With 4 zero stress comp. on p. 24-2 obtain Eq. (2) on p. 27-1

HW5: Do the other 2 eqs of equil, i.e., $j = 2, 3$ in Eq. (4), p. 27-2.

Derivation of Eq. (4), p. 27-2. (3-D case)

Do 1-D model pb 1st.

p. 24-3: $\sum F_x = 0 = -\sigma(x) A + \sigma(x+dx) A + f(x)\, dx$

$$\Rightarrow 0 = A \underbrace{\left[ \sigma(x+dx) - \sigma(x) \right]}_{= \frac{d\sigma(x)}{dx} dx + \underbrace{h.o.t.}_{\text{higher order terms}}} + f(x)\, dx$$

Taylor series

Recall: $f(x+dx) = f(x) + \frac{df(x)}{dx} dx + \frac{1}{2} \frac{d^2 f(x)}{dx^2} (dx)^2 + \ldots$

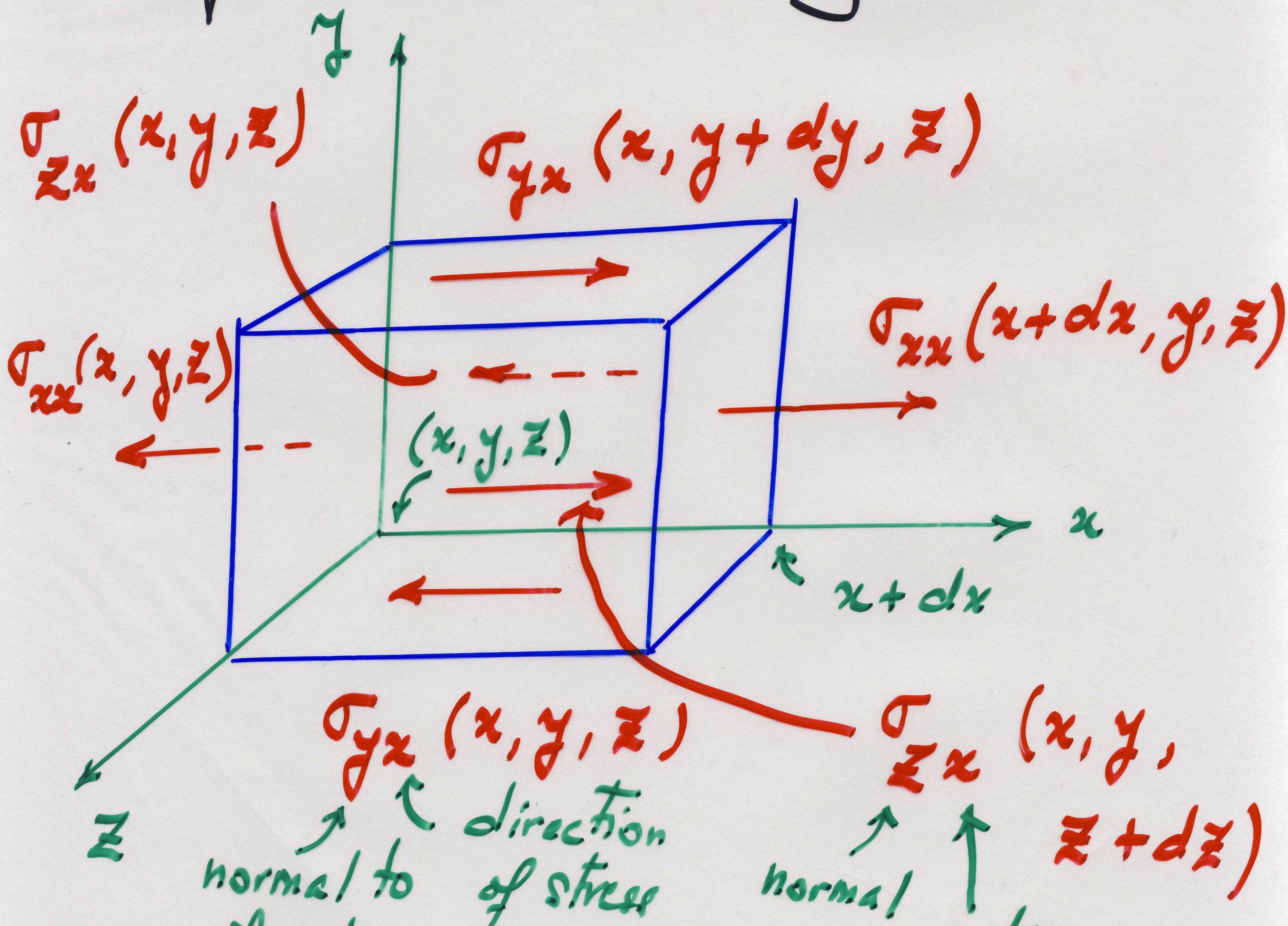
Now, neglect h.o.t.

$$\Rightarrow \boxed{\frac{d\sigma}{dx} + \frac{f(x)}{A} = 0}$$

body force = force/vol. (Jared)

$f(x)$ = force/length. applied load.

Now, nonuniform stress field in 3-D, but w/o applied load, and focusing on the x dir. only (i.e., w/o the other stress comp. to avoid cluttering fig.)




(Fig. 2.9 for complete pict.)

$$\sum F_x = 0 = dy\,dz\,[-\sigma_{xx}(x,y,z)$$

$$+ \sigma_{xx}(x+dx, y, z)]$$ facets w/ normal $x$

$$+ dz\,dx\,[-\sigma_{yx}(x,y,z) +$$

$$+ \sigma_{yx}(x, y+dy, z)]$$ facets w/ normal $y$

$$+ dx\,dy\,[-\sigma_{zx}(x,y,z)$$

$$+ \sigma_{zx}(x, y, z+dz)]$$ facets w/ normal $z$

$$\Rightarrow 0 = (dx\,dy\,dz)\left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right]$$

$\Rightarrow$ Eq. (3) p. 27.2.

---

Roadmap: p. 16.2

D. Prandtl stress function $\phi$.

Mtg 29 : Mon, 3 Nov 08

→ Plate buckling + dimen. Anal. + HW6.

