Line Integrals (4A)

- Line Integral
- Path Independence

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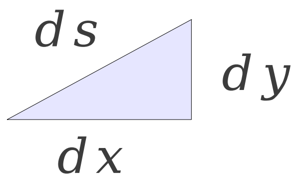
Line Integral In the Plane

$$x = f(t) \quad \Rightarrow \quad \frac{dx}{dt} = f'(t) \quad \Rightarrow \quad dx = f'(t)\,dt$$

$$y = g(t) \quad \Rightarrow \quad \frac{dy}{dt} = g'(t) \quad \Rightarrow \quad dy = g'(t)\,dt$$

$$ds = \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

Curve C $\quad a \le t \le b$



$$\int_C G(x, y)\,dx = \int_a^b G(f(t), g(t))\,f'(t)\,dt$$

$$\int_C G(x, y)\,dy = \int_a^b G(f(t), g(t))\,g'(t)\,dt$$

$$\int_C G(x, y)\,ds = \int_a^b G(f(t), g(t))\,\sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

Line Integral In Space

$x = f(t)$ ⟹ $\dfrac{dx}{dt} = f'(t)$ ⟹ $dx = f'(t)\,dt$

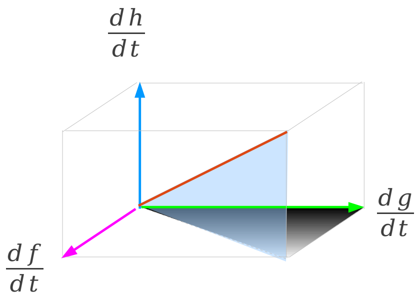
$y = g(t)$ ⟹ $\dfrac{dy}{dt} = g'(t)$ ⟹ $dy = g'(t)\,dt$

$z = h(t)$ ⟹ $\dfrac{dz}{dt} = h'(t)$ ⟹ $dz = h'(t)\,dt$

Curve C $\quad a \leq t \leq b$

$$ds = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$$

$$\int_C G(x, y, z)\,dz = \int_a^b G(f(t), g(t), h(t))\,h'(t)\,dt$$



$$\int_C G(x, y, z)\,ds = \int_a^b G(f(t), g(t), h(t))\sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$$

Line Integral using ***r***(t)

Arc Length Parameter

s increases in the direction of increasing t

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau = \int_{t_0}^{t} \|\mathbf{r}'(\tau)\|\, d\tau = \int_{t_0}^{t} \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2 + [h'(\tau)]^2}\, d\tau$$

$$ds = |\mathbf{v}(t)|\, dt \qquad ds = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

$$\int_C G(x, y, z)\, ds = \int_a^b G(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt$$

$$= \int_a^b G(f(t), g(t), h(t))\, |\mathbf{v}(t)|\, dt$$

$$= \int_a^b G(f(t), g(t), h(t)) \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

Line Integral with an Explicit Curve Function

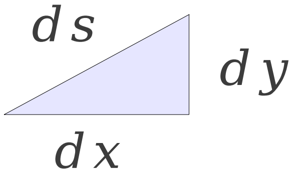
$y = f(x)$ ➡ $\dfrac{dy}{dx} = f'(x)$ ➡ $dy = f'(x)\,dx$

