

# Surface Integrals (6A)

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- Surface Integral
- Stokes' Theorem

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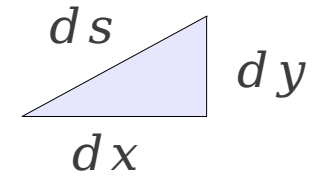
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# Arc Length In the Plane

$$y = f(x)$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



# Surface Area In the Space

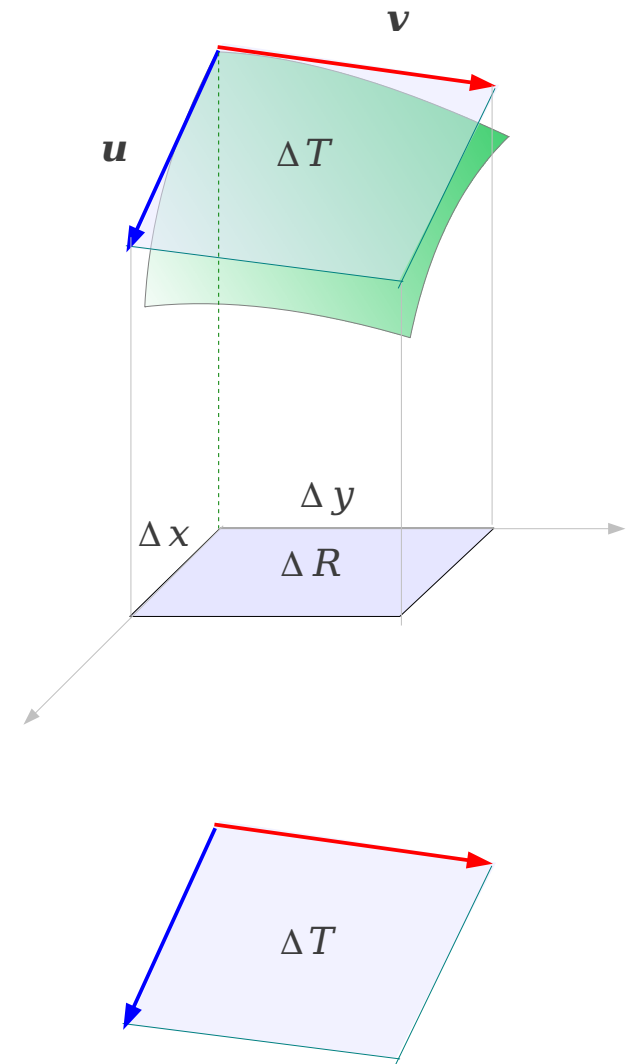
$$z = f(x, y)$$

Area of the surface over  $R$

$$\begin{aligned} A(S) &= \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA \\ &= \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dA \end{aligned}$$

Differential of surface area

$$dS = \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$



# Differential of Surface Area (1)

Differential of surface area

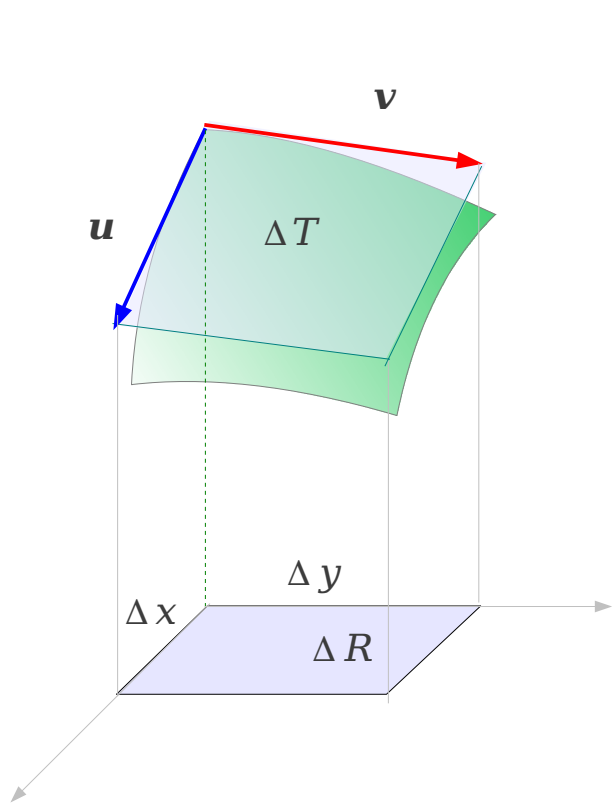
$$dS = \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$

Slope along  
x direction

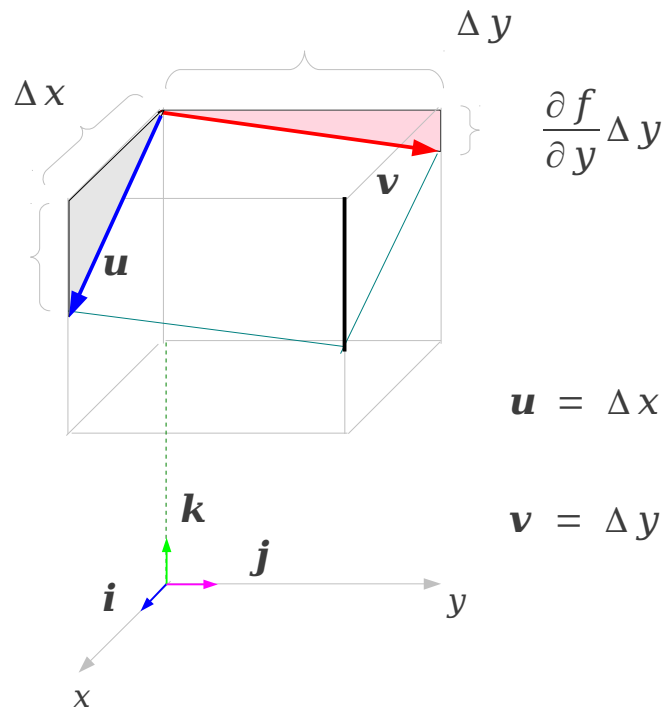
$$\frac{\partial f}{\partial x} = -0.4$$

Slope along  
y direction

$$\frac{\partial f}{\partial y} = -0.2$$



$$\frac{\partial f}{\partial x} \Delta x$$



$$\mathbf{u} = \Delta x \mathbf{i} + \frac{\partial f}{\partial x} \Delta x \mathbf{k}$$

$$\mathbf{v} = \Delta y \mathbf{j} + \frac{\partial f}{\partial y} \Delta y \mathbf{k}$$

