Correlation (1A)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Probability Density Function

Probability Density Function $$f(x;\ t) = \frac{\partial}{\partial x} F(x;\ t)$$

$$P[\,a \le X(t) \le b\,] = \int_a^b f(x;\ t)\,dx$$

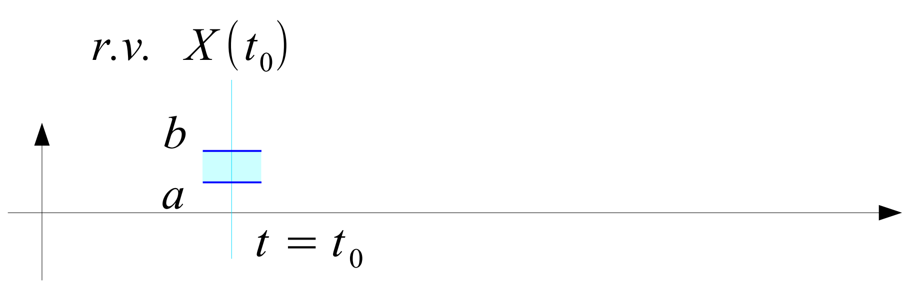
Cumulative Distribution Function $$F(x;\ t) = \int_{-\infty}^{x} f(x;\ t)\,dx$$

$$P[\,X(t) \le b\,] = \int_{-\infty}^{b} f(x;\ t)\,dx = F(b;\ t)$$

$X(t)$	*random variable X(t) at a given time t*
$f(x;\ t)$	*The probability density function of random variable X(t)*

$$P[\,x \le X(t) \le x + dt\,] = f(x;\ t)\,dx$$

$$P[\,X(t) = x\,] \Rightarrow f(x;\ t)$$



Probability Density Function

$X(t)$	*random variable X(t) at a given time t*
$f(x;\ t)$	*The probability density function of random variable X(t)*

$$P[\, x \le X(t) \le x + dt \,] = f(x;\ t)\, dx$$

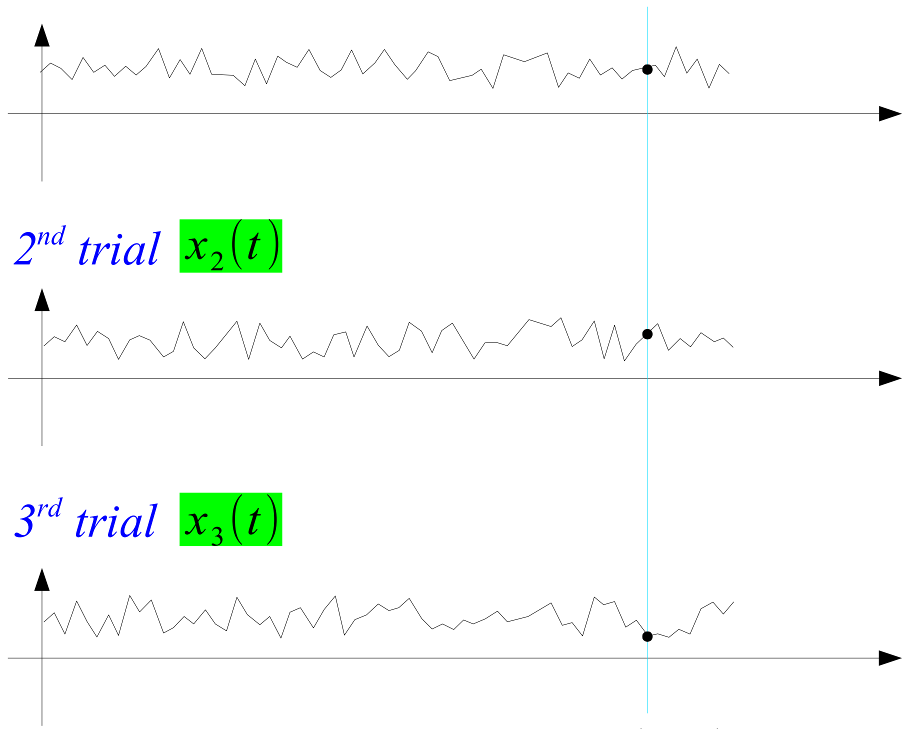
$$P[\, X(t) = x \,] \Rightarrow f(x;\ t)$$

