# Fourier Integrals (3A)

- Continuous Time Fourier Transform

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

# CTFT of a Rect(t/T) function (1)

**Continuous Time Fourier Transform** Aperiodic Continuous Time Signal

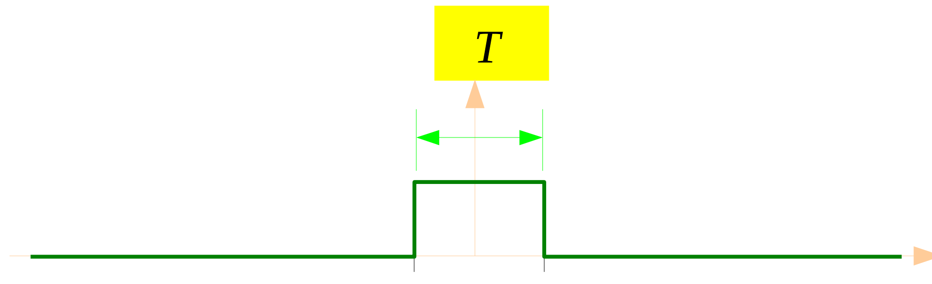
$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-T/2}^{+T/2} e^{-j\omega t}\, dt$$

$$= \left[\frac{-1}{j\omega} e^{-j\omega t}\right]_{-T/2}^{+T/2} = -\frac{e^{-j\omega T/2} - e^{+j\omega T/2}}{j\omega}$$

$$= \frac{\sin(\omega T/2)}{\omega/2}$$

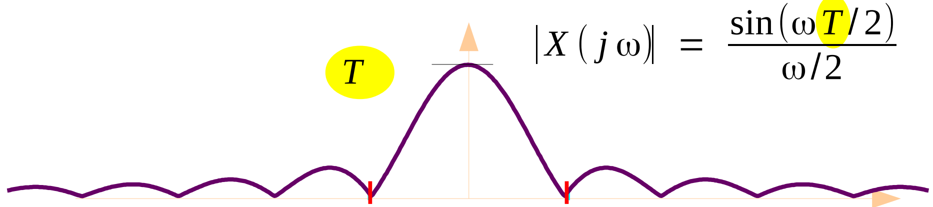
$$X(j\,0) = \lim_{\omega \to 0} \frac{\sin(\omega T/2)}{\omega/2} = \lim_{\omega \to 0} \frac{T}{2}\frac{\cos(\omega T/2)}{1/2} = T$$




$\sin(\omega T/2) = 0 \quad \Rightarrow \quad \omega T/2 = \pi n$

$\Rightarrow \quad \omega = \frac{2\pi}{T} n$

$\Rightarrow \quad \omega = \pm\frac{2\pi}{T}, \pm\frac{4\pi}{T}, \pm\frac{6\pi}{T}, \cdots$



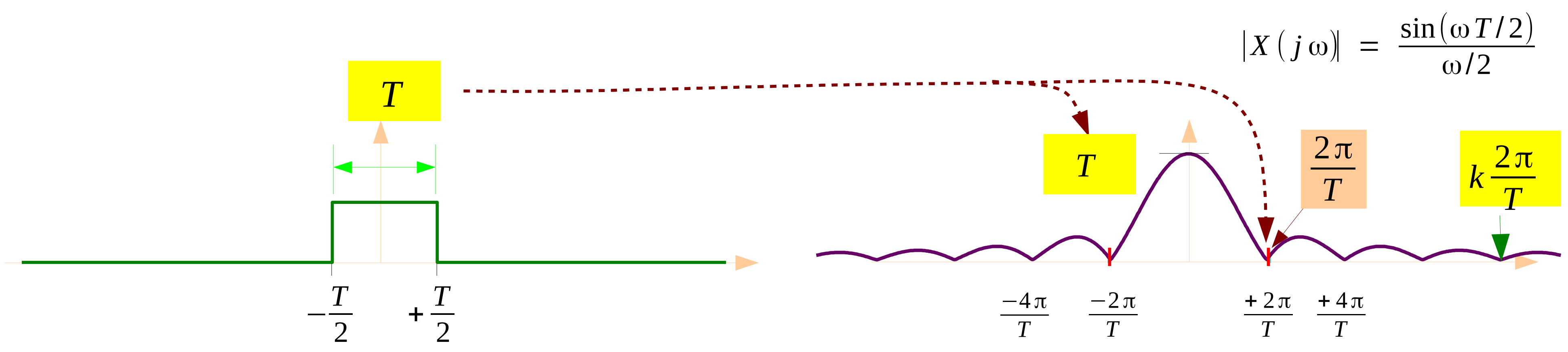
# CTFT of a Rect(t/T) function (2)