- Dimensional analysis for eq. of equil in terms of stresses

- Continue elasticity theory for torsional Anal

Roarks', Formula for stress and Strain, 7th ed., Young & Budynas, p. 730:

$$\boxed{\sigma' = K \frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2}$$

For $\frac{a}{b} = 1 \Rightarrow K = 3.29.$

MIT + my wiki:

$$\sigma_{cr} = k_c \frac{\pi^2 D}{b^2 h} = \left(m\frac{b}{a} + \frac{a}{mb}\right)^2 \pi^2 \left(\frac{1}{b^2 \cancel{h}}\right)\left(\frac{E h^{\cancel{3}\,2}}{1-\nu^2}\right)$$

$$= \underbrace{\left(\frac{mb}{a} + \frac{a}{mb}\right)^2 \frac{\pi^2}{12}}_{K} \left(\frac{E}{1-\nu^2}\right)\left(\frac{h}{b}\right)^2$$

For $\frac{a}{b} = 1$, $m = 1 \Rightarrow \underbrace{\frac{(1+1)^2 \pi^2}{12}}_{"\frac{\pi^2}{3}} = K = 3.29$

## Eq. of equil (cont'd)

Recall 1-D case  p. 28-2 : Dimen. Anal.

$$[f] = \frac{F}{L}$$

Force

length

"Dimen. of"

$$[A] = L^2$$

$$\left[\frac{f}{A}\right] = \frac{F}{L^3}$$

$$[\sigma] = \frac{F}{L^2} \Rightarrow \left[\frac{d\sigma}{dx}\right] = \frac{[d\sigma]}{[dx]} = \frac{F/L^2}{L}$$

$$[dx] = L \qquad = \underbrace{F/L^3}$$

force / vol or body force.

HW6: Plate buckling wiki page.

$$\left.\begin{array}{l} \varepsilon = \dfrac{du}{dx} \\ \varepsilon = \dfrac{\Delta L}{L} \end{array}\right\} [\varepsilon] = \frac{[du]}{[dx]} = \frac{L}{L} = 1$$ non. dim

Steve

$$\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} \Rightarrow [\nu] = \frac{[\varepsilon_{yy}]}{[\varepsilon_{xx}]} = 1$$ Josh

Torsional Anal. Cont'd.

p. 27-2
p. 29-1 } $\left[ \dfrac{\partial \sigma_{ij}}{\partial x_i} \right] = \dfrac{F}{L^3}$ = force/vol.

**HW6:** Read & Report (R&R)

(3.14) p. 27-1

---

Roadmap p. 16-2: D. Prandtl stress func. $\phi$

(1) $$\boxed{\sigma_{yx} = \frac{\partial \phi}{\partial z} \quad , \quad \sigma_{zx} = - \frac{\partial \phi}{\partial y}}$$ cf. (3.15)

$\phi$ plays role of a potential func.,

$(\sigma_{yx}, \sigma_{zx})$ comp. of "gradient" of $\phi$ wrt $(y, z)$.

Recall: Scalar func. $f(x, y, z)$

$$\underbrace{\text{grad } f(x,y,z)}_{\text{a vector}} = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

(Math) $f$ = potential func.

Eq. (1) on p. 30-1 satisfies automatically Eq. (1) p. 27-1.

MaH:
$$\frac{\partial}{\partial y}\underbrace{\left(\frac{\partial \phi}{\partial \cancel{x}\, z}\right)}_{\sigma_{yx}} + \frac{\partial}{\partial z}\underbrace{\left(-\frac{\partial \phi}{\partial y}\right)}_{\sigma_{zx}}$$

$$= \frac{\partial^2 \phi}{\partial y\,\partial z} - \frac{\partial^2 \phi}{\partial z\,\partial y} = 0,$$ since

$\phi$ cont. & smooth $\Rightarrow$ 2nd mixed deriv. interchangeable.

i.e.,
$$\frac{\partial^2 \phi}{\partial y\,\partial z} = \frac{\partial^2 \phi}{\partial z\,\partial y}.$$

HW6: R & R (3.15) $\rightarrow$ (3.19)

$$\boxed{\underbrace{\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}}_{\nabla^2 \phi \text{ (Laplacian of } \phi)} = -2G\theta}$$

R & R until end of Sec. 3.2.

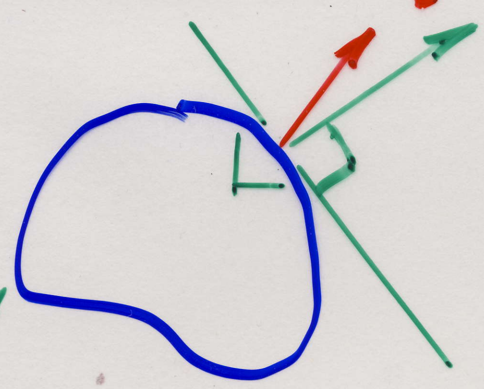
Need to explain (derive):

(2.30)
$$\underbrace{\{t\}}_{3\times1} = \underbrace{[\sigma]}_{3\times3}\ \underbrace{\{n\}}_{3\times1}$$
(eq. below (3.19)) chap. 2

comp. of traction force $\vec{t}$ — comp. of stress tensor — comp. of normal vector $\vec{n}$

