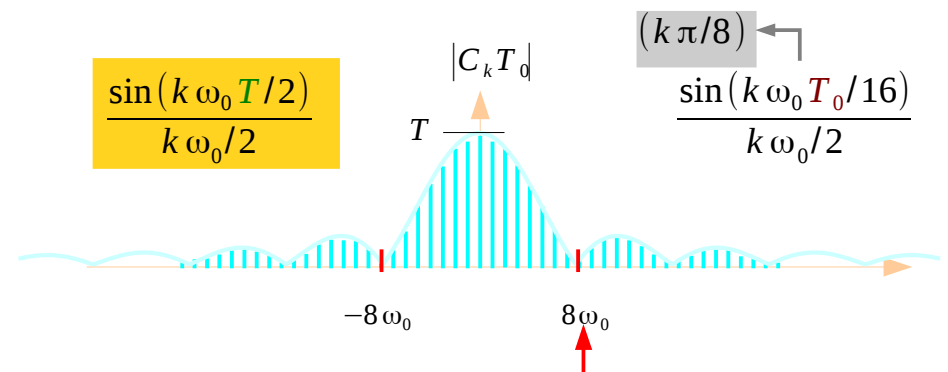
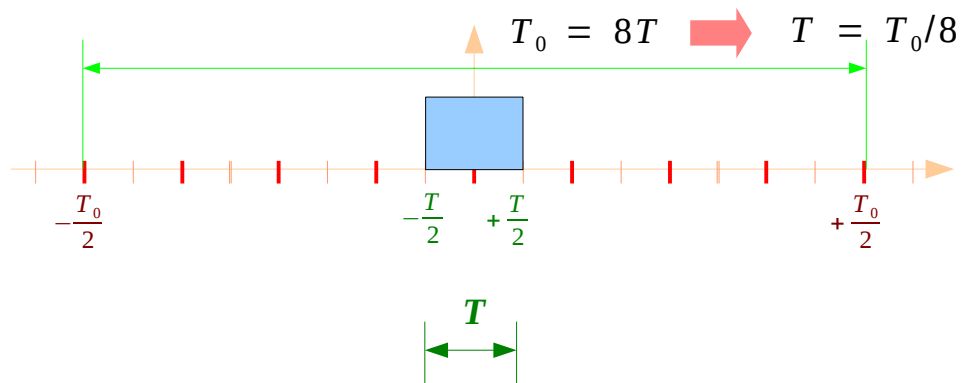
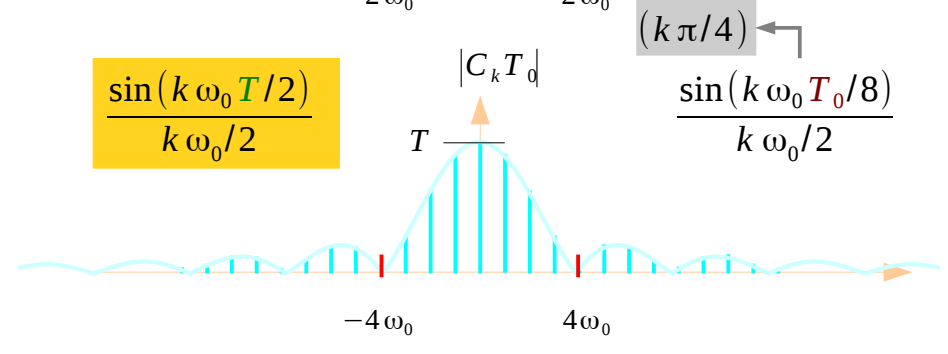
