Group Delay and Phase Delay (1A)

Young Won Lim
7/16/12

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Phase Shift and Time Shift

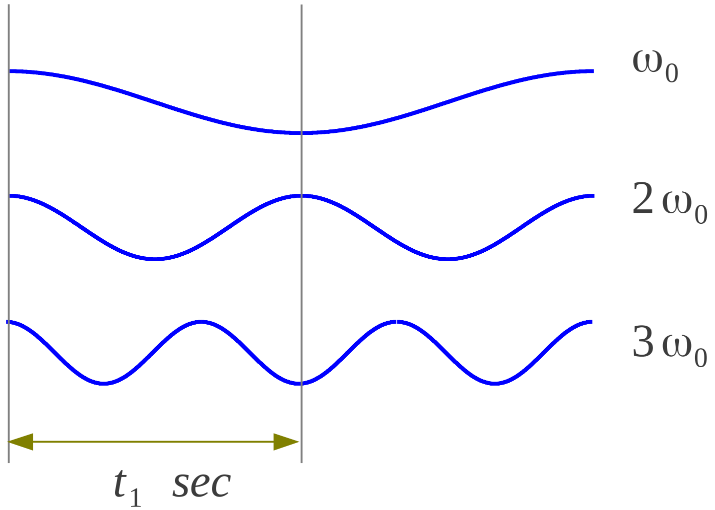
measure phase shift not in second
*but **in portions** of a cosine wave cycle*

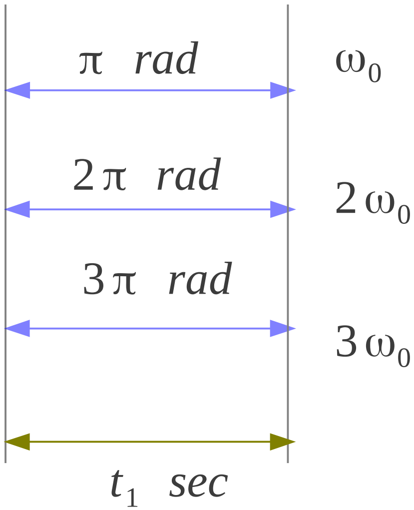
within phase change in one cycle

Phase Shift → in radians, degrees
Delay → in seconds (time)

Given time shift (delay) t_1 sec

The actual phase shift is different according to the frequency $\pi, 2\pi, 3\pi$ rad



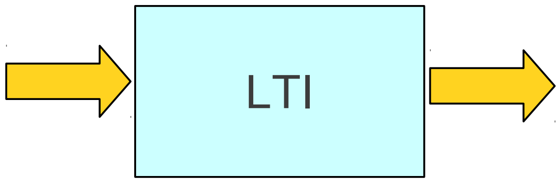


The same delay applied to all frequencies → The different phase shift to the different frequency

