All Pass Filter (2A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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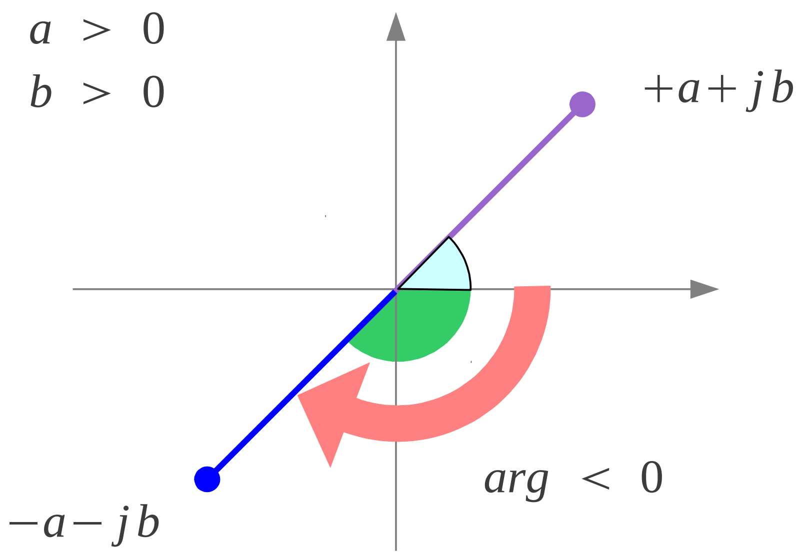
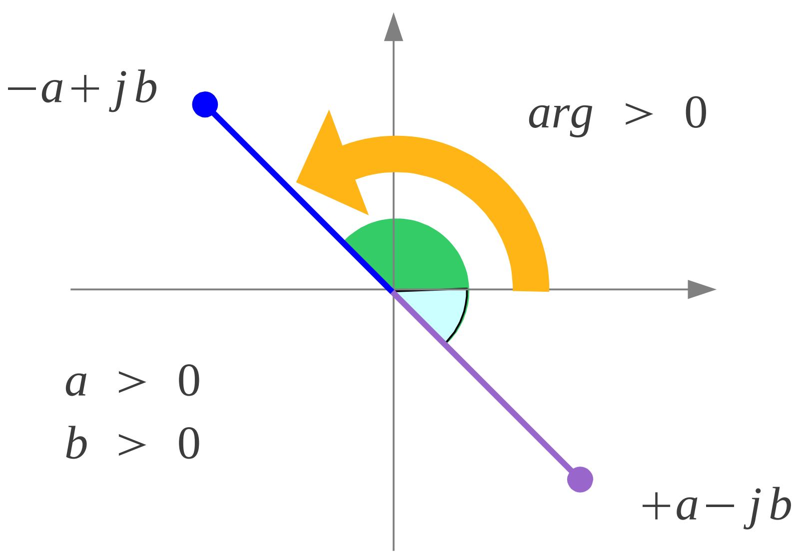
Phase and $\tan^{-1}(b/a)$ (1)





$$\tan^{-1}\left(\frac{-b}{-a}\right) = \tan^{-1}\left(\frac{+b}{+a}\right)$$

$$arg\{-a-jb\} = arg\{+a+jb\} - \pi$$

$$\tan^{-1}\left(\frac{+b}{-a}\right) = \tan^{-1}\left(\frac{-b}{+a}\right)$$

$$arg\{-a+jb\} = arg\{+a-jb\} + \pi$$

Phase and $\tan^{-1}(b/a)$ (2)

$x < 0$, $y > 0$	$x > 0$, $y > 0$
$x < 0$, $y < 0$	$x > 0$, $y < 0$

$\tan^{-1}\left(\frac{y}{x}\right) + \pi$	$\tan^{-1}\left(\frac{y}{x}\right)$
$\tan^{-1}\left(\frac{y}{x}\right) - \pi$	$\tan^{-1}\left(\frac{y}{x}\right)$