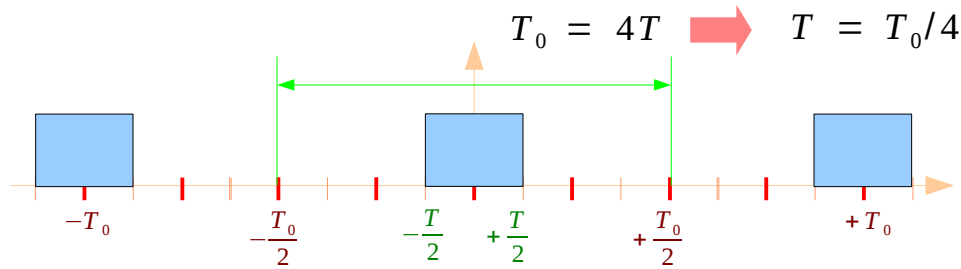
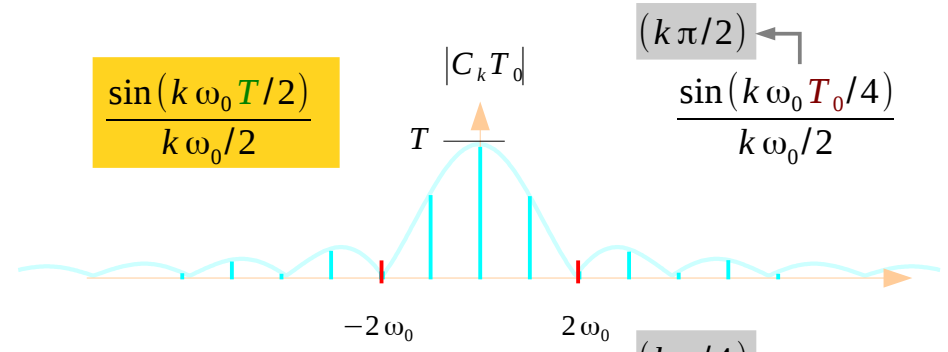
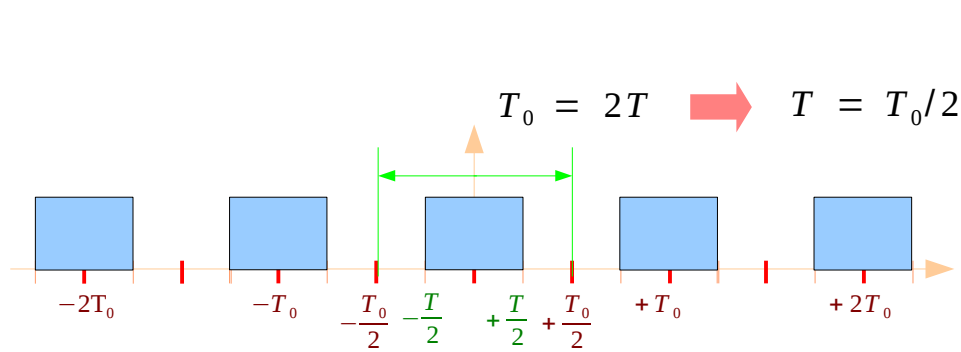
As  $T_0 \rightarrow \infty$ ,

