CTFS (1B)

- Continuous Time Fourier Series

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Fourier Series

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx\, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx\, dx$$

$$k = 1,\ 2,\ 3,\ \ldots$$

→ **one-sided spectrum**

only positive frequencies

Trigonometric Identities

$$\cos\theta\,\cos\phi = \frac{1}{2}\left(\cos(\theta-\phi) + \cos(\theta+\phi)\right)$$

$$\sin\theta\,\sin\phi = \frac{1}{2}\left(\cos(\theta-\phi) - \cos(\theta+\phi)\right)$$

$$\sin\theta\,\cos\phi = \frac{1}{2}\left(\sin(\theta+\phi) + \sin(\theta-\phi)\right)$$

$$\cos\theta\,\sin\phi = \frac{1}{2}\left(\sin(\theta+\phi) - \sin(\theta-\phi)\right)$$

$$\frac{1}{2}\left(1 + \cos(\theta+\phi)\right) \quad \textit{when} \quad \theta=\phi$$

$$\frac{1}{2}\left(1 - \cos(\theta+\phi)\right) \quad \textit{when} \quad \theta=\phi$$

$$\frac{1}{2}\left(\sin(\theta+\phi)\right) \quad \textit{when} \quad \theta=\phi$$

$$\frac{1}{2}\left(\sin(\theta+\phi)\right) \quad \textit{when} \quad \theta=\phi$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = \pi \quad (n = m)$$

Trigonometric Orthogonality

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx \, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx \, dx$$

$$k = 1, 2, 3, \ldots$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = \pi \quad (n = m)$$

$$a_{\boldsymbol{k}} \leftarrow f(x) \cdot \cos \boldsymbol{k} x = a_0 \cdot \cos \boldsymbol{k} x + \sum_{m=1}^{\infty} (a_m \cos \boldsymbol{m} x \cdot \cos \boldsymbol{k} x + b_m \sin \boldsymbol{m} x \cdot \cos \boldsymbol{k} x)$$

$$b_{\boldsymbol{k}} \leftarrow f(x) \cdot \sin \boldsymbol{k} x = a_0 \cdot \sin \boldsymbol{k} x + \sum_{m=1}^{\infty} (a_m \cos \boldsymbol{m} x \cdot \sin \boldsymbol{n} x + b_m \sin \boldsymbol{m} x \cdot \sin \boldsymbol{k} x)$$

Any Period $p = 2L$

$$g(v) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kv + b_k \sin kv)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} g(v)\, dv$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \cos kv\, dv$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \sin kv\, dv$$

$$k = 1,\ 2,\ \ldots$$

$$v : [-\pi,\ +\pi]$$

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{L} x + b_k \sin \frac{k\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\, dx$$

$$a_k = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{k\pi x}{L}\, dx$$

$$b_k = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{k\pi x}{L}\, dx$$

$$k = 1,\ 2,\ 3,\ \ldots$$

$$x : [-L,\ +L]$$

$$v = \frac{\pi}{L} x$$

$$dv = \frac{\pi}{L} dx$$

Time and Frequency

$$f(x) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi}{L}x + b_k \sin\frac{k\pi}{L}x\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{+L} f(x)\,dx$$

$$a_k = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\frac{k\pi x}{L}\,dx$$

$$b_k = \frac{1}{L}\int_{-L}^{+L} f(x)\sin\frac{k\pi x}{L}\,dx$$

$$k = 1,\ 2,\ 3,\ \ldots$$

$$x:\ [-L,\ +L]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{2\pi k}{T}t + b_k \sin\frac{2\pi k}{T}t\right)$$

$$a_0 = \frac{1}{T}\int_{0}^{T} x(t)\,dt$$

$$a_k = \frac{2}{T}\int_{0}^{T} x(t)\cos\frac{2\pi k t}{T}\,dt$$

$$b_k = \frac{2}{T}\int_{0}^{T} x(t)\sin\frac{2\pi k t}{T}\,dt$$

$$k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

$$2L = T$$

Continuous Time Periodic Signal $x(t)$

Harmonic Frquency

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{2\pi k}{T}t + b_k \sin\frac{2\pi k}{T}t\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos\frac{2\pi k t}{T}\,dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin\frac{2\pi k t}{T}\,dt$$

$$k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos(2\pi \boldsymbol{k} f_0 t) + b_k \sin(2\pi \boldsymbol{k} f_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(2\pi \boldsymbol{k} f_0 t)\,dt \qquad k = 1,\ 2,\ \ldots$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(2\pi \boldsymbol{k} f_0 t)\,dt \qquad k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

resolution frequency $\quad f_0 = \frac{1}{T}$

n-th harmonic frequency $\quad f_n = \boldsymbol{n} f_0 = \boldsymbol{n}\frac{1}{T}$

