

Correlation (1A)

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Moments of a Random Process

The n-th Moment

$$E[X^n(t)] = \int_{-\infty}^{+\infty} x^n f(x, t) dx$$

1st Moment

$$E[X(t)] = \int_{-\infty}^{+\infty} x f(x, t) dx = \mu(t)$$

2nd Moment

$$E[X^2(t)] = \int_{-\infty}^{+\infty} x^2 f(x, t) dx$$

The n-th Central Moment

$$E[(X(t) - \mu(t))^n] = \int_{-\infty}^{+\infty} (x - \mu(t))^n f(x, t) dx$$

1st Central Moment

$$E[(X(t) - \mu(t))] = \int_{-\infty}^{+\infty} (x - \mu(t)) f(x, t) dx$$

2nd Central Moment

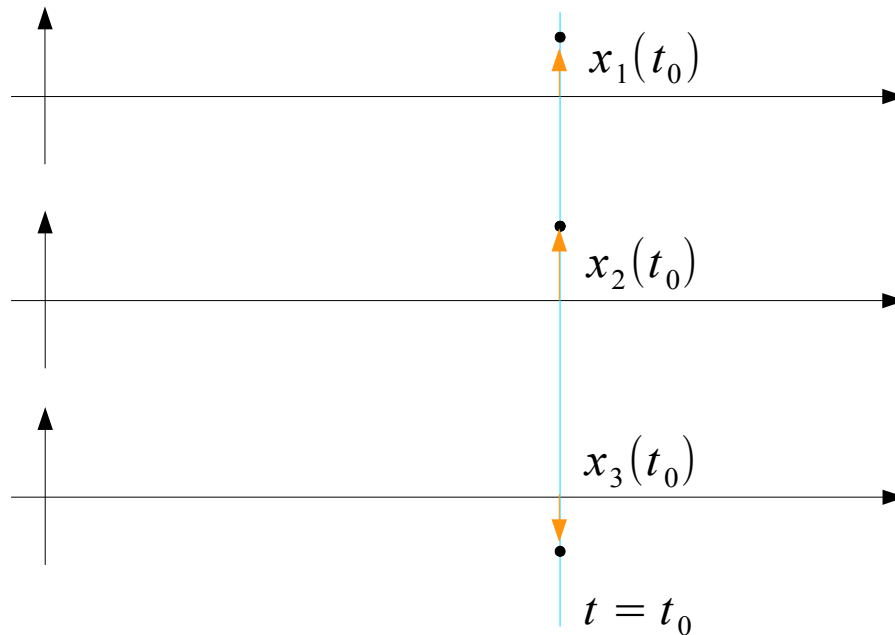
$$\begin{aligned} E[(X(t) - \mu(t))^2] &= \int_{-\infty}^{+\infty} (x - \mu(t))^2 f(x, t) dx \\ &= \sigma^2(t) \end{aligned}$$