$a \le x \le b$

Curve C

$$ds = \sqrt{[dx]^2 + [dy]^2}\,dt$$

$$ds = \sqrt{1 + [f'(x)]^2}\,dx$$



$$\int_C G(x, y)\,dx = \int_a^b G(x, f(x))\,dx$$

$$\int_C G(x, y)\,dy = \int_a^b G(x, f(x))\,f'(x)\,dx$$

$$\int_C G(x, y)\,ds = \int_a^b G(x, f(x))\sqrt{1 + [f'(x)]^2}\,dx$$

Line Integral Notation

In many applications

$$\int_C G(x, y)\, ds = \int_C P(x, y)\, dx + \int_C Q(x, y)\, dy$$

$$= \int_C P(x, y)\, dx + Q(x, y)\, dy$$

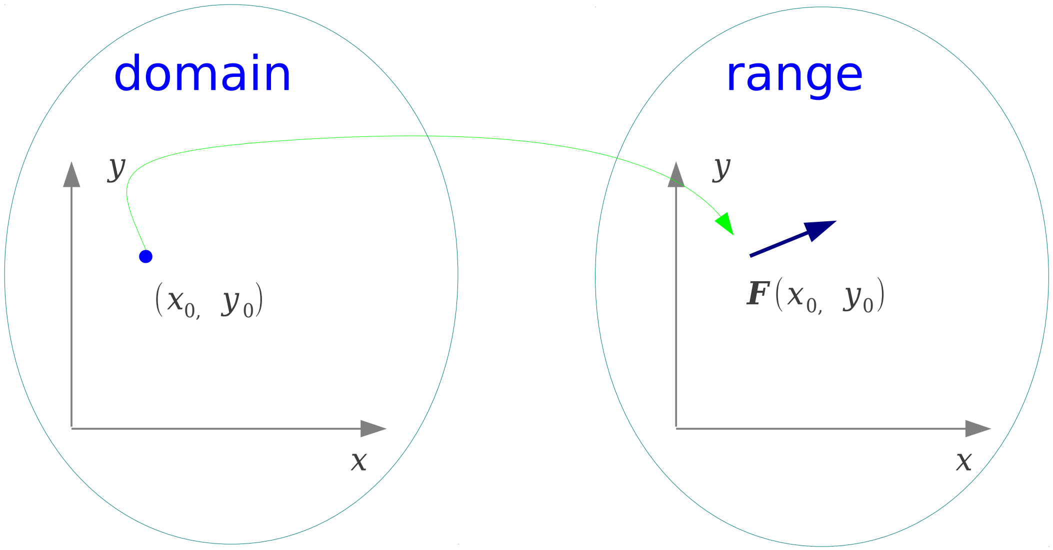
$$= \int_C P\, dx + Q\, dy$$

$$\int_C G(x, y, z)\, ds = \int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$$

Line Integral over a 2-D Vector Field (1)

a given point in a 2-d space ➡ A vector

$(x_{0,} \;\; y_0)$ $\langle\, P(x_{0,} y_0),\;\; Q(x_{0,} y_0)\,\rangle$



2 functions

$(x_{0,} \;\; y_0) \longrightarrow P(x_{0,} y_0)$

$(x_{0,} \;\; y_0) \longrightarrow Q(x_{0,} y_0)$

$(x_{0,} \;\; y_0)$

only points that are
on the curve

$\boldsymbol{r}(t) \;=\; f(t)\boldsymbol{i} \;+\; g(t)\boldsymbol{j} \longrightarrow \boldsymbol{F}(x_{0,} \;\; y_0) \;=\; P(x_{0,} y_0)\boldsymbol{i} \;+\; Q(x_{0,} y_0)\boldsymbol{j}$

$x = f(t) \quad y = g(t) \quad a \;\le\; t \;\le\; b$

Line Integral over a 2-D Vector Field (2)

$$\boldsymbol{r}(t) = f(t)\,\boldsymbol{i} + g(t)\,\boldsymbol{j}$$

$$\frac{d\boldsymbol{r}}{dt} = f'(t)\,\boldsymbol{i} + g'(t)\,\boldsymbol{j} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j}$$

$$\frac{d\boldsymbol{r}}{dt}dt = \left(\frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j}\right)dt = dx\,\boldsymbol{i} + dy\,\boldsymbol{j}$$

