Baseband (3A)

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10/30/12

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Bit Time Slot

Codeword Time Slot

Bits / PCM Word

L : number of quantization levels $L = 2^l$

Bits / Symbol

M: size of a set of message symbols $M = 2^k$

M-ary Pulse Modulation Waveforms

PAM (Pulse Amplitude Modulation)

PPM (Pulse Position Modulation)

PDM (Pulse Duration Modulation)

PWM (Pulse Width Modulation)

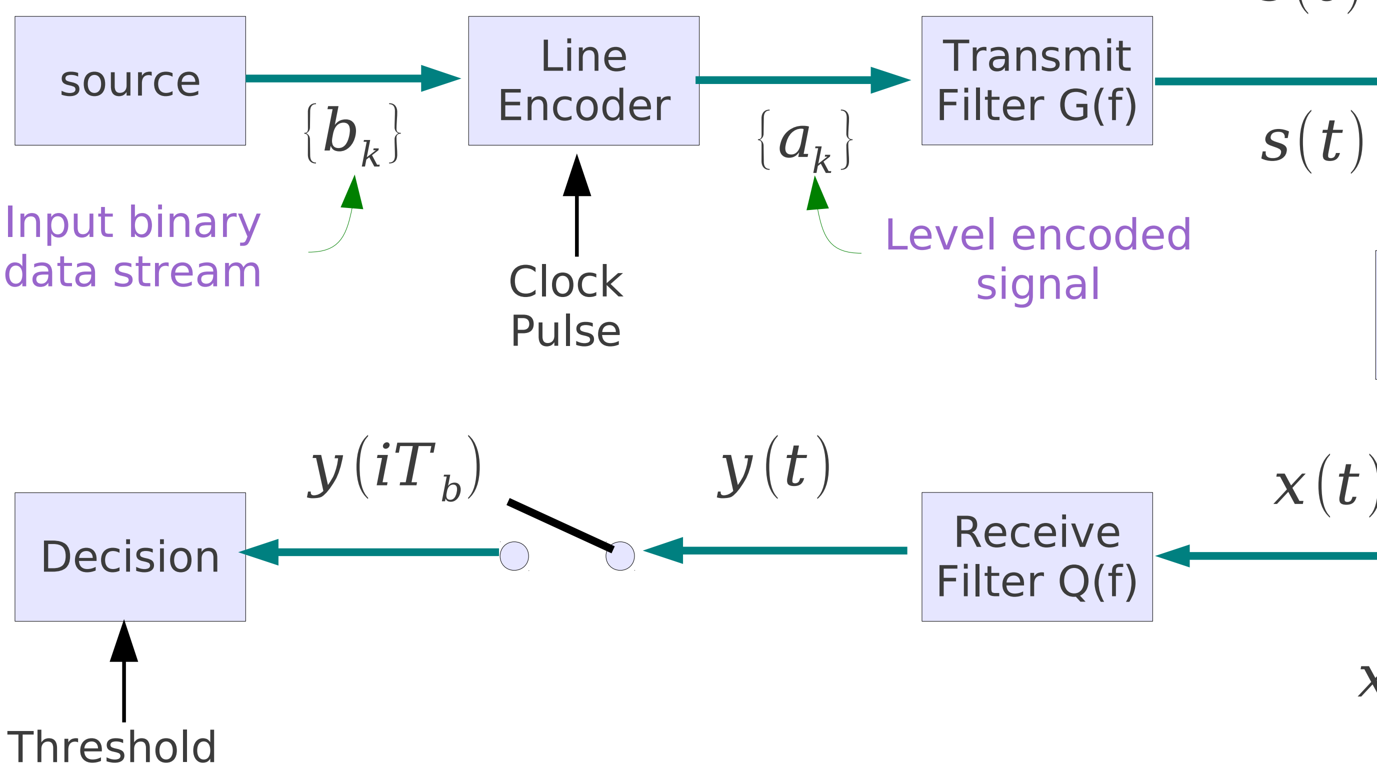
M-ary Pulse Modulation M-ary alphabet set

M-ary PAM : M allowable amplitude levels are assigned to each of the M possible symbol values.

PAM

The amplitude of transmitted pulses is varied in a discrete manner in accordance with an input stream of digital data

$$s(t) = \sum_{k=-\infty}^{+\infty} a_k g(t - kT_b)$$



$$x(t) = s(t) * h(t)$$

$$y(t) = x(t) * q(t)$$

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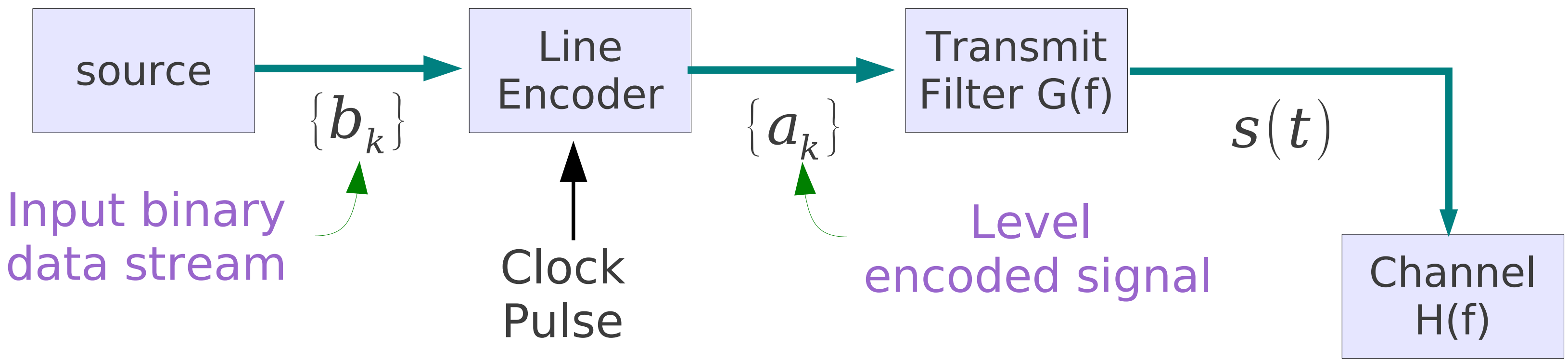
10/30/12

M-ary PAM

The amplitude of transmitted pulses is varied in a discrete manner in accordance with an input stream of digital data



