CORDIC Background (2A)

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CORDIC Background

1. A survey of CORDIC algorithms for FPGAs, Ray Andraka, www.andraka.com/cordic.htm

Vector Rotation (1)

$$x' = x\cos\phi - y\sin\phi$$
$$y' = y\cos\phi + x\sin\phi$$

$$x' = \cos\phi \cdot [x - y\tan\phi]$$
$$y' = \cos\phi \cdot [y + x\tan\phi]$$

$$x_{i+1} = K_i \cdot [x_i - y_i \cdot d_i \cdot 2^{-i}]$$
$$y_{i+1} = K_i \cdot [y_i + x_i \cdot d_i \cdot 2^{-i}]$$

$$K_i = \cos\phi_i = \cos(\tan^{-1}(2^{-i}))$$
$$= \frac{1}{\sqrt{1 + 2^{-2i}}}$$

$$d_i = \pm 1$$

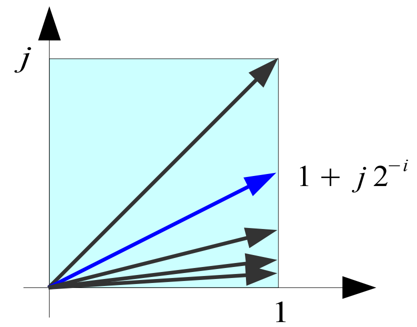
Restrict rotation angle ➡ $\tan\phi = \pm 2^{-i}$

Multiplication ➡ simple shift

$y \cdot \tan\phi$ $\quad$ $y \cdot 2^{-i}$

$x \cdot \tan\phi$ $\quad$ $x \cdot 2^{-i}$

regardless of direction ➡ $\cos(\phi) = \cos(-\phi)$



$K_i \le 1$

$\tan\phi$ ➡ 2^{-i}

$\cos\phi$ ➡ $\frac{1}{\sqrt{1+2^{-2i}}}$

Vector Rotation (2)

$$x_{i+1} = K_i \cdot \left[x_i - y_i \cdot d_i \cdot 2^{-i} \right]$$
$$y_{i+1} = K_i \cdot \left[y_i + x_i \cdot d_i \cdot 2^{-i} \right]$$

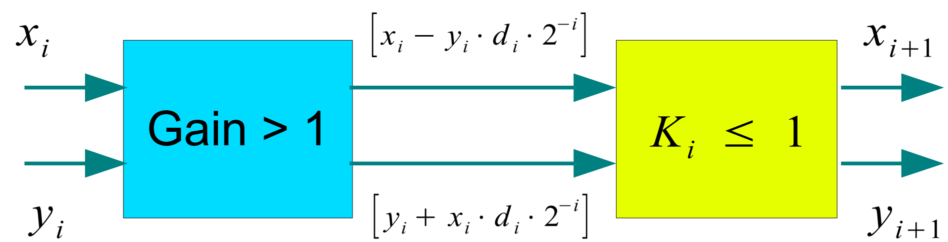
$$K_i = \cos\phi_i = \cos\left(\tan^{-1}\left(2^{-i}\right) \right)$$
$$= \frac{1}{\sqrt{1 + 2^{-2i}}}$$

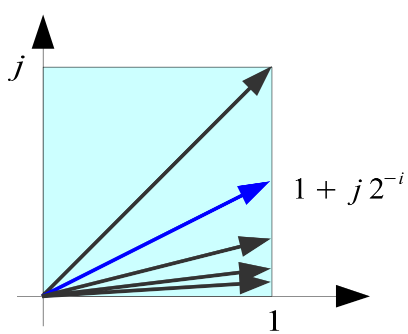
$$d_i = \pm 1$$

$$x_{i+1}{}^2 = K_i{}^2 \cdot \left[x_i{}^2 + y_i{}^2 \cdot 2^{-2i} - 2 x_i y_i d_i \cdot 2^{-i} \right]$$
$$y_{i+1}{}^2 = K_i{}^2 \cdot \left[y_i{}^2 + x_i{}^2 \cdot 2^{-2i} + 2 x_i y_i d_i \cdot 2^{-i} \right]$$