$$x_{T_0}(t) \rightarrow x(t)$$

$$\omega_0 = \frac{2\pi}{T_0} \rightarrow 0$$

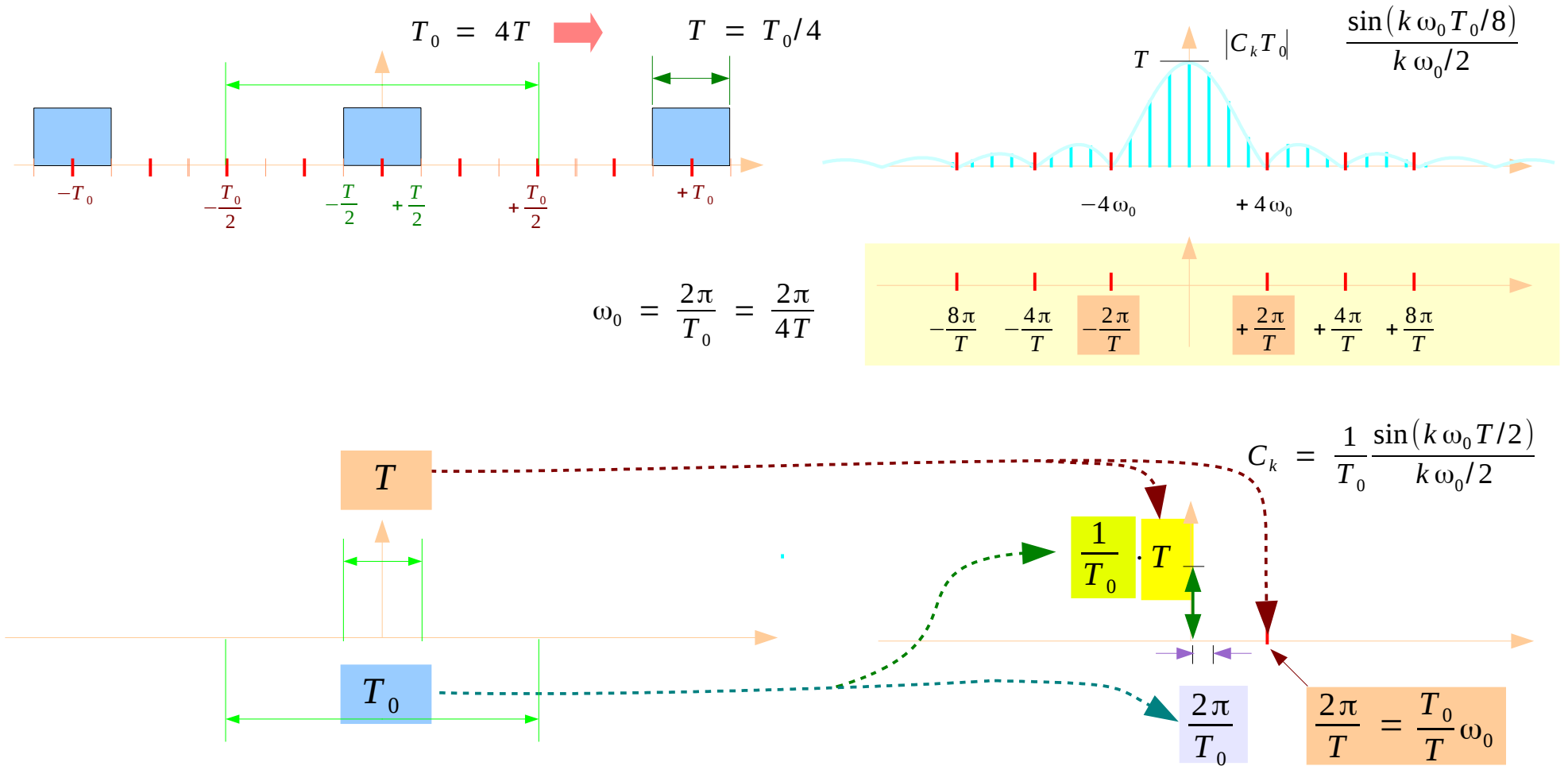
$$x_{T_0}(t) = \sum_{n=-\infty}^{+\infty} x(t - nT_0)$$

# CTFT and CTFS as $T_0 \rightarrow \infty$ (1)



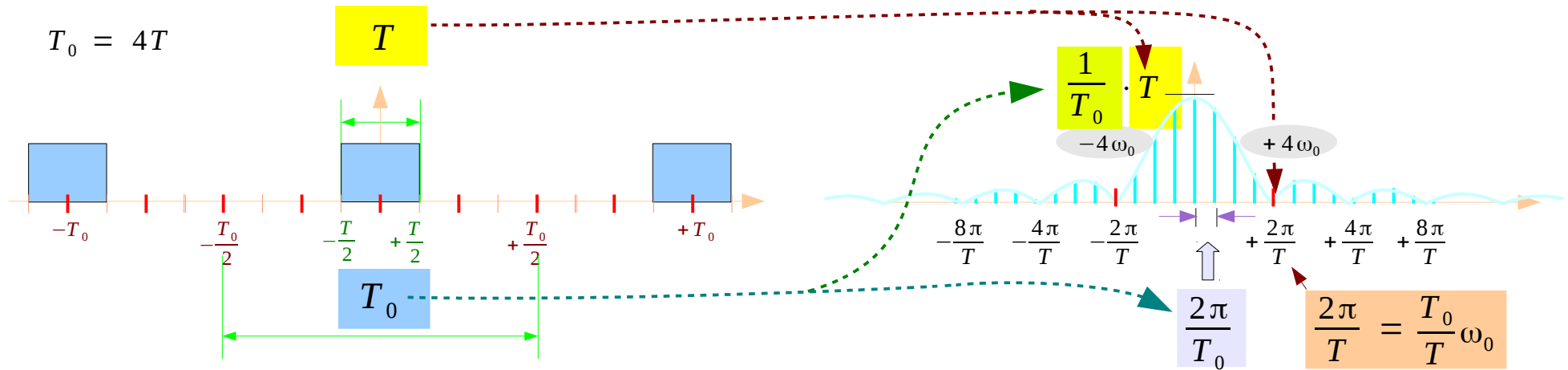
$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{8T} \rightarrow \frac{T_0}{T} \omega_0 = \frac{2\pi}{T}$$