M-ary PAM Bit Rate

$$T = T_b \log_2 M$$

M possible amplitude level (M>2)
M symbols
Transmits sequence of symbols

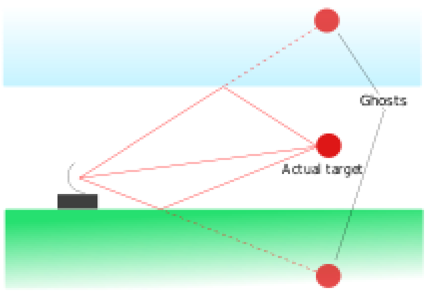
T: Symbol duration
1/T : Symbol rate

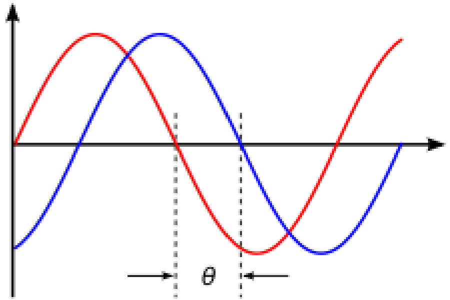
Binary PAM
Tb : Bit duration
1/Tb : Bit rate

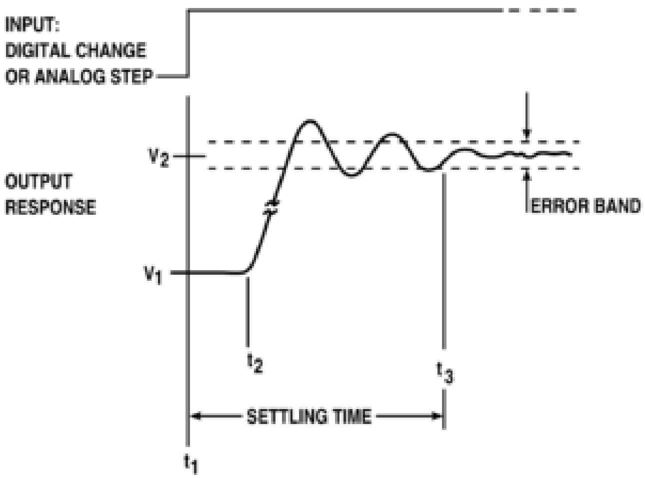
Inter-Symbol Interference

distortion of a signal
in which one symbol interferes with subsequent symbols.
multipath propagation
inherent non-linear filter → long tail, smear, blur …

- adaptive equalization
- error correcting codes

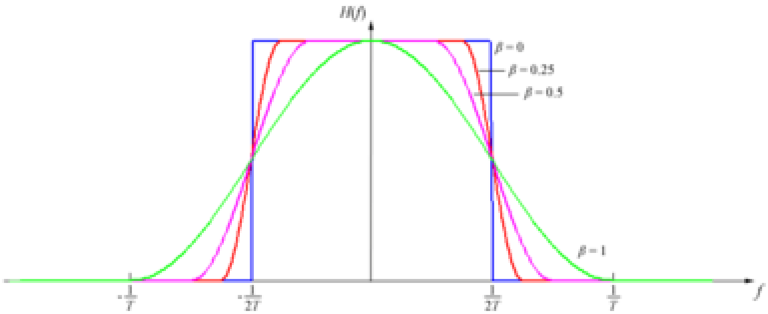




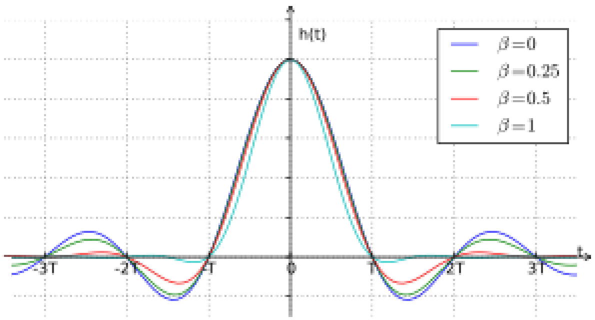


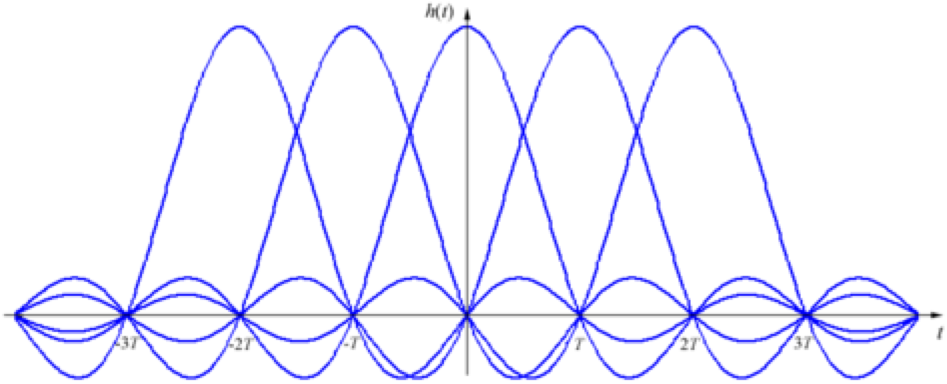
Pulse Shaping

Changing the waveform of transmitted p
Bandwidth constraints
Control ISI (inter-Symbol Interference)



- Sinc Filter
- Raised Cosine Filter
- Gaussian Filter





Signal Space

N-dim orthogonal space

Characterized by a set of N linearly independent functions

Basis functions $\Psi_j(t)$

Independent → not interfering in detection

$$\int_0^T \Psi_j(t)\Psi_k(t)\,dt = K_j\delta_{jk} \qquad 0 \le t \le T \qquad j,k = 1,\cdots,N$$