$$x_{i+1}{}^2 + y_{i+1}{}^2 = K_i{}^2 \cdot \left(1 + 2^{-2i}\right) \cdot \left(x_i{}^2 + y_i{}^2\right)$$

$$K_i = \frac{1}{\sqrt{1 + 2^{-2i}}} \qquad K_i \le 1$$





$\tan\phi \Rightarrow 2^{-i}$

$\cos\phi \Rightarrow \dfrac{1}{\sqrt{1 + 2^{-2i}}}$

CORDIC Gain : *growing in magnitude*

$$A_n = \prod_{i=1}^{n} \frac{1}{K_i} = \prod_{i=1}^{n} \sqrt{1 + 2^{-2i}} \rightarrow 1.647$$

Vector Rotation (3)

$$x_{i+1} = K_i \cdot \left[x_i - y_i \cdot d_i \cdot 2^{-i}\right]$$
$$y_{i+1} = K_i \cdot \left[y_i + x_i \cdot d_i \cdot 2^{-i}\right]$$

$$K_i = 1 / \sqrt{1 + 2^{-2i}} \quad \Leftarrow \quad \cos(\phi_i)$$
$$d_i = \pm 1$$

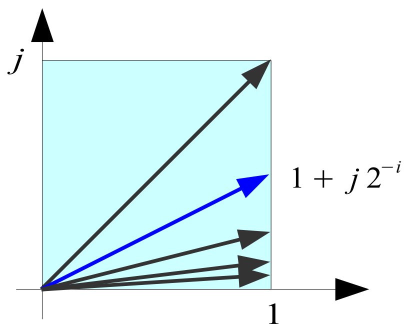
Without Scale Constants K_i

$$x_{i+1} = \left[x_i - y_i \cdot d_i \cdot 2^{-i}\right]$$
$$y_{i+1} = \left[y_i + x_i \cdot d_i \cdot 2^{-i}\right]$$
$$d_i = \pm 1$$

CORDIC Gain : *growing in magnitude*

$$A_n = \prod_{i=1}^{n} \frac{1}{K_i} = \prod_{i=1}^{n} \sqrt{1 + 2^{-2i}} \rightarrow 1.647$$

$$1 / K_i = \sqrt{1 + 2^{-2i}}$$



$\tan\phi \Rightarrow 2^{-i}$

$\cos\phi \Rightarrow \dfrac{1}{\sqrt{1 + 2^{-2i}}}$

For correction

Multiplying K_i*'s as a processing gain*

$$\prod_{i=1}^{n} K_i = \prod_{i=1}^{n} \frac{1}{\sqrt{1 + 2^{-2i}}} \rightarrow 0.6073$$

Angle Accumulator

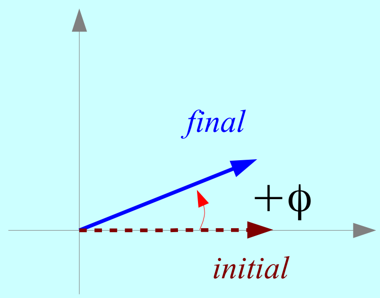
Rotation Mode

$$z_0 \leftarrow \phi \quad (desired\ angle)$$
$$z_n \rightarrow 0$$

$$\boxed{z_{i+1} = z_i - d_i \cdot \tan^{-1}(2^{-i})}$$

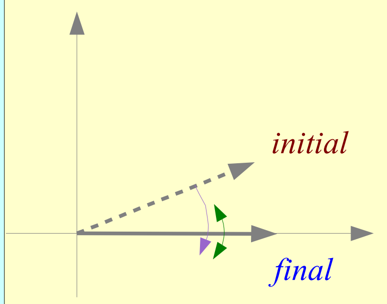
$$d_i = -1 \quad if \quad z_i < 0$$
$$d_i = +1 \quad otherwise$$