$$d\boldsymbol{r} = dx\,\boldsymbol{i} + dy\,\boldsymbol{j}$$

$$\boldsymbol{F}(x, y) = P(x, y)\,\boldsymbol{i} + Q(x, y)\,\boldsymbol{j}$$

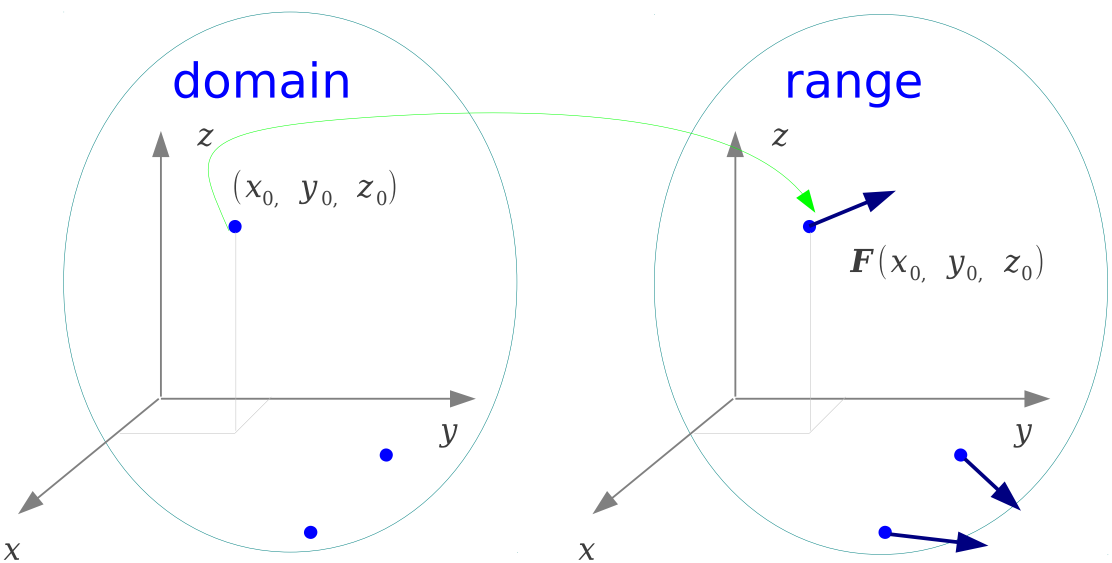
$$\boldsymbol{F}\cdot d\boldsymbol{r} = P(x, y)\,dx + Q(x, y)\,dy$$

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_C P(x, y)\,dx + Q(x, y)\,dy$$

Line Integral over a 3-D Vector Field (1)

A given point in a 3-d space ➡ A vector

$(x_{0,}\ \ y_{0,}\ \ z_0)$ $\langle\, P(x_0, y_0, z_0),\ Q(x_0, y_0, z_0),\ R(x_0, y_0, z_0)\,\rangle$



3 functions

$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow P(x_{0,} y_{0,} Z_0)$

$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow Q(x_{0,} y_{0,} z_0)$

$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow R(x_{0,} y_{0,} z_0)$

$(x_{0,}\ \ y_{0,}\ \ z_0)$

only points that are on the curve $\longrightarrow$ $\boldsymbol{F}(x_{0,}\ \ y_{0,}\ \ z_0) \ = \ P(x_{0,} y_{0,} z_0)\boldsymbol{i} \ + \ Q(x_{0,} y_{0,} z_0)\boldsymbol{j} \ + \ R(x_{0,} y_{0,} z_0)\boldsymbol{k}$

$\boldsymbol{r}(t) \ = \ f(t)\boldsymbol{i} \ + \ g(t)\boldsymbol{j} \ + \ h(t)\boldsymbol{j}$

$x = f(t) \quad y = g(t) \quad z = h(t) \quad a \ \le \ t \ \le \ b$

Line Integral over a 3-D Vector Field (2)

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$$

$$\frac{d\mathbf{r}}{dt} = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

$$\frac{d\mathbf{r}}{dt}dt = \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right)dt = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$$

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$$

$$\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$$

$$\mathbf{F}\cdot d\mathbf{r} = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

$$\int_c \mathbf{F}\cdot d\mathbf{r} = \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dy$$