Kronecker delta functions

$$\delta_{jk} = \begin{cases} 1 & for\ j = k \\ 0 & otherwise \end{cases}$$

N-dim orthonormal space $K_j = 1$

$$E_j = \int_0^T \Psi_j^2(t)\,dt = K_j$$

Linear Combination

Any finite set of waveform $\{s_i(t)\}$ $\quad i = 1, \cdots, M$

Characterized by a set of N linearly independent functions

$$
\begin{aligned}
s_1(t) &= a_{11}\Psi_1(t) + a_{12}\Psi_2(t) + \cdots + a_{1n}\Psi_N(t) \\
s_2(t) &= a_{21}\Psi_1(t) + a_{22}\Psi_2(t) + \cdots + a_{2n}\Psi_N(t) \\
&\vdots \\
s_M(t) &= a_{M1}\Psi_1(t) + a_{M2}\Psi_2(t) + \cdots + a_{MN}\Psi_N(t)
\end{aligned}
$$

$$
s_1(t) = \sum_{j=1}^{N} a_{ij}\Psi_j(t) \qquad i = 1, \cdots, M
$$

$$
N \le M
$$

Linear Combination

Any finite set of waveform $\{s_i(t)\}$ $\quad i = 1, \cdots, M$

Characterized by a set of N linearly independent functions

$$s_i(t) = \sum_{j=1}^{N} a_{ij} \Psi_j(t) \qquad i = 1, \cdots, M$$

$$N \le M$$

$$a_{ij} = \frac{1}{K_j} \int_0^T s_i(t) \Psi_j(t)\, dt \qquad i = 1, \cdots, M \qquad 0 \le t \le T$$

$$j = 1, \cdots, N$$

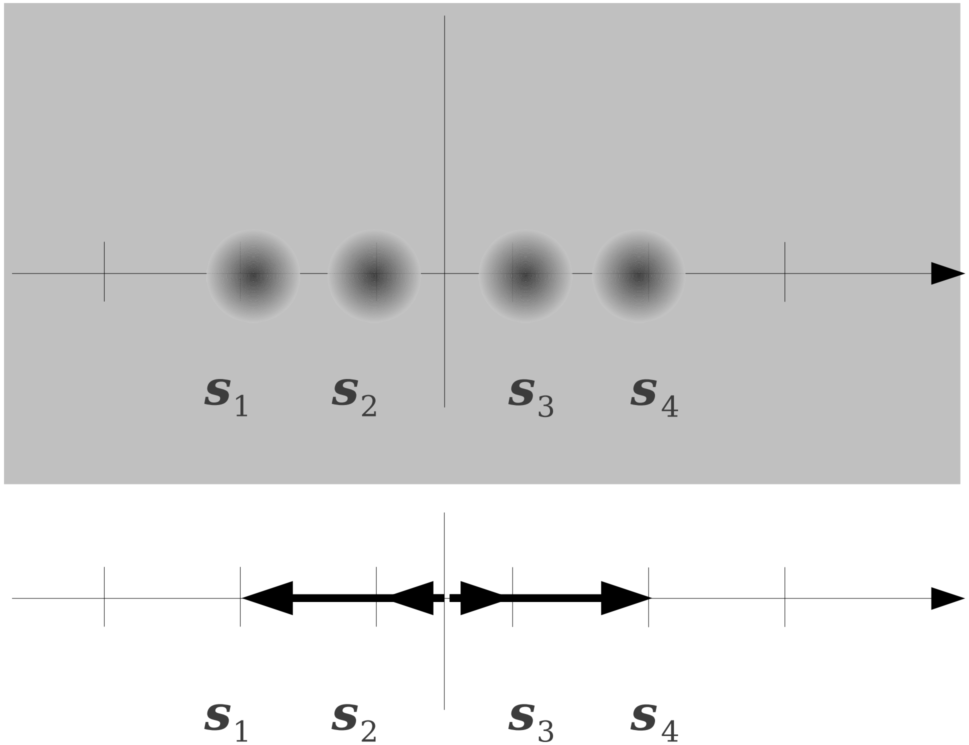
$$\{s_i(t)\} \Longleftrightarrow \{\mathbf{s}_i\} = \{a_{i1}, a_{i2}, \cdots, a_{iN}\} \qquad i = 1, \cdots, M$$

Signals and Noise

Any finite set of waveform $\{s_i(t)\}$ $\quad i = 1, \cdots, M$

Characterized by a set of N linearly independent functions

$$\{s_i(t)\} \Longleftrightarrow \{\boldsymbol{s}_i\} = \{a_{i1}, a_{i2}, \cdots, a_{iN}\} \quad i = 1, \cdots, M$$



4-ary PAM

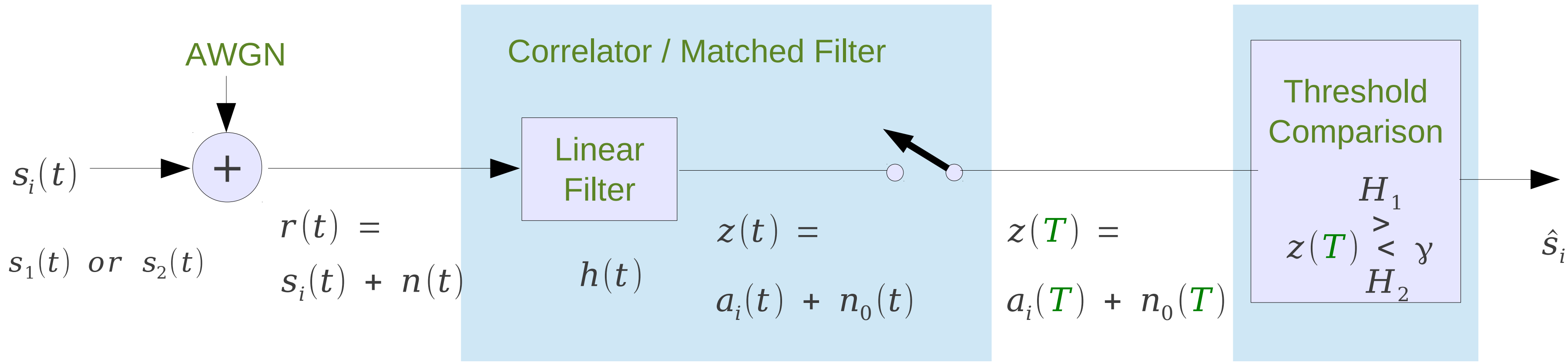
Detection of Binary Signals

Transmitted Signal

$$s_i(t) = \begin{cases} s_1(t) & 0 \le t \le T \quad \text{for a binary } 1 \\ s_2(t) & 0 \le t \le T \quad \text{for a binary } 0 \end{cases}$$