# Differential of Surface Area (2)

Differential of surface area

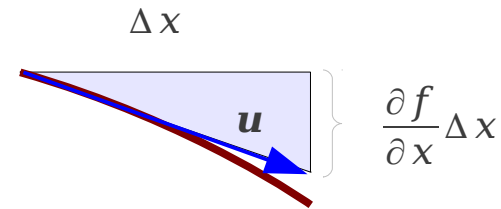
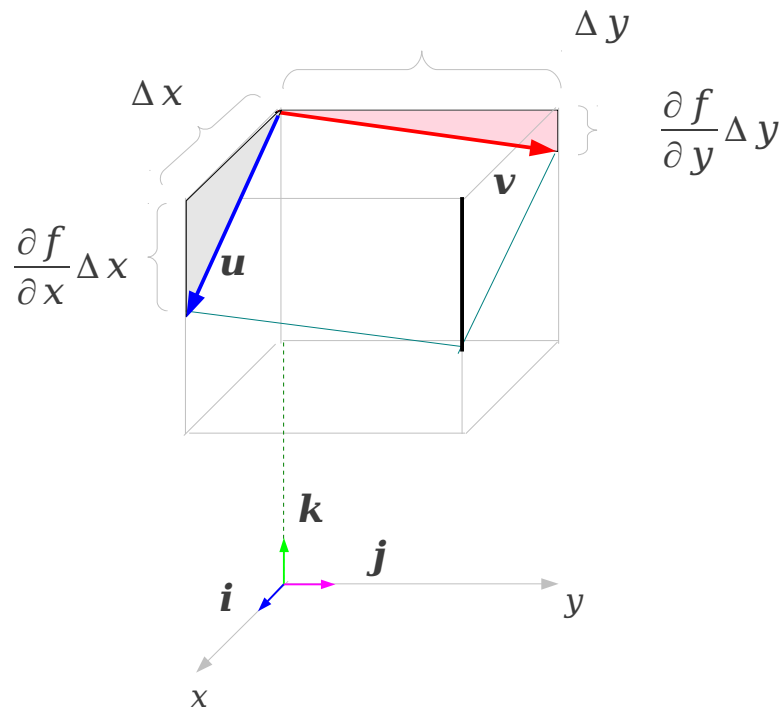
$$dS = \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$

Slope along  
x direction

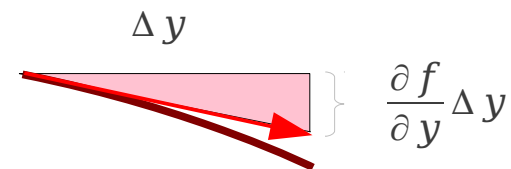
$$\frac{\partial f}{\partial x} = -0.4$$

Slope along  
y direction

$$\frac{\partial f}{\partial y} = -0.2$$



$$\mathbf{u} = \Delta x \mathbf{i} + \frac{\partial f}{\partial x} \Delta x \mathbf{k}$$



$$\mathbf{v} = \Delta y \mathbf{j} + \frac{\partial f}{\partial y} \Delta y \mathbf{k}$$

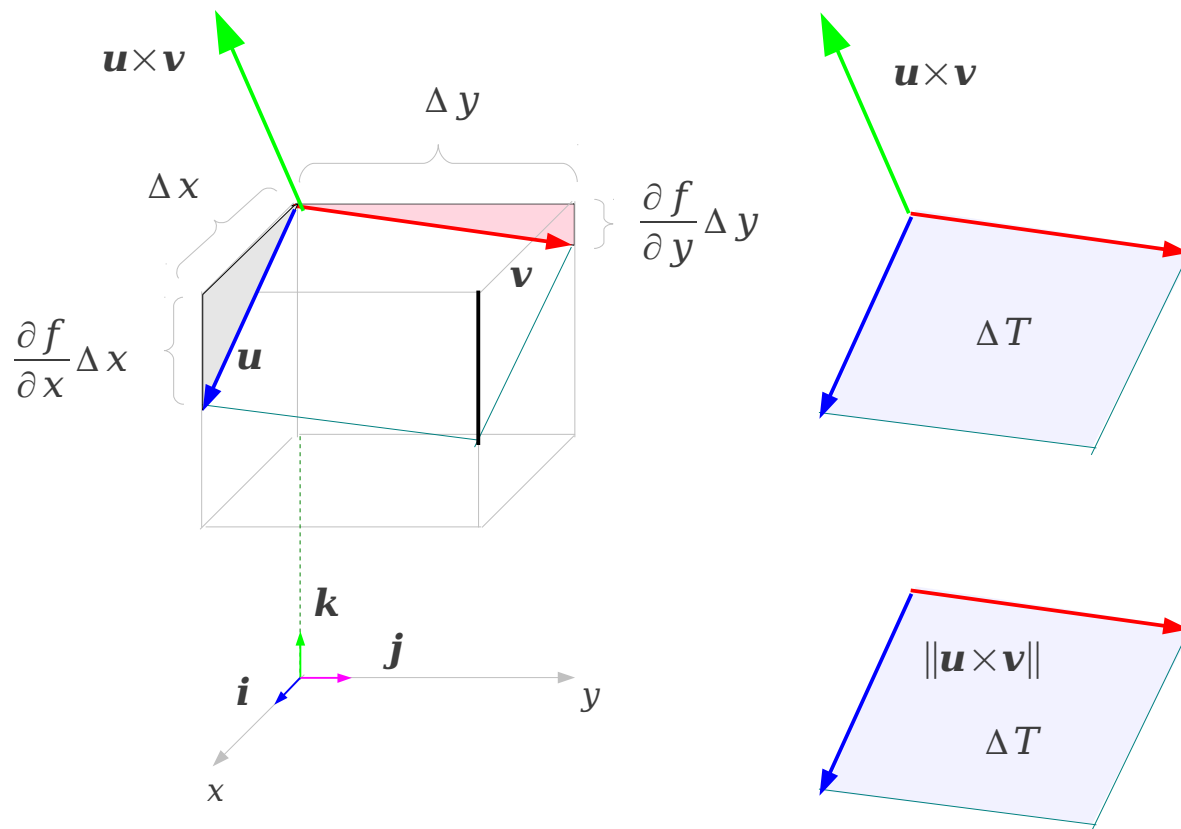
# Differential of Surface Area (3)

Differential of surface area

$$dS = \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$

$$\mathbf{u} = \Delta x \mathbf{i} + \frac{\partial f}{\partial x} \Delta x \mathbf{k}$$

$$\mathbf{v} = \Delta y \mathbf{j} + \frac{\partial f}{\partial y} \Delta y \mathbf{k}$$



$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & \frac{\partial f}{\partial x} \Delta x \\ 0 & \Delta y & \frac{\partial f}{\partial y} \Delta y \end{vmatrix}$$