Radial Frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_k \cos(k 2\pi f_0 t) + b_k \sin(k 2\pi f_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k 2\pi f_0 t)\,dt \qquad k = 1,\ 2,\ \ldots$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k 2\pi f_0 t)\,dt \qquad k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_k \cos(\boldsymbol{k}\,\omega_0 t) + b_k \sin(\boldsymbol{k}\,\omega_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(\boldsymbol{k}\,\omega_0 t)\,dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(\boldsymbol{k}\,\omega_0 t)\,dt$$

$$k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

linear frequency f

angular (radial) frequency $\omega = 2\pi f$

Complex Fourier Series Coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt$$

$$k = 1, 2, \ldots$$

$$t : [0, T]$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t}\right)$$

$$A_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$A_k = \frac{1}{T}\int_0^T x(t)\,e^{-jk\omega_0 t}\,dt$$

$$B_k = \frac{1}{T}\int_0^T x(t)\,e^{+jk\omega_0 t}\,dt$$

$$t : [0, T]$$

Real coefficients

$$a_0, \ a_k, \ b_k, \quad k = 1, 2, \ldots$$

→ **one-sided spectrum**
only positive frequencies

Complex coefficients

$$A_0, \ A_k, \ B_k, \quad k = 1, 2, \ldots$$

→ **two-sided spectrum**
Both pos and neg frequencies

Euler Equation (1)

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$= a_k \frac{1}{2} \left(e^{jk\omega_0 t} + e^{-jk\omega_0 t} \right) + b_k \frac{1}{2j} \left(e^{jk\omega_0 t} - e^{-jk\omega_0 t} \right)$$

$$= \frac{(a_k - jb_k)}{2} e^{jk\omega_0 t} + \frac{(a_k + jb_k)}{2} e^{-jk\omega_0 t}$$

$$= A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t}$$

$$e^{+j\omega t} = \cos\omega t + j\sin\omega t$$

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

zero freq → $A_0 = a_0$

pos freq → $A_k = \frac{1}{2}(a_k - jb_k)$

neg freq → $B_k = \frac{1}{2}(a_k + jb_k)$

only positive frequencies

Euler Equation (2)

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

zero freq → $A_0 = a_0$

pos freq → $A_k = \frac{1}{2}(a_k - j b_k)$

neg freq → $B_k = \frac{1}{2}(a_k + j b_k)$

only positive frequencies

$$A_k = \frac{1}{T} \int_0^T x(t) \left(\cos(k\omega_0 t) - j \sin(k\omega_0 t) \right) dt$$

$$B_k = \frac{1}{T} \int_0^T x(t) \left(\cos(k\omega_0 t) + j \sin(k\omega_0 t) \right) dt$$

↓

$$A_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$B_k = \frac{1}{T} \int_0^T x(t)\, e^{+jk\omega_0 t}\, dt$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

↓

$$x(t) = \sum_{k=0}^{\infty} \left(A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

Complex Fourier Series

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T}\int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T}\int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1, 2, \ldots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t}\right)$$

$$A_0 = a_0$$

$$A_k = \frac{1}{2}(a_k - j b_k)$$

$$B_k = \frac{1}{2}(a_k + j b_k)$$

$$k = 1, 2, \ldots$$

$$x(t) = \sum_{k=0}^{\infty} \left(A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t}\right)$$

$$A_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = 0, 1, 2, \ldots$$

$$B_k = \frac{1}{T}\int_0^T x(t)\, e^{+jk\omega_0 t}\, dt$$

$$k = 1, 2, \ldots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} A_0 & (k = 0) \\ A_k & (k > 0) \\ B_k & (k < 0) \end{cases}$$

Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$\boldsymbol{k} = +1, +2, \ldots$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k} \right)$$

$\boldsymbol{k} = +1, +2, \ldots$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$g_k \cos(k\omega_0 t + \phi_k) = g_k \cos(\phi_k)\cos(k\omega_0 t) - g_k \sin(\phi_k)\sin(k\omega_0 t)$$

$$a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$a_k = g_k \cos(\phi_k)$$

$$-b_k = g_k \sin(\phi_k)$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

Phasor Representation (1)