Actual Vectors



***Subtract** angles*
at each step

Angle Accumulator



Minimize the
residual angle

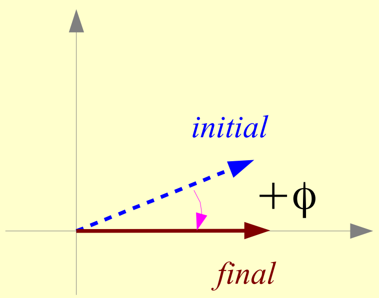
Vectoring Mode

$$z_0 \leftarrow 0$$
$$z_n \rightarrow z_0 + \tan^{-1}(y_0 / x_0)$$

$$\boxed{z_{i+1} = z_i - d_i \cdot \tan^{-1}(2^{-i})}$$

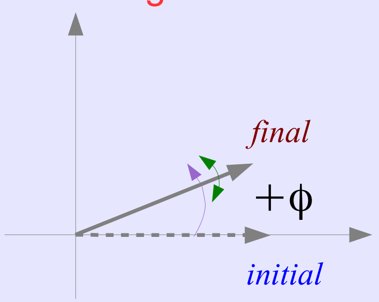
$$d_i = +1 \quad if \quad y_i < 0$$
$$d_i = -1 \quad otherwise$$

Actual Vectors



***Add** angles*
at each step

Angle Accumulator



Minimize the
residual y comp.

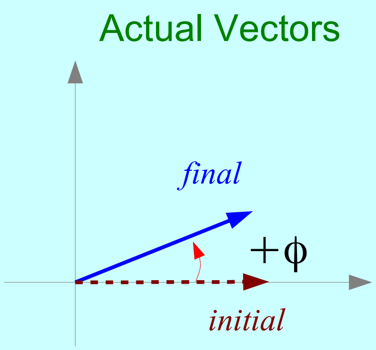
Angle Accumulator – Rotation Mode

Rotation Mode

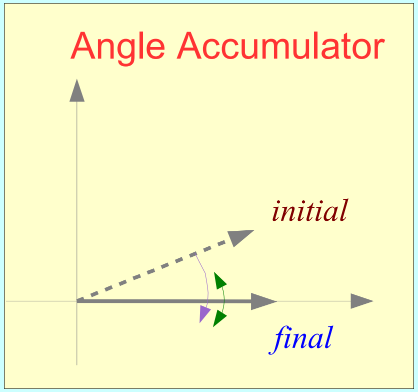
$$z_0 \leftarrow \phi \quad (\textit{desired angle})$$
$$z_n \rightarrow 0$$

$$\boxed{z_{i+1} = z_i - d_i \cdot \tan^{-1}(2^{-i})}$$

$$d_i = -1 \quad \textit{if} \quad z_i < 0$$
$$d_i = +1 \quad \textit{otherwise}$$