Octave function

atan2(y, x)

$$arg\{x + jy\} = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & (x > 0) \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & (x < 0,\ y > 0) \\ \tan^{-1}\left(\frac{y}{x}\right) - \pi & (x < 0,\ y < 0) \end{cases}$$

atan(y / x)

atan(y / x) + 3.14

atan(y / x) – 3.14

All Pass Filter Example (1)

$$H_{all}(s) = \frac{1 - 2s}{1 + 2s}$$

Flat Magnitude

$$\left|\frac{1 - j2\omega}{1 + j2\omega}\right| = \frac{|1 - j2\omega|}{|1 + j2\omega|}$$

$$= \frac{\sqrt{1 + 4\omega^2}}{\sqrt{1 + 4\omega^2}} = 1$$

$$|H_{all}(j\omega)| = \frac{\sqrt{1 + 4\omega^2}}{\sqrt{1 + 4\omega^2}} = 1$$

$$\frac{1 - j2\omega}{1 + j2\omega} = \frac{1 - j2\omega}{1 + j2\omega} \cdot \frac{1 - j2\omega}{1 - j2\omega}$$

$$= \frac{(1 - 4\omega^2) - j4\omega}{1 + 4\omega^2}$$

$$rg\{H_{all}(j\omega)\} = \tan^{-1}\left(\frac{-4\omega}{1 - 4\omega^2}\right)$$

A Pure Phase Shifter

$$arg\left\{\frac{1 - j2\omega}{1 + j2\omega}\right\} = arg\{1 - j2\omega\} - arg\{1 + j2\omega\}$$

$$= \tan^{-1}\left(\frac{-2\omega}{1}\right) - \tan^{-1}\left(\frac{2\omega}{1}\right) = -2\tan^{-1}(2\omega)$$

$$arg\{H_{all}(j\omega)\} = -2\tan^{-1}(2\omega)$$

All Pass Filter Example (2)

$$H_{all}(s) = -\frac{s - 0.5}{s + 0.5} \quad \Leftarrow \quad H_{all}(s) = \frac{1 - 2s}{1 + 2s}$$

Flat Magnitude

$$\left|\frac{j\omega - 0.5}{j\omega + 0.5}\right| = \frac{|\ j\omega - 0.5\ |}{|\ j\omega + 0.5\ |}$$

$$= \frac{\sqrt{\omega^2 + 0.25}}{\sqrt{\omega^2 + 0.25}} = 1$$

$$|H_{all}(j\omega)| = \frac{\sqrt{1 + 4\omega^2}}{\sqrt{1 + 4\omega^2}} = 1$$

$$\frac{-j\omega + 0.5}{+j\omega + 0.5} = \frac{-j\omega + 0.5}{+j\omega + 0.5} \cdot \frac{-j\omega + 0.5}{-j\omega + 0.5}$$

$$= \frac{(-\omega^2 + 0.25) - j\omega}{\omega^2 + 0.25}$$

$$arg\{H_{all}(j\omega)\} = \tan^{-1}\left(\frac{-\omega}{0.25 - 4\omega^2}\right)$$

A Pure Phase Shifter

$$arg\left\{\frac{-j\omega + 0.5}{+j\omega + 0.5}\right\} = arg\{-j\omega + 0.5\} - arg\{+j\omega + 0.5\}$$

$$= \tan^{-1}\left(\frac{-\omega}{+0.5}\right) - \tan^{-1}\left(\frac{+\omega}{+0.5}\right) = -2\tan^{-1}(2\omega)$$

$$arg\{H_{all}(j\omega)\} = -2\tan^{-1}(2\omega)$$

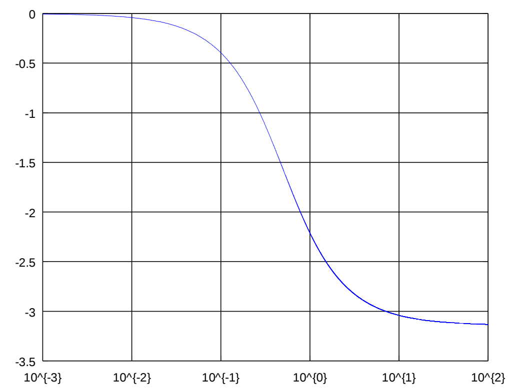
$$-arg\left\{\frac{j\omega - 0.5}{j\omega + 0.5}\right\} = -arg\{j\omega - 0.5\} + arg\{j\omega + 0.5\}$$