Frequency Response

Frequency Response $H(e^{j\omega})$

$e^{j\omega t}$

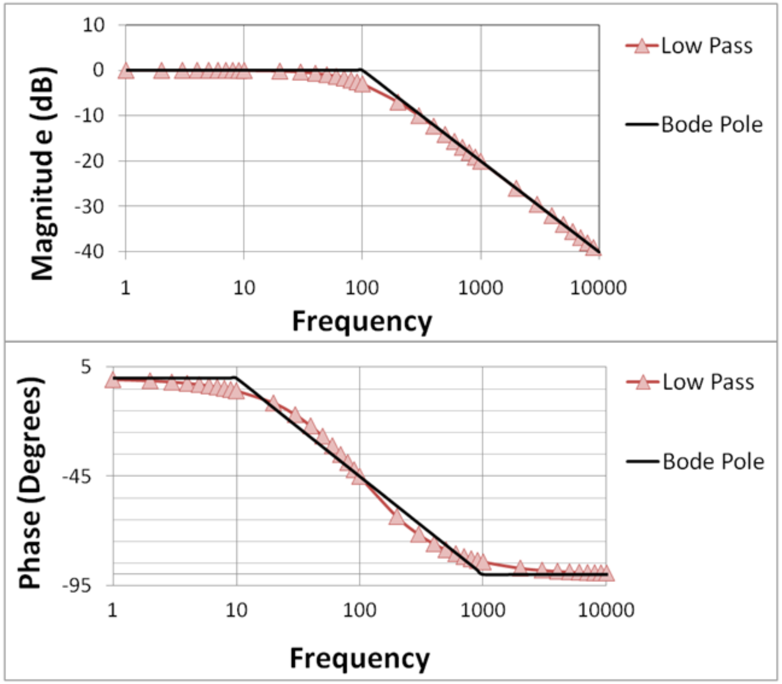


$H(e^{j\omega})\, e^{j\omega t}$

LPF example

$|H(e^{j\omega})|$ Magnitude Response

$\angle H(e^{j\omega})$ Phase Response

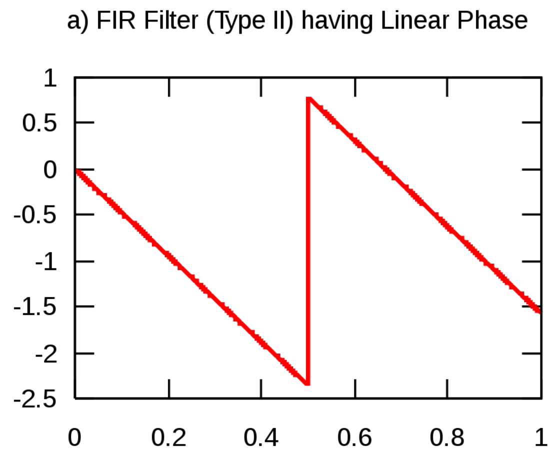


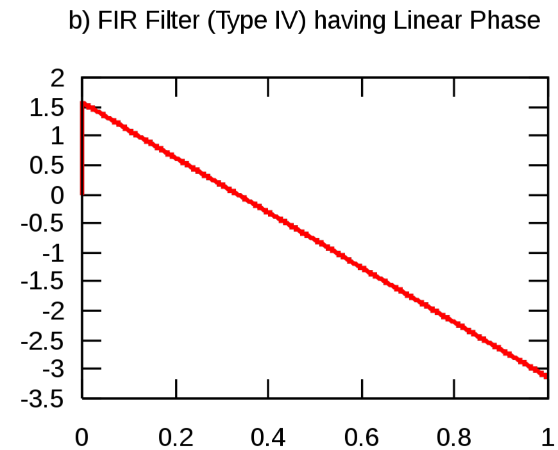
Young Won Lim
7/16/12

Linear Phase System

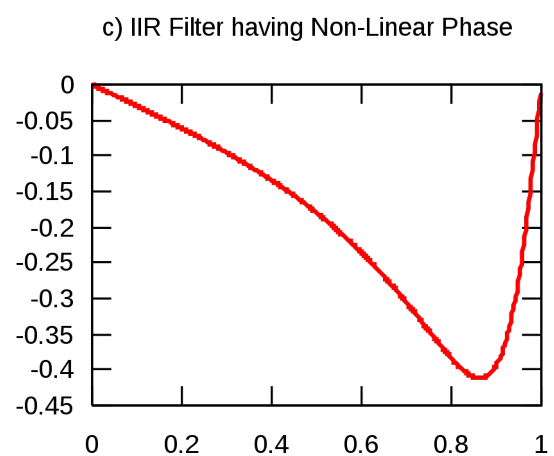
Phase Shift $\propto$ Frequency

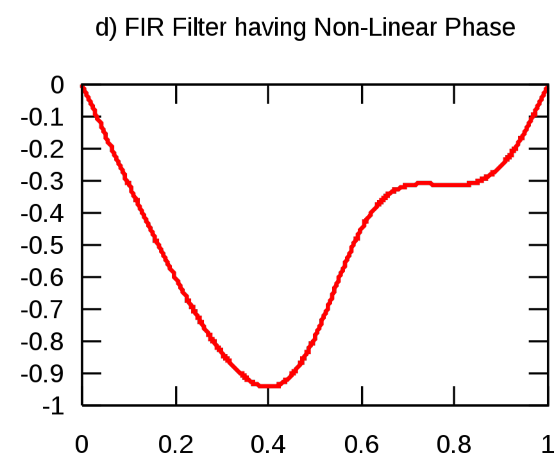
$$\angle H\left(e^{j\omega}\right) \propto \omega$$





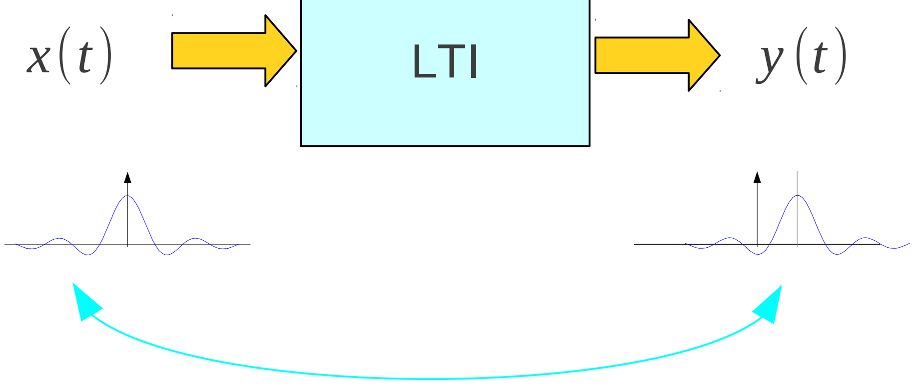
Non-Linear Phase System





Uniform Time Delay (1)