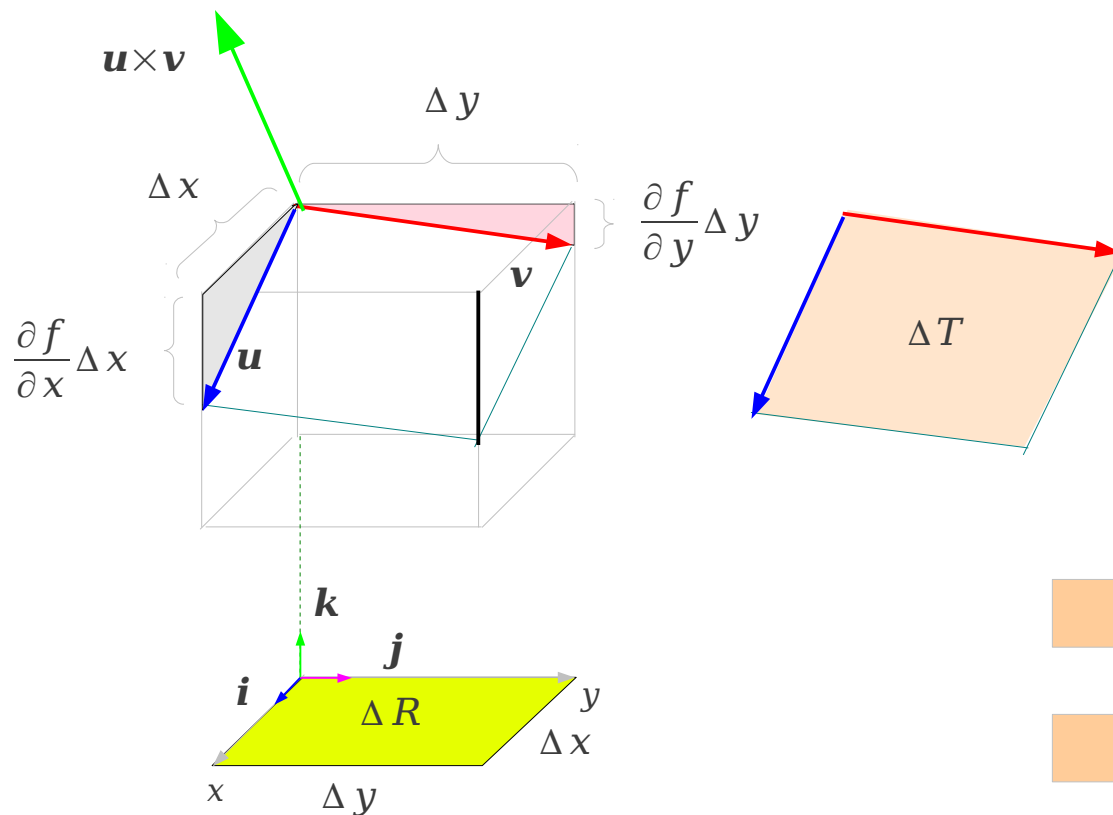
$$= \left[ -\frac{\partial f}{\partial x} \mathbf{i} - \frac{\partial f}{\partial y} \mathbf{j} + \mathbf{k} \right] \Delta x \Delta y$$

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \Delta x \Delta y$$

# Differential of Surface Area (4)

Differential of surface area

$$dS = \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$



$$\mathbf{u} = \Delta x \mathbf{i} + \frac{\partial f}{\partial x} \Delta x \mathbf{k}$$

$$\mathbf{v} = \Delta y \mathbf{j} + \frac{\partial f}{\partial y} \Delta y \mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \left[ -\frac{\partial f}{\partial x} \mathbf{i} - \frac{\partial f}{\partial y} \mathbf{j} + \mathbf{k} \right] \Delta x \Delta y$$

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \Delta x \Delta y$$

**S**

**R**

$$\Delta T = \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \Delta x \Delta y$$

$$\Delta S = \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \Delta A$$



# Line Integral with an Explicit Curve Function

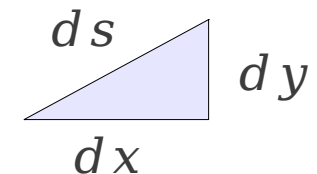
$$y = f(x) \quad \longrightarrow \quad \frac{dy}{dx} = f'(x) \quad \longrightarrow \quad dy = f'(x) dx$$