Line Integral in Vector Fields

$$\boldsymbol{r}(t) = f(t)\,\boldsymbol{i} + g(t)\,\boldsymbol{j}$$

$$d\boldsymbol{r} = dx\,\boldsymbol{i} + dy\,\boldsymbol{j}$$

$$\boldsymbol{F}(x, y, z) = P\,\boldsymbol{i} + Q\,\boldsymbol{j}$$

$$\int_c \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C P\,dx + Q\,dy$$

$$\begin{cases} P(x, y) \\ Q(x, y) \end{cases}$$

$$\boldsymbol{r}(t) = f(t)\,\boldsymbol{i} + g(t)\,\boldsymbol{j} + h(t)\,\boldsymbol{k}$$

$$d\boldsymbol{r} = dx\,\boldsymbol{i} + dy\,\boldsymbol{j} + dz\,\boldsymbol{k}$$

$$\boldsymbol{F}(x, y, z) = P\,\boldsymbol{i} + Q\,\boldsymbol{j} + R\,\boldsymbol{k}$$

$$\int_c \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C P\,dx + Q\,dy + R\,dz$$

$$\begin{cases} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{cases}$$

Work (1)

$$W = \boldsymbol{F} \cdot \boldsymbol{d}$$

A force field $\boldsymbol{F}(x, y) = P(x, y)\boldsymbol{i} + Q(x, y)\boldsymbol{j}$

A smooth curve $C : x = f(t), \; y = g(t), \; a \le t \le b$

Work done by **F** along C $W = \int_c \boldsymbol{F}(x, y) \cdot d\boldsymbol{r}$

$$= \int_C P(x, y)\, dx + Q(x, y)\, dy$$

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} \qquad d\boldsymbol{r} = \frac{d\boldsymbol{r}}{ds} ds \qquad d\boldsymbol{r} = \boldsymbol{T}\, ds$$

$$W = \int_c \boldsymbol{F} \cdot d\boldsymbol{r} = \int_c \boldsymbol{F} \cdot \boldsymbol{T}\, ds$$

Work (2)

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} \qquad d\mathbf{r} = \frac{d\mathbf{r}}{ds}ds \qquad d\mathbf{r} = \mathbf{T}\,ds$$

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

$$= \int_{t_1}^{t_0} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt$$

$$= \int_{t_0}^{t_1} \left(P\frac{df}{dt} + Q\frac{dg}{dt} + R\frac{dh}{dt}\right)dt$$

$$= \int_{t_0}^{t_1} \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right)dt$$

$$= \int_{t_0}^{t_1} P\,dx + Q\,dy + R\,dz$$

$$\mathbf{F}(x,y,z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$
$$= P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t)$$

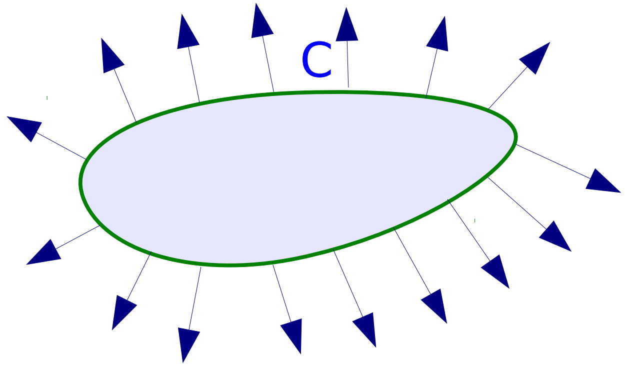
Circulation

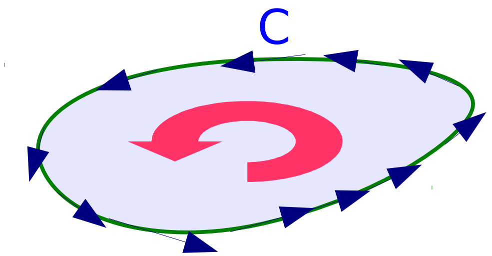
A Simple Closed Curve C → Circulation

$$circulation = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_C \boldsymbol{F} \cdot \boldsymbol{T}\, d\boldsymbol{s}$$

Assume **F** is a velocity field of a fluid

Circulation : a measure of the amount by which the fluid tends to turn the curve C by rotating around it