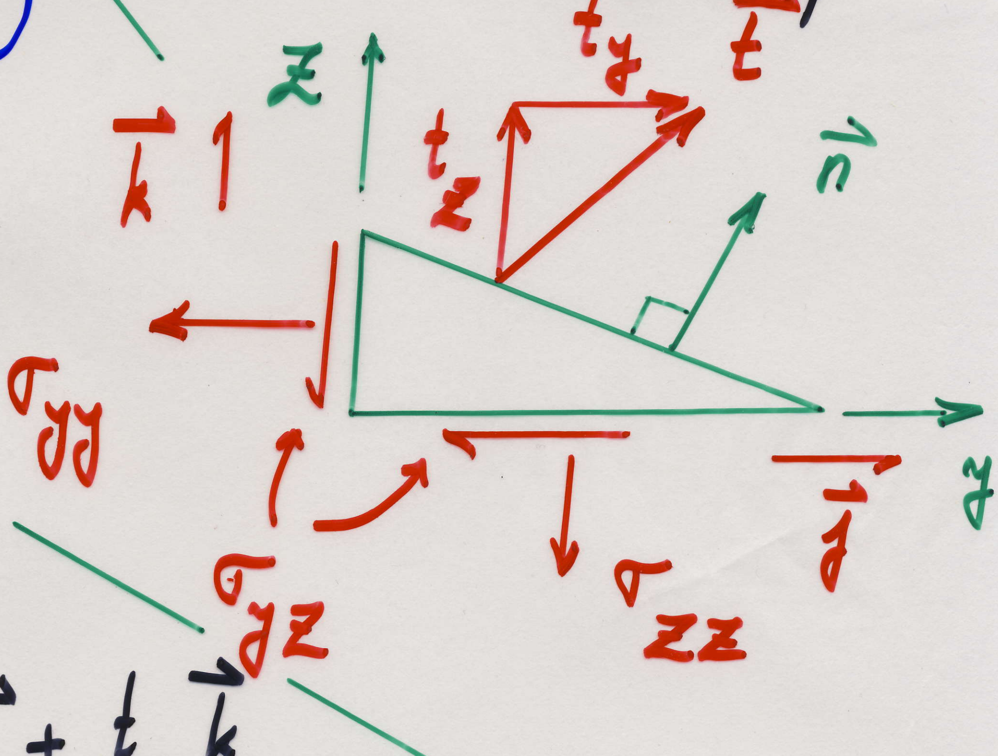
2-D case first: $\vec{t}$ p. 29

$\vec{n}$ (normal)

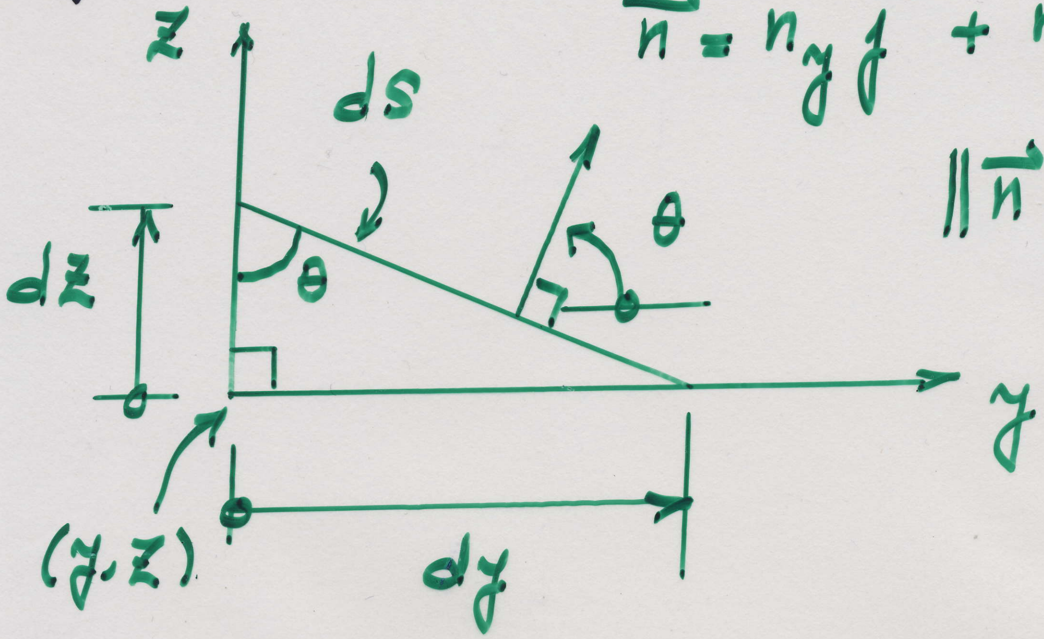


S (lateral surf. of bar)

p.26-1: Example of $\vec{t}$ in aircraft.




$$\vec{t} = t_y\vec{j} + t_z\vec{k}$$




$\vec{n} = n_y \vec{j} + n_z \vec{k}$

$\|\vec{n}\| = 1$

$\vec{n}$: unit normal vect.

$$\begin{cases} dz = ds \cdot \cos\theta \\ dy = ds \cdot \sin\theta \end{cases} \qquad \begin{cases} n_y = \cos\theta \\ n_z = \sin\theta \end{cases}$$

$$\sum F_y = 0 = -\sigma_{yy} \cdot (dz \cdot 1)$$

$$- \sigma_{yz} \cdot (dy \cdot 1)$$

$$+ t_y \cdot (ds \cdot 1)$$

← unit depth along x axis

$$\Rightarrow 0 = -\sigma_{yy} \cdot \cancel{ds} \cdot n_y - \sigma_{yz} \cdot \cancel{ds} \cdot n_z$$

$$+ t_y \cdot \cancel{ds}$$

$$\Rightarrow \boxed{t_y = \sigma_{yy} \cdot n_y + \sigma_{yz} \cdot n_z} \quad (1)$$

Note: $[t_y] = \dfrac{F}{L^2}$ force/area

$\vec{t}$ : traction vector (distr. surf. force)

$[t_y] = [\sigma]$

$\sum F_z = 0$ HW6

$$\Rightarrow \boxed{t_z = \sigma_{yz} \cdot n_y + \sigma_{zz} \cdot n_z} \quad (2)$$

(1) & (2) :

$$\boxed{\begin{Bmatrix} t_y \\ t_z \end{Bmatrix} = \begin{bmatrix} \sigma_{yy} & \sigma_{yz} \\ \underset{\displaystyle \parallel \atop \displaystyle \sigma_{yz}}{\sigma_{zy}} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} n_y \\ n_z \end{Bmatrix}} \quad (3)$$

Generalization to 3D:

$$\boxed{\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}} \quad (4)$$

$$t_i = \sum_{j=1}^{3} \sigma_{ij} \, n_j \quad , \quad i = 1,2,3$$

row col.

$$\Rightarrow \boxed{\underset{3\times 1}{\{t_i\}} = \underset{3\times 3}{[\sigma_{ij}]} \underset{3\times 1}{\{n_j\}}}$$

Col. mat Col. mat.