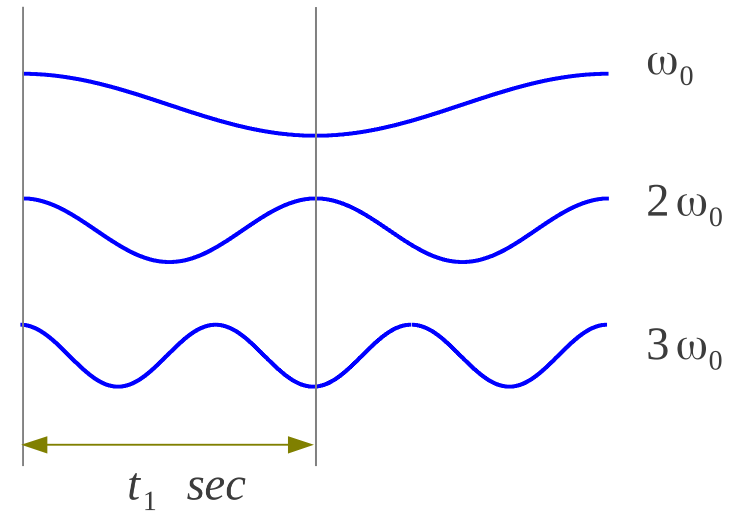
Frequency Response $H(e^{j\omega})$

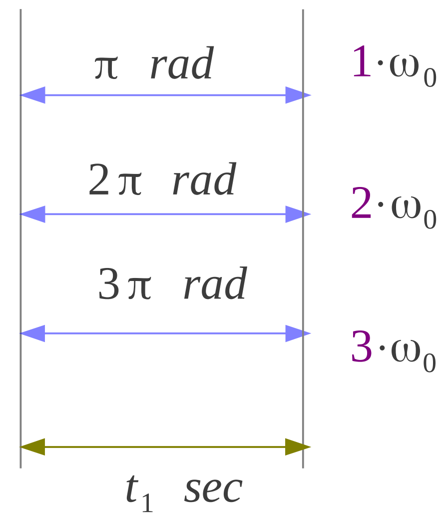


The waveform shape can be preserved.

- uniform magnitude $|H(e^{j\omega})| = c$
- uniform time delay

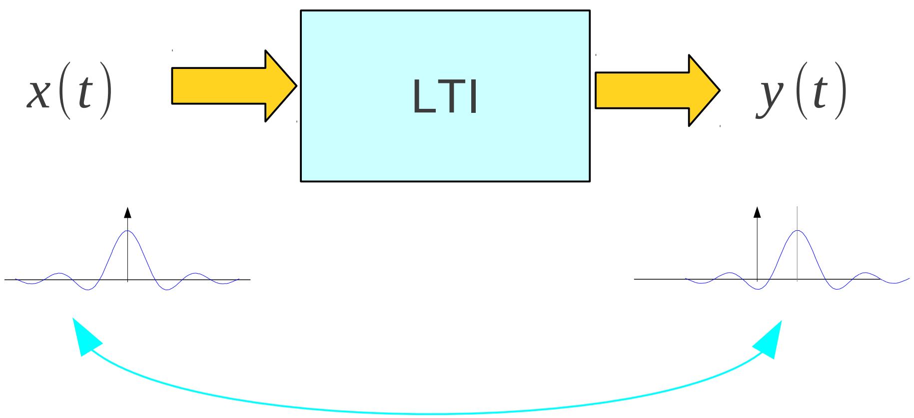
linear phase $\angle H(e^{j\omega}) = k\omega$





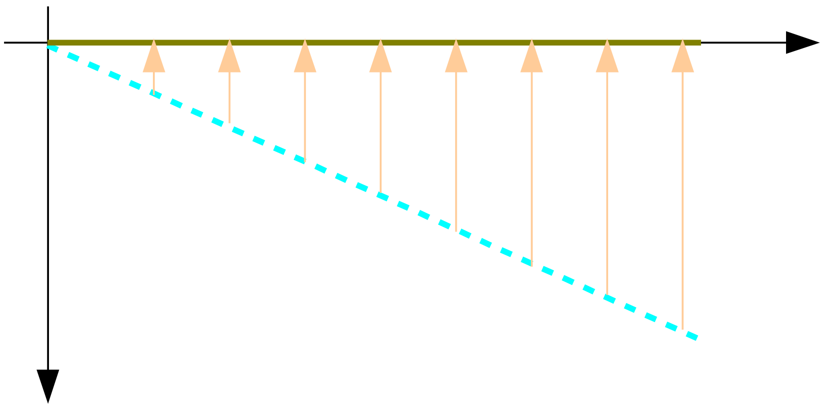
Uniform Time Delay (2)

Frequency Response $H(e^{j\omega})$



Uniform Time Delay

Could remove delay
from the phase response
to achieve a horizontal line at
***zero degree** (No delay)*



The waveform shape can be preserved.