Subtract *angles at each step*



Minimize the residual angle

$z_i < 0$

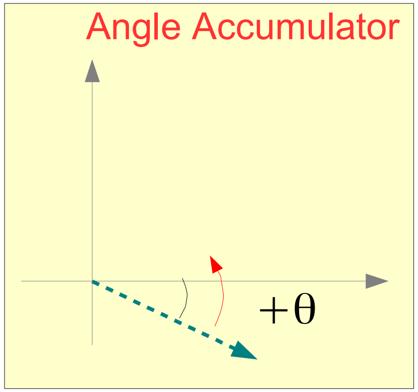
Increase Angle $\quad d_i = -1$

$$z_{i+1} = z_i + \tan^{-1}(2^{-i})$$

$z_i \geq 0$

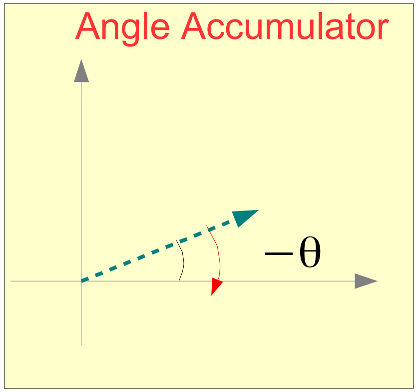
Decrease Angle $\quad d_i = +1$

$$z_{i+1} = z_i - \tan^{-1}(2^{-i})$$



$z_i < 0$

Increase Angle $+\theta$



$z_i \geq 0$

Decreases Angle $-\theta$

Angle Accumulator – Vectoring Mode

Vectoring Mode

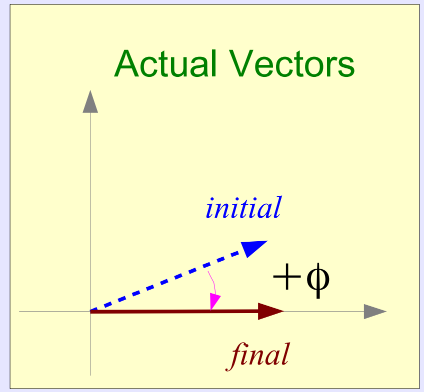
$$z_0 \leftarrow 0$$

$$z_n \rightarrow z_0 + \tan^{-1}(y_0 / x_0)$$

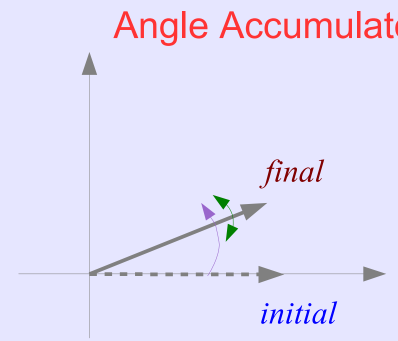
$$\boxed{z_{i+1} = z_i - d_i \cdot \tan^{-1}(2^{-i})}$$

$$d_i = +1 \quad \textit{if} \quad y_i < 0$$

$$d_i = -1 \quad \textit{otherwise}$$



Add *angles at each step*



Minimize the residual y comp.

$y_i < 0$

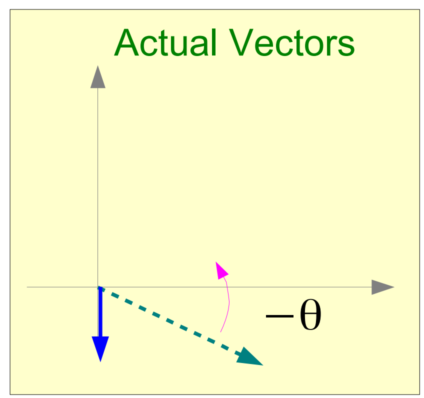
Decrease Angle $d_i = +1$

$$z_{i+1} = z_i - \tan^{-1}(2^{-i})$$

$y_i > 0$

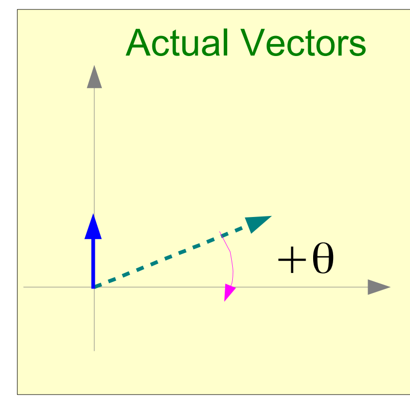
Increase Angle $d_i = -1$

$$z_{i+1} = z_i + \tan^{-1}(2^{-i})$$



$y_i < 0$

Decrease Angle $-\theta$



$y_i > 0$

Increases Angle $+\theta$

Initial Rotation $\pm\pi/2$

Positive Phase (y > 0) ➡ *Rotate by –90 degrees*

Negative Phase (y < 0) ➡ *Rotate by +90 degrees*

Resulting Phase ➡ [-90, +90]