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k} \right)$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k\, \Re\left\{ e^{+j(k\omega_0 t + \phi_k)} \right\}$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \Re\left\{ g_k \cdot e^{+j\phi_k} \cdot e^{+jk\omega_0 t} \right\}$$

$$x(t) = X_0 + \sum_{k=1}^{\infty} \Re\left\{ X_k e^{+jk\omega_0 t} \right\}$$

$$X_0 = g_0$$

$$X_k = g_k \cdot e^{+j\phi_k}$$

$$k = 1,\ 2,\ \ldots$$

Phasor Representation (2)

$$x(t) = g_0 + \sum_{k=1}^{\infty} \frac{g_k}{2} \cdot \left(e^{+j(k\omega_0 t + \phi_k)} + e^{-j(k\omega_0 t + \phi_k)}\right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \left(\frac{g_k}{2} e^{+j\phi_k} e^{+jk\omega_0 t} + \frac{g_k}{2} e^{-j\phi_k} e^{-jk\omega_0 t} \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \left(\frac{g_k e^{+j\phi_k}}{2} e^{+jk\omega_0 t} + \frac{g_k e^{-j\phi_k}}{2} e^{-jk\omega_0 t} \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k}\right)$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{g_k\, e^{+j\phi_k}}{2} \qquad (k > 0)$$

$$C_{-k} = \frac{g_k\, e^{-j\phi_k}}{2} \qquad (k < 0)$$

$$k = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$x(t) = X_0 + \sum_{k=1}^{\infty} \Re\{X_k e^{+jk\omega_0 t}\}$$

$$X_0 = g_0$$

$$X_k = g_k \cdot e^{+j\phi_k}$$

$$k = 1,\ 2,\ \ldots$$

Two-Sided Spectrum

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} \tan^{-1}(-b_k/a_k) & (k > 0) \\ \tan^{-1}(+b_k/a_k) & (k < 0) \end{cases}$$

Power Spectrum   *Two-Sided*

$$|C_k|^2 + |C_{-k}|^2 = \frac{1}{2}|g_k|^2 = \frac{1}{2}(a_k^2 + b_k^2)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2} g_k \, e^{+j\phi_k} & (k > 0) \\ \frac{1}{2} g_k \, e^{-j\phi_k} & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} +\phi_k & (k > 0) \\ -\phi_k & (k < 0) \end{cases}$$

Periodogram   *One-Sided*

$$2 \cdot |C_k| = |g_k| = \sqrt{a_k^2 + b_k^2}$$

CTFS of Impulse Train (1)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Fourier Series Expansion of Impulse Train

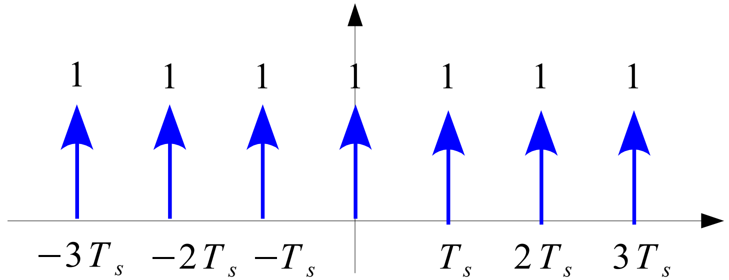
$$p(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t}$$

Fourier Series Coefficients

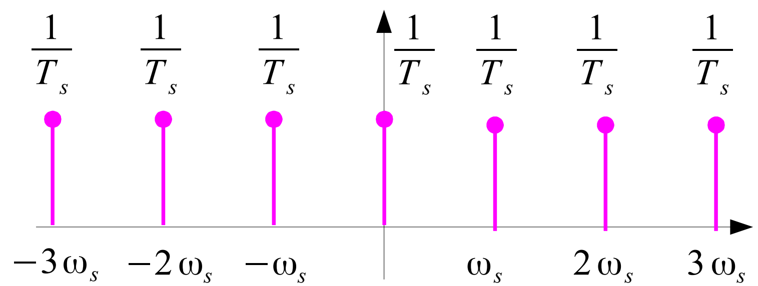
$$a_k = \frac{1}{T_s}\int_{-T_s/2}^{+T_s/2} \delta(t)\, e^{-jk\omega_s t}\, dt$$