**Continuous Time Fourier Transform** **Transform**

Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-T/2}^{+T/2} e^{-j\omega t}\, dt = \frac{\sin(\omega T/2)}{\omega/2}$$

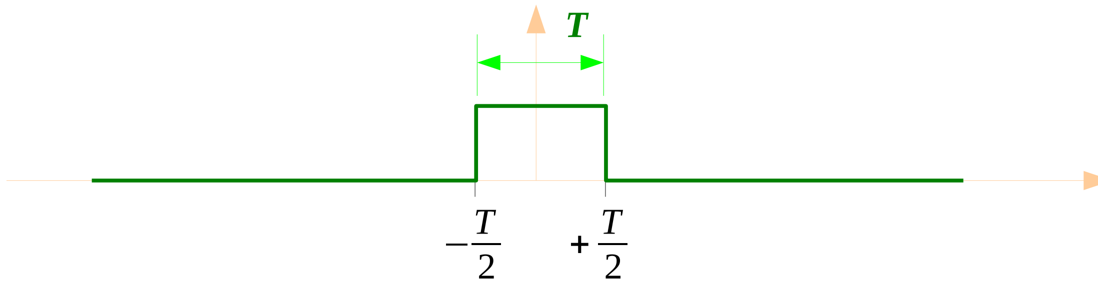


# CTFT and CTFS

**Continuous Time Fourier Transform** — Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

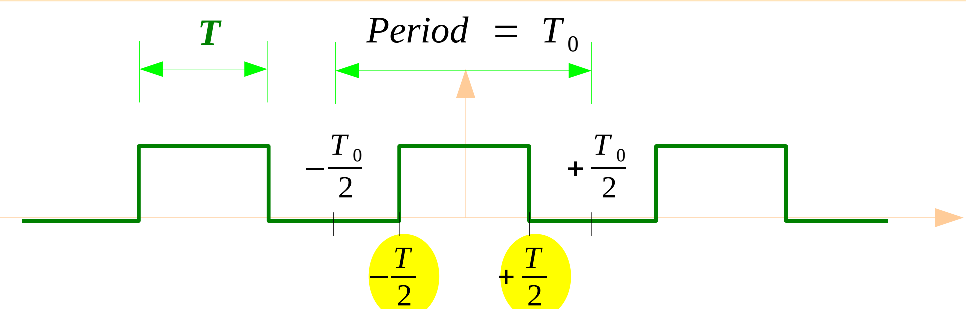
$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$




**Continuous Time Fourier Series** — Periodic Continuous Time Signal

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{+jk\omega_0 t}$$

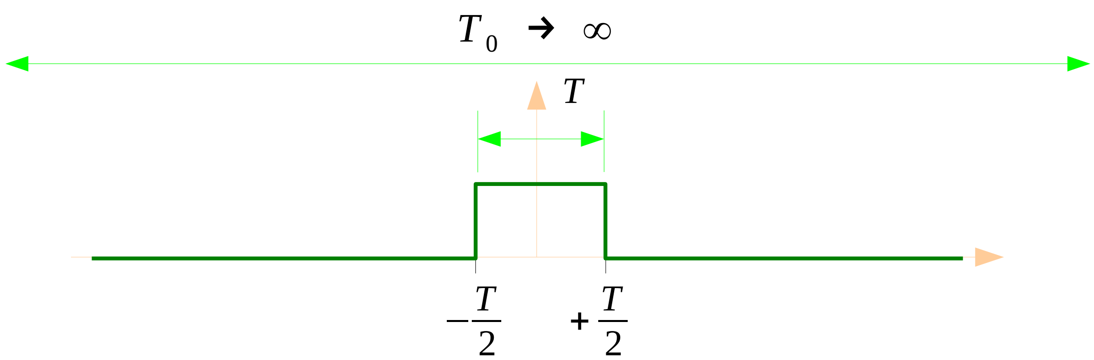
$$C_k = \frac{1}{T_0}\frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$



