

# DTFT (3A)

---

- Discrete Time Fourier Transform

Copyright (c) 2009-2011 Young W. Lim.

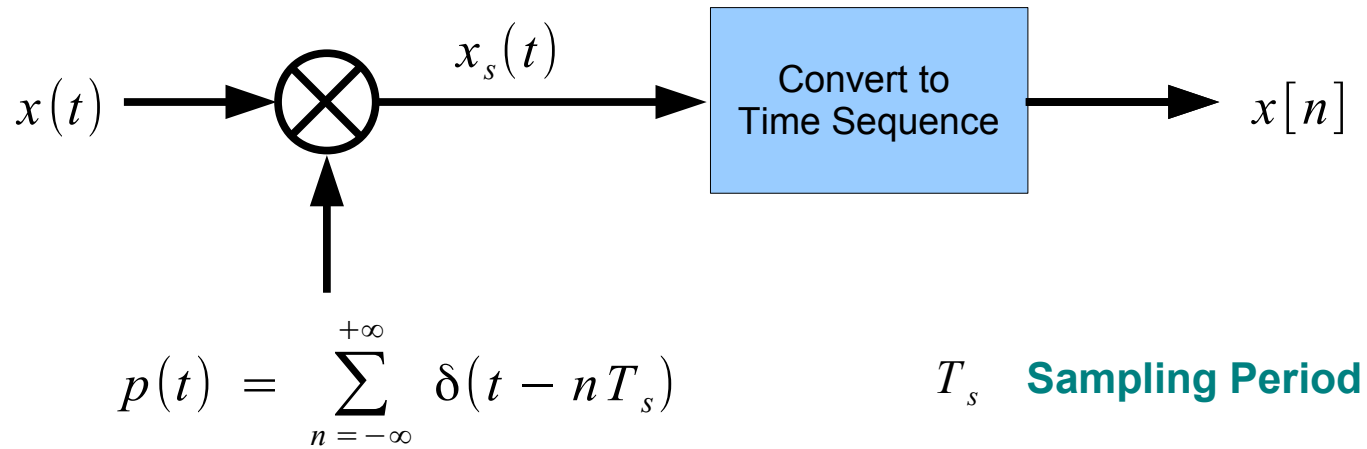
Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

Please send corrections (or suggestions) to [youngwlim@hotmail.com](mailto:youngwlim@hotmail.com).

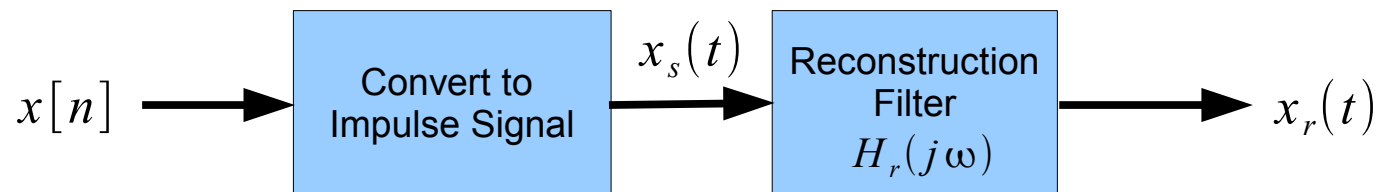
This document was produced by using OpenOffice and Octave.