$$= \frac{1}{T_s}\int_{-T_s/2}^{+T_s/2} \delta(t)\, e^{-jk\omega_s 0}\, dt$$

$$= \frac{1}{T_s}\int_{-T_s/2}^{+T_s/2} \delta(t)\, dt = \frac{1}{T_s}$$



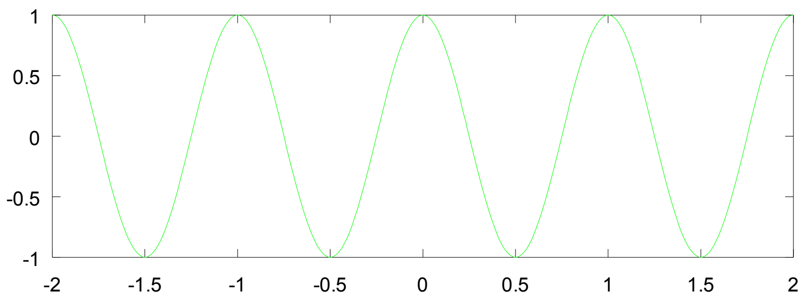
$$\omega_s = \frac{2\pi}{T_s}$$

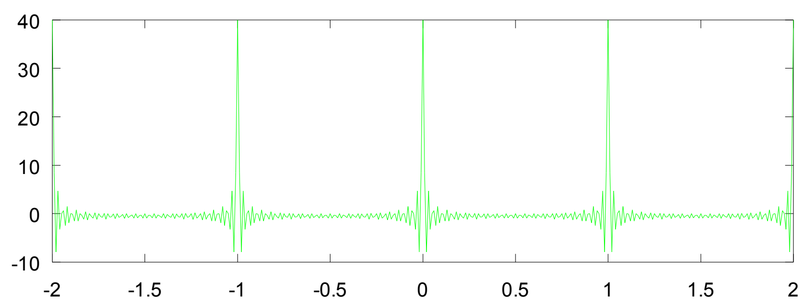


CTFS of Impulse Train (2)

$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} (\cos k\omega_s t - j \sin k\omega_s t)$$

