solutions_BVP_odd_extension.mw

> *restart* : *assume*(*n*, *integer*) : $L := 2$;

$$L := 2 \quad (1)$$

To solve the BVP $y'' + 4\,y = 4\,x$ with $y(0) = 0, y(L) = 0$,

1) Find the homogeneous solution y_h

2) Any solution to the inhomogeneous problem, y_p. Use a fourier series to represent the solution.

$$y_p = \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right)$$

3) Create the solution $y = y_h + y_p$.

4) Use boundary conditions to determine coefficients.

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Next find a solution that satisfies the nonhomogeneous problem $y'' + 4 \cdot y = 4\,x$, this is y_p.

First, use a even extension of $f(x) = 4 \cdot x$ and compute the fourier coefficients.

1) Find the general solution, y_h, to the homogeneous problem.

In the notes we found $y_h = c_1 \cdot \cos(2 \cdot x) + c_2 \cdot \sin(2 \cdot x)$ for the homogeneous equation $y'' + 4 \cdot y = 0$.

> *restart* :

> $yh := c1 \cdot \cos(2 \cdot x) + c2 \cdot \sin(2 \cdot x)$

$$yh := c1 \cos(2\,x) + c2 \sin(2\,x) \quad (2)$$

> $L := 2$

$$L := 2 \quad (3)$$

2) Find any solution, y_p, to the inhomogeneous problem. For the BVP use a fourier series to represent the solution.

$$y_p = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right) + b_n \cdot \sin\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right)$$

To determine the coefficients for the preceding series, it is necessary to determine the Fourier series representation of the function $f(x) = 4 \cdot x$.

Find the FS of f(x)=4x.

A first attempt is to extend $f(x)$ as an odd function with period $T = 2 \cdot L$. For this problem $L = 2$.

The even extension of $f(x)$ is represented by a cosine series only.

$$f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right)$$

the coefficients are found with the following integrals.

> $assume(n :: integer)$:

> $f := x \rightarrow 4 \cdot x;$

$$f := x \rightarrow 4\,x \qquad \textbf{(4)}$$

$$b_n = \frac{2}{L} \cdot \int_0^L f(x) \cdot \sin\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right) \mathrm{d}x$$

> $bn := \frac{2}{L} \cdot int\left(f(x) \cdot \sin\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right), x = 0..L\right);$

$$bn := \frac{16\,(-1)^{1+n\sim}}{n\sim\,\pi} \qquad \textbf{(5)}$$

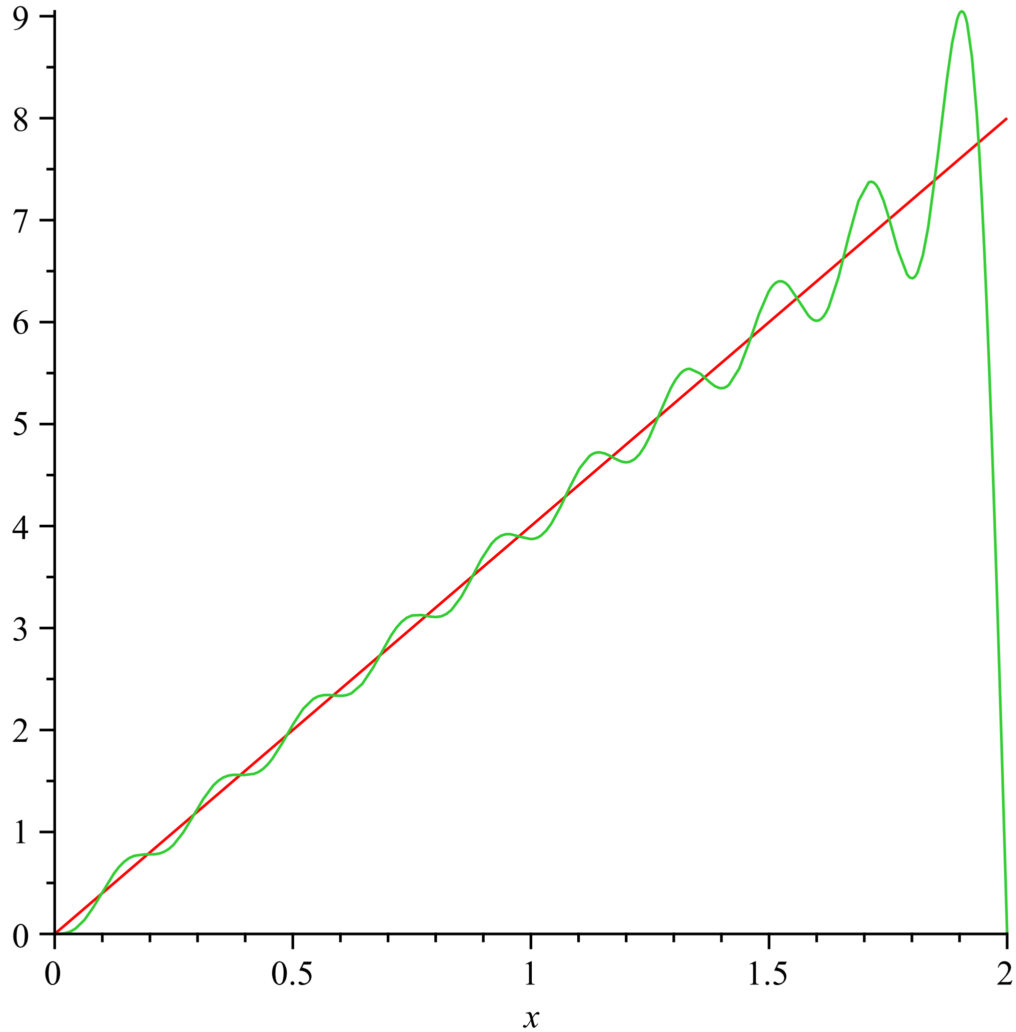
To check the FS approximation a plot of the original function $f(x) = 4 \cdot x$ and an approximation using