$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_C \boldsymbol{F} \cdot \boldsymbol{T}\, d\boldsymbol{s} = 0$$

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_C \boldsymbol{F} \cdot \boldsymbol{T}\, d\boldsymbol{s} > 0$$

Path Independence

$C1 \neq C2$ ⇒ $\int_{C1} \boldsymbol{F} \cdot d\boldsymbol{r} \neq \int_{C2} \boldsymbol{F} \cdot d\boldsymbol{r}$ In general

⇒ $\int_{C1} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C2} \boldsymbol{F} \cdot d\boldsymbol{r}$ Exceptional Case

Path Independence

for a special kind of a vector field $\boldsymbol{F}$

⇓

Conservative Vector Field

If we can find a scalar function Φ

that satisfies $\boldsymbol{F} = \nabla \Phi$

$\boldsymbol{F}$ is a gradient field of a scalar function Φ

Conservative Vector Field

$\boldsymbol{F}$ can be written as the gradient of a scalar function Φ $\quad \boldsymbol{F} = \nabla \Phi$

⇒ A vector function $\boldsymbol{F}$ in 2-d or 3-d space is conservative

$$\boldsymbol{F}(x, y) = P(x, y)\,\boldsymbol{i} + Q(x, y)\,\boldsymbol{j}$$

If $\Phi(x, y)$ satisfies

$$\nabla \Phi(x, y) = \frac{\partial \Phi}{\partial x}\boldsymbol{i} + \frac{\partial \Phi}{\partial y}\boldsymbol{j}$$

$$= P(x,y)\boldsymbol{i} + Q(x,y)\boldsymbol{j}$$

$$\begin{cases} \dfrac{\partial}{\partial x}\Phi(x,y) = P(x,y) \\ \dfrac{\partial}{\partial y}\Phi(x,y) = Q(x,y) \end{cases}$$

$$\boldsymbol{F}(x,y,z) = P(x,y,z)\boldsymbol{i} + Q(x,y,z)\boldsymbol{j} + R(x,y,z)\boldsymbol{k}$$

If $\Phi(x, y, z)$ satisfies

$$\nabla \Phi(x, y, z) = \frac{\partial \Phi}{\partial x}\boldsymbol{i} + \frac{\partial \Phi}{\partial y}\boldsymbol{j} + \frac{\partial \Phi}{\partial z}\boldsymbol{k}$$

$$= P(x,y,z)\boldsymbol{i} + Q(x,y,z)\boldsymbol{j} + R(x,y,z)\boldsymbol{k}$$

$$\begin{cases} \dfrac{\partial}{\partial x}\Phi(x,y,z) = P(x,y,z) \\ \dfrac{\partial}{\partial y}\Phi(x,y,z) = Q(x,y,z) \\ \dfrac{\partial}{\partial z}\Phi(x,y,z) = R(x,y,z) \end{cases}$$

Fundamental Line Integral Theorem (1)

$\boldsymbol{F}$ can be written as the gradient of a scalar function Φ $\quad \boldsymbol{F} = \nabla \Phi$

⇒ A vector function $\boldsymbol{F}$ in 2-d or 3-d space is conservative

Conservative vector field $\quad \boldsymbol{F}(x, y) = P(x, y)\,\boldsymbol{i} + Q(x, y)\,\boldsymbol{j}$

⇒ $\int_C \nabla \Phi \cdot d\boldsymbol{r} = \Phi(B) - \Phi(A)$ $\quad A = (f(a), g(a)), \quad B = (f(b), g(b))$

$\int_a^b f'(x)\, dx = f(b) - f(a)$

think as a differentiation

$\boldsymbol{F} = \nabla \Phi$ $\quad \Phi^{\nabla}(x, y)$

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \nabla \Phi \cdot d\boldsymbol{r} = \Phi(B) - \Phi(A)$$

Fundamental Line Integral Theorem (2)

Conservative vector field $\boldsymbol{F}(x, y) = P(x, y)\,\boldsymbol{i} + Q(x, y)\,\boldsymbol{j}$