$$a \leq x \leq b$$

Curve C

$$ds = \sqrt{[dx]^2 + [dy]^2}$$

$$ds = \sqrt{1 + [f'(x)]^2} dx$$



$$\int_C G(x, y) dx = \int_a^b G(x, f(x)) dx$$

$$\int_C G(x, y) dy = \int_a^b G(x, f(x)) f'(x) dx$$

$$\int_C G(x, y) ds = \int_a^b G(x, f(x)) \sqrt{1 + [f'(x)]^2} dx$$

# Surface Integral with an Explicit Surface Function

$$z = f(x, y) \quad \rightarrow \quad \frac{df}{dx} = f_x(x, y)$$

$$a \leq x \leq b$$

$$c \leq y \leq d$$

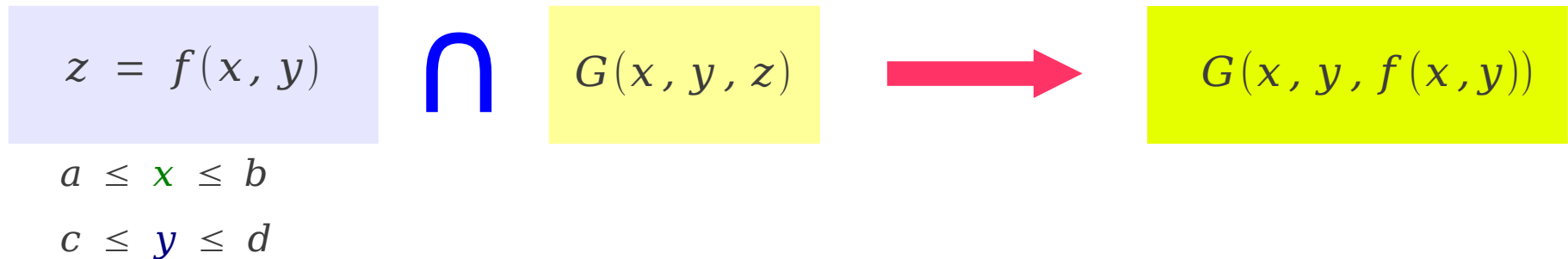
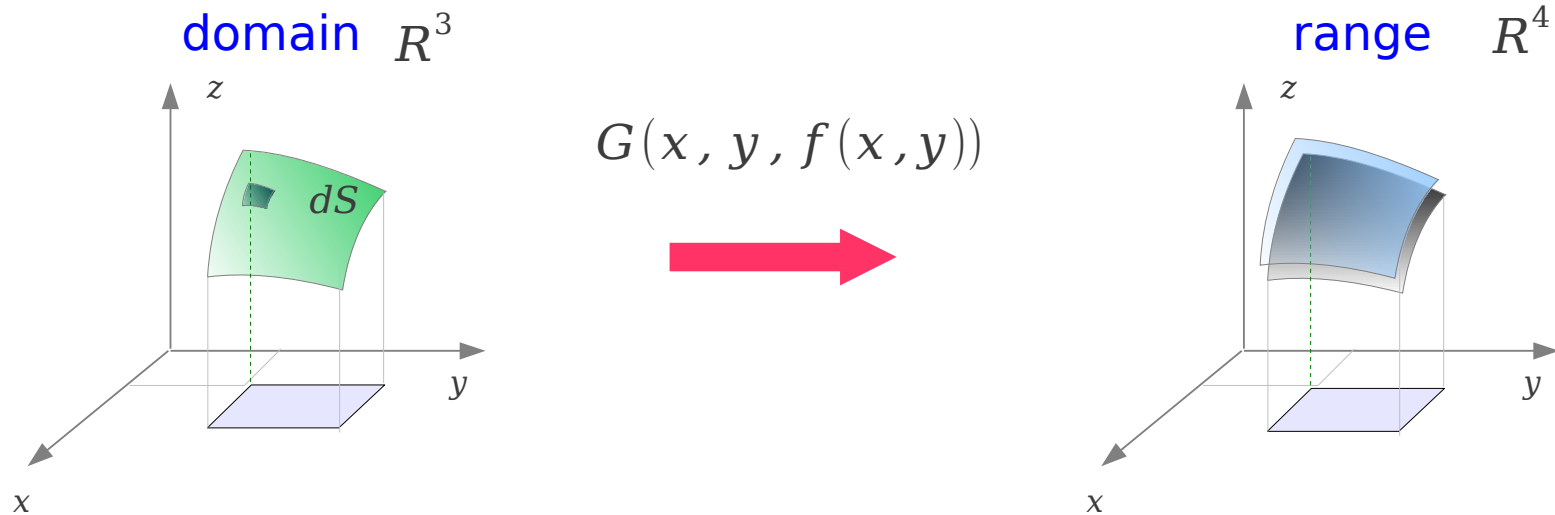
$$\frac{df}{dy} = f_y(x, y)$$

Region R

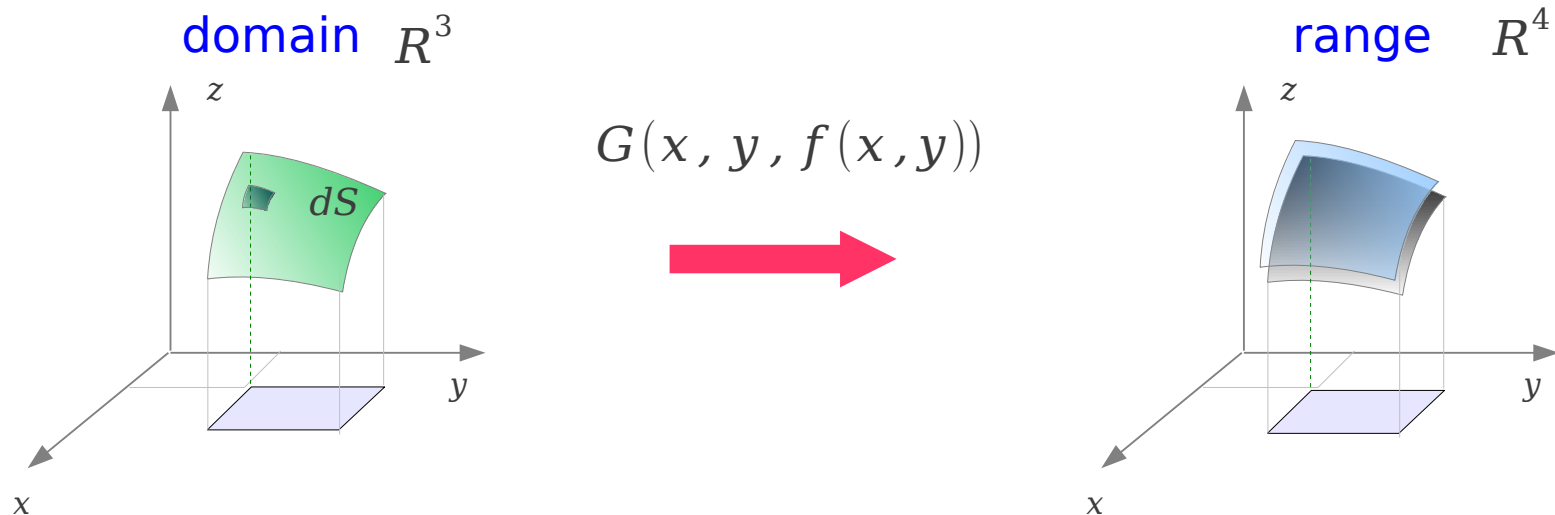
$$\rightarrow \quad dS = \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$

$$\iint_S G(x, y, z) dS = \iint_R G(x, y, f(x, y)) \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$