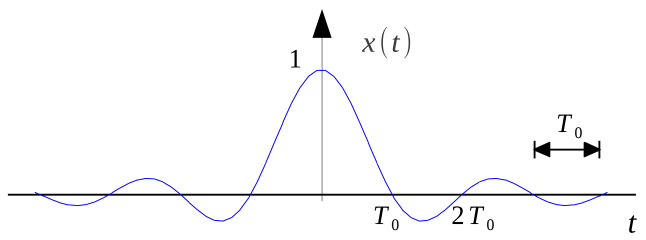
- uniform magnitude $|H(e^{j\omega})| = c$
- uniform time delay

⇕

linear phase $\angle H(e^{j\omega}) = k\omega$

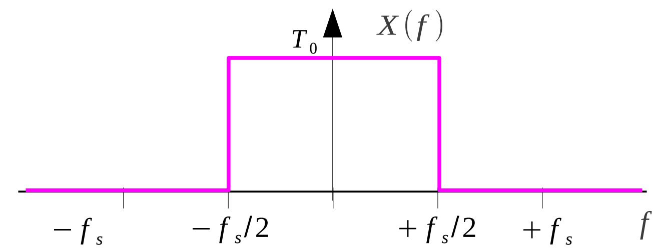
CTFT of Sinc Function

$t = \pm T_0,\ \pm 2T_0,\ \pm 3T_0,\ \cdots \Rightarrow x(t) = 0$



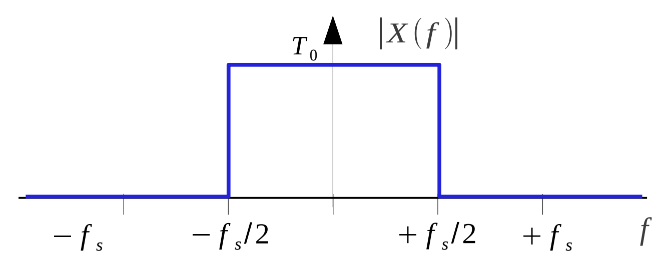
$$\frac{1}{T_0} \equiv f_s$$

CTFT

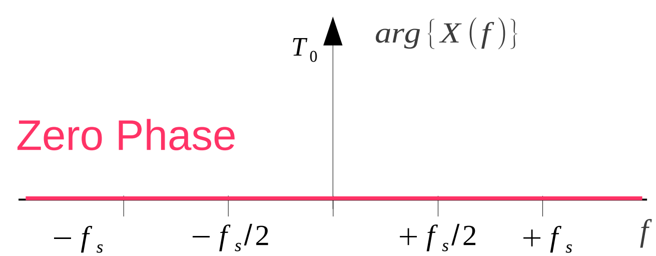


$$x(t) = \frac{\sin(\pi t/T_0)}{\pi t/T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

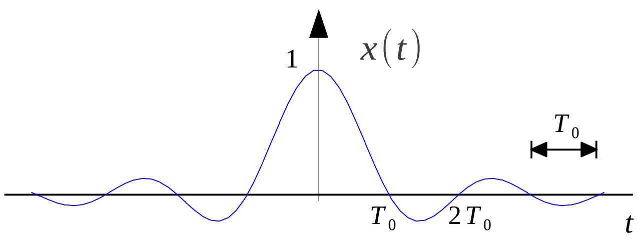
$$H(f) = \begin{cases} T_0, & |f| \le f_s/2 \\ 0, & \textit{otherwise} \end{cases}$$



Real Symmetric Signal

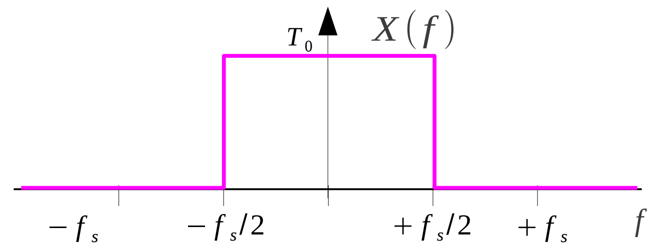


CTFT Time Shifting Property



$$\frac{1}{T_0} \equiv f_s$$

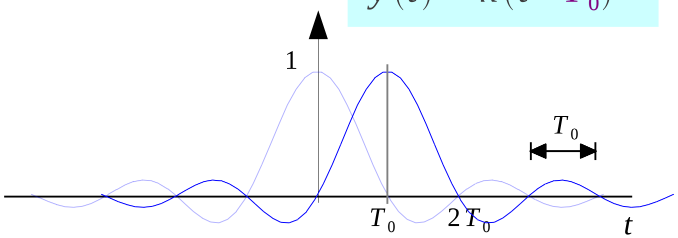
CTFT



$$x(t) = \frac{\sin(\pi t/T_0)}{\pi t/T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

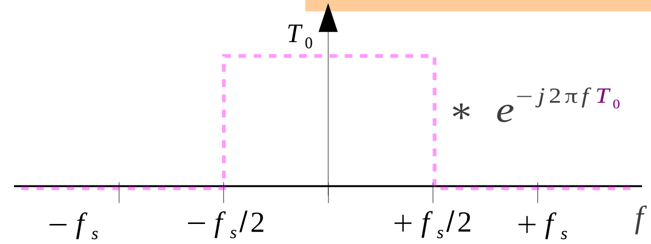
$$X(f) = \begin{cases} T_0, & |f| \le f_s/2 \\ 0, & otherwise \end{cases}$$