10 terms in the fourier series is plotted below.

> $fapprox := sum\left(bn\cdot\sin\left(\frac{n\cdot\mathrm{Pi}\cdot x}{L}\right), n=1..20\right)$

$$fapprox := \frac{16\sin\left(\frac{1}{2}\pi x\right)}{\pi} - \frac{8\sin(\pi x)}{\pi} + \frac{16}{3}\frac{\sin\left(\frac{3}{2}\pi x\right)}{\pi} - \frac{4\sin(2\pi x)}{\pi} + \frac{16}{5}\frac{\sin\left(\frac{5}{2}\pi x\right)}{\pi} - \frac{8}{3}\frac{\sin(3\pi x)}{\pi} + \frac{16}{7}\frac{\sin\left(\frac{7}{2}\pi x\right)}{\pi} - \frac{2\sin(4\pi x)}{\pi} + \frac{16}{9}\frac{\sin\left(\frac{9}{2}\pi x\right)}{\pi} - \frac{8}{5}\frac{\sin(5\pi x)}{\pi} + \frac{16}{11}\frac{\sin\left(\frac{11}{2}\pi x\right)}{\pi} - \frac{4}{3}\frac{\sin(6\pi x)}{\pi} + \frac{16}{13}\frac{\sin\left(\frac{13}{2}\pi x\right)}{\pi} - \frac{8}{7}\frac{\sin(7\pi x)}{\pi} + \frac{16}{15}\frac{\sin\left(\frac{15}{2}\pi x\right)}{\pi} - \frac{\sin(8\pi x)}{\pi} + \frac{16}{17}\frac{\sin\left(\frac{17}{2}\pi x\right)}{\pi} - \frac{8}{9}\frac{\sin(9\pi x)}{\pi} + \frac{16}{19}\frac{\sin\left(\frac{19}{2}\pi x\right)}{\pi} - \frac{4}{5}\frac{\sin(10\pi x)}{\pi} \quad (6)$$

> $plot([4\cdot x, fapprox], x=0..L);$



The plots are fairly close over the interval $(0, L)$. However at the boundaries there is a worsening match.

Next assume the solution is of the same form as $f(x)$. This gives $y_p = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n \cdot \pi \cdot x}{L}\right)$.

Substitute this into the differential equation,

$$y_p' = \sum_{n=1}^{\infty} B_n \cdot \left(\frac{n \cdot \pi}{L}\right) \cos\left(\frac{n \cdot \pi \cdot x}{L}\right)$$

$$y_p'' = \sum_{n=1}^{\infty} -B_n \cdot \left(\frac{n \cdot \pi}{L}\right)^2 \sin\left(\frac{n \cdot \pi \cdot x}{L}\right)$$