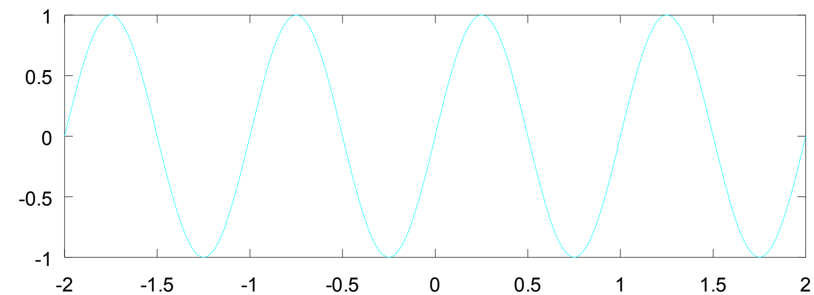
$\cos 2\pi \cdot 1 \cdot t$

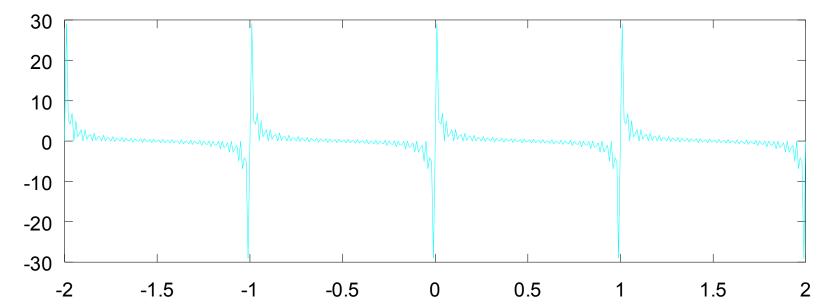




$\sum_{k=1}^{40} \cos 2\pi \cdot k \cdot t$

$\sin 2\pi \cdot 1 \cdot t$



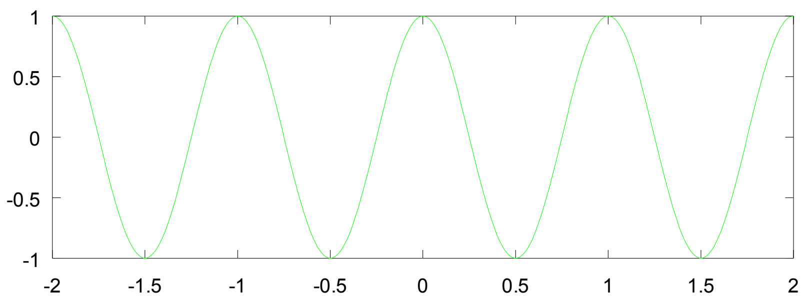


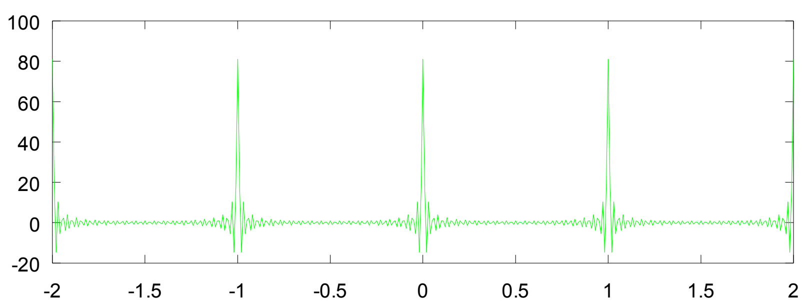
$\sum_{k=1}^{40} \sin 2\pi \cdot k \cdot t$

CTFS of Impulse Train (3)

$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} (\cos k\omega_s t - j \sin k\omega_s t)$$

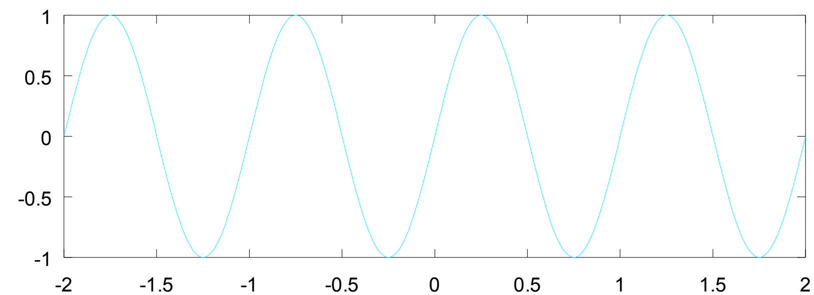
$\cos 2\pi \cdot 1 \cdot t$

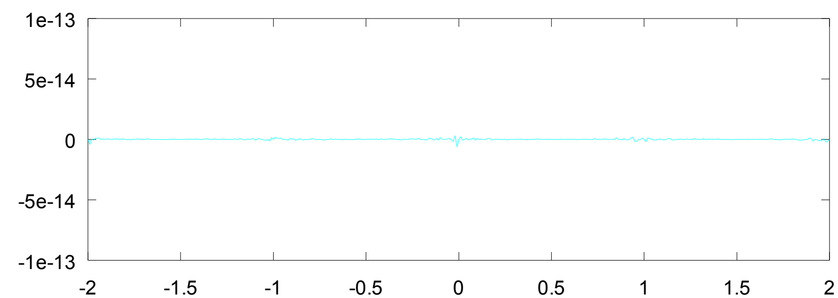




$$\sum_{k=-40}^{40} \cos 2\pi \cdot k \cdot t$$

$\sin 2\pi \cdot 1 \cdot t$





$$\sum_{k=-40}^{40} \sin 2\pi \cdot k \cdot t$$

Inner Product Space

Hilber Space

$$\langle f , g \rangle = \int_a^b f(t)\overline{g(t)}\,dt$$

Complex Vector Inner Product

Hermitian inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^H \cdot \boldsymbol{y} = \sum x_i^* y_i$$