# CTFT and CTFS as $T_0 \rightarrow \infty$ (2)





# CTFT of a Rect(t/T) function (3)



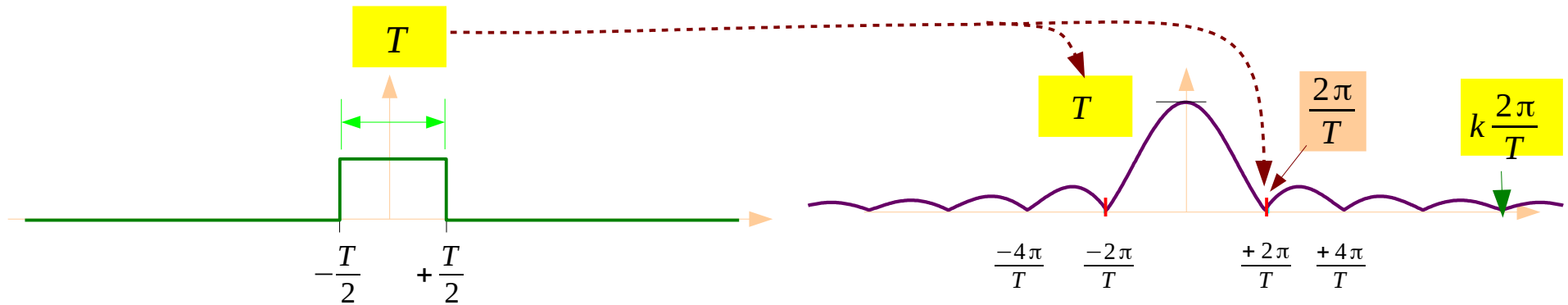
$$C_k T_0 = \frac{\sin(k \omega_0 T/2)}{k \omega_0/2}$$

$$X(j\omega) = \lim_{k \omega_0 \rightarrow \omega} \frac{\sin(k \omega_0 T/2)}{k \omega_0/2} = \frac{\sin(\omega T/2)}{\omega/2}$$

$$C_k = \frac{1}{T_0} \frac{\sin(k \omega_0 T/2)}{k \omega_0/2}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$



# From CTFS to CTFT

## Continuous Time Fourier Series

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt \quad \longleftrightarrow \quad x(t) = \sum_{k=-\infty}^{\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(t) e^{-jk\omega_0 t} dt$$

$$x_{T_0}(t) = \sum_{n=0}^{\infty} C_k e^{+jk\omega_0 t} \cdot \frac{2\pi}{2\pi} \cdot \frac{T_0}{T_0}$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t) e^{-jk\omega_0 t} dt$$

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=0}^{\infty} C_k T_0 e^{+jk\omega_0 t} \cdot \frac{2\pi}{T_0}$$

$$T_0 \rightarrow \infty \quad \Rightarrow \quad C_k T_0 \rightarrow X(j\omega) \quad x_{T_0} \rightarrow x(t) \quad \omega_0 = \frac{2\pi}{T_0} \rightarrow d\omega$$

## Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

# Other Convention

## Continuous Time Fourier Transform {unitary, angular frequency}

$$X(j\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow$$

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

## Continuous Time Fourier Transform {non-unitary, angular frequency}

$$X(j\omega) = 1 \cdot \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

## References

[1] <http://en.wikipedia.org/>

[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003