This gives the equation ,

$$y_p'' + 4\, y_p = 4 \cdot x$$

$$\sum_{n=1}^{\infty} -B_n \cdot \left(\frac{n\cdot\pi}{L}\right)^2 \sin\left(\frac{n\cdot\pi\cdot x}{L}\right) + 4\cdot\sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\cdot\pi\cdot x}{L}\right) = \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\cdot \mathrm{Pi}\cdot x}{L}\right)$$

$$\sum_{n=1}^{\infty} B_n \cdot \left(4 - \left(\frac{n\cdot\pi}{L}\right)^2\right) \sin\left(\frac{n\cdot\pi\cdot x}{L}\right) = \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\cdot \mathrm{Pi}\cdot x}{L}\right)$$

Equate the coefficients of like terms on each side and solve for B_n in terms of b_n

Similarly $B_n \cdot \left(4 - \left(\frac{n\cdot\pi}{L}\right)^2\right) = b_n$ so $B_n = \dfrac{b_n}{4 - \left(\frac{n\cdot\pi}{L}\right)^2}$ and again a_n was found above.

The calculations follow.

>

> $Bn := \dfrac{bn}{4 - \left(\frac{n\cdot \mathrm{Pi}}{L}\right)^2}$

$$Bn := \frac{16\,(-1)^{1+n\sim}}{n\sim\,\pi\left(4 - \frac{1}{4}\,n\sim^2\,\pi^2\right)} \tag{7}$$

> Bn

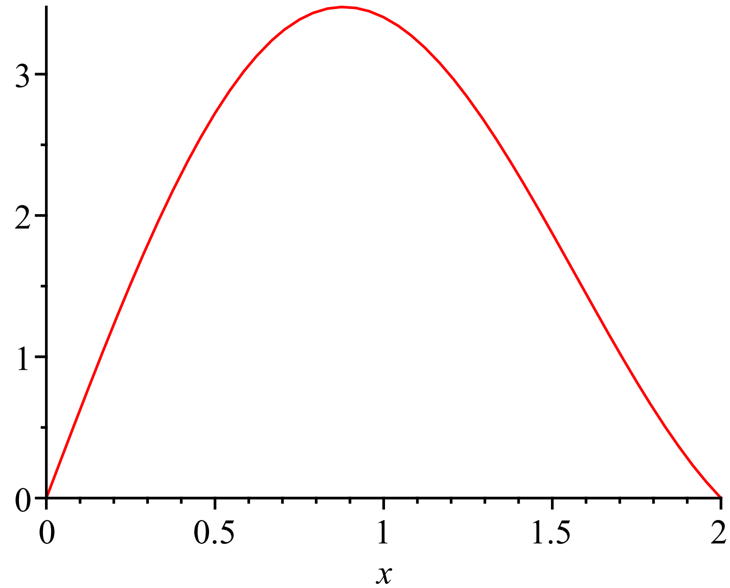
$$\frac{16\,(-1)^{1+n\sim}}{n\sim\,\pi\left(4 - \frac{1}{4}\,n\sim^2\,\pi^2\right)} \tag{8}$$

Let's look at an approximation of y_p using the fourier sum with 5 terms.

> $yp := sum\left(Bn\cdot\sin\left(\frac{n\cdot \mathrm{Pi}\cdot x}{L}\right), n = 1\,..30\right):$

A plot of the approximation of y_p. This certainly fulfills the boundary conditions.

> $plot(yp, x = 0\,..L);$



And it does satisfy the ODE.

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> plot([diff(yp, x, x) + 4·yp, 4·x], x = 0 ..L);
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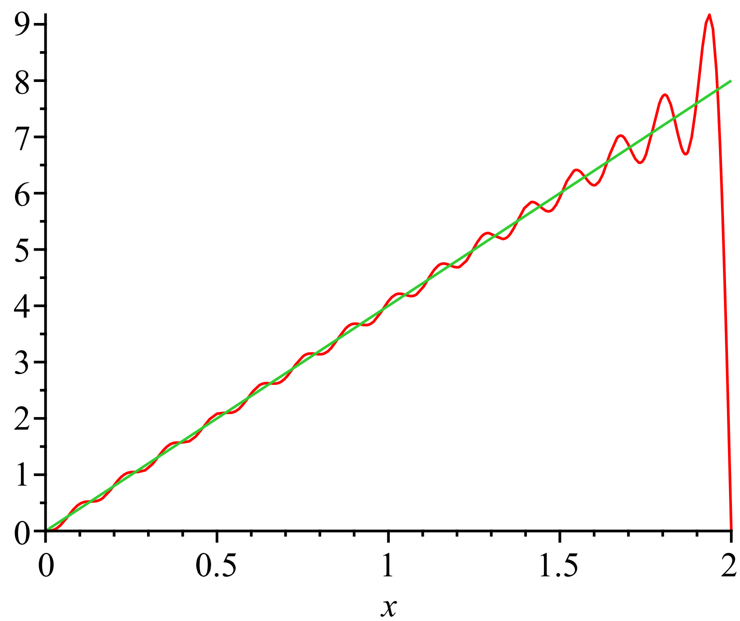