$\boldsymbol{x}^H$: conjugate transpose

Norm of Hermitian inner products

$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} = \sqrt{\boldsymbol{x}^H \cdot \boldsymbol{x}} = \sqrt{\sum x_i^* x_i}$$

the length of a vector

$$\boldsymbol{x} = \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix}$$

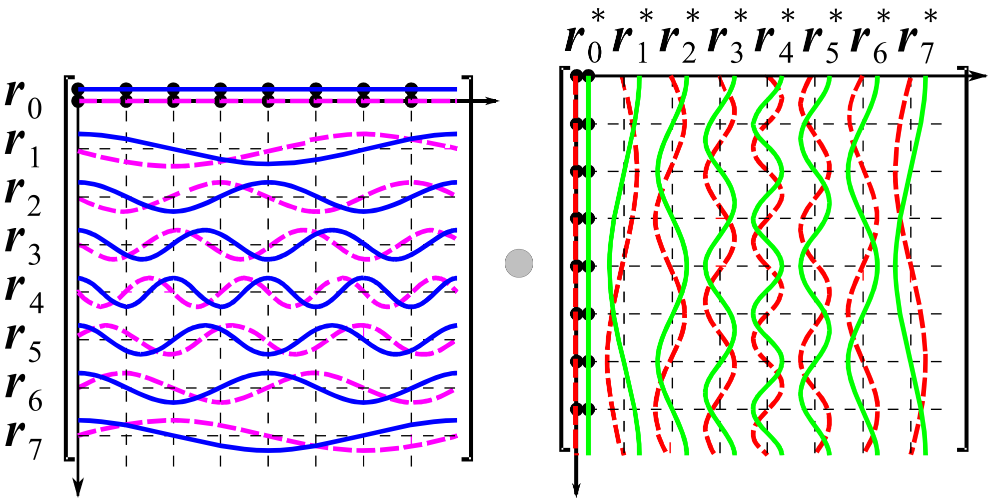
$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = \boldsymbol{x}^H \cdot \boldsymbol{x} = \sum x_i^* x_i$$

$$\begin{pmatrix} a_1 - j b_1 & a_2 - j b_2 & \cdots & a_n - j b_n \end{pmatrix} \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix} = \sum_{i=1}^{n} a_i^2 + b_i^2$$

Orthogonality

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$

$$\begin{cases} \boldsymbol{A}^H = \boldsymbol{B} \\ \boldsymbol{B}^H = \boldsymbol{A} \end{cases} \quad \begin{cases} \boldsymbol{AB} = N\,\boldsymbol{I} \\ \boldsymbol{BA} = N\,\boldsymbol{I} \end{cases} \Rightarrow \begin{cases} \boldsymbol{A}^H \boldsymbol{A} = \boldsymbol{A}\,\boldsymbol{A}^H = N\,\boldsymbol{I} \\ \boldsymbol{B}^H \boldsymbol{B} = \boldsymbol{B}\,\boldsymbol{B}^H = N\,\boldsymbol{I} \end{cases}$$