# Surface Integral Over XY (1)



# Surface Integral Over XY (2)

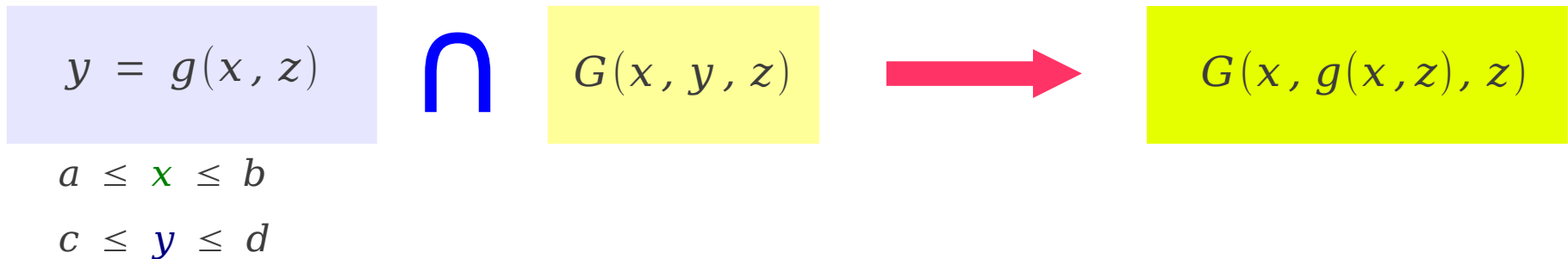
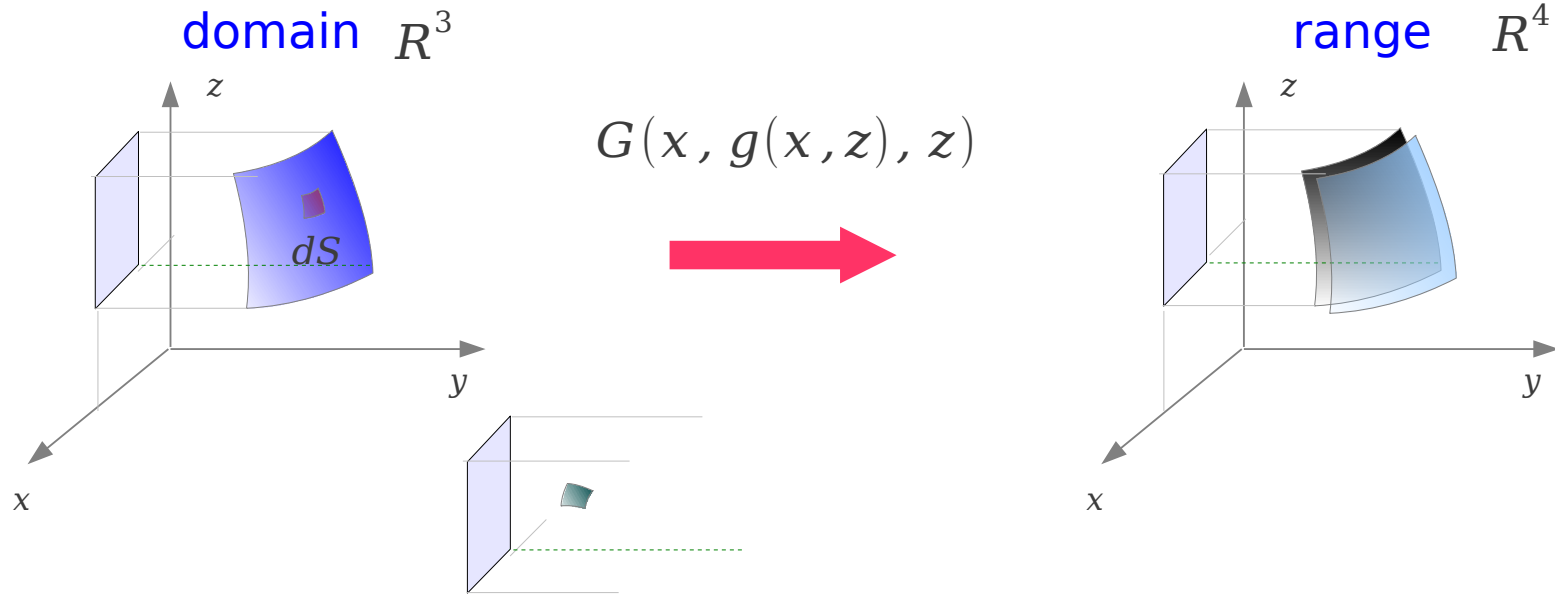


$$z = f(x, y)$$

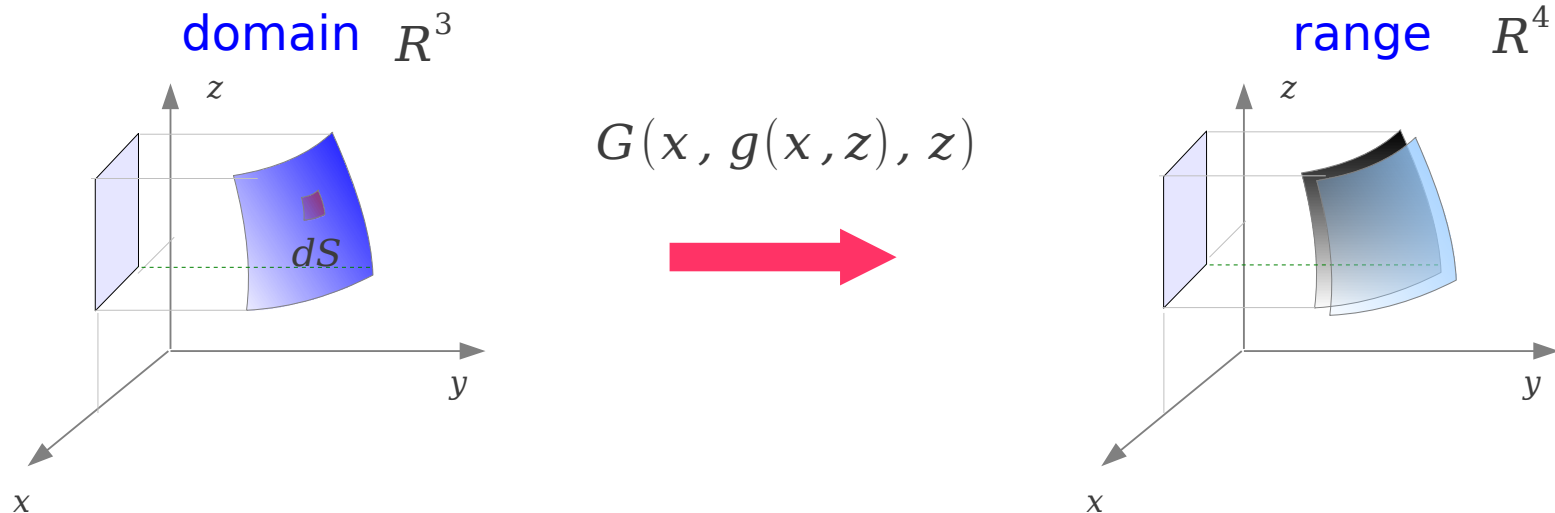
$$\iint_S G(x, y, z) dS$$

$$= \iint_R G(x, y, f(x, y)) \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$

# Surface Integral Over XZ (1)



# Surface Integral Over XZ (2)



$$y = g(x, z)$$

$$\iint_S G(x, y, z) dS$$

$$= \iint_R G(x, g(x, z), z) \sqrt{1 + [g_x(x, z)]^2 + [g_z(x, z)]^2} dA$$

# Surface Integral with an Explicit Surface Function

$$z = f(x, y) \quad \Rightarrow \quad \frac{df}{dx} = f_x(x, y) \quad \Rightarrow \quad dS = \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$

