

Complex Series (3A)

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Analytic Functions

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \rightarrow 0} \frac{\Delta f}{\Delta z}$$


$$\Delta f = f(z + \Delta z) - f(z)$$

$$\Delta z = \Delta x + i\Delta y$$

$f(z)$: **analytic** in a region 

$f(z)$ has a (**unique**) **derivative** at every point of the region

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \rightarrow 0} \frac{\Delta f}{\Delta z}$$

$f(z)$: **analytic** at a point $z = a$ 

$f(z)$ has a (**unique**) **derivative** at every point of some small circle about $z = a$

Singular Point

Regular point of $f(z)$ \Leftrightarrow
a point at which $f(z)$ is **analytic**

Singular point of $f(z)$ \Leftrightarrow
a point at which $f(z)$ is not **analytic**

Isolated Singular point of $f(z)$ \Leftrightarrow
a point at which $f(z)$ is **analytic** everywhere
else inside some small circle about the singular point

Cauchy-Riemann Condition

$$f(z) = u(x, y) + iv(x, y) \quad : \text{analytic in a region}$$



in that region

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

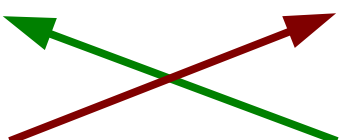
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$f(z) = u(x, y) + iv(x, y)$$


$$\frac{\partial}{\partial x}$$


$$\frac{\partial}{\partial y}$$

$$f(z) = u(x, y) + iv(x, y)$$



$$\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y}$$

Analytic

$$f(z) = u(x, y) + i v(x, y)$$

$$u(x, y), v(x, y), \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \quad : \text{continuous}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

 $f(z) = u(x, y) + i v(x, y)$

: **analytic** at all points **inside** a region
not necessarily on the boundary

Derivatives

$f(z) = u(x, y) + iv(x, y)$: **analytic** in a region R



derivatives of all orders at points inside region

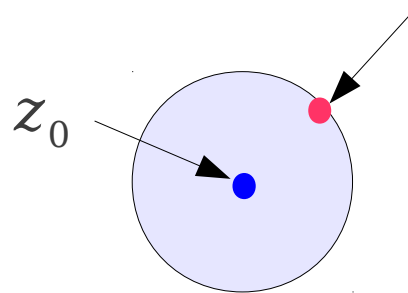
$f'(z_0), f''(z_0), f^{(3)}(z_0), f^{(4)}(z_0), f^{(5)}(z_0), \dots$



Taylor series expansion about any point z_0 inside the region

The power series converges inside the circle about z_0

This circle extends to the nearest **singular point**



Laplace Equation

$f(z) = u(x, y) + i v(x, y)$: **analytic** in a region R

➡ $u(x, y), v(x, y)$ satisfy Laplace's equation in the region
harmonic functions

$u(x, y), v(x, y)$ satisfy Laplace's equation in simply connected region

➡ Real / imaginary part of an analytic function $f(z)$

Cauchy's Theorem

$$f(z) : \text{analytic on and inside } C \quad \Rightarrow \quad \oint_{\text{around } C} f(z) dz = 0$$

simple closed curve

a continuously turning tangent

except possibly at a finite number of points

allow a finite number of corners (not smooth)

Cauchy's Integral Formula

$f(z)$: **analytic** on and inside simple close curve C

➔
$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$

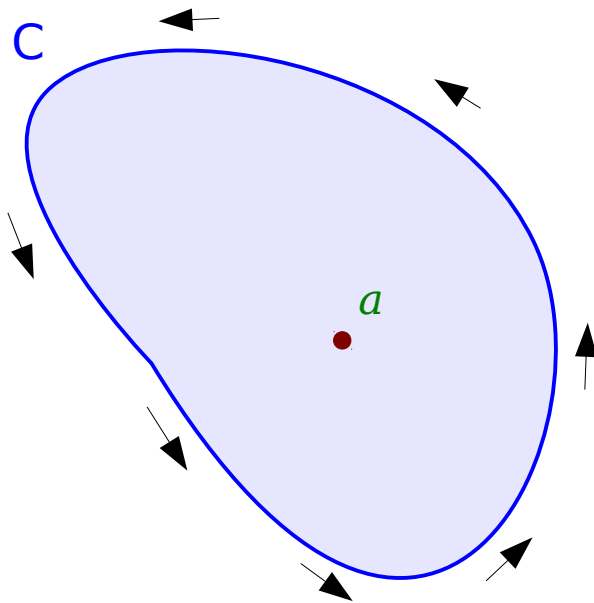
the value of $f(z)$
at a point $z = a$ inside C