Use the homogeneous solution y_h that was found earlier and form the general solution

$y = y_h + y_p$

The complete solution is $y = y_h + y_p$

> $y := yh + yp$

$$y := c1\cos(2x) + c2\sin(2x) + \frac{16\sin\left(\frac{1}{2}\pi x\right)}{\pi\left(4-\frac{1}{4}\pi^2\right)} + \frac{16}{11}\frac{\sin\left(\frac{11}{2}\pi x\right)}{\pi\left(4-\frac{121}{4}\pi^2\right)} - \frac{2}{3}\frac{\sin(12\pi x)}{\pi(4-144\pi^2)} - \frac{2\sin(4\pi x)}{\pi(4-16\pi^2)} - \frac{8}{7}\frac{\sin(7\pi x)}{\pi(4-49\pi^2)} + \frac{16}{7}\frac{\sin\left(\frac{7}{2}\pi x\right)}{\pi\left(4-\frac{49}{4}\pi^2\right)} - \frac{4}{3}\frac{\sin(6\pi x)}{\pi(4-36\pi^2)} - \frac{4}{5}\frac{\sin(10\pi x)}{\pi(4-100\pi^2)} + \frac{16}{9}\frac{\sin\left(\frac{9}{2}\pi x\right)}{\pi\left(4-\frac{81}{4}\pi^2\right)} - \frac{\sin(8\pi x)}{\pi(4-64\pi^2)} + \frac{16}{23}\frac{\sin\left(\frac{23}{2}\pi x\right)}{\pi\left(4-\frac{529}{4}\pi^2\right)} + \frac{16}{17}\frac{\sin\left(\frac{17}{2}\pi x\right)}{\pi\left(4-\frac{289}{4}\pi^2\right)} - \frac{8}{9}\frac{\sin(9\pi x)}{\pi(4-81\pi^2)} + \frac{16}{19}\frac{\sin\left(\frac{19}{2}\pi x\right)}{\pi\left(4-\frac{361}{4}\pi^2\right)} + \frac{16}{3}\frac{\sin\left(\frac{3}{2}\pi x\right)}{\pi\left(4-\frac{9}{4}\pi^2\right)} - \frac{4\sin(2\pi x)}{\pi(4-4\pi^2)} + \frac{16}{27}\frac{\sin\left(\frac{27}{2}\pi x\right)}{\pi\left(4-\frac{729}{4}\pi^2\right)} + \frac{16}{13}\frac{\sin\left(\frac{13}{2}\pi x\right)}{\pi\left(4-\frac{169}{4}\pi^2\right)} + \frac{16}{15}\frac{\sin\left(\frac{15}{2}\pi x\right)}{\pi\left(4-\frac{225}{4}\pi^2\right)} - \frac{8}{3}\frac{\sin(3\pi x)}{\pi(4-9\pi^2)} + \frac{16}{29}\frac{\sin\left(\frac{29}{2}\pi x\right)}{\pi\left(4-\frac{841}{4}\pi^2\right)} + \frac{16}{21}\frac{\sin\left(\frac{21}{2}\pi x\right)}{\pi\left(4-\frac{441}{4}\pi^2\right)} - \frac{4}{7}\frac{\sin(14\pi x)}{\pi(4-196\pi^2)} - \frac{8}{11}\frac{\sin(11\pi x)}{\pi(4-121\pi^2)} - \frac{8}{5}\frac{\sin(5\pi x)}{\pi(4-25\pi^2)} - \frac{8\sin(\pi x)}{\pi(4-\pi^2)} + \frac{16}{25}\frac{\sin\left(\frac{25}{2}\pi x\right)}{\pi\left(4-\frac{625}{4}\pi^2\right)} + \frac{16}{5}\frac{\sin\left(\frac{5}{2}\pi x\right)}{\pi\left(4-\frac{25}{4}\pi^2\right)} - \frac{8}{13}\frac{\sin(13\pi x)}{\pi(4-169\pi^2)} - \frac{8}{15}\frac{\sin(15\pi x)}{\pi(4-225\pi^2)} \quad (9)$$

Use boundary conditions to find c_1, c_2.

