DTFT (3A)

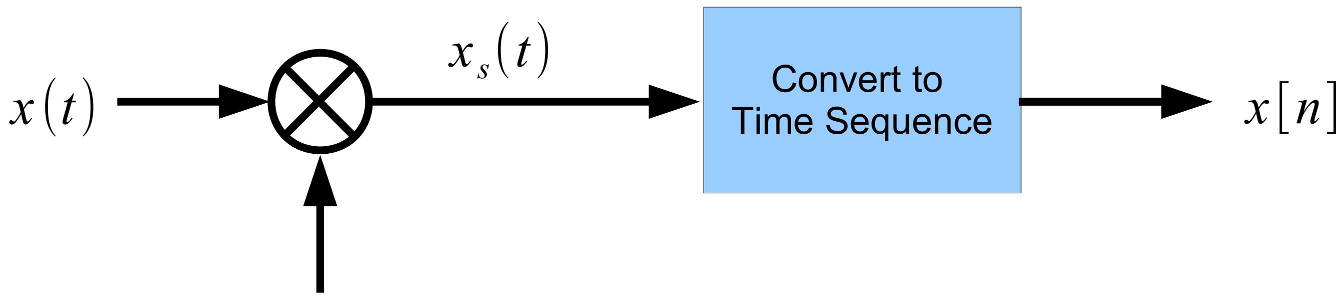
- Discrete Time Fourier Transform

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Sampling and Reconstruction

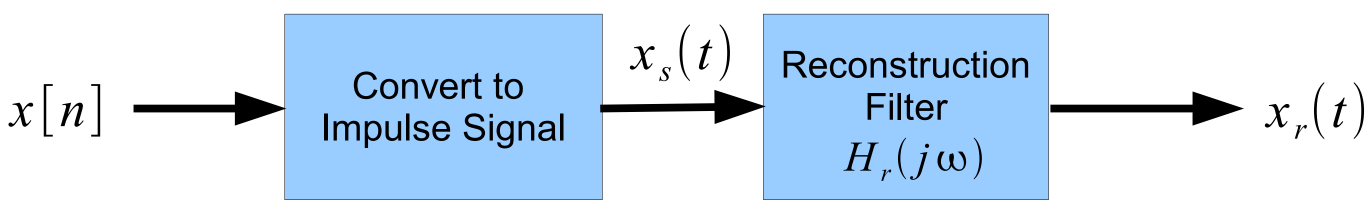
Ideal Sampling



$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

T_s **Sampling Period**

Ideal Reconstruction



CTFS of Impulse Train (1)

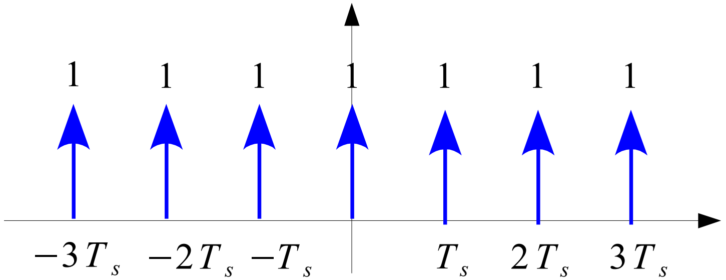
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Fourier Series

$$p(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{+jk\omega_s t}$$

Fourier Series Expansion

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{+jk\omega_s t}$$



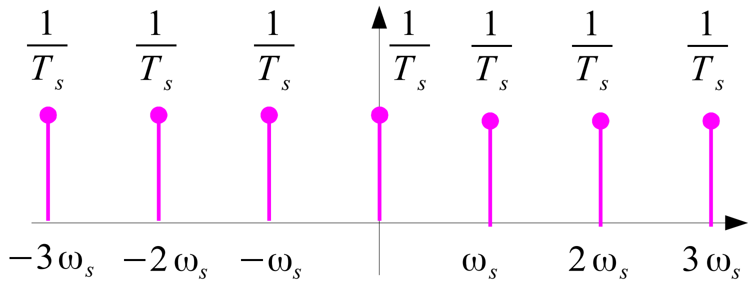
Fourier Series Coefficients

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t)\, e^{-jk\omega_s t}\, dt$$

$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t)\, e^{-jk\omega_s 0}\, dt$$