$$= -\tan^{-1}\left(\frac{\omega}{-0.5}\right) + \tan^{-1}\left(\frac{\omega}{0.5}\right) = -(-\tan^{-1}(2\omega) + \pi) + \tan^{-1}(2\omega)$$

$$arg\{H_{all}(j\omega)\} = -\pi + 2\tan^{-1}(2\omega)$$

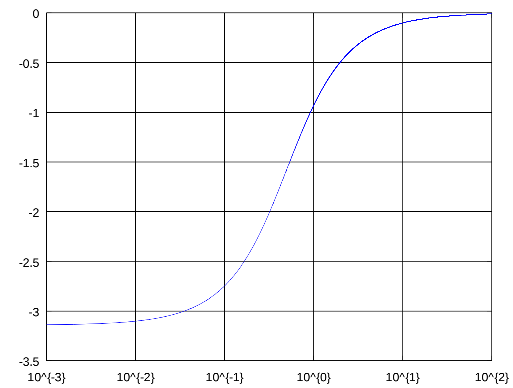
All Pass Filter Example (3)

$$\arg\{H_{all}(j\omega)\} = -2\tan^{-1}(2\omega)$$



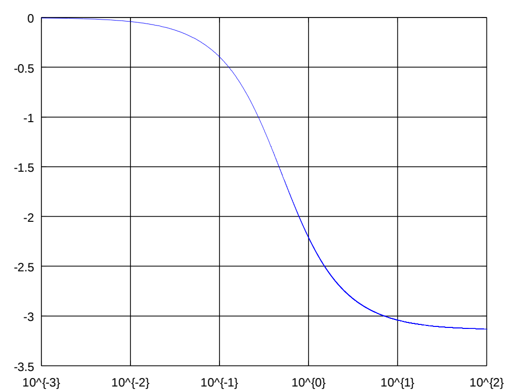
semilogx(...)

$$\arg\{H_{all}(j\omega)\} = -\pi + 2\tan^{-1}(2\omega)$$



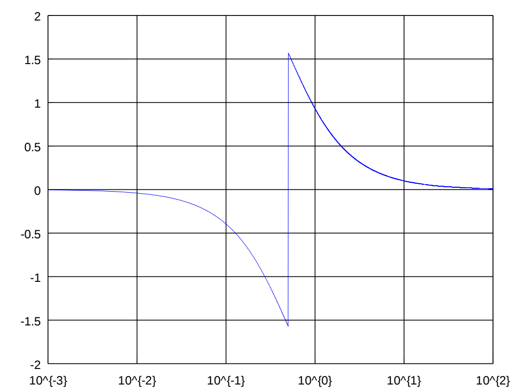
semilogx(...)

$$\arg\{H_{all}(j\omega)\} = -\mathbf{atan2}(4\omega, 1-4\omega^2)$$



semilogx(...)

$$\arg\{H_{all}(j\omega)\} = -\mathbf{atan}\left(\frac{4\omega}{1-4\omega^2}\right)$$



semilogx(...)

All Pass Filter Example (2)

$$H_{all}(s) = \frac{s - 0.5}{s + 0.5} \quad \Leftrightarrow \quad H_{all}(s) = \frac{1 - 2s}{1 + 2s}$$