<u>Note:</u> row mat.

$$\{t_i\}^T = \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix}^T = \underset{1\times 3}{\lfloor t_1 \;\; t_2 \;\; t_3 \rfloor}$$

$\equiv$

Roadmap p. 16-2:

G. Bound. cond. for $\phi$ (3.24)

$$\boxed{\phi = \text{const on lateral surf. of the bar.}}$$

(3.24a)

- Rube-Goldberg device: How to run an airplane; Quotations (inspiring): Narrative
- Best of HW5: Team VQ Crew's narrative of computed results
- Plate buckling wiki page: (Sum Sec 7.7)
    * continue to improve / add material
    * $k_c^*$, $m^*$
    * Clamped rect. plate: Roark's formula

  HW6: Verify for SS plate, K for $\frac{a}{b} = 1$ and $m = 1$

- Torsional analysis of bar w/ circular cross section (Sec. 3.3)
    * no warping: HW 6.

  Roadmap p. 16-2:

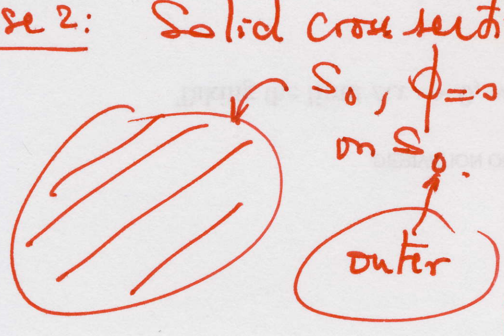
  G. Bound. cond. for $\phi$

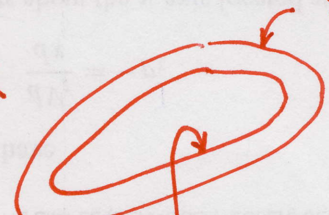
  Case 1: Thin-walled cross section

  $S_o$, $\phi = C_o$ on $S_o$

  outer

Case 2: Solid cross section



$S_i \equiv S_1$, $\phi = C_1$ on $S_1$

inner

**Mtg 32:** Mon, 10 Nov 08. EAS4200C

- Rube-Goldberg device
- Plate buckling: cont'd

HW6: Find rel. betw. $K$ and $k_c$

Verify $K$ for $a/b = 1$, $m = 1$.

---

**Torsional Anal: cont'd** p. 31-3

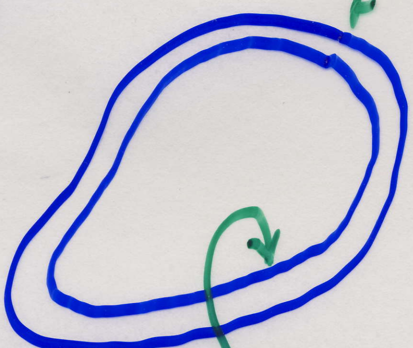
**Case 1:** Thin-walled cross section (closed)

e.g. NACA airfoil

cf. Fig. 3.11

$S_o$, $\phi = C_o$ on $S_o$ (3.40)

↑ outer



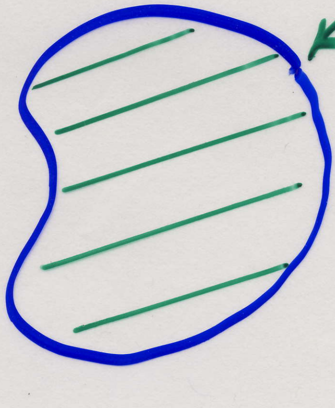
$S_i$, $\phi = C_1$ on $S_1$ (3.41)

$S_1 \equiv$ ↑ inner

**Case 2:** Solid cross section

$S_o$, $\phi = 0$ on $S_o$

(3.24b)



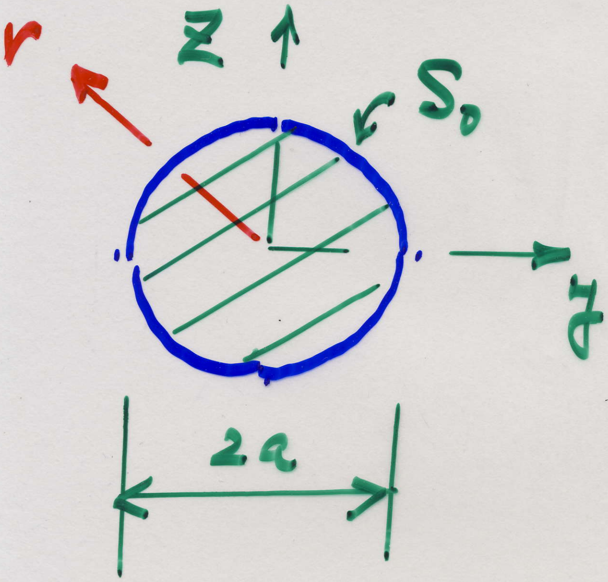
Uniform bar w/ solid, circular cross sect

p. 16-3: Roadmap, expr. for (Sec 3.3)

T and J in terms of $\phi$.

$$\phi(y,z) = C\left(\frac{y^2}{a^2} + \frac{z^2}{\underset{\overset{\shortparallel}{(b=a)}}{b^2}} - 1\right)$$