$$= \begin{bmatrix}
\boldsymbol{r}_0\cdot\boldsymbol{r}_0^* & \boldsymbol{r}_0\cdot\boldsymbol{r}_1^* & \boldsymbol{r}_0\cdot\boldsymbol{r}_2^* & \boldsymbol{r}_0\cdot\boldsymbol{r}_3^* & \boldsymbol{r}_0\cdot\boldsymbol{r}_4^* & \boldsymbol{r}_0\cdot\boldsymbol{r}_5^* & \boldsymbol{r}_0\cdot\boldsymbol{r}_6^* & \boldsymbol{r}_0\cdot\boldsymbol{r}_7^* \\
\boldsymbol{r}_1\cdot\boldsymbol{r}_0^* & \boldsymbol{r}_1\cdot\boldsymbol{r}_1^* & \boldsymbol{r}_1\cdot\boldsymbol{r}_2^* & \boldsymbol{r}_1\cdot\boldsymbol{r}_3^* & \boldsymbol{r}_1\cdot\boldsymbol{r}_4^* & \boldsymbol{r}_1\cdot\boldsymbol{r}_5^* & \boldsymbol{r}_1\cdot\boldsymbol{r}_6^* & \boldsymbol{r}_1\cdot\boldsymbol{r}_7^* \\
\boldsymbol{r}_2\cdot\boldsymbol{r}_0^* & \boldsymbol{r}_2\cdot\boldsymbol{r}_1^* & \boldsymbol{r}_2\cdot\boldsymbol{r}_2^* & \boldsymbol{r}_2\cdot\boldsymbol{r}_3^* & \boldsymbol{r}_2\cdot\boldsymbol{r}_4^* & \boldsymbol{r}_2\cdot\boldsymbol{r}_5^* & \boldsymbol{r}_2\cdot\boldsymbol{r}_6^* & \boldsymbol{r}_2\cdot\boldsymbol{r}_7^* \\
\boldsymbol{r}_3\cdot\boldsymbol{r}_0^* & \boldsymbol{r}_3\cdot\boldsymbol{r}_1^* & \boldsymbol{r}_3\cdot\boldsymbol{r}_2^* & \boldsymbol{r}_3\cdot\boldsymbol{r}_3^* & \boldsymbol{r}_3\cdot\boldsymbol{r}_4^* & \boldsymbol{r}_3\cdot\boldsymbol{r}_5^* & \boldsymbol{r}_3\cdot\boldsymbol{r}_6^* & \boldsymbol{r}_3\cdot\boldsymbol{r}_7^* \\
\boldsymbol{r}_4\cdot\boldsymbol{r}_0^* & \boldsymbol{r}_4\cdot\boldsymbol{r}_1^* & \boldsymbol{r}_4\cdot\boldsymbol{r}_2^* & \boldsymbol{r}_4\cdot\boldsymbol{r}_3^* & \boldsymbol{r}_4\cdot\boldsymbol{r}_4^* & \boldsymbol{r}_4\cdot\boldsymbol{r}_5^* & \boldsymbol{r}_4\cdot\boldsymbol{r}_6^* & \boldsymbol{r}_4\cdot\boldsymbol{r}_7^* \\
\boldsymbol{r}_5\cdot\boldsymbol{r}_0^* & \boldsymbol{r}_5\cdot\boldsymbol{r}_1^* & \boldsymbol{r}_5\cdot\boldsymbol{r}_2^* & \boldsymbol{r}_5\cdot\boldsymbol{r}_3^* & \boldsymbol{r}_5\cdot\boldsymbol{r}_4^* & \boldsymbol{r}_5\cdot\boldsymbol{r}_5^* & \boldsymbol{r}_5\cdot\boldsymbol{r}_6^* & \boldsymbol{r}_5\cdot\boldsymbol{r}_7^* \\
\boldsymbol{r}_6\cdot\boldsymbol{r}_0^* & \boldsymbol{r}_6\cdot\boldsymbol{r}_1^* & \boldsymbol{r}_6\cdot\boldsymbol{r}_2^* & \boldsymbol{r}_6\cdot\boldsymbol{r}_3^* & \boldsymbol{r}_6\cdot\boldsymbol{r}_4^* & \boldsymbol{r}_6\cdot\boldsymbol{r}_5^* & \boldsymbol{r}_6\cdot\boldsymbol{r}_6^* & \boldsymbol{r}_6\cdot\boldsymbol{r}_7^* \\
\boldsymbol{r}_7\cdot\boldsymbol{r}_0^* & \boldsymbol{r}_7\cdot\boldsymbol{r}_1^* & \boldsymbol{r}_7\cdot\boldsymbol{r}_2^* & \boldsymbol{r}_7\cdot\boldsymbol{r}_3^* & \boldsymbol{r}_7\cdot\boldsymbol{r}_4^* & \boldsymbol{r}_7\cdot\boldsymbol{r}_5^* & \boldsymbol{r}_7\cdot\boldsymbol{r}_6^* & \boldsymbol{r}_7\cdot\boldsymbol{r}_7^*
\end{bmatrix}$$

$$\langle \boldsymbol{r}_i^H , \boldsymbol{r}_i^H \rangle = \boldsymbol{r}_i \cdot \boldsymbol{r}_i^* = N$$

$$\langle \boldsymbol{r}_i^H , \boldsymbol{r}_j^H \rangle = \boldsymbol{r}_i \cdot \boldsymbol{r}_j^* = 0 \quad (i \neq j)$$

Cauchy-Schwartz Inequality

For all vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of an inner product space

$$\left|\langle \boldsymbol{x}, \boldsymbol{y}\rangle\right|^2 \le \langle \boldsymbol{x}, \boldsymbol{x}\rangle \cdot \langle \boldsymbol{y}, \boldsymbol{y}\rangle$$

$$\left|\langle \boldsymbol{x}, \boldsymbol{y}\rangle\right| \le \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\|$$

The equality holds if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly dependent ⇨ maximum

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \boldsymbol{x}^H \cdot \boldsymbol{y} \le \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\| \qquad \boldsymbol{x} = \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix} \qquad \boldsymbol{y} = k \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix}$$

Inner product is maximum
when $\boldsymbol{y} = k\,\boldsymbol{x}$

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle \le k\left(\sum_{i=1}^{n} a_i^2 + b_i^2\right)$$

References

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