$X(t_0)$ *random variable at a given time t_0*

1st trial $x_1(t)$

2nd trial $x_2(t)$

3rd trial $x_3(t)$



$t = t_0$

$$X(t_0) = x_1(t_0)$$

r.v. $X(t_0)$ *has a value* $x_1(t_0)$ *at time* t_0 *in the* ***1st trial***

$$X(t_0) = x_2(t_0)$$

r.v. $X(t_0)$ *has a value* $x_2(t_0)$ *at time* t_0 *in the* ***2nd trial***

$$X(t_0) = x_3(t_0)$$

r.v. $X(t_0)$ *has a value* $x_3(t_0)$ *at time* t_0 *in the* ***3rd trial***

Joint Probability Density Function

Joint Probability Density Function $$f(x_1, x_2;\ t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2;\ t_1, t_2)$$

$$P\left[\{a \le X(t_1) \le b\} \cap \{c \le X(t_2) \le d\}\right] = \int_a^b \int_c^d f(x_1, x_2;\ t_1, t_2)\, dx_1\, dx_2$$

Joint Cumulative Distribution Function $$F(x_1, x_2;\ t_1, t_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(x_1, x_2;\ t_1, t_2)\, dx_1\, dx_2$$

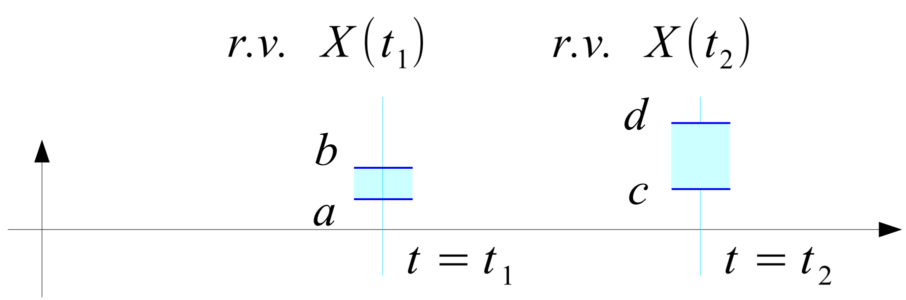
$$P\left[\{X(t_1) \le b\} \cap \{X(t_2) \le d\}\right] = \int_{-\infty}^{b} \int_{-\infty}^{d} f(x_1, x_2;\ t_1, t_2)\, dx_1\, dx_2 = F(b, d;\ t_1, t_2)$$

$X(t_1),\ X(t_2)$ *random variables at time t_1 and t_2*

$f(x_1, t_1;\ x_2, t_2)$ *joint probability density function*

$$P\left[\{X(t_1) = x_1\} \cap \{X(t_2) = x_2\}\right]$$

$$\Rightarrow\ f(x_1, t_1;\ x_2, t_2)$$



Probability Density Function

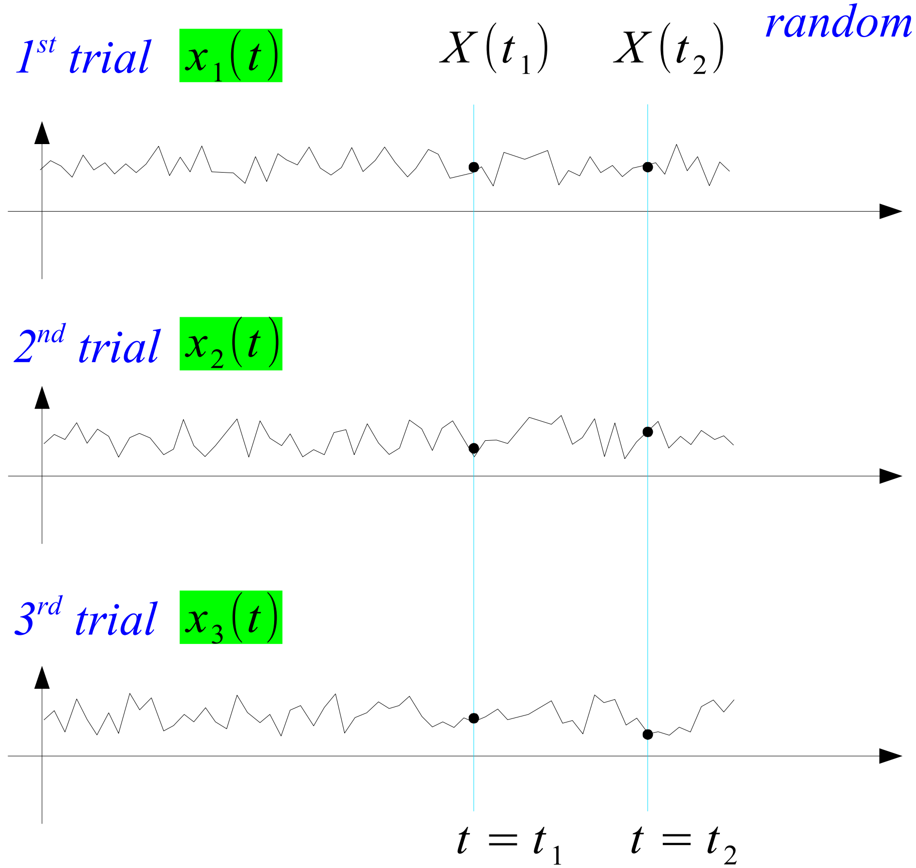
$X(t_1),\ \ X(t_2)$ *random variables at time t_1 and t_2*