Region R

$$\frac{df}{dy} = f_y(x, y)$$

$$\iint_S G(x, y, z) dS = \iint_R G(x, y, f(x, y)) \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$

$$y = g(x, z) \quad \Rightarrow \quad \frac{dg}{dx} = g_x(x, z) \quad \Rightarrow \quad dS = \sqrt{1 + [g_x(x, z)]^2 + [g_z(x, z)]^2} dA$$

Region R

$$\frac{dg}{dz} = g_z(x, z)$$

$$\iint_S G(x, y, z) dS = \iint_R G(x, g(x, z), z) \sqrt{1 + [g_x(x, z)]^2 + [g_z(x, z)]^2} dA$$

$$x = h(y, z) \quad \Rightarrow \quad \frac{dh}{dy} = h_y(y, z) \quad \Rightarrow \quad dS = \sqrt{1 + [h_y(y, z)]^2 + [h_z(y, z)]^2} dA$$

Region R

$$\frac{dh}{dz} = h_z(y, z)$$

$$\iint_S G(x, y, z) dS = \iint_R G(h(y, z), y, z) \sqrt{1 + [h_y(y, z)]^2 + [h_z(y, z)]^2} dA$$

# Mass of a Surface

density  $\rho(x, y, z)$

mass  $m = \iint_S \rho(x, y, z) dS$



# Orientation of a Surface

Oriented Surface      there exists a continuous **unit normal vector function**  $\mathbf{n}$  defined at each point  $(x,y,z)$  on the surface

Orientation      the vector field  $\mathbf{n}(x, y, z)$  or  $-\mathbf{n}(x, y, z)$

Upward orientation       $z = f(x,y)$  : **positive**  $\mathbf{k}$  component

Downward orientation       $z = f(x,y)$  : **negative**  $\mathbf{k}$  component

Surface       $g(x, y, z) = 0$

Unit Normal Vector       $\mathbf{n} = \frac{1}{\|\nabla g\|} \nabla g$        $\nabla g = \frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial g}{\partial z} \mathbf{k}$

Surface       $z = f(x, y)$

$$g(x, y, z) = z - f(x, y) = 0$$

$$g(x, y, z) = f(x, y) - z = 0$$

# Level Curve and Surface

Functions of two variables

$$z = F(x, y) \quad \mathbb{R}^3$$

$$G(x, y, z) = 0$$

Level Curve

$$c_i = F(x, y) \quad \mathbb{R}^2$$

$$F(x, y) = 0$$

when  $z = 0$

$$0 = \nabla F(x, y) \cdot \mathbf{r}'(t)$$

Functions of three variables

$$w = G(x, y, z) \quad \mathbb{R}^4$$

$$H(x, y, z, w) = 0$$

Level Surface

$$c_i = G(x, y, z) \quad \mathbb{R}^3$$

$$G(x, y, z) = 0$$

when  $w = 0$

$$0 = \nabla G(x, y, z) \cdot \mathbf{r}'(t)$$

# Gradient & Normal Vector

Functions of three variables

$$w = F(x, y, z) \quad \mathbb{R}^4$$

Level Surface

$$c_0 = F(x, y, z) \quad \mathbb{R}^3 \quad x = f(t)$$

$$y = g(t)$$

$$z = h(t)$$

$$\frac{dc_0}{dt} = \frac{dF}{dt}(x, y, z)$$

$$0 = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt}$$

$$0 = \left( \frac{\partial F}{\partial x} \mathbf{i} + \frac{\partial F}{\partial y} \mathbf{j} + \frac{\partial F}{\partial z} \mathbf{k} \right) \cdot \left( \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \right)$$

$$0 = \nabla F(x, y, z) \cdot \mathbf{r}'(t) \quad \begin{array}{l} \text{tangent} \\ \text{vector} \end{array}$$

$\nabla F$  **normal** to the level surface at  $P(x_0, y_0, z_0)$

# Surface Integral over a 3-D Vector Field (1)

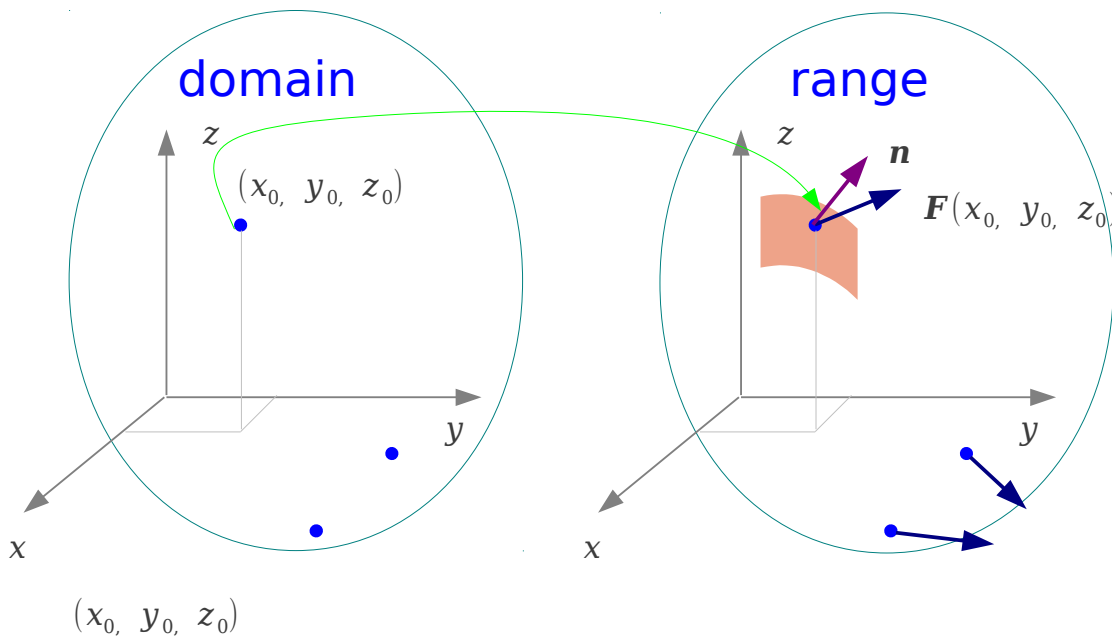
A given point in a 3-d space



A vector

$$(x_0, y_0, z_0)$$

$$\langle P(x_0, y_0, z_0), Q(x_0, y_0, z_0), R(x_0, y_0, z_0) \rangle$$



3 functions

$$(x_0, y_0, z_0) \longrightarrow P(x_0, y_0, z_0)$$

$$(x_0, y_0, z_0) \longrightarrow Q(x_0, y_0, z_0)$$

$$(x_0, y_0, z_0) \longrightarrow R(x_0, y_0, z_0)$$

only points that are on the surface

$$\longrightarrow \mathbf{F}(x_0, y_0, z_0) = P(x_0, y_0, z_0)\mathbf{i} + Q(x_0, y_0, z_0)\mathbf{j} + R(x_0, y_0, z_0)\mathbf{k}$$

consider only the component of  $\mathbf{F}$  along  $\mathbf{n} \longrightarrow \mathbf{F} \cdot \mathbf{n}$