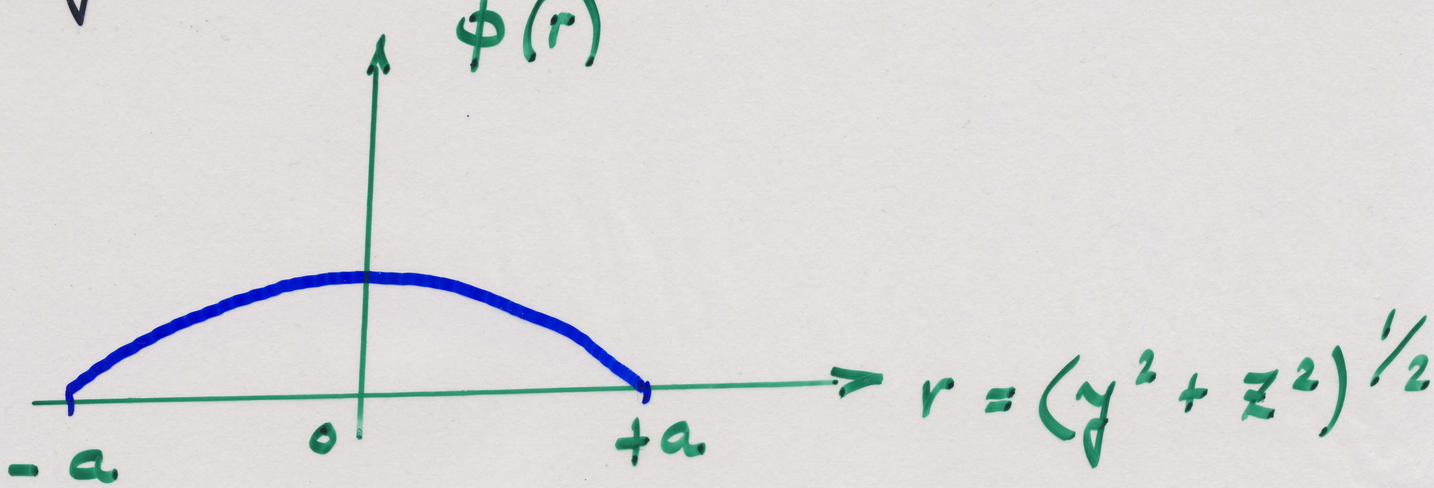
(circle)




$\Rightarrow \phi = 0$ on $S_0$ HW6

$\cancel{\Rightarrow}$ $\nabla^2\phi = -2G\theta$

$\Rightarrow C = -\frac{1}{2}a^2 G\theta$




$$T = 2\int_A \phi \, dA = 2C\left(\frac{J}{a^2} - \underset{\pi a^2}{\underset{\parallel}{A}}\right)$$

$$J = \int_A r^2 dA = \frac{1}{2}\pi a^4$$

$$\Rightarrow \boxed{T = GJ\theta} \qquad (3.28)$$

$$\sigma_{yx} = \frac{\partial \phi}{\partial z} = -G\theta z \qquad (3.29)$$

$$\sigma_{zx} = -\frac{\partial \phi}{\partial y} = G\theta y \qquad (3.30)$$

R+R thru end Sec 3.3.

Also: $\tau = \dfrac{Tr}{J}$ p. 14-2

Using (3.16), (3.29), (3.30) and $\sigma$-$\varepsilon$ rel., show that $\boxed{u_x(y,z)=0}$

i.e., no warping. (ad hoc assump. p.14-2)

**Hint:**

(3.16ab) $$\gamma_{yx} = \frac{\sigma_{yx}}{G} = \frac{\partial u_x}{\partial y} - \theta z$$

$$\gamma_{zx} = \frac{\sigma_{zx}}{G} = \frac{\partial u_x}{\partial z} + \theta y$$

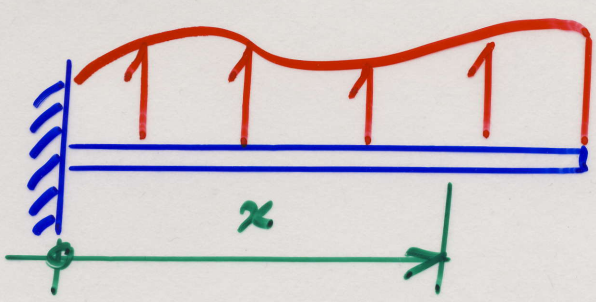
(3.29) $$\sigma_{yx} = - G\theta z$$

(3.30) $$\sigma_{zx} = G\theta y$$

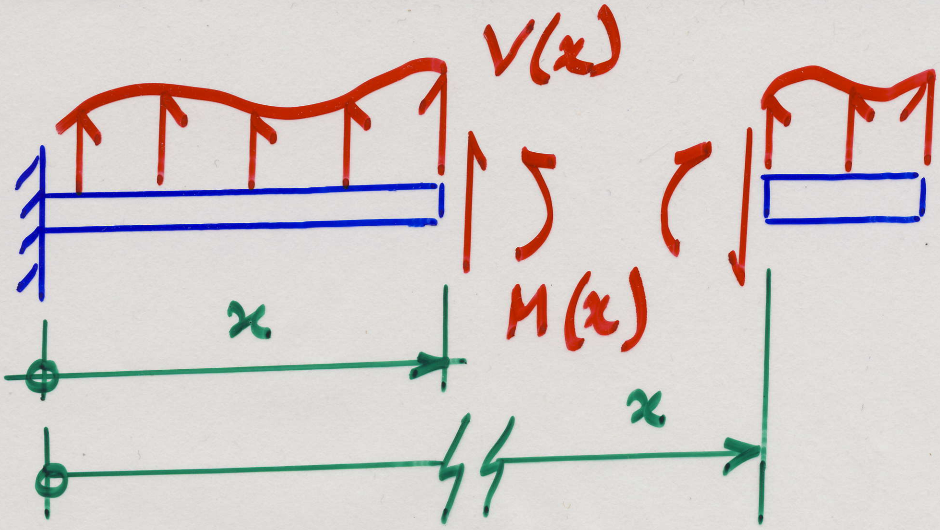
---

## Flexural Shear flow in thin-walled Sections

(Chap. 5) 5.1, 5.1.1, 5.1.2, 5.1.3, 5.3, 5.3.1, 5.3.2, 5.4




p. 26-1



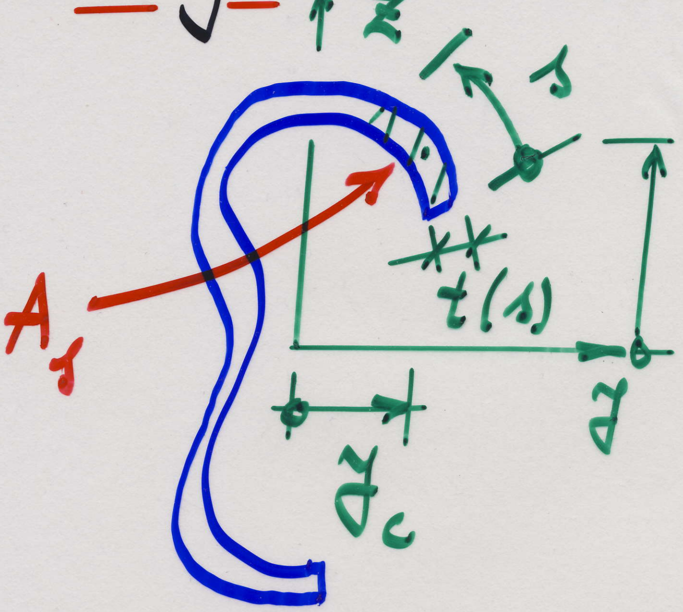

HW5: Effect of $M(x)$ on NACA airfoil

HW6: Plate buckling, compressive in-plane load
NACA airfoil

HW7: Effects of $V(x)$ on NACA airfoil
Plate buckling, shear load.