# CTFT ← CTFS

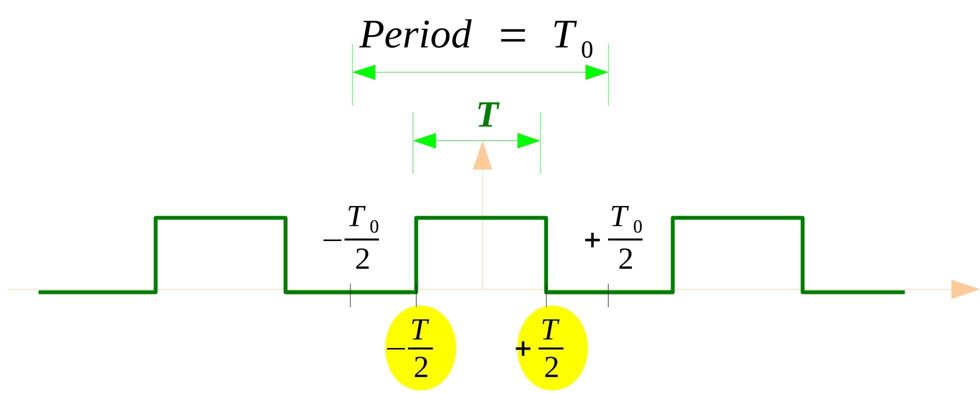
Aperiodic Continuous Time Signal

**Continuous Time Fourier Transform**




Periodic Continuous Time Signal

**Continuous Time Fourier Series**




$$x(t)$$

↑