$$y(t) = x(t - T_0)$$

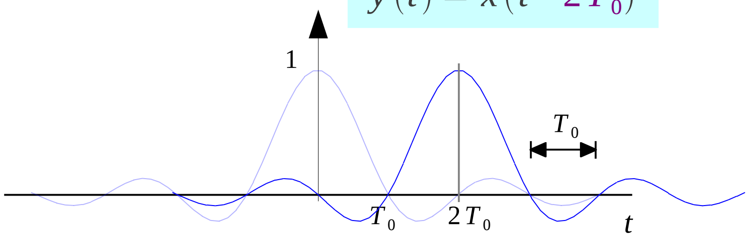


CTFT

$$Y(f) = X(f) \cdot e^{-j2\pi f T_0}$$

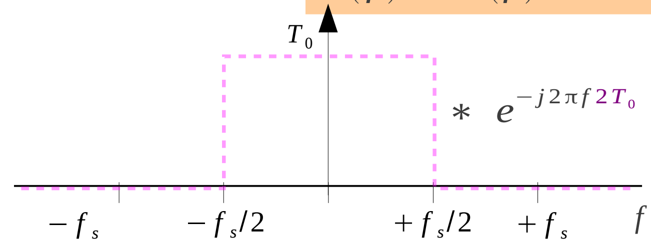


$$y(t) = x(t - 2T_0)$$

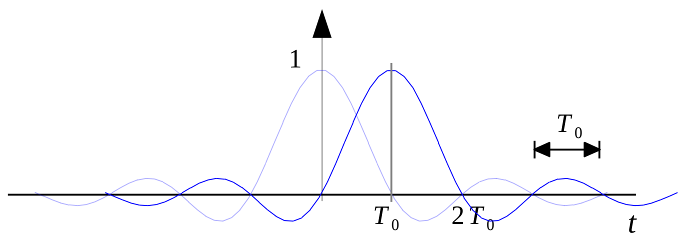


CTFT

$$Y(f) = X(f) \cdot e^{-j2\pi f 2T_0}$$

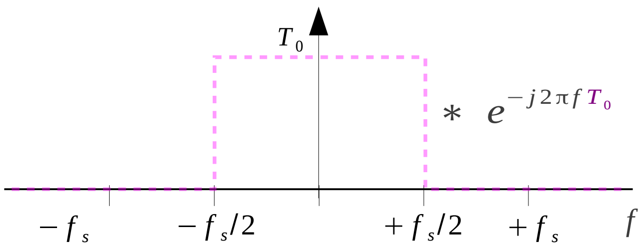


CTFT of Sinc Function Shifted by T_0



$$\frac{1}{T_0} \equiv f_s$$

CTFT



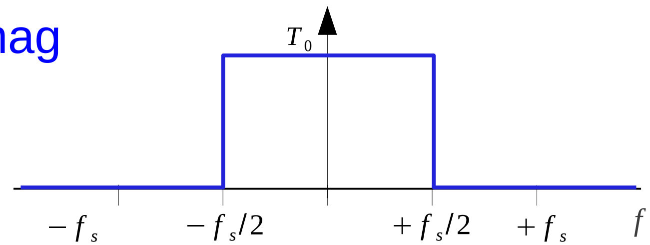
$$x(t) = \frac{\sin(\pi t/T_0)}{\pi t/T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

$$X(f) = \begin{cases} T_0, & |f| \le f_s/2 \\ 0, & otherwise \end{cases}$$

$$y(t) = x(t - T_0)$$

$$Y(f) = X(f) \cdot e^{-j2\pi f T_0}$$

mag

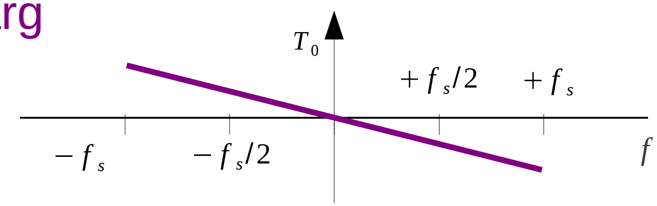


$Arg \Rightarrow \Phi(f)$

$$slope = \frac{d\Phi}{df} = -2\pi T_0 \Rightarrow \frac{d\Phi}{d\omega} = -T_0$$