Secs 5.1, 5.1.1, 5.1.2

Unsym. thin walled cross sect.




$$\boxed{\int_{A_s} \frac{d\sigma_{xx}}{dx}\, dA = -q_s \quad (5.1)}$$

$q(s)$

general for unsym. cross sect.

**Sym. about y axis**

$$\sigma_{xx} = \frac{M_y z}{I_y}$$

$$q(s) = -\frac{V_z Q_y}{I_y} \quad (5.2)$$

$$Q_y = \int_{A_s} z\, dA = z_c A_s$$

**Unsym. cross sect.**

$$\sigma_{xx} = (k_y M_z - k_{yz} M_y)\, y + (k_z M_y - k_{yz} M_z)\, z \quad (5.3)$$

$$k_y = \frac{I_y}{D} \qquad \text{p. 26-2}$$

$$k_{yz} = I_{yz} / D$$

$$k_z = I_z / D$$

$$\underset{1\times 1}{\sigma_{xx}} = \underset{1\times 2}{\lfloor y \quad z \rfloor} \underbrace{\begin{bmatrix} k_y & -k_{yz} \\ -k_{yz} & k_z \end{bmatrix} \begin{Bmatrix} M_z \\ M_y \end{Bmatrix}}_{\underset{2\times 1}{\begin{Bmatrix} k_y M_z - k_{yz} M_y \\ -k_{yz} M_z + k_z M_y \end{Bmatrix}}}$$

$$\boxed{\sigma_{xx} = \lfloor z \quad y \rfloor \begin{bmatrix} k_z & -k_{yz} \\ -k_{yz} & k_y \end{bmatrix} \begin{Bmatrix} M_y \\ M_z \end{Bmatrix}}$$

Particularize to sym. cross sect. $\boxed{I_{yz} = 0}$ Why? HW6

Consider $M_z = 0$.

$I_{yz} = 0 \Rightarrow \quad D = I_y I_z$

$$k_y = \frac{1}{I_z}, \quad k_{yz} = 0, \quad k_z = \frac{1}{I_y}$$

$$\Rightarrow \quad \sigma_{xx} = \frac{z M_y}{I_y}$$

Sym. | Non-sym.

$$q(s) = -\left(k_y V_y - k_{yz} V_z\right) Q_z - \left(k_z V_z - k_{yz} V_y\right) Q_y \qquad (5.5)$$

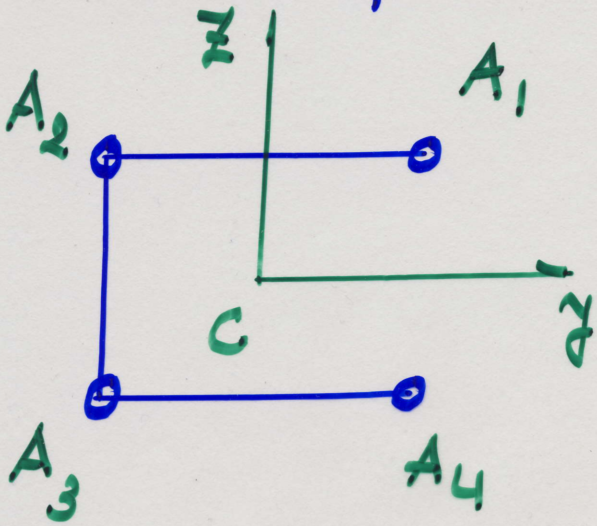
$$Q_z = \int_{A_s} y \, dA \ , \quad Q_y = \int_{A_s} z \, dA$$

HW6: Put (5.5) in matrix form and recover (5.2) as particular case.

Stringer - web sections: thickness $t$ of skin and spar webs very small $\Rightarrow$ neglect in comp. $I_y$, $I_z$, $I_{yz}$, $Q_y$, $Q_z$ (Use only areas of stringers).

Sym. wrt y axis

Ex 5.2 p. 154




$A_3 = A_2$

$A_4 = A_1$

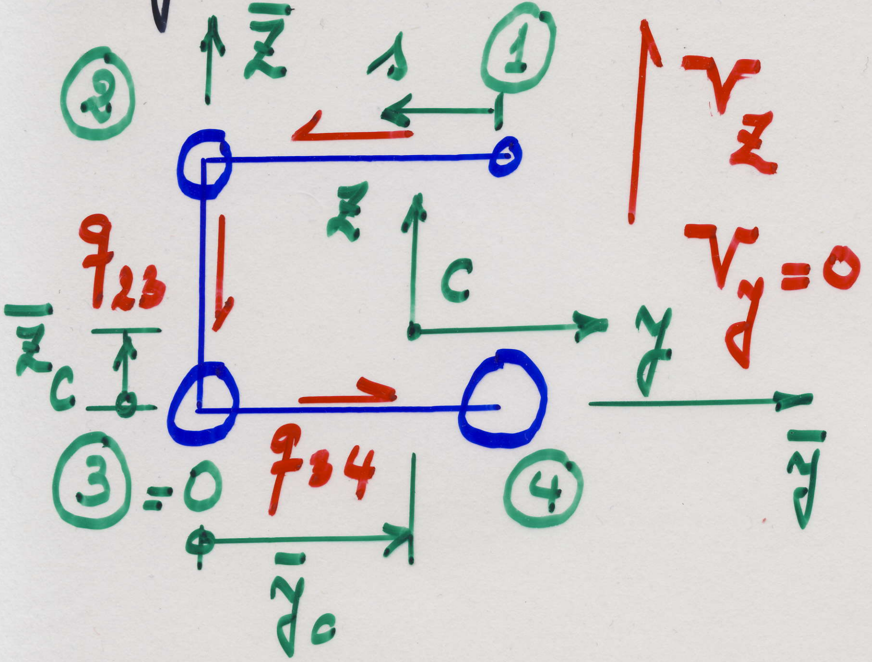
Non-sym

$A_1 = A$

$A_2 = 2A$

$A_3 = 3A$

$A_4 = 4A$




Mean Value Thm "Average" (MVT)