$$x' = -d \cdot y$$
$$y' = +d \cdot x$$
$$z' = z + d \cdot \frac{\pi}{2}$$

$$d = +1 \quad \textit{if} \quad y < 0$$
$$d = -1 \quad \textit{otherwise}$$

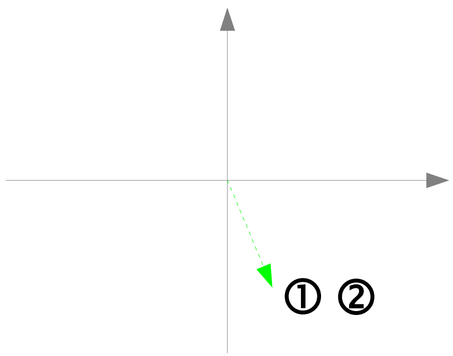
No magnitude change

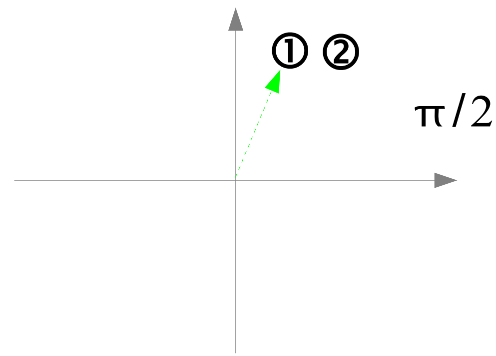
$$x' \leftarrow y$$
$$y' \leftarrow x$$

Consistent

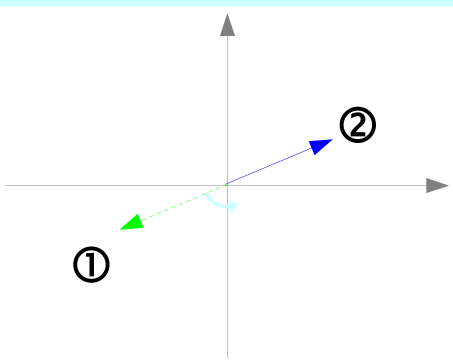
Initial Rotation $0, \; +\pi$

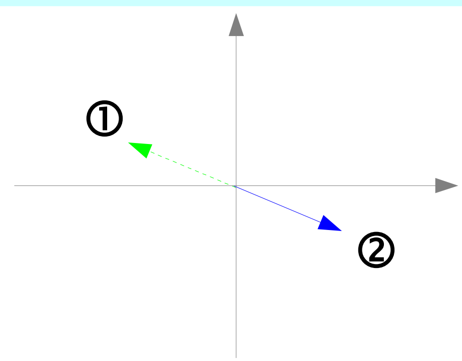
Positive x (x > 0) ➡ *Rotate by –90 degrees*





Negative x (x > 0) ➡ *Rotate by +90 degrees*





Resulting Phase ➡ [-90, +90]

$$x' = +d \cdot x$$
$$y' = +d \cdot y$$
$$z' = z \quad \text{if} \quad d = 1$$
$$z' = \pi - z \quad \text{if} \quad d = -1$$

$$d = -1 \quad \text{if} \quad x < 0$$
$$d = +1 \quad \text{otherwise}$$

No magnitude change

$$x' \leftarrow y$$
$$y' \leftarrow x$$

Convenient wiring in FPGA

A. Sine and Cosine (1)

Rotation Mode

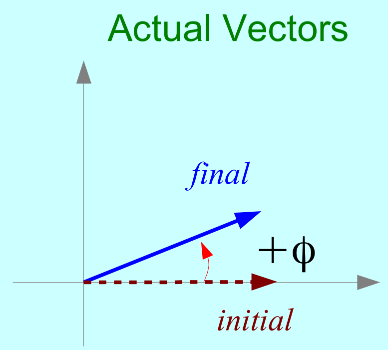
$$z_0 \leftarrow \phi \quad (\textit{desired angle})$$