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw$$

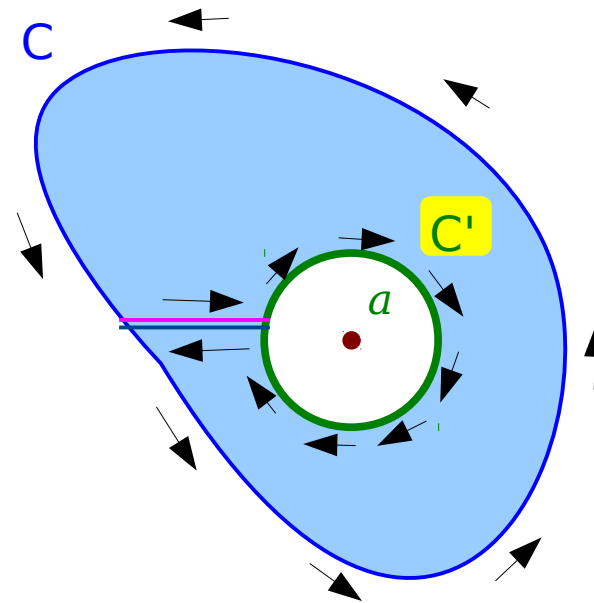
Cauchy's Integral Formula

$f(z)$: **analytic** on and inside simple close curve C

➔
$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$



$$\oint_C f(z) dz = 0$$



$$\oint_{\text{ccw } C} \frac{f(z)}{z-a} dz + \oint_{\text{cw } C'} \frac{f(z)}{z-a} dz = 0$$

$$\begin{aligned} & \oint_{\text{ccw } C} \frac{f(z)}{z-a} dz \\ &= \oint_{\text{ccw } C'} \frac{f(z)}{z-a} dz \end{aligned}$$

Cauchy's Integral Formula

$f(z)$: **analytic** on and inside simple close curve C

➔ $f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$

along C' $z - a = \rho e^{i\theta}$

$z = a + \rho e^{i\theta}$

$dz = i\rho e^{i\theta} d\theta$

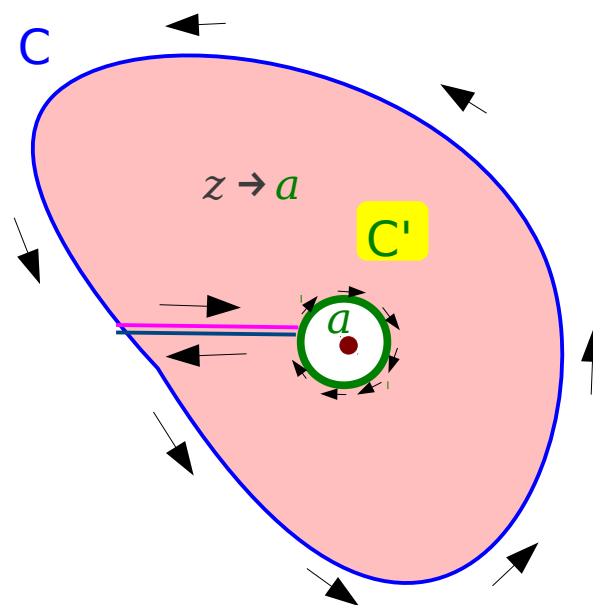
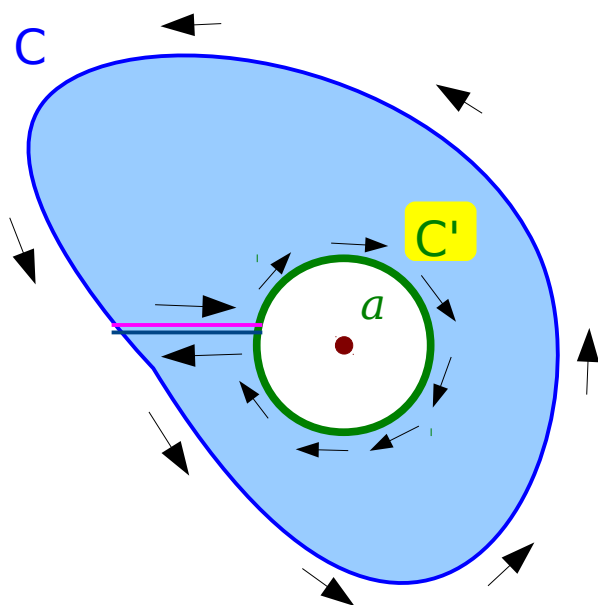
$\frac{dz}{z-a} = \frac{i\rho e^{i\theta} d\theta}{\rho e^{i\theta}}$