Received Signal

$$r(t) = s_i(t) + n(t) \qquad i = 1,2; \qquad 0 \le t \le T$$



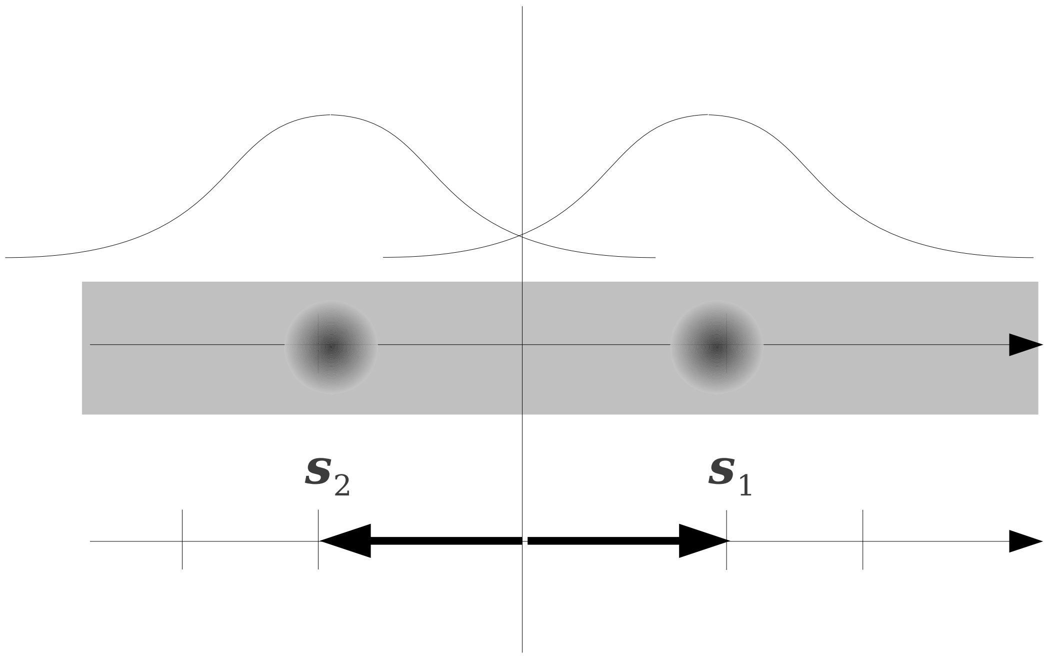
Detection of Binary Signals

$$z(T) = a_i(T) + n_0(T) \Rightarrow z = a_i + n_0$$

$$p(n_0) = \frac{1}{\sigma_0\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{n_0}{\sigma_0}\right)^2\right]$$

$$p(z|s_1) = \frac{1}{\sigma_0\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{z-a_1}{\sigma_0}\right)^2\right]$$

$$p(z|s_2) = \frac{1}{\sigma_0\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{z-a_2}{\sigma_0}\right)^2\right]$$



$$z(T) \underset{H_2}{\overset{H_1}{\gtrless}} \gamma$$

$$\frac{p(z|s_1)}{p(z|s_2)} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(s_2)}{P(s_1)}$$

$$\frac{p(z|s_1)}{p(z|s_2)} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{a_1+a_2}{2} = \gamma_0$$

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Error Probability

error e

$$p(e|s_1) = p(H_2|s_1) = \int_{-\infty}^{\gamma_0} p(z|s_1)\,dz$$

$$p(e|s_2) = p(H_1|s_2) = \int_{\gamma_0}^{-\infty} p(z|s_2)\,dz$$

probability of bit error P_B

$$P_B = P(e|s_1)P(s_1) + P(e|s_2)P(s_2)$$
$$= P(H_2|s_1)P(s_1) + P(H_1|s_2)P(s_2)$$

equal a priori probabilities

$$P_B = \tfrac{1}{2}P(H_2|s_1) + \tfrac{1}{2}P(H_1|s_2)$$
$$= P(H_2|s_1) = P(H_1|s_2)$$

$$P_B = \int_{\gamma_0 = (a_1+a_2)/2}^{+\infty} p(z|s_2)\,dz$$

$$= \int_{\gamma_0 = (a_1+a_2)/2}^{+\infty} \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a_2}{\sigma_0}\right)^2\right] dz$$

$u = (z - a_2)/\sigma_0$ $\quad$ $\sigma_0\,du = dz$

$$= \int_{u = (a_1-a_2)/2\sigma_0}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right) du$$

$$= Q\left(\frac{a_1-a_2}{2\sigma_0}\right)$$

complementary error function
(co-error function)

$$Q(x) = \int_{x}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) dx$$

ISI

Gaussian Random Process

Thermal Noise — zero-mean white Gaussian random process

$n(t)$ random function
the value at time t is characterized by
Gaussian probability density function

$$z(t) = a + n(t)$$

$$p(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{n}{\sigma}\right)^2\right]$$

$$p(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a}{\sigma}\right)^2\right]$$

σ^2 variance of n

$\sigma = 1$ normalized (standardized)
Gaussian function

Central Limit Theorem

sum of statistically independent random variables
approaches Gaussian distribution
regardless of individual distribution functions

White Gaussian Noise

$n($

Thermal Noise

power spectral density is the same for all frequencies

$$G_n(f) = \frac{N_0}{2}$$ watts / hertz

equal amount of noise power per unit bandwidth

uniform spectral density ➡ White Noise

average power

$$P_n = \int_{-\infty}^{+\infty} \frac{N_0}{2}\, df = \infty$$

$$P_x^T = \frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)\, dt = \int_{-\infty}^{+\infty} G_x(f)\, df$$

$$R_n(t) = \frac{N_0}{2}\delta(t) \quad \Longleftrightarrow \quad G_n(f) = \frac{N_0}{2}$$

$\delta(t)$ totally uncorrelated, noise samples are independent

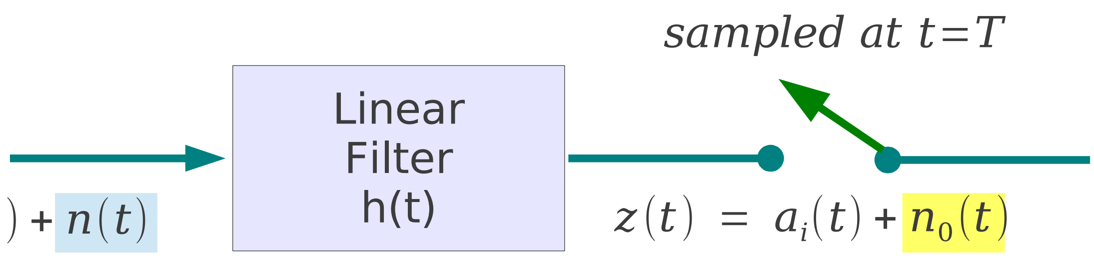
memoryless channel

additive and no multiplicative mechanism

Additive White Gaussian Noise (AWGN)