Moments of a Random Process

$X(t)$ Random Variable at a given time t

$x_i(t)$ outcome of i^{th} realization at a given time t

Ensemble



Ensemble Average

$$\bar{x}(t) = \frac{1}{N} \sum_{n=1}^N X_n(t)$$

↓ $N \rightarrow \infty$

$$E[X(t)] = \int_{-\infty}^{+\infty} x f(x, t) dx = \mu(t)$$

$$E[X(t_0)] = \int_{-\infty}^{+\infty} x f(x, t_0) dx = \mu(t_0)$$

Moments of a Random Process

The n-th Moment

$$E[X^n(t)] = \int_{-\infty}^{+\infty} x^n f(x, t) dx$$



$f(x, t) = f(x)$
if f does not change with time

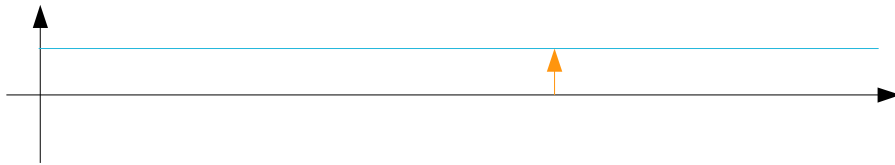
$$E[X^n(t)] = E[X^n]$$

The n-th Central Moment

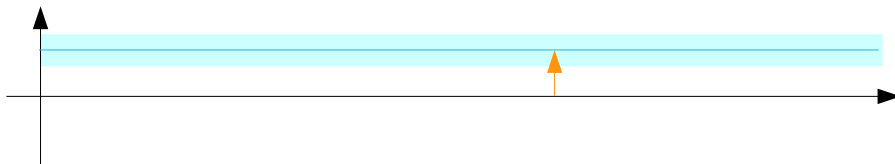
$$E[(X(t) - \mu(t))^n] = \int_{-\infty}^{+\infty} (x - \mu(t))^n f(x, t) dx$$



$$E[(X(t) - \mu(t))^n] = E[(X - \mu)^n]$$



$$E[X(t)] = E[X] = \mu$$



$$E[(X(t) - \mu(t))^2] = E[(X - \mu)^2] = \sigma^2$$

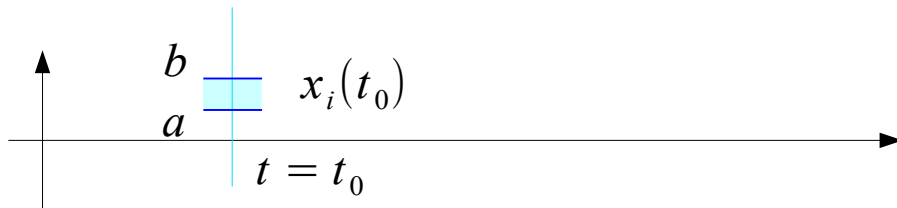
Probability Density Function

Probability Density Function $f(x, t) = \frac{\partial}{\partial x} F(x, t)$

$$P[a \leq X(t) \leq b] = \int_a^b f(x, t) dx$$

Cumulative Distribution Function $F(x, t) = \int_{-\infty}^x f(x, t) dx$

$$P[X(t) \leq b] = \int_{-\infty}^b f(x, t) dx = F(b, t)$$



Joint Probability Density Function

Joint Probability Density Function

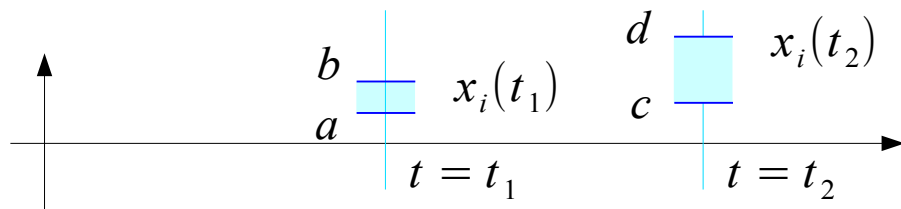
$$f(x_1, t_1; x_2, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, t_1; x_2, t_2)$$

$$P[\{a \leq X(t_1) \leq b\} \cap \{c \leq X(t_2) \leq d\}] = \int_a^b \int_c^d f(x_1, t_1; x_2, t_2) dx_1 dx_2$$

Joint Cumulative Distribution Function

$$F(x_1, t_1; x_2, t_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(x_1, t_1; x_2, t_2) dx_1 dx_2$$

$$P[\{X(t_1) \leq b\} \cap \{X(t_2) \leq d\}] = \int_{-\infty}^b \int_{-\infty}^d f(x_1, t_1; x_2, t_2) dx_1 dx_2 = F(b, t_1; d, t_2)$$