$\oint_{\text{ccw } C} \frac{f(z) dz}{z-a}$

$= \int_0^{2\pi} f(z) i d\theta$

$= 2\pi i f(a)$

as $z \rightarrow a$ ➔ $\rho \rightarrow 0$



$\oint_{\text{ccw } C} \frac{f(z) dz}{z-a} = \oint_{\text{ccw } C'} \frac{f(z) dz}{z-a}$

$= 2\pi i f(a)$

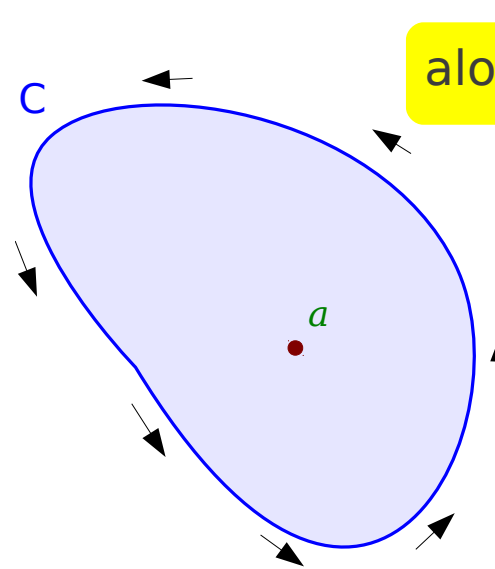
Cauchy's Integral Formula

$$\frac{dz}{(z-a)^2} = \frac{i\rho e^{i\theta} d\theta}{(\rho e^{i\theta})^2}$$

$$\oint_{\text{ccw } C} \frac{f(z) dz}{(z-a)^2} = \int_0^{2\pi} \frac{f(z)i}{\rho e^{i\theta}} d\theta$$

$$= \int_0^{2\pi} \frac{f(z)}{\rho} i e^{-i\theta} d\theta = \left[-\frac{f(z)}{\rho} e^{-i\theta} \right]_0^{2\pi}$$

$$= -\frac{f(z)}{\rho} (e^{-i2\pi} - e^{-i0}) = 0$$



along C' $z - a = \rho e^{i\theta}$

$$z = a - \rho e^{i\theta}$$

$$dz = i\rho e^{i\theta} d\theta$$

$$dz = i\rho e^{i\theta} d\theta$$

$$\oint_{\text{ccw } C} f(z) dz = \int_0^{2\pi} f(z) i\rho e^{i\theta} d\theta$$

$$= \left[f(z) \rho e^{i\theta} \right]_0^{2\pi}$$

$$= f(z) \rho (e^{-i2\pi} - e^{-i0}) = 0$$

$$(z-a) dz = \rho e^{i\theta} i\rho e^{i\theta} d\theta$$

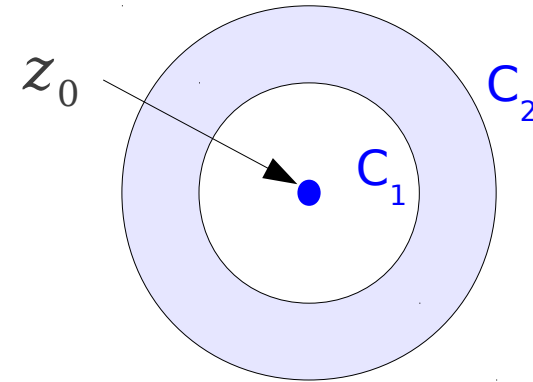
$$\oint_{\text{ccw } C} (z-a) f(z) dz = \int_0^{2\pi} f(z) i(\rho e^{i\theta})^2 d\theta$$