$$\int_A \bar{y}\, dA = \bar{y}_c \int_A dA = \bar{y} A$$

$$\int_A \bar{z}\, dA = \bar{z}_c A$$

"average"

$$A = \sum_{i=1}^{4} A_i$$ (neglect skin + spar webs)

cf. HW5

(5.5) p. 34.3

$$q(s) = \left( k_{yz} Q_z^{(s)} - k_z Q_y^{(s)} \right) V_z$$

Since:
1) $V_z$
2) $k_{yz}$, $k_z$
3) $\underbrace{Q_z^{(s)}, Q_y^{(s)}}_{\text{all areas concentr. on stringers.}}$ ← const betw 2 stringers

indep. of s

$\Rightarrow$ shear flow $q(s)$ const. betw. 2 stringers but $q(s)$ would increment (jump) when crossing a stringer.

step 1: Find $(\bar{y}_c, \bar{z}_c)$ and $(y_i, z_i)$, $i=1,\dots,4$ HW7

step 2: Find $I_y$, $I_z$, $I_{yz}$ (Gabriel)

step 3: Find $k_y$, $k_z$, $k_{yz}$

step 4: Follow path "s" to find

$$\underbrace{q_{12}}_{\text{shear flow in skin panel 12}}, \quad q_{23}, \quad q_{34}$$

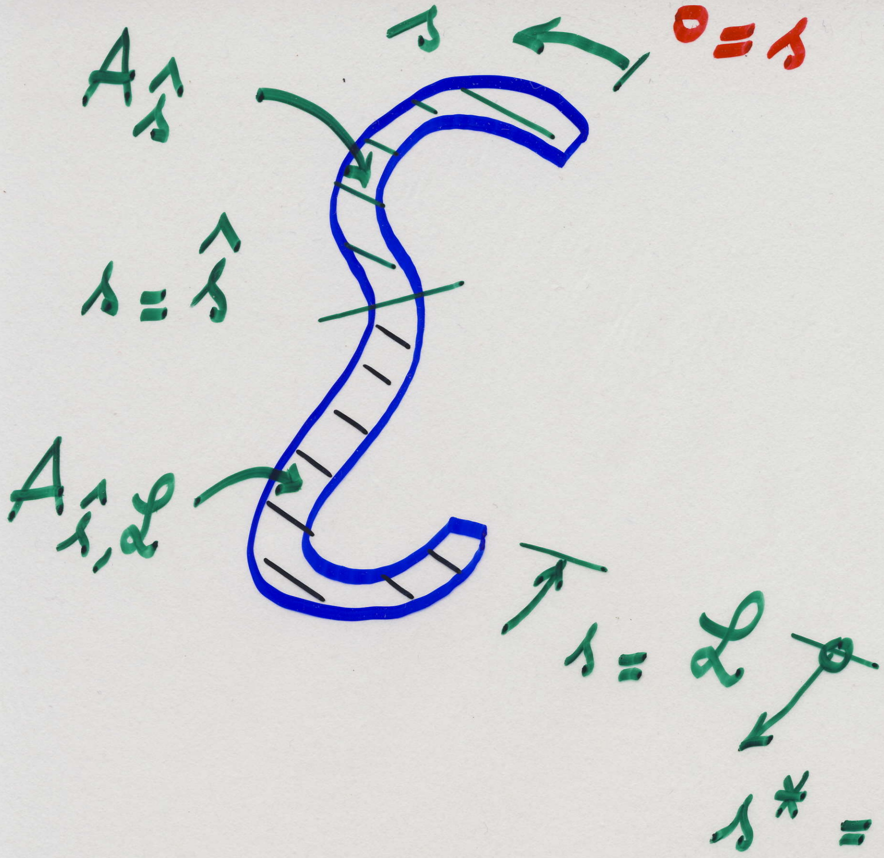
$$q_{12} = \left(k_{yz} Q_z^{12} - k_z Q_y^{12}\right) V_z$$

$$Q_z^{12} = y_1 A_1$$

$y_1$: y coord. of stringer 1

$$Q_y^{12} = z_1 A_1$$

Note: $Q_z^{23} = y_1 A_1 + y_2 A_2$




$$A = \underbrace{A_{\hat{s}}}_{= A_{0,\hat{s}}} + A_{\hat{s},\mathcal{L}}$$

(Jared)

To be cont'd soon

Mini plan: S) Single-cell sections

S.1) Without stringers

S.2) w/ stringers

M) Multi-cell sections

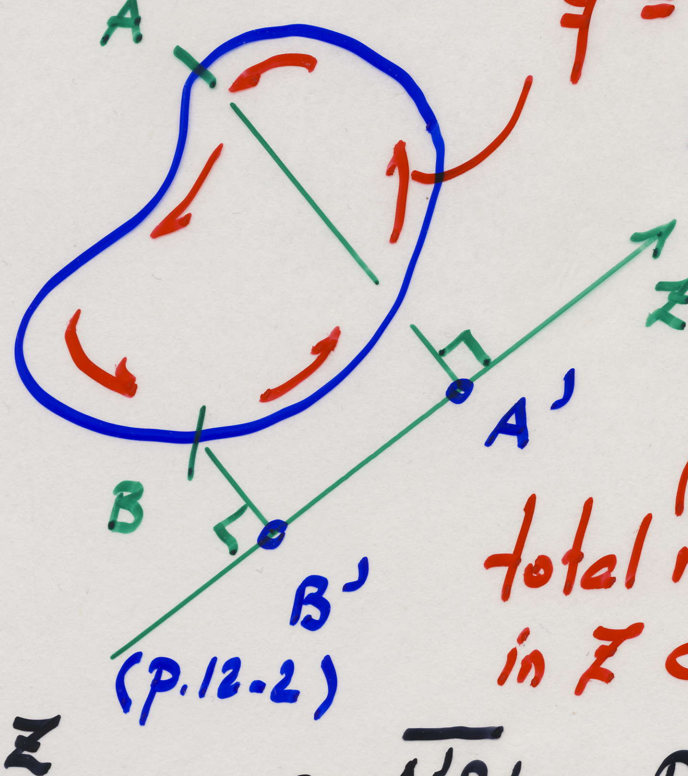
M.1) w/o stringers

M.2) w/ stringers

---

S) Single-cell

S.1) w/o stringers




(p.12-2)

q = const. shearflow

Q: Can this set up resist $V_z$ ?