Group Delay $\quad -\frac{d\Phi}{d\omega} = T_0$

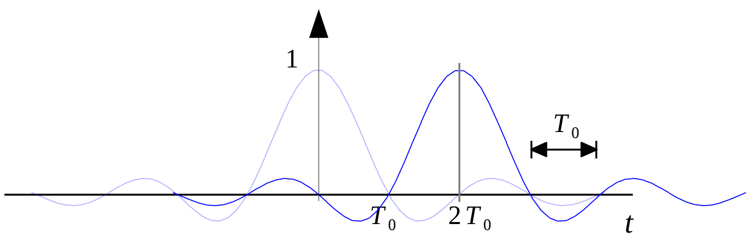
arg



Pure Delay (No Dispersion) ⇒ Linear Phase Change

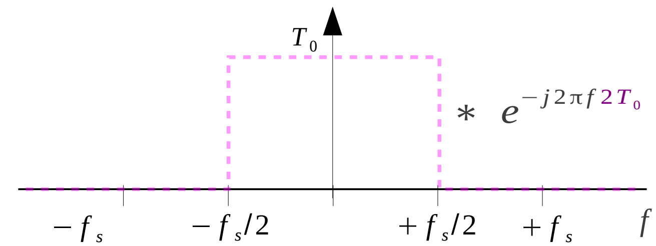
$slope = -2\pi T_0$

CTFT of Sinc Function Shifted by $2T_0$



$$\frac{1}{T_0} \equiv f_s$$

CTFT



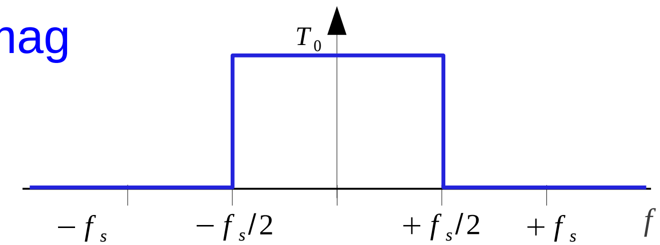
$$x(t) = \frac{\sin(\pi t/T_0)}{\pi t/T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

$$X(f) = \begin{cases} T_0, & |f| \le f_s/2 \\ 0, & otherwise \end{cases}$$

$$y(t) = x(t - 2T_0)$$

$$Y(f) = X(f) \cdot e^{-j2\pi f 2T_0}$$

mag

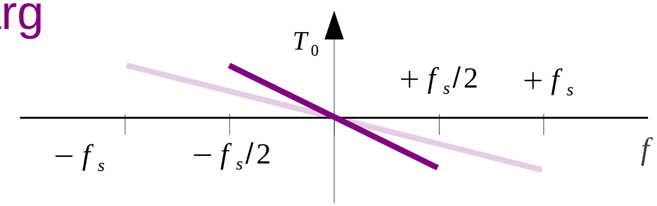


$Arg \Rightarrow \Phi(f)$

$$slope = \frac{d\Phi}{df} = -2\pi 2T_0 \Rightarrow \frac{d\Phi}{d\omega} = -2T_0$$

Group Delay $\quad -\dfrac{d\Phi}{d\omega} = 2T_0$

arg



Pure Delay (No Dispersion) → Linear Phase Change

$slope = -2\pi 2T_0$

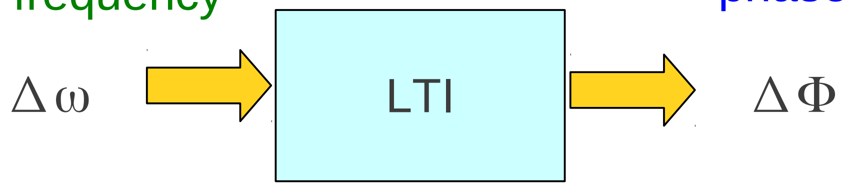
Group Delay (1)

Consider the cosine components
at *closely spaced frequencies* and
their phase shifts in relation to each other

Group Delay:
The phase shift changes
for small changes in frequency

small changes in frequency

phase shift changes



A uniform, waveform preserving
phase response → linear

Constant Group Delay → Uniform Time Delay (linear phase)

Group Delay (2)

Constant slope ➡ Constant Group Delay

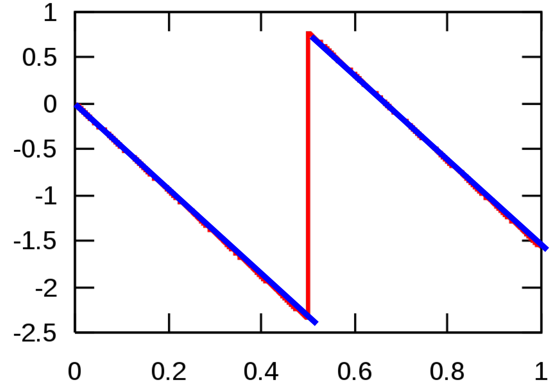
Linear Phase System

Phase Shift $\propto$ Frequency

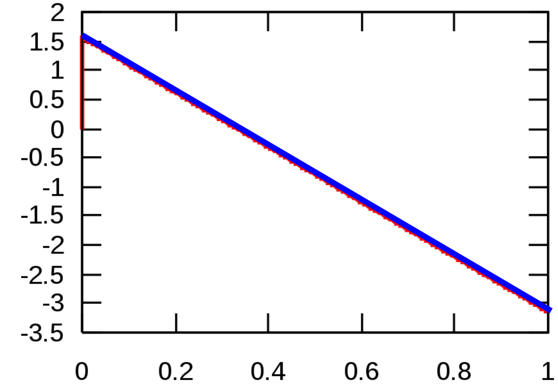
$$\angle H(e^{j\omega}) \propto \omega$$