Matched Filter (1)

to find a filter h(t) that gives max signal-to-noise ratio



$r(t) = s_i(t) + n(t)$

$z(t) = a_i(t) + n_0(t)$

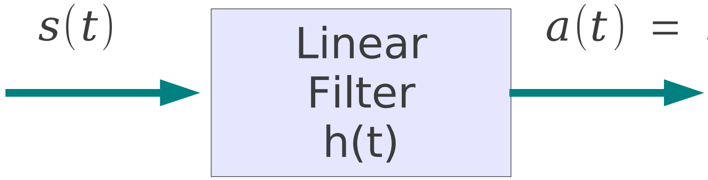
variance of $n_0(t)$ ➡ σ_0^2 avg noise power

$$\frac{\text{instantaneous signal power}}{\text{average noise power}} \Rightarrow \left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2}$$

assume $H_0(f)$ a filter transfer function that maximizes $\left(\frac{S}{N}\right)_T$

Matched Filter (2)

$s(t)$

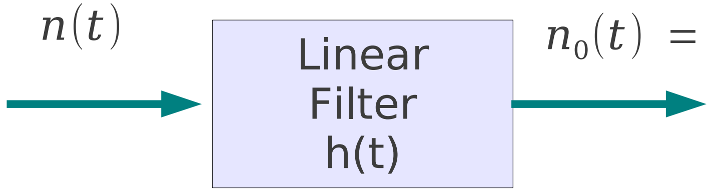


$$a(t) = s(t) * h(t)$$

$S(f)$

$$A(f) = S(f)H(f) \quad \Longleftrightarrow \quad a(t) = \int_{-\infty}^{+\infty} S(f)H(f)e^{j2\pi f t}\,df$$

$n(t)$



$$n_0(t) = n(t) * h(t)$$

$$G_n(f) = \frac{N_0}{2}$$

$$G_{n0}(f) = G_n(f)\,|H(f)|^2 = \begin{cases} \frac{N_0}{2}|H(f)|^2 & \text{for } |f| < f_u \\ 0 & \text{otherwise} \end{cases}$$

Average output noise power $\quad \sigma_0 = \frac{N_0}{2}\int_{-\infty}^{+\infty} |H(f)|^2\,df$

Matched Filter (3)

instantaneous signal power a_i^2 ← $a(t) = \int_{-\infty}^{+\infty} S(f) H(f) e^{j2\pi f t}\, df$

average output noise power $\sigma_0 = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H(f)|^2\, df$

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{\left|\int_{-\infty}^{+\infty} H(f) S(f) e^{+j2\pi f T}\, df\right|^2}{N_0/2 \int_{-\infty}^{+\infty} |H(f)|^2\, df}$$

Does not depend on the particular shape of the waveform

Cauchy Schwarz's Inequality

$$\left|\int_{-\infty}^{+\infty} f_1(x) f_2(x)\, dx\right|^2 \le \int_{-\infty}^{+\infty} |f_1(x)|^2\, dx \int_{-\infty}^{+\infty} |f_2(x)|^2\, dx \qquad \text{'=' } \textit{holds when } f_1(x) = k f_2^*(x)$$

$$\left|\int_{-\infty}^{+\infty} H(f) S(f) e^{+j2\pi ft}\, dx\right|^2 df \le \int_{-\infty}^{+\infty} |H(f)|^2\, df \int_{-\infty}^{+\infty} |S(f) e^{+j2\pi f T}|^2\, df$$

$$|e^{+j2\pi f T}| = 1$$

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{\left|\int_{-\infty}^{+\infty} H(f) S(f) e^{+j2\pi f T}\, df\right|^2}{N_0/2 \int_{-\infty}^{+\infty} |H(f)|^2\, df} \le \frac{\boxed{\int_{-\infty}^{+\infty} |H(f)|^2\, df} \int_{-\infty}^{+\infty} |S(f) e^{+j2\pi f T}|^2\, df}{N_0/2 \boxed{\int_{-\infty}^{+\infty} |H(f)|^2\, df}} = \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2\, df$$

Matched Filter (4)

Two-sided power spectral density of input noise ➡ $\dfrac{N_0}{2}$

Average noise power $\sigma_0 = \dfrac{N_0}{2} \int_{-\infty}^{+\infty} |H(f)|^2 \, df$

$$\left(\frac{S}{N}\right)_T = \frac{{a_i}^2}{{\sigma_0}^2} = \frac{\left| \int_{-\infty}^{+\infty} H(f) S(f) e^{+j2\pi f T} \, df \right|^2}{N_0/2 \int_{-\infty}^{+\infty} |H(f)|^2 \, df}$$

Cauchy Schwarz's Inequality

$$\left(\frac{S}{N}\right)_T \le \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 \, df$$

input signal energy

$$max \left(\frac{S}{N}\right)_T = \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 \, df = \frac{2E}{N_0}$$

power spectral density of input noise

does not depend on the particular shape of the waveform

Matched Filter (5)

$$\left|\int_{-\infty}^{+\infty} H(f)S(f)e^{+j2\pi ft}\,dx\right|^2 df \le \int_{-\infty}^{+\infty} |H(f)|^2\,df \int_{-\infty}^{+\infty} |S(f)e^{+j2\pi fT}|^2\,df$$

$$\left(\frac{S}{N}\right)_T \le \frac{2}{N_0}\int_{-\infty}^{+\infty} |S(f)|^2\,df$$

$$max\left(\frac{S}{N}\right)_T = \frac{2}{N_0}\int_{-\infty}^{+\infty} |S(f)|^2\,df = \frac{2E}{N_0}$$

when complex conjugate relationship exists

$$H(f) = H_0(f) = k S^*(f) e^{-j2\pi fT}$$

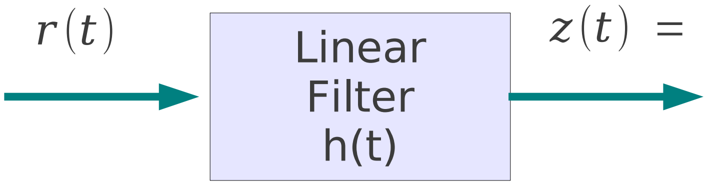
⬌

$$h(t) = h_0(t) = \begin{cases} k s(T-t) & 0 \le t \le T \\ 0 & elsewhere \end{cases}$$