$$z_n \rightarrow 0$$

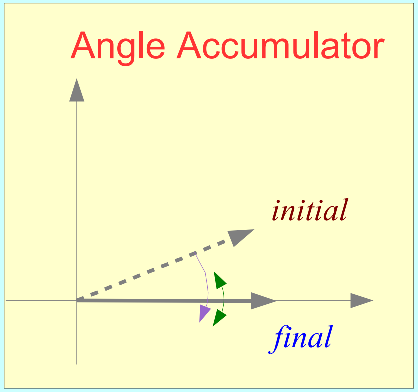
$$\boxed{z_{i+1} = z_i - d_i \cdot \tan^{-1}(2^{-i})}$$

$$d_i = -1 \quad \textit{if} \quad z_i < 0$$

$$d_i = +1 \quad \textit{otherwise}$$



Subtract *angles at each step*



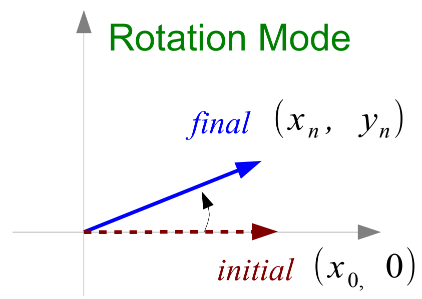
Minimize the residual angle

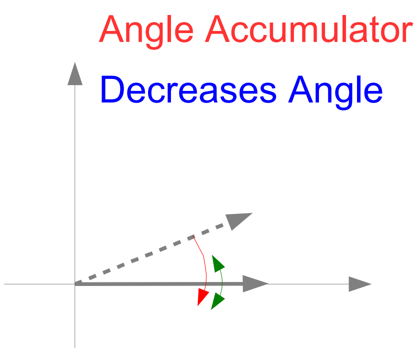
Finding Sine and Cosine

$$(x_0,\ 0) \rightarrow (x_n,\ y_n)$$

$$x_n = A_n \cdot x_0 \cos z_0$$

$$y_n = A_n \cdot x_0 \sin z_0$$





A. Sine and Cosine (2)

Finding Sine and Cosine

$$(x_{0,}\ \ 0) \ \rightarrow \ (x_n,\ \ y_n)$$

$$x_n \ = \ A_n \cdot x_0 \cos z_0$$
$$y_n \ = \ A_n \cdot x_0 \sin z_0$$

Unscaled Sine and Cosine

$$if \quad x_0 \ \leftarrow \ 1/A_n$$

$$x_n \ = \ \boxed{\cos z_0}$$
$$y_n \ = \ \boxed{\sin z_0}$$

Modulated Sine and Cosine

$$K_n \cdot x_n \ = \ K_n \cdot A_n \cdot x_0 \cos z_0$$
$$K_n \cdot y_n \ = \ K_n \cdot A_n \cdot x_0 \sin z_0$$

$$K_n \cdot x_n \ = \ \boxed{x_0 \cos z_0}$$
$$K_n \cdot y_n \ = \ \boxed{x_0 \sin z_0}$$

Look Up Table
→ a pair of MULT

CORDIC method
→ MULT as a part of rotation

B. Polar to Rectangular

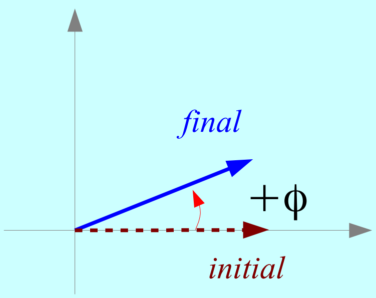
Rotation Mode

$$z_0 \leftarrow \phi \quad (\textit{desired angle})$$
$$z_n \rightarrow 0$$

$$\boxed{z_{i+1} = z_i - d_i \cdot \tan^{-1}(2^{-i})}$$

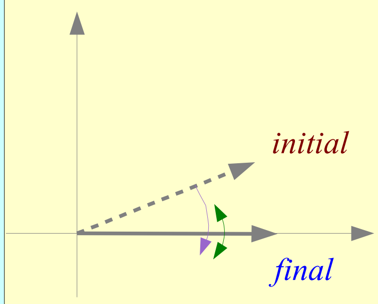
$$d_i = -1 \quad \textit{if} \quad z_i < 0$$
$$d_i = +1 \quad \textit{otherwise}$$