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_C \nabla\Phi\cdot d\boldsymbol{r} = \Phi(B) - \Phi(A)$$

$$A = (f(a), g(a))$$
$$B = (f(b), g(b))$$

$$\boldsymbol{F}(x, y) = \nabla\Phi(x, y) = \frac{\partial\Phi}{\partial x}\boldsymbol{i} + \frac{\partial\Phi}{\partial y}\boldsymbol{j}$$

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_C \boldsymbol{F}\cdot\frac{d\boldsymbol{r}(t)}{dt}dt = \int_C \boldsymbol{F}\cdot\boldsymbol{r}'(t)dt$$

$$= \int_C \left(\frac{\partial\Phi}{\partial x}\boldsymbol{i} + \frac{\partial\Phi}{\partial y}\boldsymbol{j}\right)\cdot\left(\frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j}\right)dt \quad = \int_C \left(\frac{\partial\Phi}{\partial x}\frac{dx}{dt} + \frac{\partial\Phi}{\partial y}\frac{dy}{dt}\right)dt$$

$$= \int_C \left(\frac{\partial\Phi}{dt}\right)dt$$

$$= \left[\Phi(x(t), y(t))\right]_a^b \quad = \Phi(x(b), y(b)) - \Phi(x(a), y(a))$$

$$= \Phi(B) - \Phi(A)$$

Connected Region (1)

Connected

Every pair of points A and B in the region can be joined by a piecewise smooth curve that lies entirely in the region

Simply Connected

Connected and every simple closed curve lying entirely within the region can be shrunk, or contracted, to a point without leaving the region

➡ The interior of the curve lies also entirely in the region

➡ No holes in the region

Disconnected

Cannot be joined by a piecewise smooth curve that lies entirely in the region

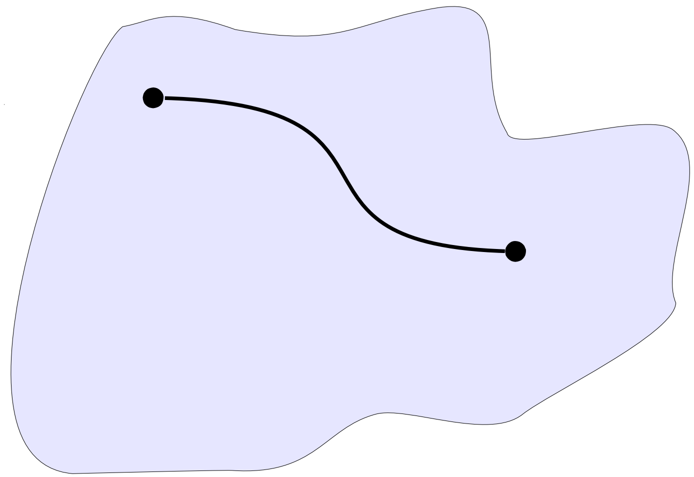
Multiply Connected Many holes within the region

Open Connected Contains no boundary points

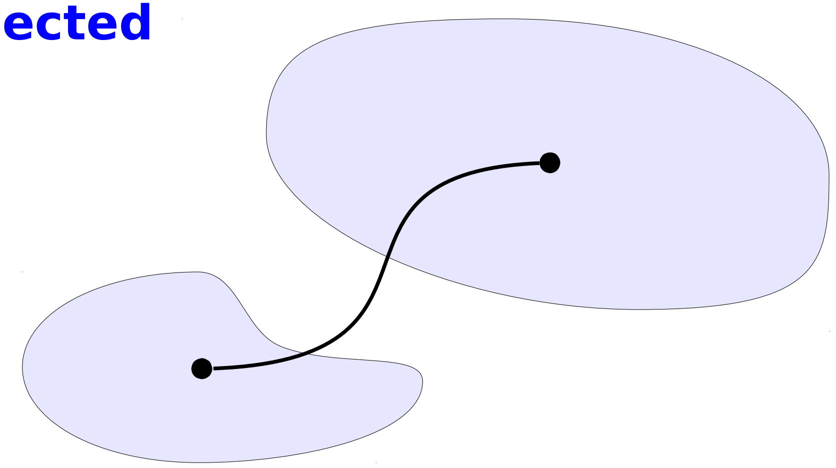
Connected Region (2)