Flat Magnitude

$$\left|\frac{j\omega - 0.5}{j\omega + 0.5}\right| = \frac{|\ j\omega - 0.5\ |}{|\ j\omega + 0.5\ |}$$

$$= \frac{\sqrt{\omega^2 + 0.25}}{\sqrt{\omega^2 + 0.25}} = 1$$

$$|H_{all}(j\omega)| = \frac{\sqrt{1 + 4\omega^2}}{\sqrt{1 + 4\omega^2}} = 1$$

$$\frac{+j\omega - 0.5}{+j\omega + 0.5} = \frac{+j\omega - 0.5}{+j\omega + 0.5} \cdot \frac{-j\omega + 0.5}{-j\omega + 0.5}$$

$$= \frac{(\omega^2 - 0.25) + j\omega}{\omega^2 + 0.25}$$

$$arg\{H_{all}(j\omega)\} = \tan^{-1}\left(\frac{\omega}{\omega^2 - 0.25}\right)$$

A Pure Phase Shifter

$$arg\left\{\frac{+j\omega - 0.5}{+j\omega + 0.5}\right\} = arg\{+j\omega - 0.5\} - arg\{+j\omega + 0.5\}$$

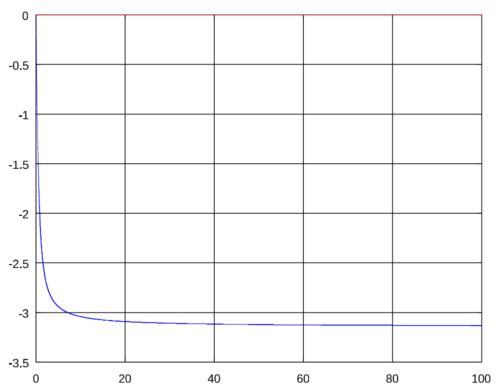
$$= \tan^{-1}\left(\frac{+\omega}{-0.5}\right) - \tan^{-1}\left(\frac{+\omega}{+0.5}\right) = \pi - 2\tan^{-1}(2\omega)$$

$$arg\{H_{all}(j\omega)\} = \pi - 2\tan^{-1}(2\omega)$$

Example - All Pass Filter (1)

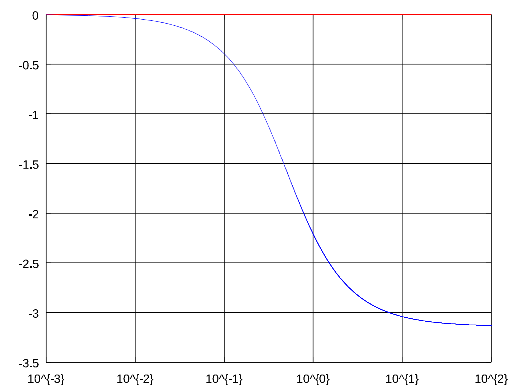
$$arg\{H_{all}(j\omega)\} = -2\tan^{-1}(2\omega)$$

0



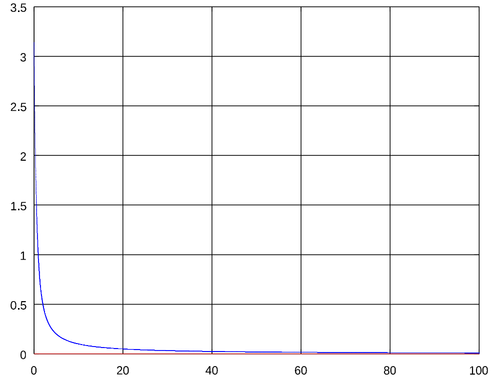
plot(...)

0



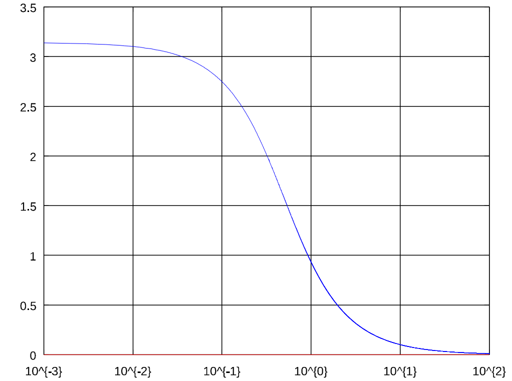
semilogx(...)

$$arg\{H_{all}(j\omega)\} = \pi - 2\tan^{-1}(2\omega)$$



0

plot(...)