Actual Vectors



Subtract *angles at each step*

Angle Accumulator



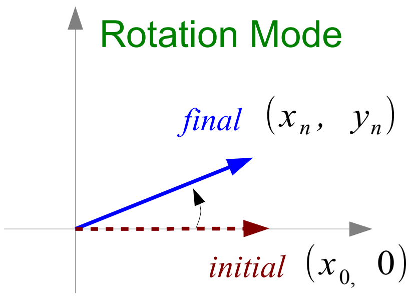
Minimize the residual angle

Finding Sine and Cosine

$$(x_{0,} \ \ 0) \rightarrow (x_n, \ \ y_n)$$

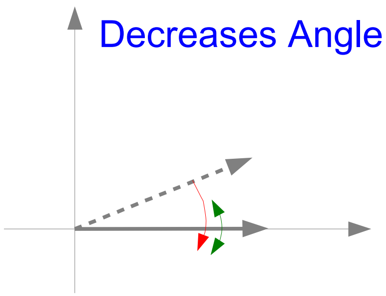
$$x_n = A_n \cdot x_0 \cos z_0$$
$$y_n = A_n \cdot x_0 \sin z_0$$

Rotation Mode



Angle Accumulator

Decreases Angle



$x_0 \leftarrow r$	$x_n \rightarrow r \cos\theta$
$z_0 \leftarrow \theta$	$y_n \rightarrow r \sin\theta$

B. Polar to Rectangular

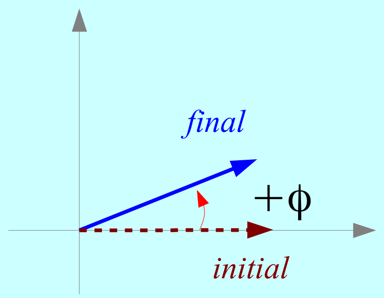
Rotation Mode

$$z_0 \leftarrow \phi \quad (\textit{desired angle})$$
$$z_n \rightarrow 0$$

$$\boxed{z_{i+1} = z_i - d_i \cdot \tan^{-1}(2^{-i})}$$

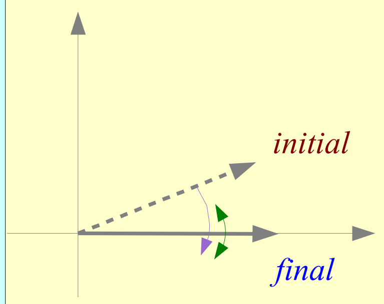
$$d_i = -1 \quad \textit{if} \quad z_i < 0$$
$$d_i = +1 \quad \textit{otherwise}$$

Actual Vectors



Subtract *angles at each step*

Angle Accumulator



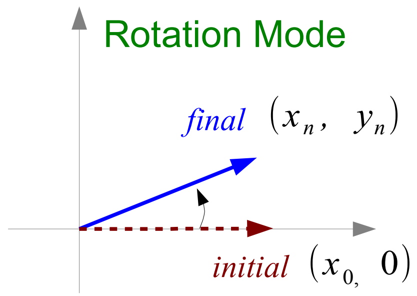
Minimize the residual angle

Finding Sine and Cosine

$$(x_0, \; 0) \rightarrow (x_n, \; y_n)$$

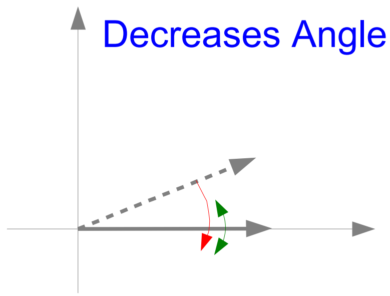
$$x_n = A_n \cdot x_0 \cos z_0$$
$$y_n = A_n \cdot x_0 \sin z_0$$