# Sampling and Reconstruction

## Ideal Sampling



## Ideal Reconstruction



# CTFS of Impulse Train (1)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Fourier Series  

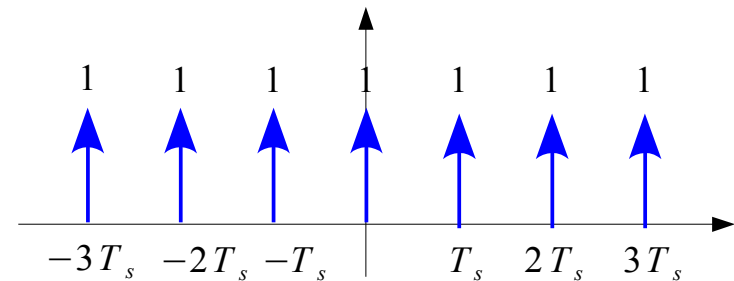

$$p(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{+jk\omega_s t}$$

## Fourier Series Expansion

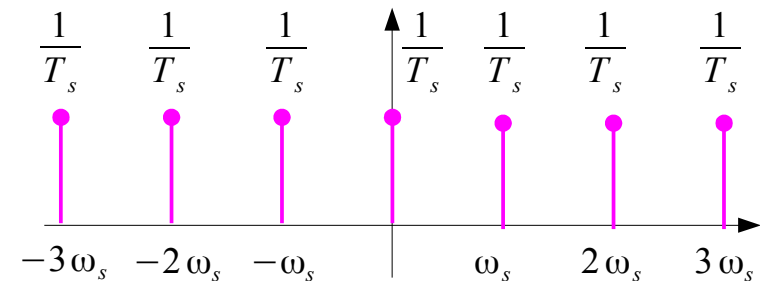
$$p(t) = \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t}$$

## Fourier Series Coefficients

$$\begin{aligned} a_k &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jk\omega_s 0} dt \\ &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) dt = \frac{1}{T_s} \end{aligned}$$



$$\omega_s = \frac{2\pi}{T_s}$$



# CTFS of Impulse Train (2)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Fourier Series



$$p(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{+jk\omega_s t}$$

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t}$$

Fourier Transform

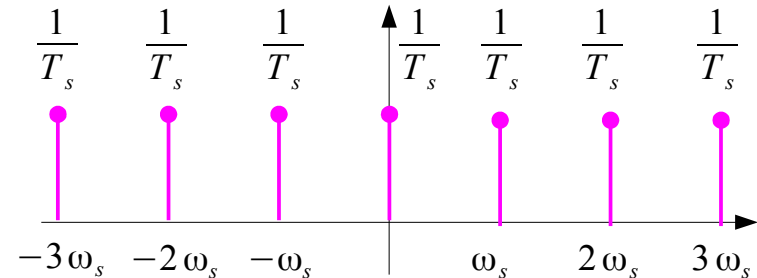


$$P(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_s)$$

Fourier Transform of impulse train

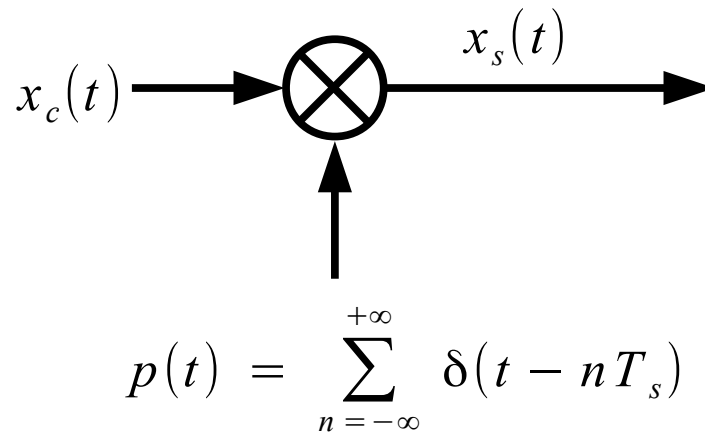
$$P(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s)$$

$$\omega_s = \frac{2\pi}{T_s}$$

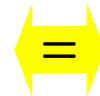


# Sampled Signal

## Ideal Sampling



$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$
$$= \sum_{n=-\infty}^{+\infty} x_c(nT_s) \delta(t - nT_s)$$



$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$
$$\omega_s = \frac{2\pi}{T_s}$$

# CTFT Frequency Shift Property

## Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

## Frequency Shift Property

$$x_c(t)$$



$$X_c(j\omega)$$

$$x_c(t) e^{jk\omega_s t}$$



$$X_c(j(\omega - k\omega_s))$$

$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$\omega_s = \frac{2\pi}{T_s}$$