# Surface Integral over a 3-D Vector Field (2)

$$\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}$$

## Line Integral over a 3-d Vector Field

$$\mathbf{r}(t) = f(t) \mathbf{i} + g(t) \mathbf{j} + h(t) \mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz$$

## Surface Integral over a 3-d Vector Field

$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|}$$

$$\iint_S (\mathbf{F} \cdot \mathbf{n}) dS = \text{flux}$$

total volume of a fluid  
passing through  $S$   
per unit time

$\mathbf{F}$

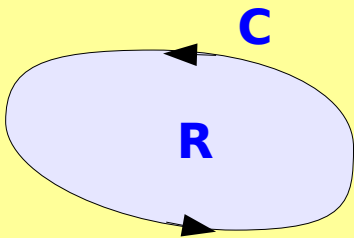
velocity field  
of a fluid

# Vector Form of Green's Theorem

A force field

A smooth curve

Work done by  $\mathbf{F}$  along  $C$



$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

$$C: x = f(t), y = g(t), a \leq t \leq b$$

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds \\ &= \int_C P(x, y) dx + Q(x, y) dy \end{aligned}$$

$$\oint_C P dx + Q dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

3-space

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

$$(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$\oint_C P dx + Q dy = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA$$

# Stokes' Theorem (1)

Work done by  $\mathbf{F}$  along  $C$

$$W = \int_c \mathbf{F} \cdot d\mathbf{r} = \int_c \mathbf{F} \cdot \mathbf{T} ds$$

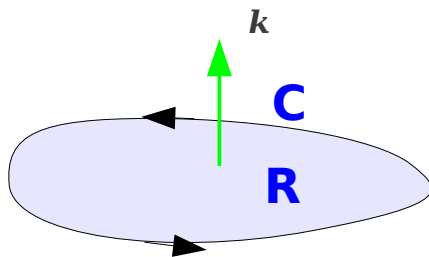
A force field (2-space)

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

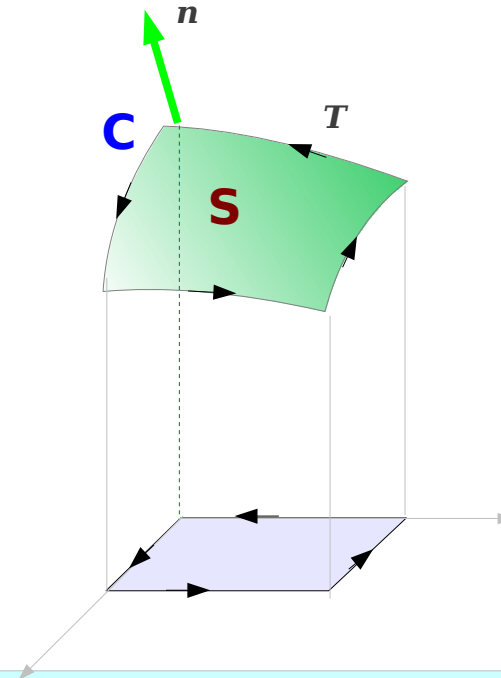
A force field (3-space)

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

2-space



3-space



$$\int_c \mathbf{F} \cdot \mathbf{T} ds = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA$$

$$\int_c \mathbf{F} \cdot \mathbf{T} ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

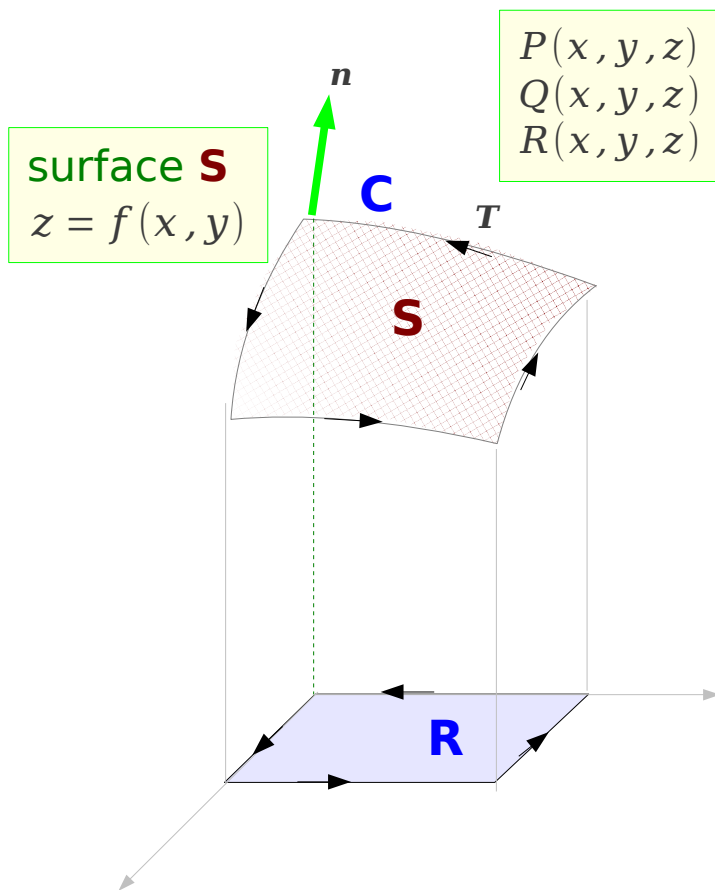
# Stokes' Theorem (2)