$f(x_1, t_1;\ \ x_2, t_2)$ *joint probability density function*

$$P[\,\{X(t_1) = x_1\} \cap \{X(t_2) = x_2\}\,]$$

$$\Rightarrow\ f(x_1, t_1;\ \ x_2, t_2)$$

random variable at a given time t_0



$X(t_1) = x_1(t_1), \quad X(t_2) = x_1(t_2)$

r.v. $X(t_1)$ *has a value* $x_1(t_1)$

r.v. $X(t_2)$ *has a value* $x_1(t_2)$

at times $t_1,\ t_2$ *in the **1st trial***

$X(t_1) = x_2(t_1), \quad X(t_2) = x_2(t_2)$

r.v. $X(t_1)$ *has a value* $x_2(t_1)$

r.v. $X(t_2)$ *has a value* $x_2(t_2)$

at times $t_1,\ t_2$ *in the **2nd trial***

$X(t_1) = x_3(t_1), \quad X(t_2) = x_3(t_2)$

r.v. $X(t_1)$ *has a value* $x_3(t_1)$

r.v. $X(t_2)$ *has a value* $x_3(t_2)$

at times $t_1,\ t_2$ *in the **3rd trial***

Moments of a Random Process

The n-th Moment

$$E[\,X^n(t)\,] = \int_{-\infty}^{+\infty} x^n \, f(x\,;\,t)\,dx$$

1st Moment

$$E[\,X(t)\,] = \int_{-\infty}^{+\infty} x \; f(x\,;\,t)\,dx = \mu(t)$$

2nd Moment

$$E[\,X^2(t)\,] = \int_{-\infty}^{+\infty} x^2 \, f(x\,;\,t)\,dx$$

The n-th Central Moment

$$E[\,(X(t) - \mu(t))^n\,] = \int_{-\infty}^{+\infty} (x - \mu(t))^n \, f(x\,;\,t)\,dx$$

1st Central Moment

$$E[\,(X(t) - \mu(t))\,] = \int_{-\infty}^{+\infty} (x - \mu(t)) \, f(x\,;\,t)\,dx$$

2nd Central Moment

$$E[\,(X(t) - \mu(t))^2\,] = \int_{-\infty}^{+\infty} (x - \mu(t))^2 \, f(x\,;\,t)\,dx$$