No dispersion

a) FIR Filter (Type II) having Linear Phase



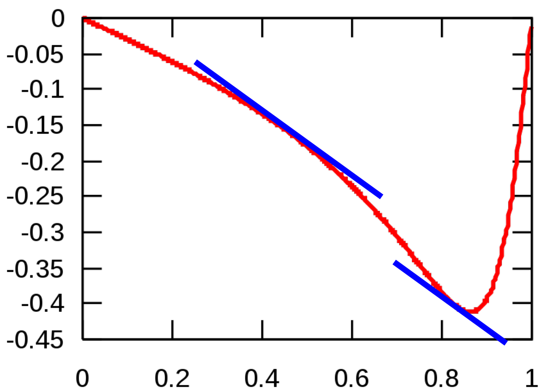
b) FIR Filter (Type IV) having Linear Phase



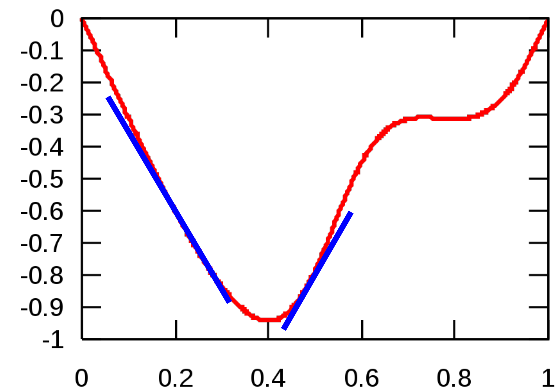
Non-Linear Phase System

Dispersion

c) IIR Filter having Non-Linear Phase



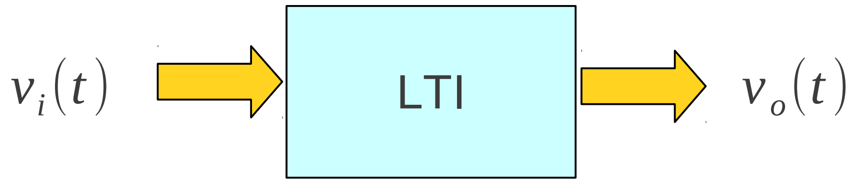
d) FIR Filter having Non-Linear Phase

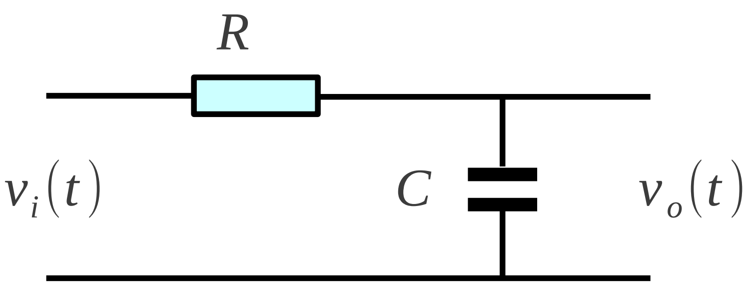


Varying slope ➡ Varying Group Delay

Simple Low Pass Filter (1)

Frequency Response





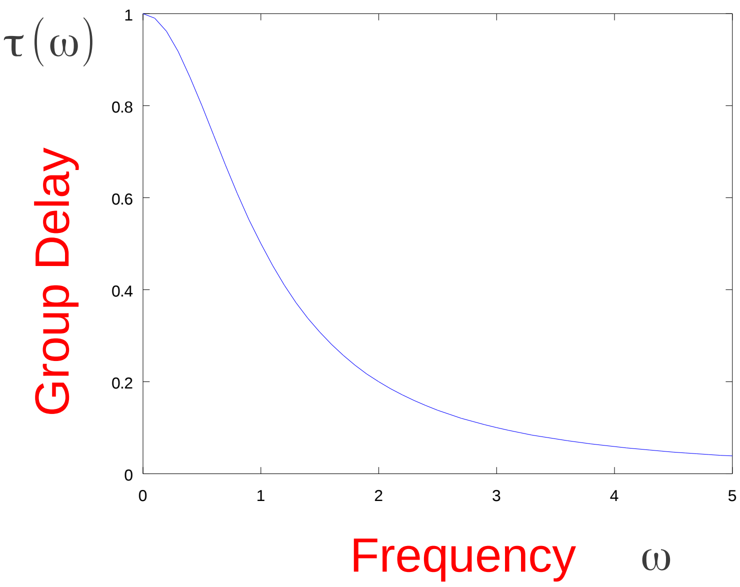
$$H(j\omega) = \frac{1}{1 + j\omega/\omega_0} \qquad \omega_0 = \frac{1}{RC}$$

$$H(j\omega) = A(j\omega)e^{j\phi(j\omega)}$$

$$A(j\omega) = \frac{1}{\sqrt{1+\omega^2/\omega_0^2}}$$

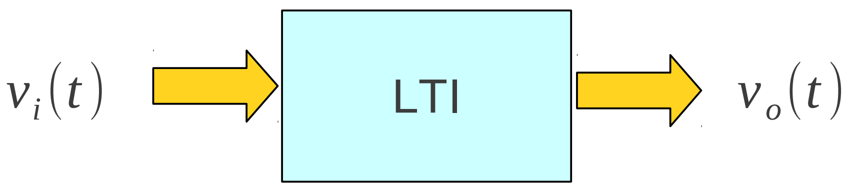
$$\phi(j\omega) = \tan^{-1}(-\omega/\omega_0)$$