$$= \int_0^{2\pi} f(z) \rho^2 i e^{i2\theta} d\theta = \left[f(z) \frac{\rho}{2} e^{i2\theta} \right]_0^{2\pi}$$

$$= f(z) \frac{\rho}{2} (e^{-i4\pi} - e^{-i0}) = 0$$

Laurent's Theorem

$f(z)$: **analytic** in the region R
between circles C_1, C_2
centered at z_0



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

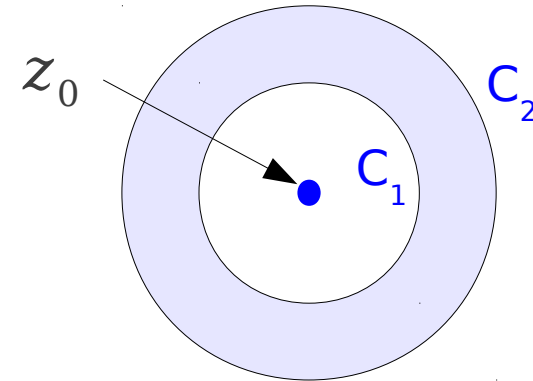
$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots$$

Principal part

: **convergent** in the region R

Laurent's Theorem - Region of Convergence

$f(z)$: **analytic** in the region R
 between circles C_1, C_2
 centered at z_0



➔ $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$

For this “a” series to converge,
 the ROC must be in the form

$|z-z_0| < const \Rightarrow$ inside C_2

$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots$

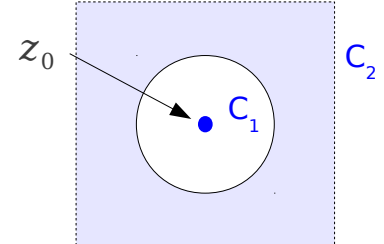
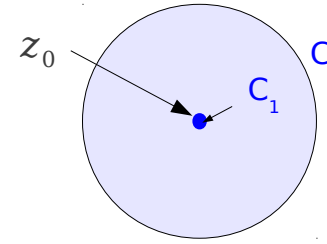
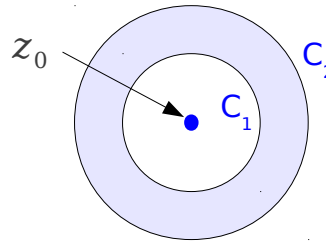
Principal part

For this “b” series to converge,
 the ROC must be in the form

$\left| \frac{1}{z-z_0} \right| < const \Rightarrow$ outside C_1

Laurent's Theorem - Coefficients

$f(z)$: **analytic** in the region R
 between circles C_1, C_2
 centered at z_0



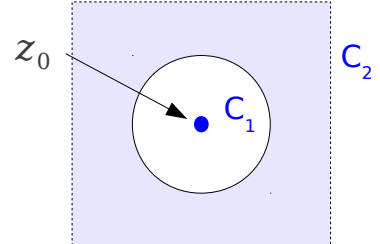
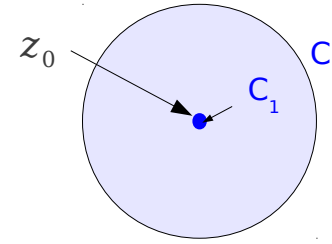
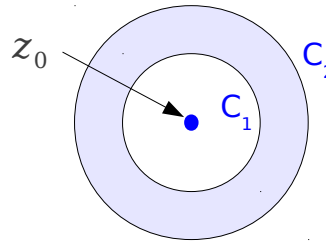
$$\begin{aligned} \Rightarrow f(z) = & a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \\ & + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots \end{aligned} \quad \left. \vphantom{f(z)} \right\} : \text{convergent in the region } R$$

$$\Rightarrow a_n = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z-z_0)^{n+1}}$$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z-z_0)^{-n+1}}$$