$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t)\, dt = \frac{1}{T_s}$$

$$\omega_s = \frac{2\pi}{T_s}$$



CTFS of Impulse Train (2)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Fourier Series

$$p(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{+jk\omega_s t}$$

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t}$$

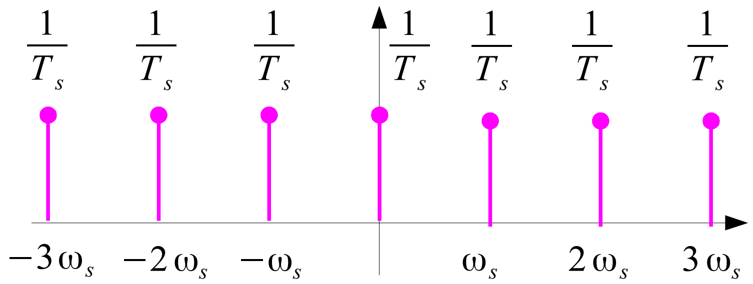
Fourier Transform

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_s)$$

Fourier Transform of impulse train

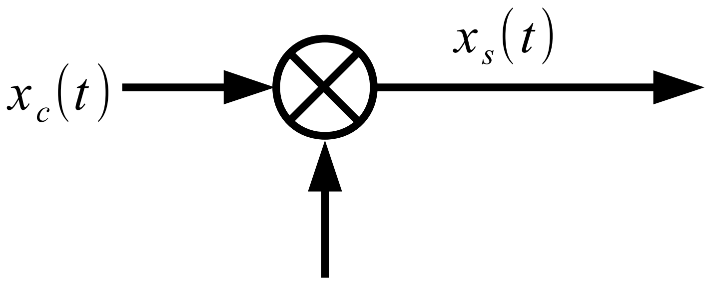
$$P(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s)$$

$$\omega_s = \frac{2\pi}{T_s}$$



Sampled Signal

Ideal Sampling



$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{+\infty} x_c(nT_s)\delta(t - nT_s)$$

$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT Frequency Shift Property

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \quad \Longleftrightarrow \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Frequency Shift Property

$$x_c(t) \quad \Longleftrightarrow \quad X_c(j\omega)$$

$$x_c(t)\, e^{jk\omega_s t} \quad \Longleftrightarrow \quad X_c(j(\omega - k\omega_s))$$

$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t} \quad \Longleftrightarrow \quad X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s} \qquad\qquad \omega_s = \frac{2\pi}{T_s}$$

CTFT Delay Property

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \iff X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Fourier Transform of an Impulse

$$\delta(t - t_d) \iff e^{j\omega t_d}$$

$$\delta(t - nT_s) \iff e^{-j\omega n T_s}$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, \delta(t - nT_s) \iff X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \quad \Longleftrightarrow \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x_s(t) = x_c(t)\sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\,\delta(t - nT_s)$$

CTFT $\Longleftrightarrow$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega nT_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega nT_s}$$

CTFS ||

DTFT ||

$$x_s(t) = x_c(t)\sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT $\Longleftrightarrow$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

z-Transform of a Sampled Signal

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$\Leftrightarrow$ =

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT of a sampled signal

$$\sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s} = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s)) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

Z-Transform of a sampled signal

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad x[n] = x_c(nT_s)$$

$$X(z)\Big|_{z = e^{j\omega T_s}} = X(e^{j\omega T_s})$$

evaluated at $z = e^{j\omega T_s}$

z-Transform and Normalized Frequency

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

=

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(z)\Big|_{z=e^{j\omega T_s}} = X(e^{j\omega T_s}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

z-Transform

$$\hat{\omega} = \omega T_s$$

Normalized Frequency

$$X(z)\Big|_{z=e^{j\hat{\omega}}} = X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

Discrete Time Fourier Transform

DTFT and CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

=

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

DTFT of a sampled signal

$$X(e^{j\hat{\omega}}) \Big|_{\hat{\omega} = \omega T_s} = X(e^{j\omega T_s}) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c\left(j\left(\omega - \frac{2\pi k}{T_s}\right)\right)$$

CTFT of a sampled signal

DTFT and CTFT

Continuous Time Fourier Transform CTFT

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Discrete Time Fourier Transform DTFT

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\hat{\omega}})\, e^{+j\hat{\omega} n}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

=

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

Dirichlet Function

Dirichlet Function

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega} L/2)}{L\sin(\hat{\omega}/2)} \qquad drcl(t, L) = \frac{\sin(\pi L t)}{L\sin(\pi t)} \qquad diric(x, N) = \frac{\sin(N x/2)}{N\sin(x/2)}$$

odd L ➡ Infinite sum of uniformly spaced sinc functions

$t = \frac{m}{L}$ ➡ $\sin(\pi L t) = 0$

$t = n$ ➡ $\sin(\pi t) = 0$

$$\lim_{t \to n} \frac{\sin(\pi L t)}{L\sin(\pi t)} = \lim_{t \to n} \frac{L\pi\cos(\pi L t)}{L\pi\cos(\pi t)} = \pm 1$$

integer n

odd L $\quad \lim_{t \to n} \frac{\cos(\pi L t)}{\cos(\pi t)} = +1$

$$\frac{\cos(-L2\pi)}{\cos(-2\pi)}, \frac{\cos(-L\pi)}{\cos(-\pi)}, \frac{\cos(0)}{\cos(0)}, \frac{\cos(L\pi)}{\cos(\pi)}, \frac{\cos(L2\pi)}{\cos(2\pi)}, \frac{\cos(L3\pi)}{\cos(3\pi)}, \cdots$$

even L $\quad \lim_{t \to n} \frac{\cos(\pi L t)}{\cos(\pi t)} = (-1)^n$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003