Solve $y(0) = 0 = c_1 \cdot \cos(0) + c_2 \cdot \sin(0) + y_p(0)$ for c_1.

> $c1 := solve(subs(x=0, y)=0, c1) : evalf(c1);$

$$0. \quad \textbf{(10)}$$

And c_2.

> $c2 := solve(subs(x=L, y)=0, c2) : evalf(c2);$

$$0. \quad \textbf{(11)}$$

Both coefficients are zero so $y_h = 0$ on the interval $[0, 2]$. The is different from the even extension where a nonzero homogeneous solution is needed to fulfill the boundary conditions.

Whoopi! THis is a really interesting outcome.

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Next an approximation of the solution, $y = y_h + y_p$, of the BVP is made by cutting off the Fourier sum at "N" terms.

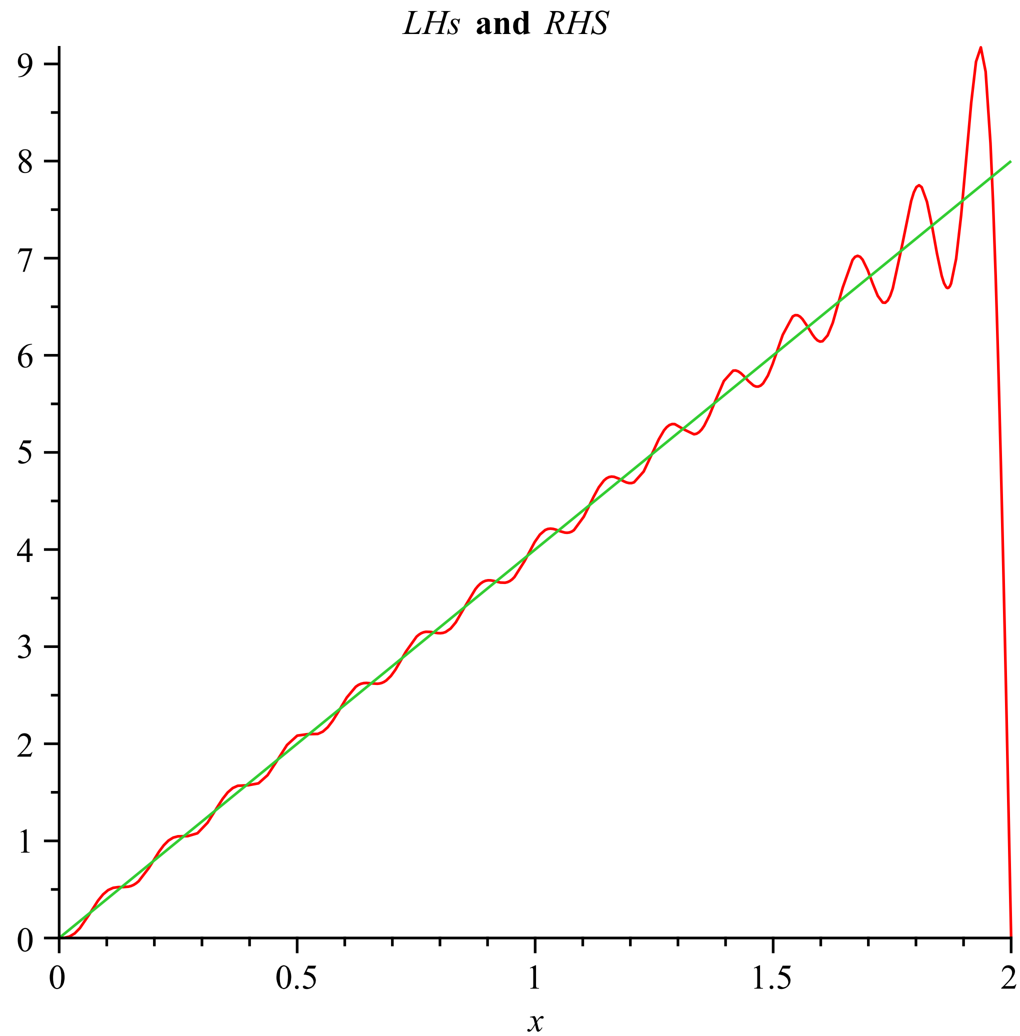
It is used to check to see if the solution satisfies the differential equations and the boundary condition.

>

1) Does the solution satisfy the ODE $y'' + 4\,y = 4\,x$ on the interval [0,2]. I could substitute our solution into the lefts and right sides of the DE and work through it by hand. Another way is to use the finite series with "N" terms to approximate y(x) and use maple to plot the left and right sides of the DE. If the plots are similar then the solution would seem to satisfy the DE.