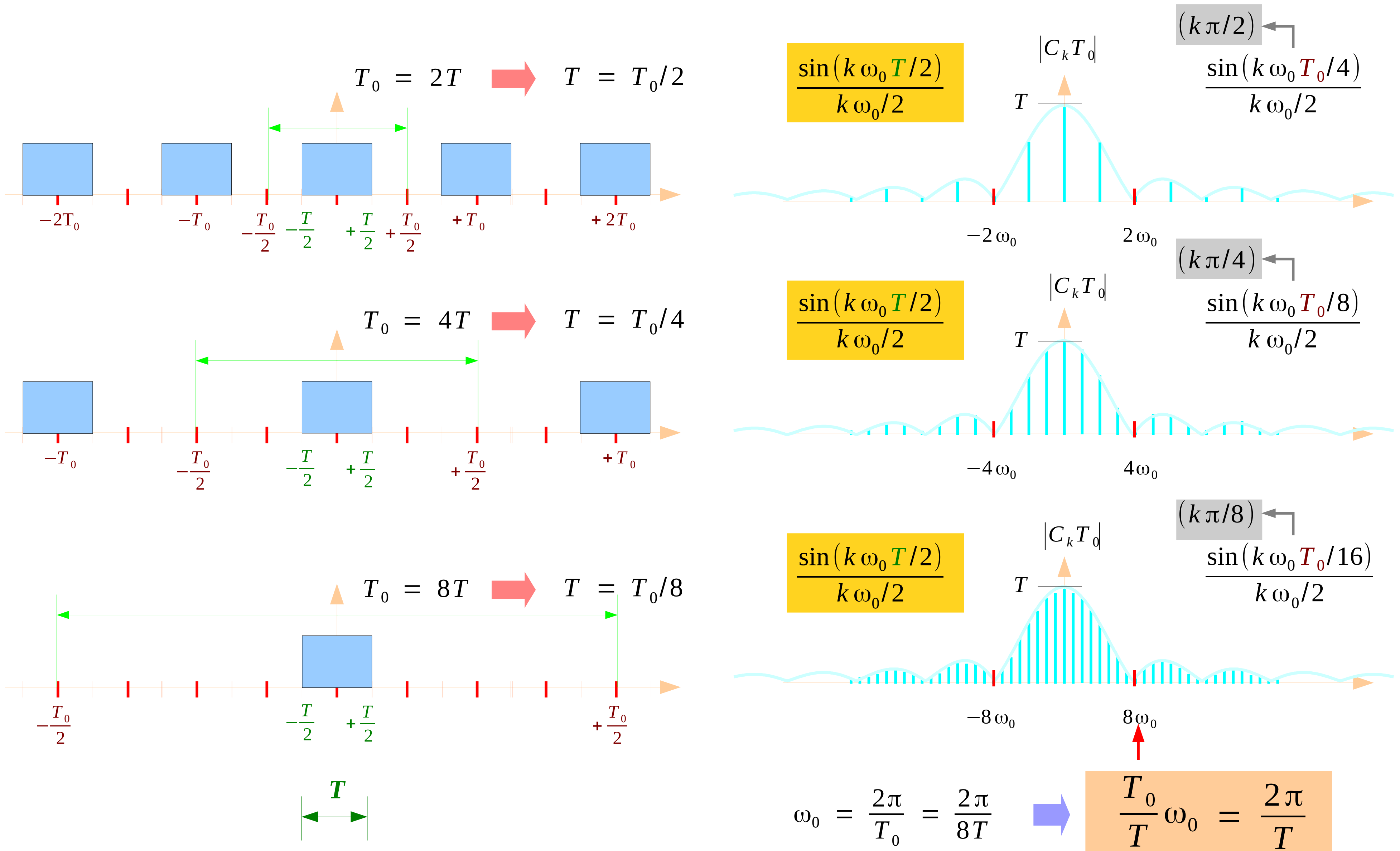
As $T_0 \rightarrow \infty$,

$$x_{T_0}(t) \rightarrow x(t)$$

$$\omega_0 = \frac{2\pi}{T_0} \rightarrow 0$$

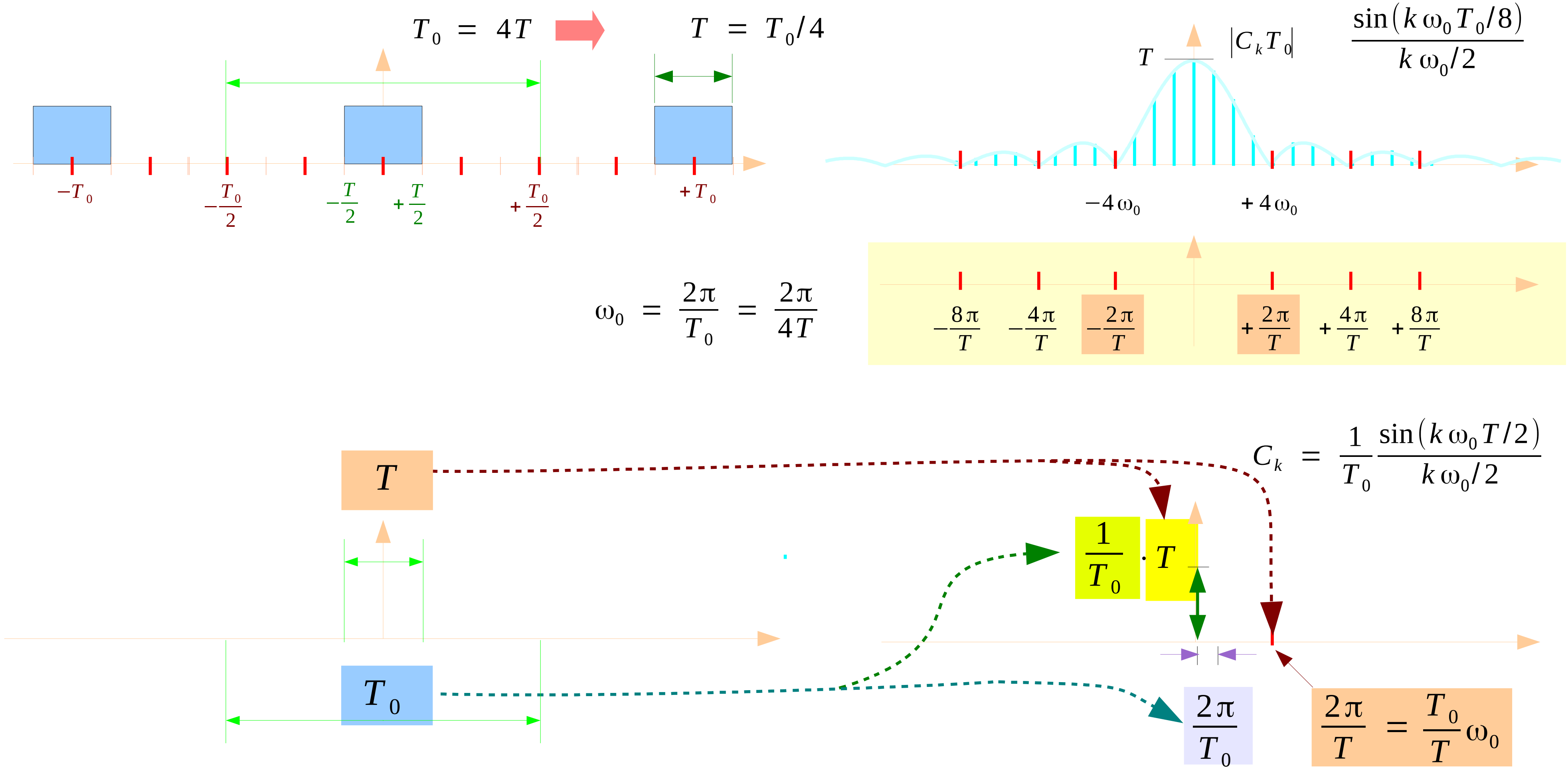
$$x_{T_0}(t) = \sum_{n=-\infty}^{+\infty} x(t - nT_0)$$