$H_0(f)$ a filter transfer function that maximizes $\left(\frac{S}{N}\right)_T$

impulse response : <u>delayed</u> version of
the <u>mirror</u> image of the <u>signal</u> waveform

Correlation Realization



$$z(t) = r(t) * h(t)$$

$$z(t) = \int_0^t r(\tau) h(t-\tau)\, d\tau$$

$$= \int_0^t r(\tau) s(T-(t-\tau))\, d\tau$$

$$= \int_0^t r(\tau) s(T-t+\tau)\, d\tau$$

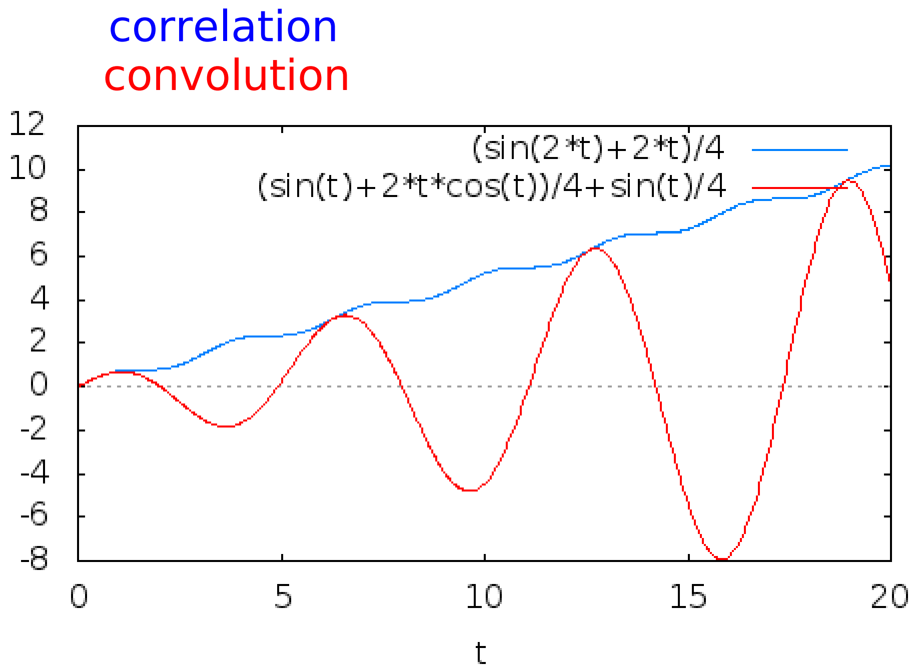
$$z(T) = \int_0^t r(\tau) s(\tau)\, d\tau$$

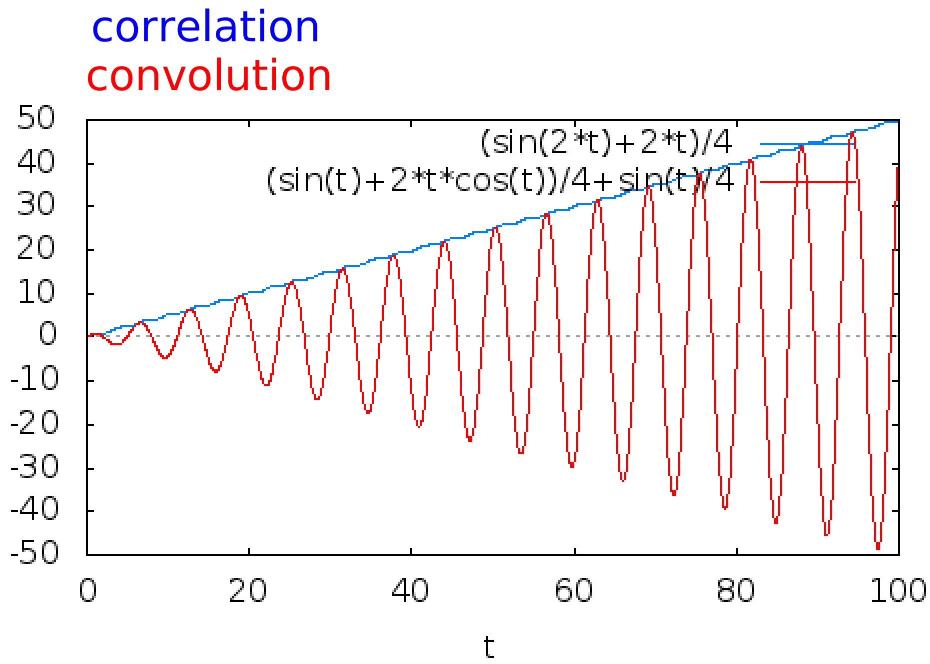
Power spectral density
of input noise

convolution $\quad z(t) = \int_0^t r(\tau) s(T-t+\tau)\, d\tau$ a sine-wave amplitude modulated by a linear ramp