# CTFT Delay Property

## Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

## Fourier Transform of an Impulse

$$\delta(t - t_d)$$



$$e^{j\omega t_d}$$

$$\delta(t - nT_s)$$



$$e^{-j\omega nT_s}$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\omega nT_s}$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) \delta(t - nT_s)$$



$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega nT_s}$$



# CTFT of a Sampled Signal

## Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

CTFT

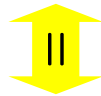


$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\omega nT_s}$$

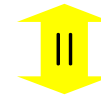
$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) \delta(t - nT_s)$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega nT_s}$$

CTFS



DTFT



$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$

CTFT



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$\omega_s = \frac{2\pi}{T_s}$$

# z-Transform of a Sampled Signal

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT of a sampled signal

$$\sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s)) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

**Z-Transform** of a sampled signal



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] = x_c(nT_s)$$

$$X(z) \Big|_{z = e^{j\omega T_s}} = X(e^{j\omega T_s})$$

*evaluated at*  $z = e^{j\omega T_s}$

# z-Transform and Normalized Frequency

$$\begin{aligned}
 X_s(j\omega) &= \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\omega n T_s} \\
 X_s(j\omega) &= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s}
 \end{aligned}
 \iff
 \begin{aligned}
 X_s(j\omega) &= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s)) \\
 \omega_s &= \frac{2\pi}{T_s}
 \end{aligned}$$

$$\begin{aligned}
 X(z) \Big|_{z = e^{j\omega T_s}} &= X(e^{j\omega T_s}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s} && \text{z-Transform} \\
 \downarrow & \hat{\omega} = \omega T_s && \text{Normalized Frequency} \\
 X(z) \Big|_{z = e^{j\hat{\omega}}} &= X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega} n} && \text{Discrete Time Fourier Transform}
 \end{aligned}$$

# DTFT and CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n}$$

**DTFT of a sampled signal**

$$X(e^{j\hat{\omega}})$$

$$\hat{\omega} = \omega T_s$$

$$= X(e^{j\omega T_s}) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

**CTFT of a sampled signal**

# DTFT and CTFT

## Continuous Time Fourier Transform

## CTFT

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

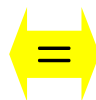
## Discrete Time Fourier Transform

## DTFT

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n} \quad \longleftrightarrow \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}}) e^{+j\hat{\omega}n} d\hat{\omega}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

# Dirichlet Function

## Dirichlet Function

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}$$

$$drcl(t, L) = \frac{\sin(\pi Lt)}{L\sin(\pi t)}$$

$$diric(x, N) = \frac{\sin(Nx/2)}{N\sin(x/2)}$$

odd L → Infinite sum of uniformly spaced sinc functions

$$t = \frac{m}{L} \rightarrow \sin(\pi Lt) = 0$$

$$t = n \rightarrow \sin(\pi t) = 0$$

$$\lim_{t \rightarrow n} \frac{\sin(\pi Lt)}{L\sin(\pi t)} = \lim_{t \rightarrow n} \frac{L\pi \cos(\pi Lt)}{L\pi \cos(\pi t)} = \pm 1 \quad \text{integer } n$$

$$\text{odd } L \quad \lim_{t \rightarrow n} \frac{\cos(\pi Lt)}{\cos(\pi t)} = +1 \quad \frac{\cos(-L2\pi)}{\cos(-2\pi)}, \frac{\cos(-L\pi)}{\cos(-\pi)}, \frac{\cos(0)}{\cos(0)}, \frac{\cos(L\pi)}{\cos(\pi)}, \frac{\cos(L2\pi)}{\cos(2\pi)}, \frac{\cos(L3\pi)}{\cos(3\pi)}, \dots$$

$$\text{even } L \quad \lim_{t \rightarrow n} \frac{\cos(\pi Lt)}{\cos(\pi t)} = (-1)^n$$

## References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003