# CTFT and CTFS as $T_0 \to \infty$ (1)

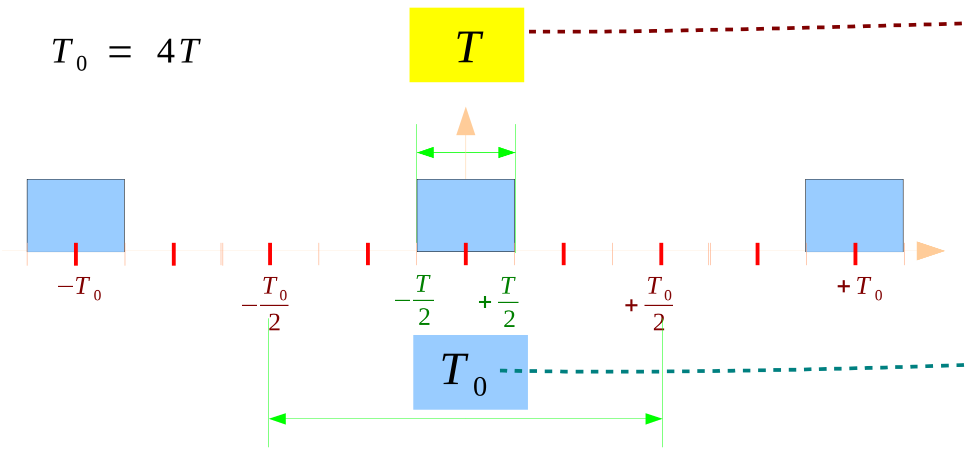


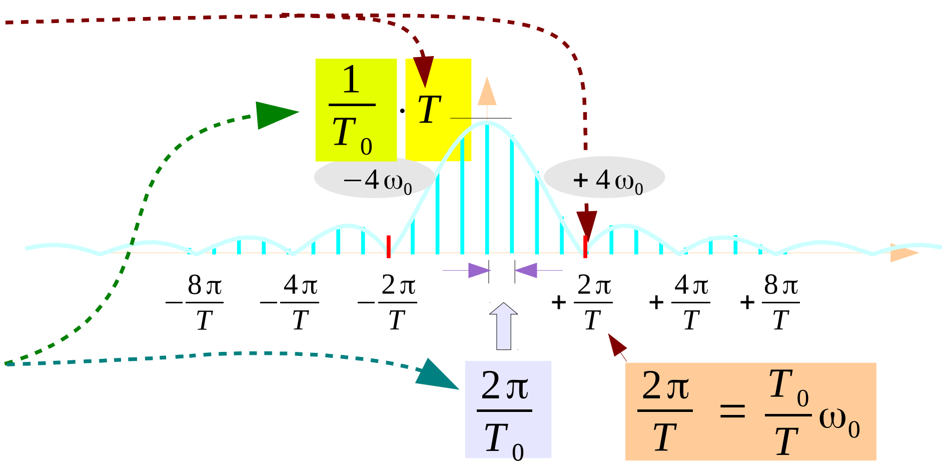

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{8T} \quad \Rightarrow \quad \frac{T_0}{T}\,\omega_0 = \frac{2\pi}{T}$$