correlation $\quad z(T) = \int_0^T r(\tau) s(\tau)\, d\tau$ a linear ramp output

Correlation and Convolution





z : integrate(cos(x)*cos(2*%pi - t +x), x, 0, t); convolution

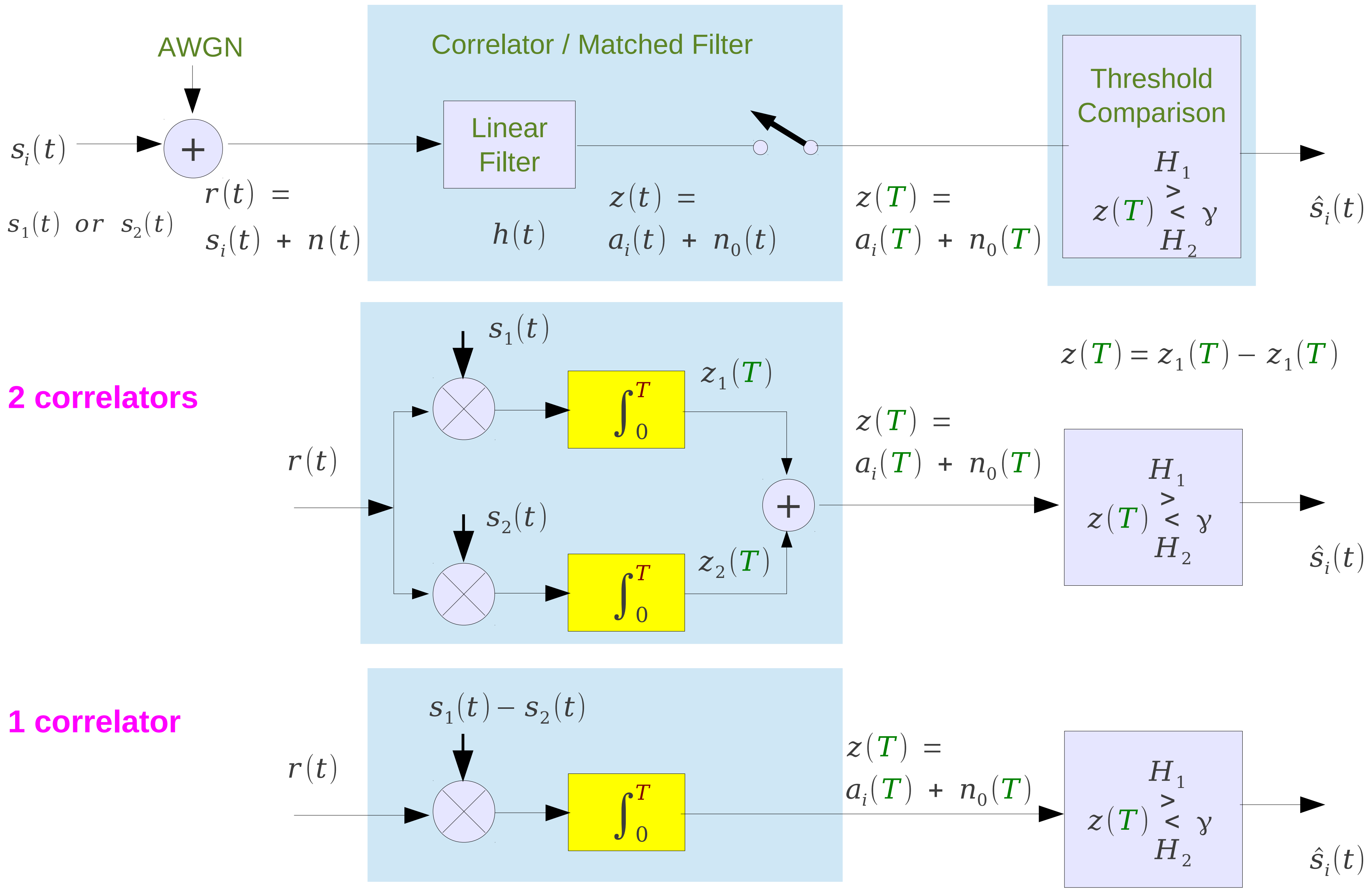
(sin(t)+2*t*cos(t))/4+sin(t)/4

correlation

z : integrate(cos(x)*cos(x), x, 0, t);

(sin(2*t)+2*t)/4

Binary Correlator Receiver



Maximum Likelihood Receiver

maximum likelihood detector

$P(s_1) = P(s_2)$ equal a priori probability

$p(z|s_1), \quad p(z|s_2)$ symmetric likelihood

$$\gamma_0 = \frac{(a_1 + a_2)}{2}$$

optimum threshold for minimizing the error probability

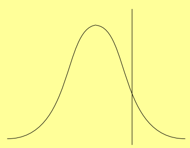
select the hypothesis with the maximum likelihood

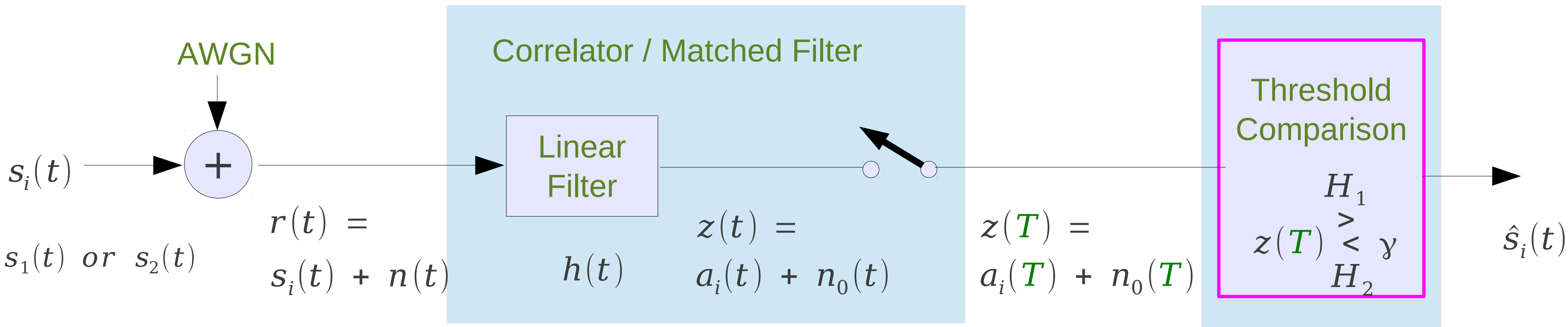
complementary error function

$$Q(x) = \int_{x}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right) du$$

$$P_B = \int_{\gamma_0 = (a_1 - a_2)/2\sigma_0}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right) du$$

$$= Q\left(\frac{a_1 - a_2}{2\sigma_0}\right)$$





Matched Filter Minimizes P_B by Maximizing SNR

Matched Filter / Correlator

maximize $\dfrac{(a_1 - a_2)^2}{\sigma_0^2}$

maximize $\dfrac{(a_1 - a_2)^2}{2\sigma_0}$

minimize

$$P_B = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right)$$

maximize $\left(\dfrac{S}{N}\right)_T = \dfrac{a_i^2}{\sigma_0^2}$ $\qquad max \left(\dfrac{S}{N}\right)_T = \dfrac{2E}{N_0}$

matched to $s_1 - s_2$

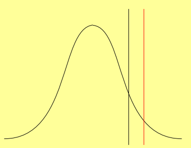
$$\left(\frac{S}{N}\right)_T = \frac{a_1 - a_2{}^2}{\sigma_0{}^2} = \frac{2E_d}{N_0}$$