0

semilogx(...)

Analog All Pass Filter (1)

$$H_{all}(s) = \frac{s-a}{s+a}$$

$$H_{all}(j\omega) = \frac{j\omega - a}{j\omega + a}$$

Impulse Response

$$H_{all}(s) = \frac{s-a}{s+a} = \frac{s+a-2a}{s+a}$$

$$H(s) = 1 - \frac{2a}{(s+a)}$$

Inverse Laplace Transform

$$h(t) = \delta(t) - e^{-at}$$

Flat Magnitude

$$|H_{all}(j\omega)| = \frac{\sqrt{\omega^2 + a^2}}{\sqrt{\omega^2 + a^2}} = 1$$

Phase Shifter

$$arg\{H_{all}(j\omega)\} = \pi - 2\tan^{-1}\left(\frac{\omega}{a}\right)$$

Positive Group Delay

$$-\frac{d}{d\omega}\left(arg\{H_{all}(j\omega)\}\right)$$

$$= -\frac{d}{d\omega}\left(\pi - 2\tan^{-1}\left(\frac{\omega}{a}\right)\right)$$

$$= \frac{4}{(1 + \omega^2/a^2)} > 0$$

Analog All Pass Filter (2)

$$G_{all}(s) = \pm \frac{(s-\bar{s}_1)(s-\bar{s}_2)\ \cdots\ (s-\bar{s}_n)}{(s-s_1)(s-s_2)\ \cdots\ (s-s_n)}$$

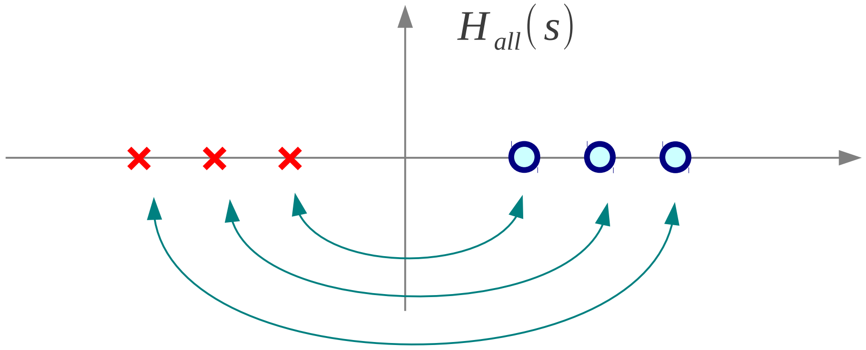
Flat Magnitude

A Pure Phase Shifter

zero $\bar{s}_i$

pole s_i

not complex conjugate



$s_i = +a + jb$

$\bar{s}_i = -a + jb$

only differ in the signs of their real parts

Digital All Pass Filter (1)

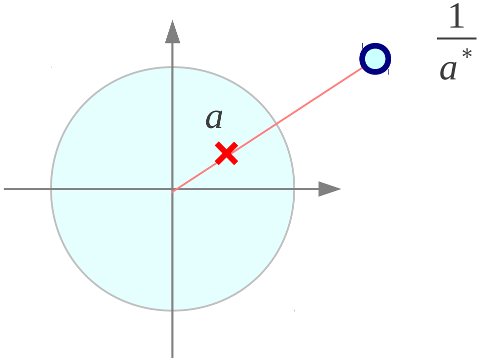
$$H(z) = \frac{z^{-1} - a^*}{1 - a z^{-1}}$$