A force field (3-space)

$$\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}$$

Work done by  $\mathbf{F}$  along  $C$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$$



$$= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

3-space

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{\frac{\partial f}{\partial x} \mathbf{i} - \frac{\partial f}{\partial y} \mathbf{j} + \mathbf{k}}{\sqrt{1 + \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2}}$$

surface

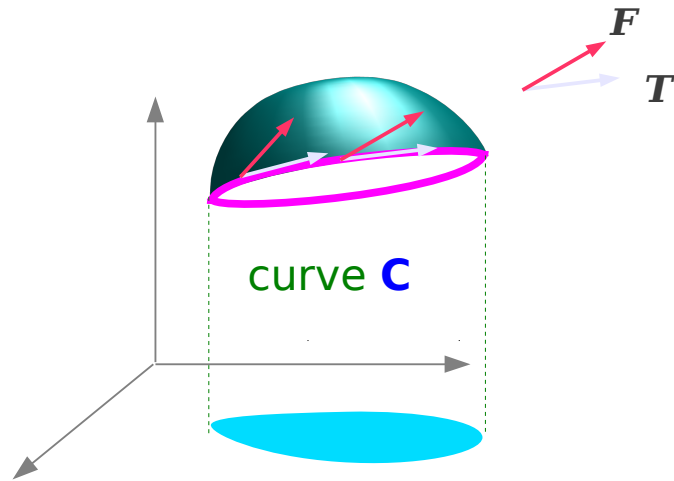
$$g(x, y, z)$$

$$= z - f(x, y)$$

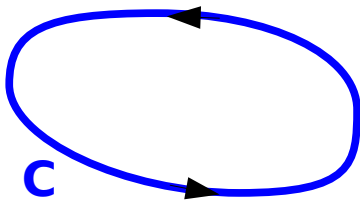


# Stokes' Theorem (3)

## Line Integral

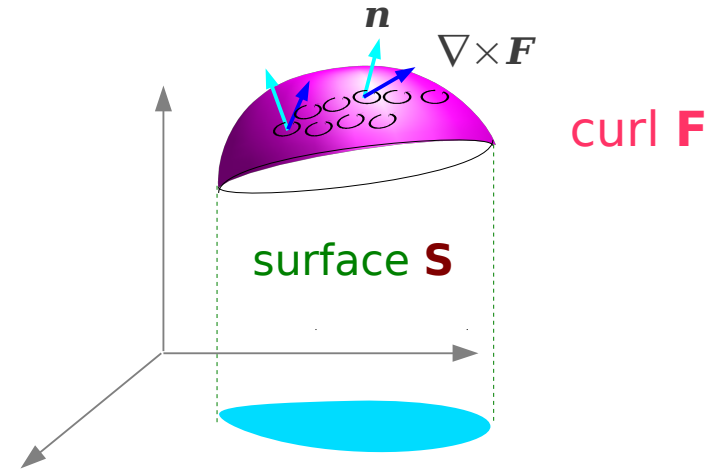


$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \oint_C P dx + Q dy + R dz$$

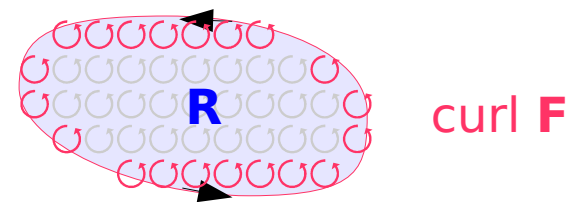


$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \oint_C P dx + Q dy$$

## Double Integral



$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

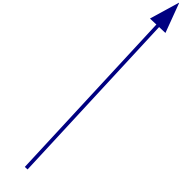


$$\iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA$$

# Chain Rule

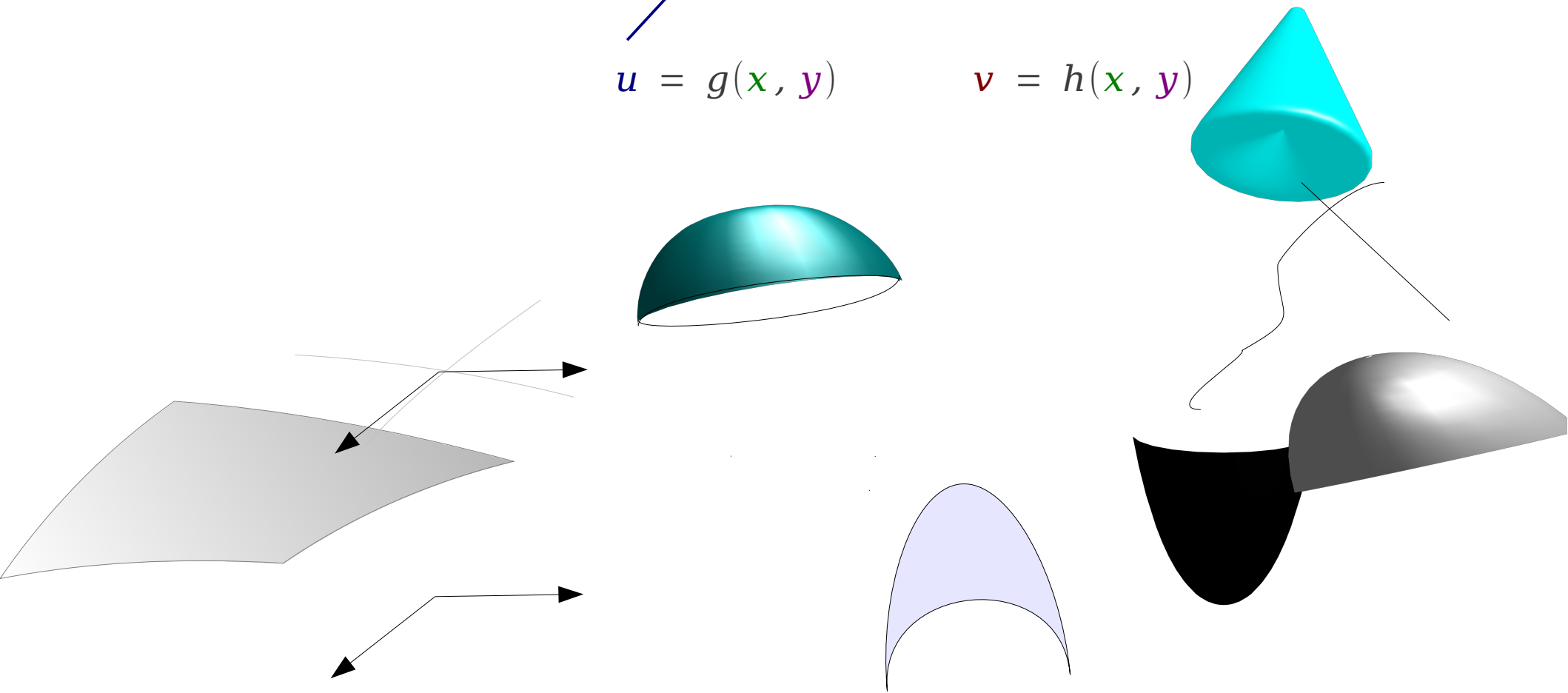
Function of two variable

$$y = f(u, v)$$



$$u = g(x, y)$$

$$v = h(x, y)$$



## References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [4] D.G. Zill, “Advanced Engineering Mathematics”