$$R^z = R^z_{AB} + R^z_{BA}$$

total result. in z dir. — result. of q in AB — result. of q in BA

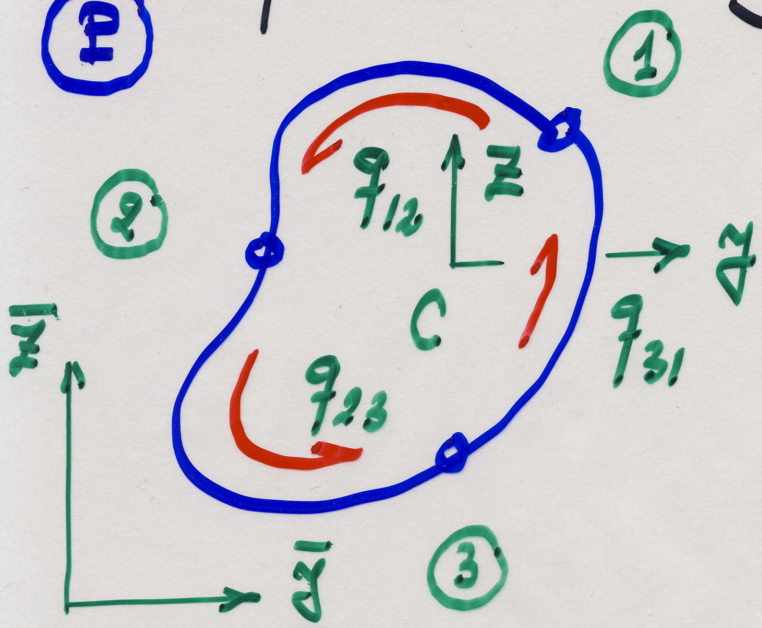
$$R^z_{AB} = -q\,\overline{A'B'} = -R^z_{BA}$$

$$\Rightarrow R^z = 0$$

S.2) w/ stringers ( neglect contrib. of web to bending. )



P




$q_{12} \neq q_{23} \neq q_{31}$

but $q_{ij}(s) = \text{const}$ within each panel.

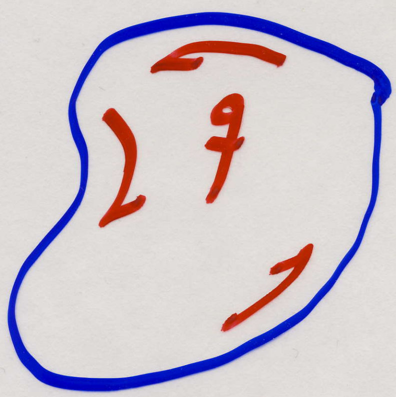
$\Rightarrow q(s)$ is piecewise const. wrt $s$

$$\boxed{R^z - V_z \neq 0}$$

Presence of stringers $\Rightarrow$ non const shear flow

||

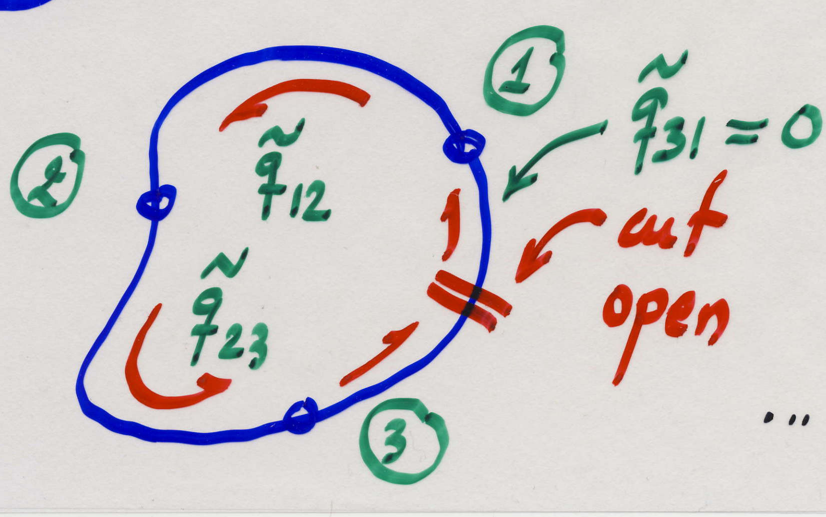
P1




+

P2

Princ. of superposition due to linearity




$q_{12} = q + \tilde{q}_{12}$

$q_{23} = q + \tilde{q}_{23}$

$$\dots \quad \boxed{q_{ij} = q + \tilde{q}_{ij}}$$

## Anal. Algo:

**Obs:** One unknown $q$ ($\tilde{q}_{ij}$ are known after solving P2) $\Rightarrow$ need 1 eq for 1 unknown.

**Method:** Data: $V_y, V_z$

1) Solve $P_2$ for $\tilde{q}_{12}$, $\tilde{q}_{23}$ ($\tilde{q}_{31} = 0$)

2) Moment eq.: Take mom. about any pt in plane $(y, z)$

   2.1) Superposition: $q_{ij} = q + \underbrace{\tilde{q}_{ij}}_{\text{known from 1)}}$

   2.2) Select pt $\bar{O}$ in plane $(y, z)$

$$\Sigma_{\bar{O}} \text{ Mom of } (V_y, V_z) = \Sigma_{\bar{O}} \text{ Mom of } q_{ij}$$

(1 eq. for 1 unknown $q$)