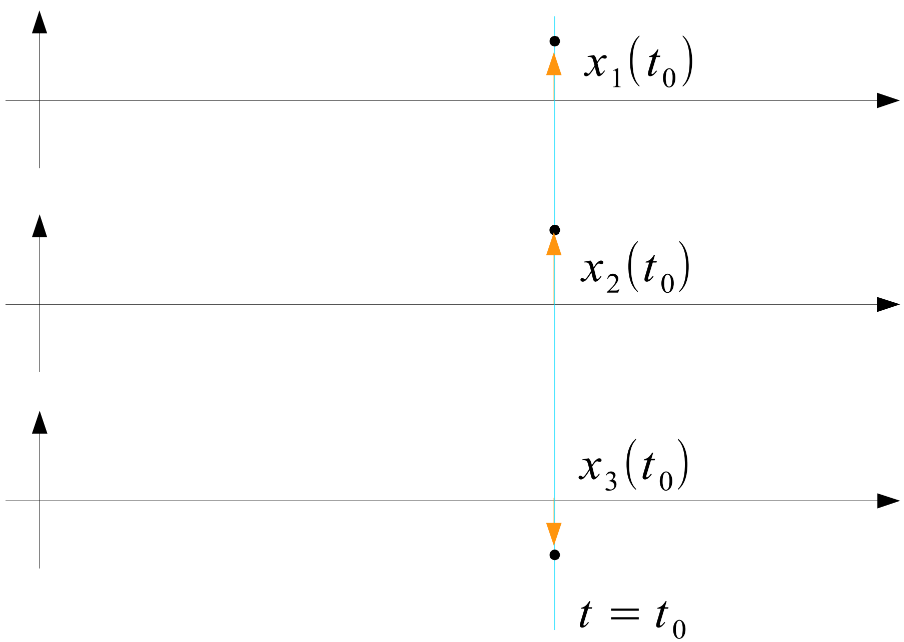
$$= \sigma^2(t)$$

Moments of a Random Process

$X(t)$ Random Variable at a given time t

$x_i(t)$ outcome of i[th] realization at a given time t

Ensemble



Ensemble Average

$$\bar{x}(t) = \frac{1}{N} \sum_{n=1}^{N} X_n(t)$$

$$N \to \infty$$

$$E[\,X(t)\,] = \int_{-\infty}^{+\infty} x\, f(x\,;\,t)\, dx = \mu(t)$$

$$E[\,X(t_0)\,] = \int_{-\infty}^{+\infty} x\, f(x\,;\,t_0)\, dx = \mu(t_0)$$

Moments of a Random Process

The n-th Moment

$$E[\,X^n(t)\,] = \int_{-\infty}^{+\infty} x^n\, f(x\,;\,t)\, dx$$

The n-th Central Moment

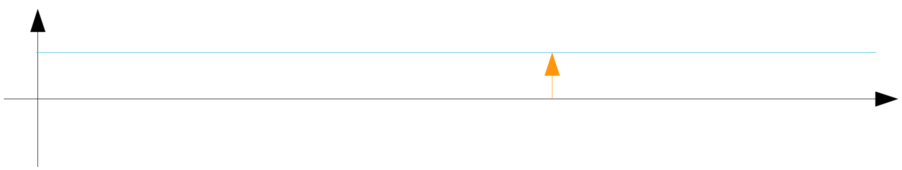
$$E[\,(X(t)-\mu(t))^n\,] = \int_{-\infty}^{+\infty} (x-\mu(t))^n\, f(x\,;\,t)\, dx$$

$$f(x,t) \;\Rightarrow\; f(x)$$

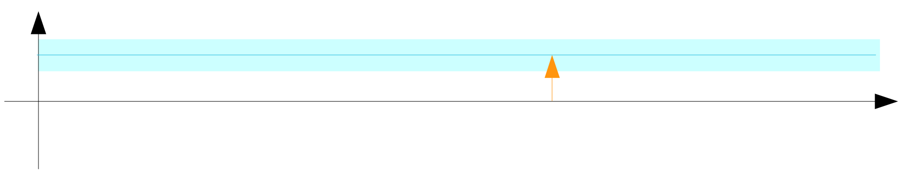
if $\boldsymbol{f}$ *does not change with time*

$$E[\,X^n(t)\,] \;\Rightarrow\; E[\,X^n\,]$$

$$E[\,(X(t)-\mu(t))^n\,] \;\Rightarrow\; E[\,(X-\mu)^n\,]$$



$$E[\,X(t)\,] \;\Rightarrow\; E[\,X\,] = \mu$$



$$E[\,(X(t)-\mu(t))^2\,] \;\Rightarrow\; E[\,(X-\mu)^2\,] = \sigma^2$$

Stationarity

First-Order Stationary Process

$$f(x_1;\; t_1) = f(x_1;\; t_1+\tau)$$

Second-Order Stationary Process

$$f(x_1, x_2;\; t_1, t_2) = f(x_1, x_2;\; t_1+\tau, t_2+\tau)$$

Nth-Order Stationary Process

$$f(x_1, x_2, \cdots, x_n;\; t_1, t_2, \cdots, t_n) = f(x_1, x_2, \cdots, x_n;\; t_1+\tau, t_2+\tau, \cdots, t_n+\tau)$$