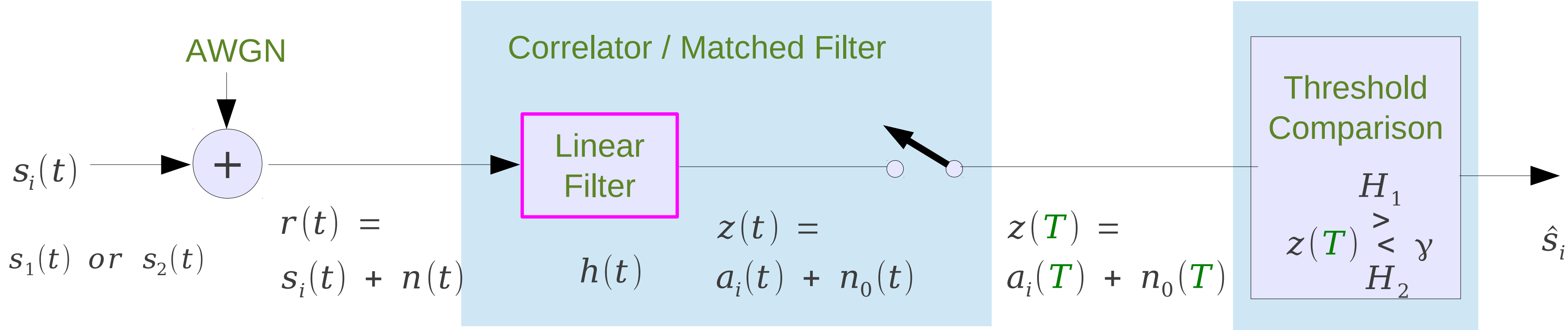
complementary error function

$$Q(x) = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right) du$$

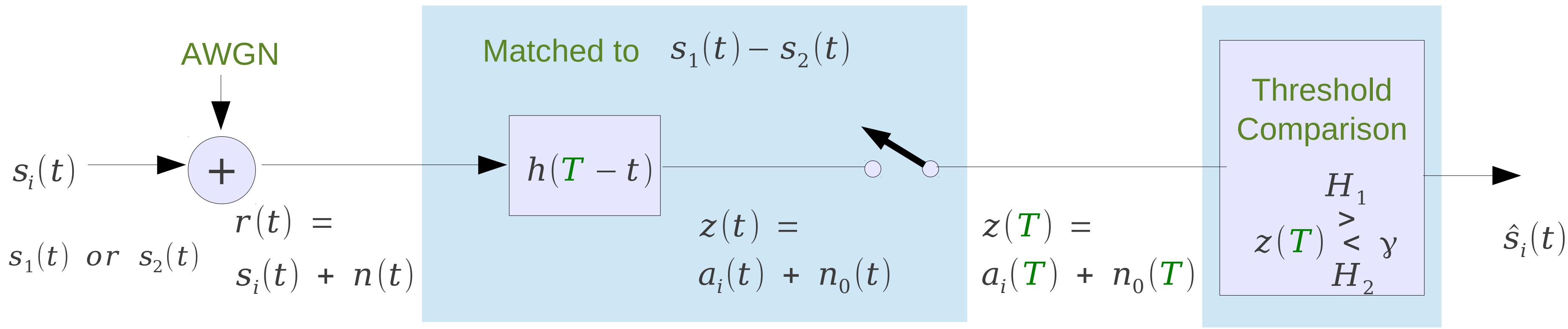
$$P_B = \int_{\gamma_0 = (a_1 - a_2)/2\sigma_0}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right) du$$

$$= Q\left(\frac{E_d}{2N_0}\right)$$

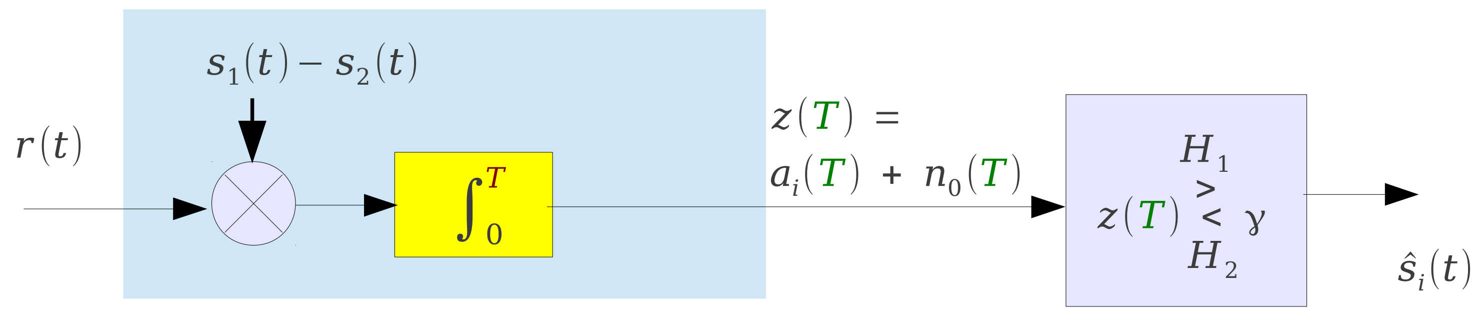




Energy Difference E_b



1 correlator



matched to $s_1 - s_2$

$$\left(\frac{S}{N}\right)_T = \frac{{a_1 - a_2}^2}{{\sigma_0}^2} = \frac{2E_d}{N_0}$$

$$\frac{1}{2}\frac{a_1 - a_2}{\sigma_0} = \sqrt{\frac{2E_d}{N_0}\frac{1}{4}}$$

Energy Difference

$$E_d = \int_0^T [s_1(t) - s_2(t)]^2 dt$$

Bit-Error Probability

$$P_B = Q\left(\frac{E_d}{2N_0}\right)$$

Time Averaging and Ergodicity

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] B. Sklar, "Digital Communications: Fundamentals and Applications"