A maple plot of the these follows

> $plot([diff(y, x, x) + 4 \cdot y, 4 \cdot x], x = 0..L, title = [LHS$ **and** $RHS]);$
 # The approximation is close .



2) Does the solution satisfy the boundary conditions. Check the boundary conditions.

At the left end $y(0) = 0$? The solution works.

> $evalf(subs(x=0, y));$

$$0. \tag{12}$$

At the right end $y(2) = 0$? The solution works.

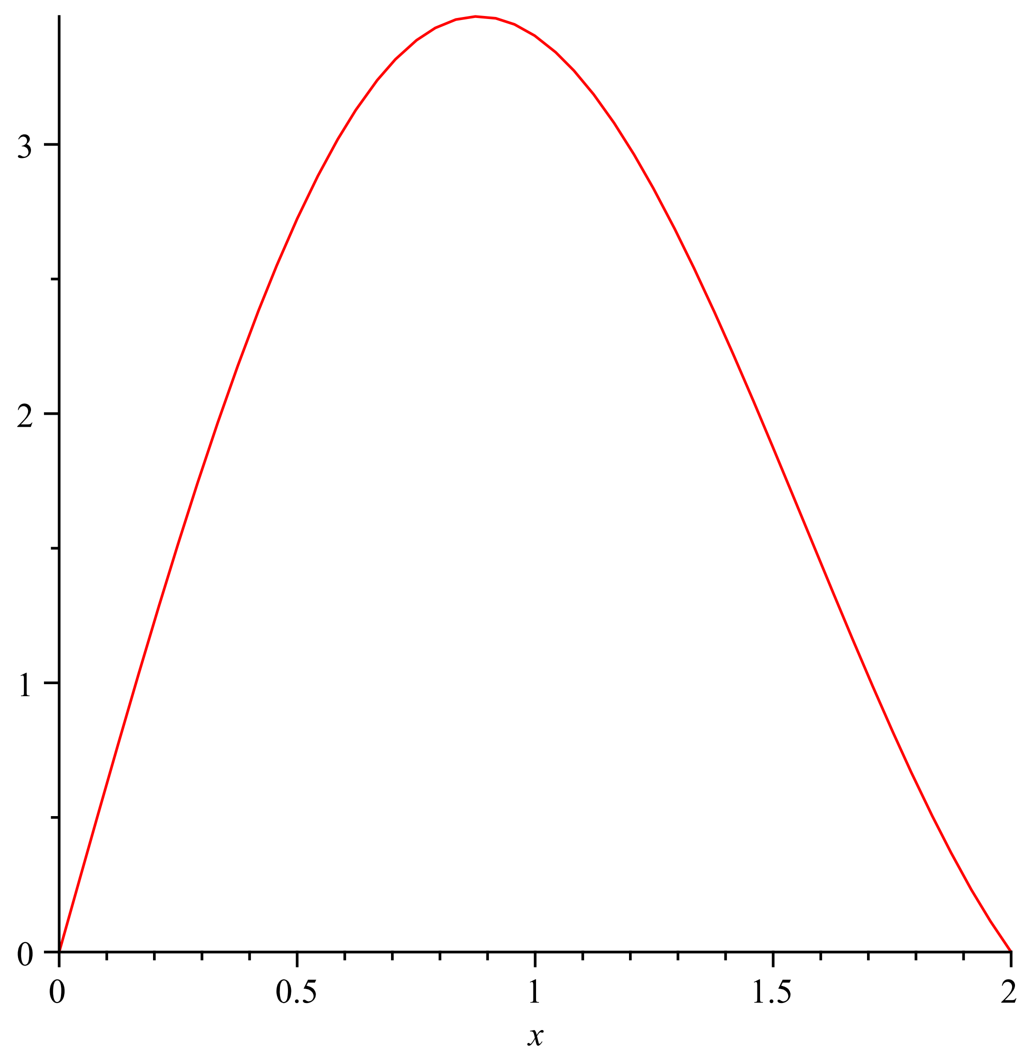
> $evalf(subs(x=L, y));$

$$-5.533469634\ 10^{-10} \tag{13}$$

These are close to "zero" , this is because we are using the approximation of the solution.

Note: The approximation of the solution on the [0,2] interval.

> $plot(y, x=0..2)$


3
2
1
0
0
0.5
1
1.5
2
x

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