Rotation Mode



Angle Accumulator

Decreases Angle



$$x_0 \leftarrow r \qquad z_0 \leftarrow \theta$$

$$x_n \rightarrow r\cos\theta \qquad y_n \rightarrow r\sin\theta$$

C. General Vector Rotation

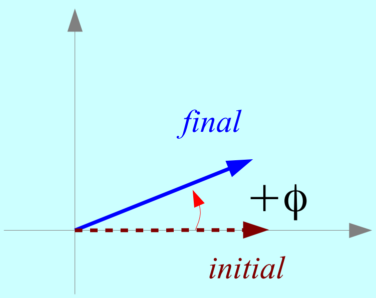
Rotation Mode

$$z_0 \leftarrow \phi \quad (\textit{desired angle})$$
$$z_n \rightarrow 0$$

$$\boxed{z_{i+1} = z_i - d_i \cdot \tan^{-1}(2^{-i})}$$

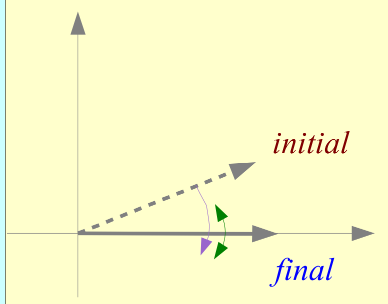
$$d_i = -1 \quad \textit{if} \quad z_i < 0$$
$$d_i = +1 \quad \textit{otherwise}$$

Actual Vectors



Subtract *angles at each step*

Angle Accumulator

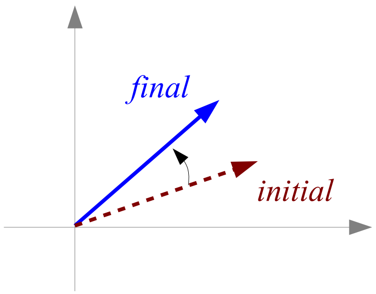


Minimize the residual angle

Motion Correction and Control System

$$x_n = A_n \left[x_0 \cdot \cos z_0 - y_0 \cdot \sin z_0 \right]$$
$$y_n = A_n \left[x_0 \cdot \sin z_0 + y_0 \cdot \cos z_0 \right]$$

Rotation Mode



Vectoring Mode

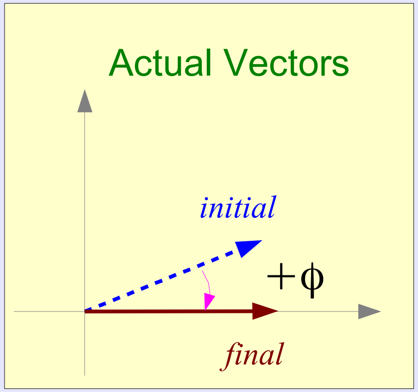
$$z_0 \leftarrow 0$$

$$z_n \rightarrow z_0 + \tan^{-1}(y_0 / x_0)$$

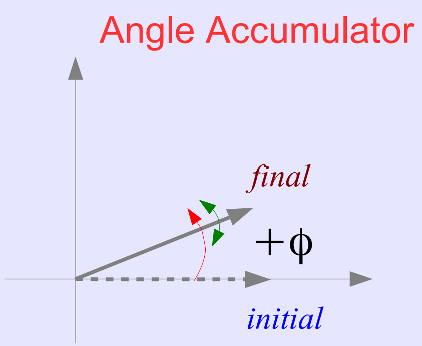
$$\boxed{z_{i+1} = z_i - d_i \cdot \tan^{-1}(2^{-i})}$$

$$d_i = +1 \quad \textit{if} \quad y_i < 0$$

$$d_i = -1 \quad \textit{otherwise}$$



Add *angles*
at each step



Minimize the
residual y comp.

References

[1] http://en.wikipedia.org/
[2] CORDIC FAQ, www.dspguru.com