Stationarity

Correlation Functions (1)

auto-covariance function

$$C_{xx}(t_1, t_2) = E \left[\underbrace{(X(t_1) - \mu_x(t_1))}_{\text{green}} \underbrace{(X(t_2) - \mu_x(t_2))}_{\text{blue}} \right]$$

$$C_{xx}(t_2 - t_1) = E \left[(X(t_1) - \mu_x(t_1)) (X(t_2) - \mu_x(t_2)) \right]$$

$$C_{xx}(\tau) = E \left[(X(t) - \mu_x) (X(t + \tau) - \mu_x) \right]$$

auto-correlation function

$$R_{xx}(t_1, t_2) = E \left[\underbrace{X(t_1)}_{\text{green}} \underbrace{X(t_2)}_{\text{blue}} \right]$$

$$R_{xx}(\tau) = E \left[X(t) X(t + \tau) \right]$$

Correlation Functions (2)

cross-covariance function

$$C_{xy}(t_1, t_2) = E \left[\underbrace{(X(t_1) - \mu_x(t_1))}_{\text{green}} \underbrace{(Y(t_2) - \mu_y(t_2))}_{\text{blue}} \right]$$

$$C_{xy}(\tau) = E \left[(X(t) - \mu_x) (Y(t + \tau) - \mu_y) \right]$$

cross-correlation function

$$R_{xy}(t_1, t_2) = E \left[\underbrace{X(t_1)}_{\text{green}} \underbrace{Y(t_2)}_{\text{blue}} \right]$$

$$R_{xy}(\tau) = E \left[X(t) Y(t + \tau) \right]$$

Ergodicity

$$\mu_x = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t) dt$$

$$\hat{\mu}_x = \frac{1}{T} \int_0^T x(t) dt = \bar{x}$$

$$\bar{\Psi}_x^2 = \frac{1}{T} \int_0^T x^2(t) dt = \bar{x}^2$$

$$C_{xy}(\tau) = E[(X(t) - \mu_x)(Y(t+\tau) - \mu_y)]$$

$$C_{xy}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (x(t) - \mu_x)(y(t+\tau) - \mu_y) dt$$

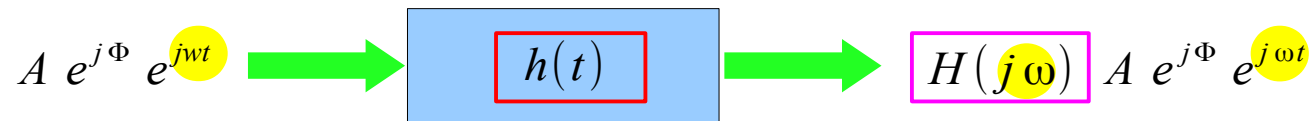
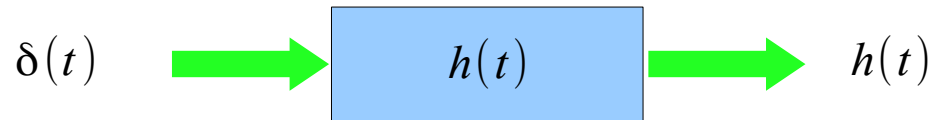
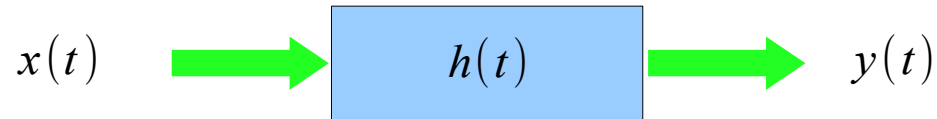
$$\hat{C}_{xy}(\tau) = \frac{1}{T-\tau} \int_0^{T-\tau} (x(t) - \bar{x})(y(t+\tau) - \bar{y}) dt \quad 0 \leq \tau < T$$

$$\hat{C}_{xy}(\tau) = \frac{1}{T-|\tau|} \int_0^{T-|\tau|} (x(t) - \bar{x})(y(t+\tau) - \bar{y}) dt \quad -T < \tau \leq 0$$

Time Average

Frequency Response

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau) x(t - \tau) d\tau$$



single frequency
component : ω

single frequency
component : ω

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau$$

References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] K.H. Shin, J.K. Hammond, Fundamentals of Signal Processing for Sound and Vibration Engineers, Wiley, 2008