$z^{-1} = a^* \rightarrow z = \frac{1}{a^*}$

$a z^{-1} = 1 \rightarrow z = a$

Flat Magnitude

A Pure Phase Shifter



$$H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - a e^{-j\omega}} = e^{-j\omega} \frac{1 - a^* e^{+j\omega}}{1 - a e^{-j\omega}}$$

$$(1 - a^* e^{+j\omega})^* = (1 - a e^{-j\omega})$$

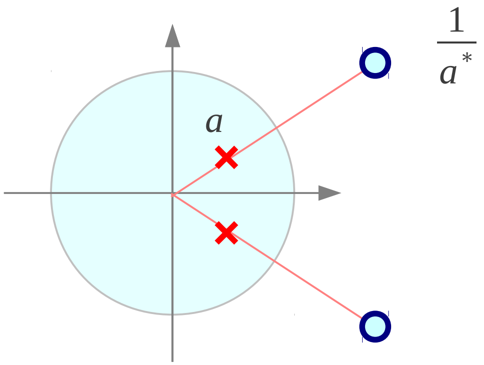
$$|H(e^{j\omega})| = |e^{-j\omega}| \left| \frac{1 - a^* e^{+j\omega}}{1 - a e^{-j\omega}} \right| = 1$$

Digital All Pass Filter (2)

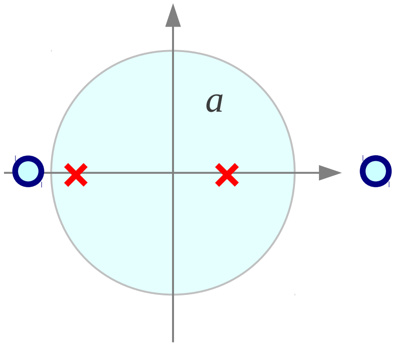
$$H_{all}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

Flat Magnitude

A Pure Phase Shifter

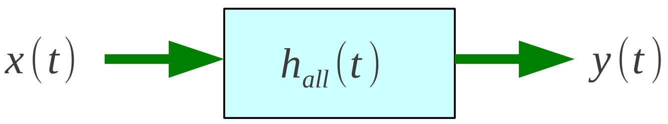


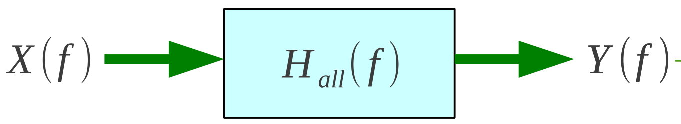
Cascade form of all pass system
for real-valued impulse response
system



Conjugate symmetric $H(e^{j\omega})$

All Pass Filter – Parseval's Theorem





Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |y(t)|^2\, dt$$

Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |X(f)|^2\, df$$

$$\int_{-\infty}^{+\infty} |y(t)|^2\, dt = \int_{-\infty}^{+\infty} |Y(f)|^2\, df$$

Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2\, dt \geq \int_{-\infty}^{t_0} |y(t)|^2\, dt$$

$$Y(f) = H_{all}(f)\, X(f)$$

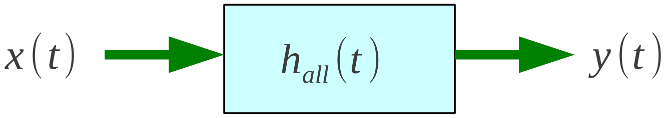
$$= \int_{-\infty}^{+\infty} |H_{all}(f)|^2 |X(f)|^2\, dt$$

Allpass Filter

$$|H_{all}(f)| = 1$$

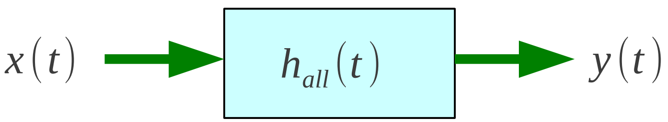
$$\int_{-\infty}^{+\infty} |Y(f)|^2\, df = \int_{-\infty}^{+\infty} |X(f)|^2\, df$$

$$\int_{-\infty}^{+\infty} |y(t)|^2\, dt = \int_{-\infty}^{+\infty} |x(t)|^2\, dt$$



The energy build-up in the input is more **rapid** than in the output

All Pass Filter – Energy Compaction

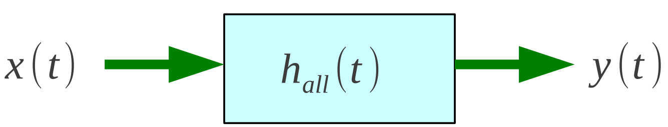


Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |y(t)|^2 \, dt$$

Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2 \, dt \geq \int_{-\infty}^{t_0} |y(t)|^2 \, dt$$