Correlation Functions (1)

auto-covariance function

$$C_{xx}(t_1, t_2) = E\big[\,\underline{(X(t_1) - \mu_x(t_1))}\ \underline{(X(t_2) - \mu_x(t_2))}\,\big]$$

$$C_{xx}(t_2 - t_1) = E\big[\,(X(t_1) - \mu_x(t_1))\ (X(t_2) - \mu_x(t_2))\,\big]$$

$$C_{xx}(\tau) = E\big[\,(X(t) - \mu_x)\ (X(t+\tau) - \mu_x)\,\big]$$

auto-correlation function

$$R_{xx}(t_1, t_2) = E\big[\,\underline{X(t_1)}\ \underline{X(t_2)}\,\big]$$

$$R_{xx}(\tau) = E\big[\,X(t)\ X(t+\tau)\,\big]$$

Correlation Functions (2)

cross-covariance function

$$C_{xy}(t_1, t_2) = E\left[\underline{(X(t_1) - \mu_x(t_1))} \; \underline{(Y(t_2) - \mu_y(t_2))} \right]$$

$$C_{xy}(\tau) = E\left[(X(t) - \mu_x) \; (Y(t+\tau) - \mu_y) \right]$$

cross-correlation function

$$R_{xy}(t_1, t_2) = E\left[\underline{X(t_1)} \; \underline{Y(t_2)} \right]$$

$$R_{xy}(\tau) = E\left[X(t) \, Y(t + \tau) \right]$$

Ergodicity

$$\mu_x = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,dt$$

$$\hat{\mu}_x = \frac{1}{T}\int_0^T x(t)\,dt = \bar{x}$$

$$\bar{\psi}_x^2 = \frac{1}{T}\int_0^T x^2(t)\,dt = \overline{x^2}$$

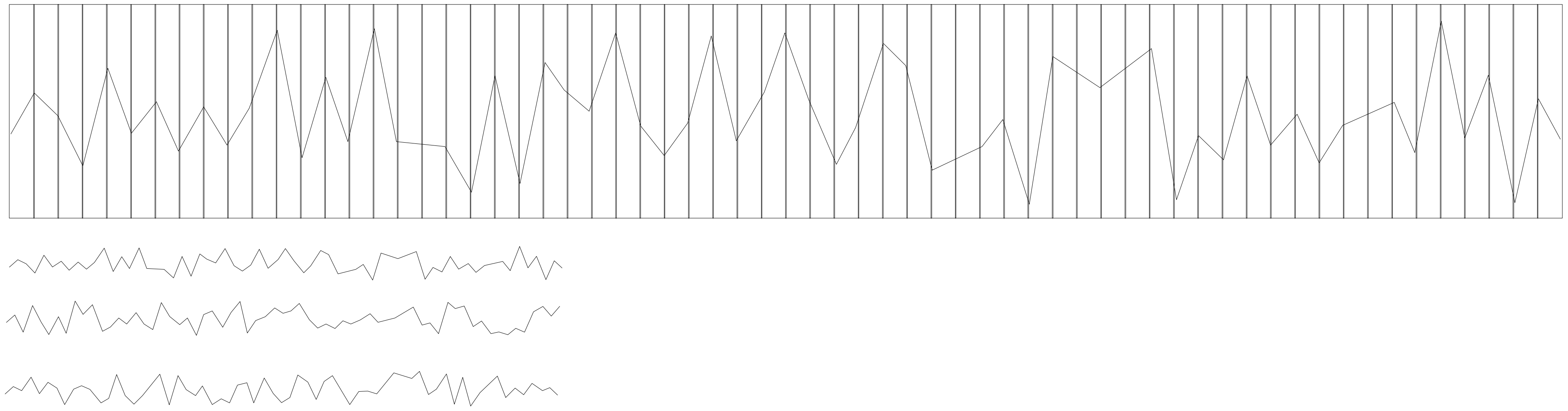
$$C_{xy}(\tau) = E\left[\,(X(t)-\mu_x)\;(Y(t+\tau)-\mu_y)\,\right]$$

$$C_{xy}(\tau) = \lim_{T\to 0} \frac{1}{T}\int_0^T (x(t)-\mu_x)\;(y(t+\tau)-\mu_y)\,dt$$

$$\hat{C}_{xy}(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} (x(t)-\bar{x})\;(y(t+\tau)-\bar{y})\,dt \qquad 0 \le \tau < T$$

$$\hat{C}_{xy}(\tau) = \frac{1}{T-|\tau|}\int_0^{T-|\tau|} (x(t)-\bar{x})\;(y(t+\tau)-\bar{y})\,dt \qquad -T < \tau \le 0$$

Time Average



Frequency Response

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow h(t)$$

$$A\, e^{j\Phi}\, e^{jwt} \longrightarrow \boxed{h(t)} \longrightarrow \boxed{H(j\omega)}\, A\, e^{j\Phi}\, e^{j\omega t}$$

single frequency component : ω

single frequency component : ω

$$\boxed{H(j\omega) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau}$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] K.H. Shin, J.K. Hammond, Fundamentals of Signal Processing for Sound and Vibration Engineers, Wiley, 2008