Connected

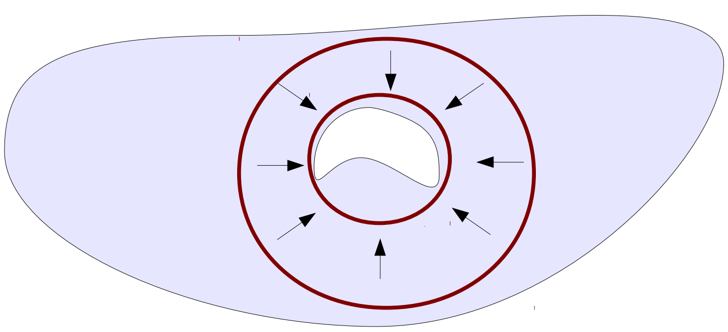
Simply Connected



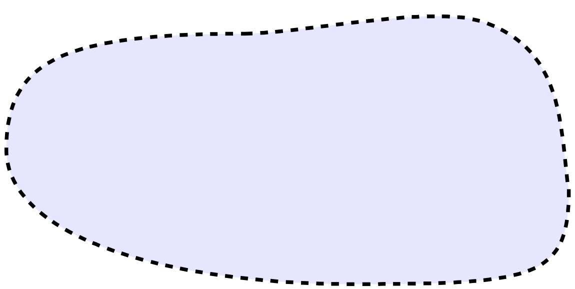
Disconnected



Multiply Connected



Open Connected



Equivalence

In an open connected region

Path Independence $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$

⟺ Conservative $\boldsymbol{F} = \nabla \Phi$

⟺ Closed path C $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$

Test for a Conservative Field

$\boldsymbol{F}(x,y) = P(x,y)\,\boldsymbol{i} + Q(x,y)\,\boldsymbol{j}$: **conservative** vector field in an **open** region R

$\Rightarrow$ $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$\boldsymbol{F}(x,y) = P(x,y)\,\boldsymbol{i} + Q(x,y)\,\boldsymbol{j}$: **conservative** vector field in R

$\Leftarrow$ $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ for all points in a **simply connected** region R

$$\boldsymbol{F} = \nabla\Phi = \frac{\partial\Phi}{\partial x}\boldsymbol{i} + \frac{\partial\Phi}{\partial y}\boldsymbol{j} \quad\Rightarrow\quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\begin{cases} P = \dfrac{\partial\Phi}{\partial x} \\[2ex] Q = \dfrac{\partial\Phi}{\partial y} \end{cases} \qquad \begin{cases} \dfrac{\partial P}{\partial y} = \dfrac{\partial^2\Phi}{\partial x\,\partial y} \\[2ex] \dfrac{\partial Q}{\partial x} = \dfrac{\partial^2\Phi}{\partial x\,\partial y} \end{cases}$$

Equivalence in 3-D

In an open connected region

Path Independence $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ ⟺

Conservative $\boldsymbol{F} = \nabla \Phi$ ⟺

Closed path C $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ ⟺

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

$$curl\,\boldsymbol{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right)\boldsymbol{k}$$

$$\boldsymbol{F} = P\,\boldsymbol{i} + Q\,\boldsymbol{j} + R\,\boldsymbol{k} \qquad \boldsymbol{F} = \nabla \Phi = \frac{\partial \Phi}{\partial x}\boldsymbol{i} + \frac{\partial \Phi}{\partial y}\boldsymbol{j} + \frac{\partial \Phi}{\partial z}\boldsymbol{k}$$

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y + \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x \quad = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y$$

Flux density

$$= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

Divergence of **F** Flux Density

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] D.G. Zill, “Advanced Engineering Mathematics”