Laurent's Theorem - Coefficients

$f(z)$: **analytic** in the region R
 between circles C_1, C_2
 centered at z_0



$$\frac{f(z)}{(z-z_0)^{n+1}} = \frac{a_0}{(z-z_0)^{n+1}} + \frac{a_1}{(z-z_0)^n} + \frac{a_2}{(z-z_0)^{n-1}} + \cdots + \frac{a_n}{(z-z_0)} + a_{n+1} + a_{n+2}(z-z_0) + \cdots$$

$$+ \frac{b_1}{(z-z_0)^{n+2}} + \frac{b_2}{(z-z_0)^{n+3}} + \cdots$$

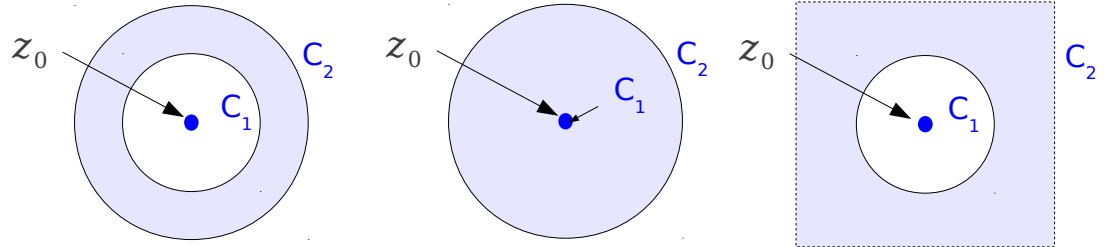


$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z-z_0)^{n+1}}$$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z-z_0)^{-n+1}}$$

Laurent's Theorem - a_n

$f(z)$: **analytic** in the region R
 between circles C_1, C_2
 centered at z_0



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots$$

$$\frac{f(z)}{(z-z_0)^{n+1}} = \frac{a_0}{(z-z_0)^{n+1}} + \frac{a_1}{(z-z_0)^n} + \frac{a_2}{(z-z_0)^{n-1}} + \dots + \frac{a_n}{(z-z_0)} + a_{n+1} + a_{n+2}(z-z_0) + \dots$$

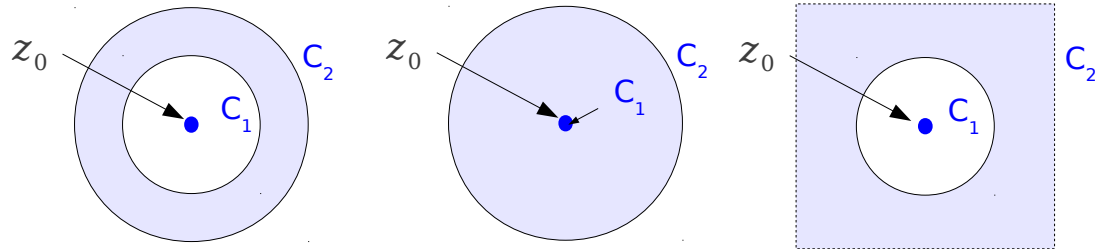
$$+ \frac{b_1}{(z-z_0)^{n+2}} + \frac{b_2}{(z-z_0)^{n+3}} + \dots$$



$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z-z_0)^{n+1}}$$

Laurent's Theorem - b_n

$f(z)$: **analytic** in the region R
 between circles C_1, C_2
 centered at z_0



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots$$

$$\frac{f(z)}{(z-z_0)^{-n+1}} = a_0(z-z_0)^{n-1} + a_1(z-z_0)^n + a_2(z-z_0)^{n+1} + \dots$$

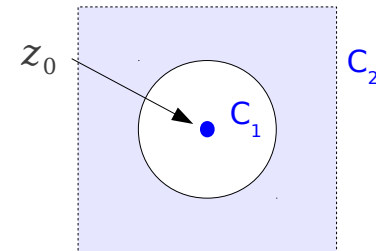
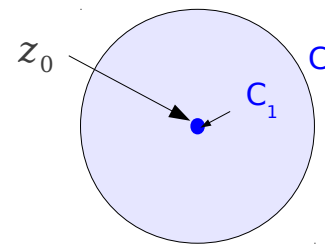
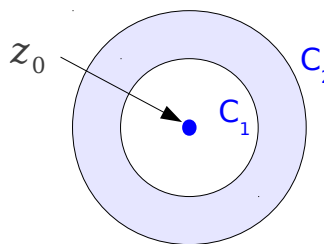
$$+ \frac{b_1}{(z-z_0)^3} + \frac{b_2}{(z-z_0)^4} + \dots + b_{n-1} + \frac{b_n}{(z-z_0)} + \frac{b_{n+1}}{(z-z_0)^2} + \dots$$