The energy build-up in the input is more **rapid** than in the output

Truncated input

$$x_1(t) = \begin{cases} x(t) & (t \leq t_0) \\ 0 & (t > t_0) \end{cases}$$

Corresponding output

$$y_1(t)$$

For $t \leq t_0$ → $x_1(t) = x(t)$ → $y_1(t) = y(t)$

$$y_1(t) = \int_{-\infty}^{t_0} h(t-\tau) x_1(\tau) \, d\tau = \int_{-\infty}^{t} h(t-\tau) x(\tau) \, d\tau = y(t)$$

For $t > t_0$ → $x_1(t) = 0$ → $y_1(t) \neq 0$

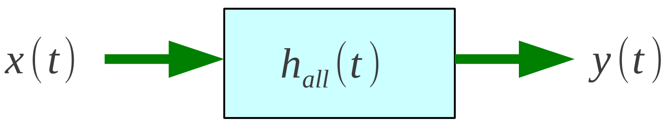
$$\int_{-\infty}^{+\infty} |x_1(t)|^2 \, dt = \int_{-\infty}^{+\infty} |y_1(t)|^2 \, dt$$

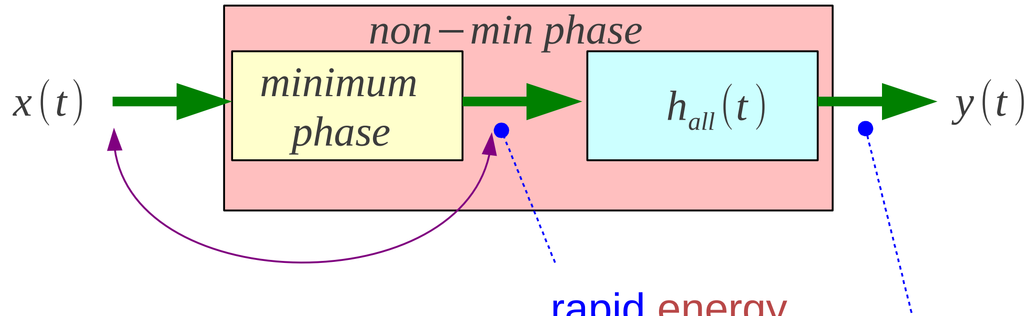
$$\int_{-\infty}^{t_0} |x_1(t)|^2 \, dt = \int_{-\infty}^{t_0} |y_1(t)|^2 \, dt + \int_{t_0}^{+\infty} |y_1(t)|^2 \, dt$$

For $t \leq t_0$

$$\int_{-\infty}^{t_0} |x(t)|^2 \, dt \geq \int_{-\infty}^{t_0} |y(t)|^2 \, dt$$

All Pass Filter (6)





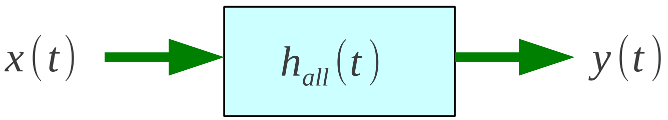
rapid energy
build-up than the output

Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |y(t)|^2 \, dt$$

Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2 \, dt \geq \int_{-\infty}^{t_0} |y(t)|^2 \, dt$$



The energy build-up in the input
is more **rapid** than in the output

The signal energy until t_0 of the minimum phase
≥ any other causal signal
with the same magnitude response

Thus minimum phase signals
➡ maximally concentrated toward time 0
when compared against all causal signals
having the same magnitude response

minimum phase signals
➡ minimum delay signals

Properties of a Minimum Phase System

Properties of a Minimum Phase System

Properties of a Minimum Phase System

Properties of a Minimum Phase System

Properties of a Minimum Phase System

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
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[4] K.Shin, J.K. Hammond, "Fundamentals of Signal Processing for Sound and Vibration Engineers"
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