# CTFT and CTFS as $T_0 \to \infty$ (2)



# CTFT of a Rect(t/T) function (3)



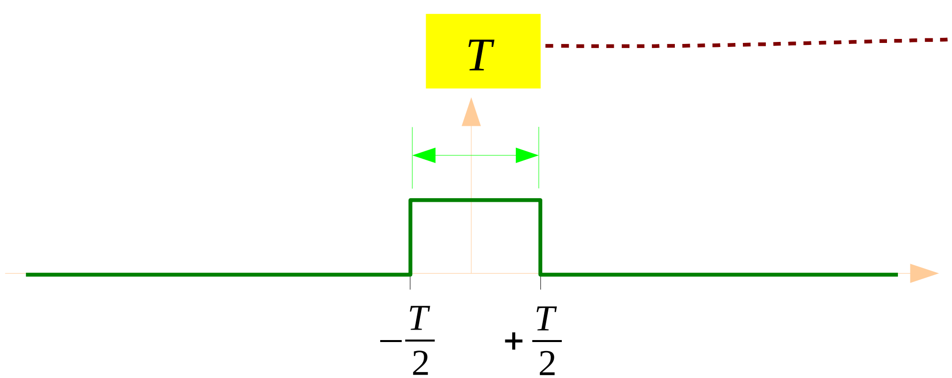

$$C_k T_0 = \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

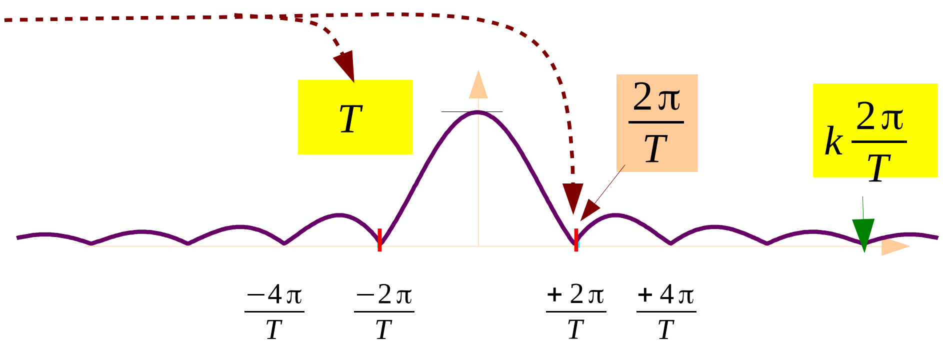
$$X(j\omega) = \lim_{k\omega_0 \to \omega} \frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{\sin(\omega T/2)}{\omega/2}$$

$$C_k = \frac{1}{T_0}\frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$



# From CTFS to CTFT

## Continuous Time Fourier **Series**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \qquad x_{T_0}(t) = \sum_{n=0}^{\infty} C_k\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{2\pi} \cdot \frac{T_0}{T_0}$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \qquad x_{T_0}(t) = \frac{1}{2\pi}\sum_{k=0}^{\infty} C_k T_0\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{T_0}$$

$$T_0 \to \infty \quad \Rightarrow \quad C_k T_0 \to X(j\omega) \qquad x_{T_0} \to x(t) \qquad \omega_0 = \frac{2\pi}{T_0} \to d\omega$$

## Continuous Time Fourier **Transform**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

# Other Convention

**Continuous Time Fourier Transform {unitary, angular frequency}**

$$X(j\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \Longleftrightarrow$$

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

**Continuous Time Fourier Transform {non-unitary, angular frequency}**

$$X(j\omega) = 1 \cdot \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \Longleftrightarrow$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

## References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003