$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z-z_0)^{-n+1}}$$

Laurent's Theorem - Some Points

$f(z)$: **analytic** in the region R
 between circles C_1, C_2
 centered at z_0



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

regular point z_0

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_n}{(z-z_0)^n}$$

pole of order n z_0

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

$$+ \frac{b_1}{(z-z_0)}$$

simple pole z_0

b_1 : *residue of* $f(z)$

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots + \dots$$

essential singularity z_0

Residue Theorem (1)

$f(z)$: **analytic** on and inside C (no singular point) 

$$\oint_C f(z) dz = 0$$

$f(z)$: **analytic** on and inside C except z_0 

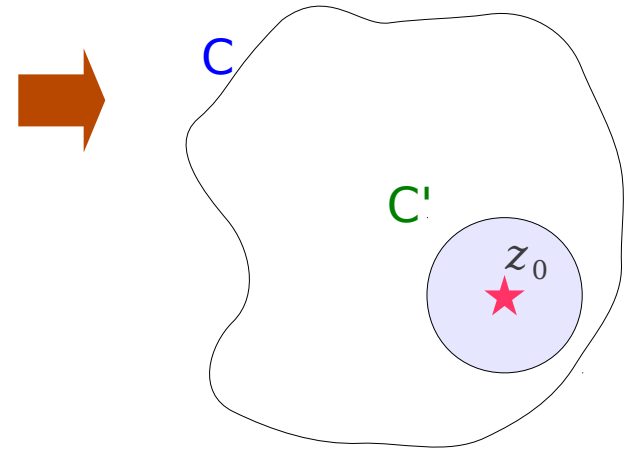
$$\oint_C f(z) dz = 2\pi i \cdot \sum \text{the residues of } f(z) \text{ inside } C$$

z_0 Isolated singular point

The integral around C is in the **counterclockwise** direction

Residue Theorem (2)

$f(z)$: **analytic** on and inside C except z_0
 z_0 Isolated singular point



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots + \dots$$

$$\oint_C a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots dz = 0$$

analytic on and inside C

along C' $z = z_0 + \rho e^{i\theta}$

$$\oint_{C'} \frac{b_1}{(z-z_0)} dz = b_1 \int_0^{2\pi} \frac{\rho i e^{i\theta} d\theta}{\rho e^{i\theta}} = b_1 \int_0^{2\pi} i d\theta = 2\pi i b_1$$

$$\oint_{C'} \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \dots dz = 0$$

$$\int_0^{2\pi} e^{ik\theta} d\theta = \left[\frac{e^{ik\theta}}{ik} \right]_0^{2\pi} = 0$$

Finding Residues (1)

$$\oint_C f(z) dz = 2\pi i \cdot \sum \text{the residues of } f(z) \text{ inside } C$$

The integral around C is in the **counterclockwise** direction

Methods of Finding Residues

Laurent Series: b_1 $1/(z - z_0)$

Simple Pole : $f(z) \cdot (z - z_0)$

Multiple Pole : $f(z) \cdot (z - z_0)^m$

Finding Residues (2)

$$\oint_C f(z) dz = 2\pi i \cdot \sum \text{the residues of } f(z) \text{ inside } C$$

The integral around C is in the **counterclockwise** direction

Laurent Series: $b_1 \frac{1}{(z-z_0)}$

Simple Pole : $f(z) \cdot (z-z_0)$

Multiple Pole : $f(z) \cdot (z-z_0)^m$

$$R(z_0) = \lim_{z \rightarrow z_0} (z-z_0) f(z)$$

$$R(z_0) = \frac{g(z_0)}{h'(z_0)} \quad \leftarrow \quad f(z) = \frac{g(z)}{h(z)} \quad \begin{array}{l} g(z_0) \neq 0 \\ h'(z_0) \neq 0 \quad h(z_0) = 0 \end{array}$$

$$R(z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$$

References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”