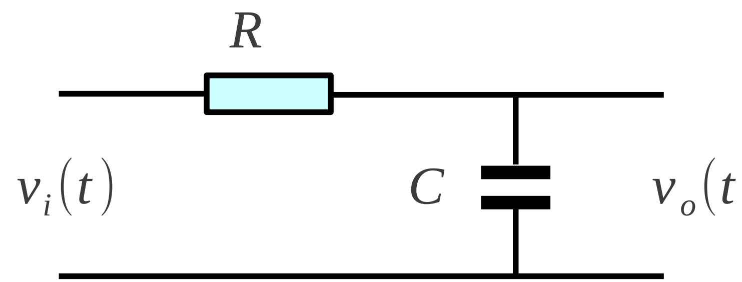
$$\tau(\omega) = -\frac{d\phi}{d\omega} = \frac{1}{1+\omega^2/\omega_0^2}$$



Simple Low Pass Filter (2)

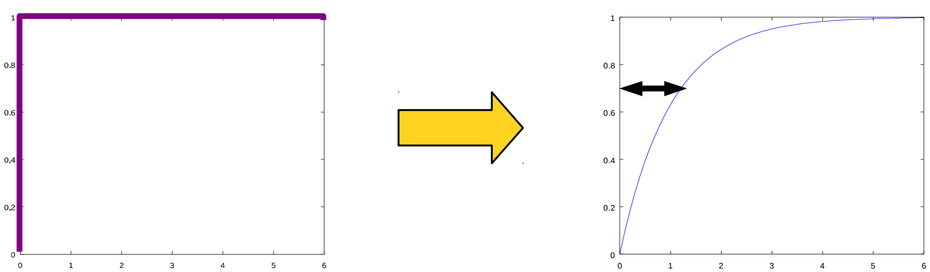
Frequency Response

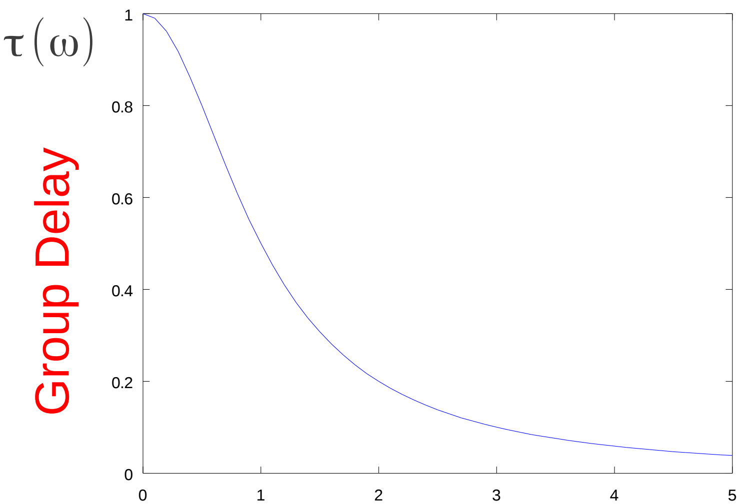




$$v_o(t) = 1 - e^{-\frac{t}{\tau}} \qquad \omega_0 = \frac{1}{RC} = \frac{1}{\tau}$$

Where is the group delay ?



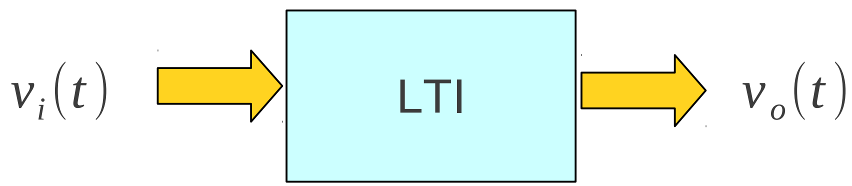


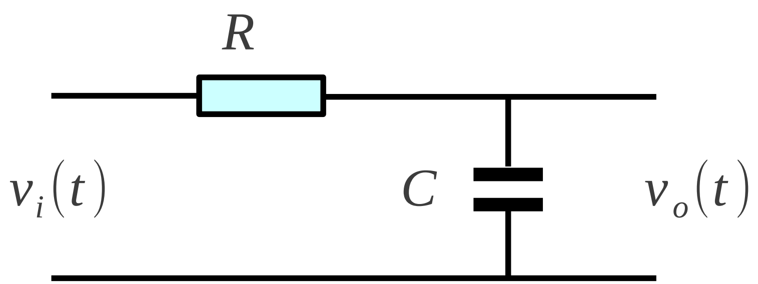
Group delay is not constant

Dispersion

Simple Low Pass Filter (3)

Frequency Response





$$v_o(t) = 1 - e^{-\frac{t}{\tau}} \qquad \omega_0 = \frac{1}{RC} = \frac{1}{\tau}$$

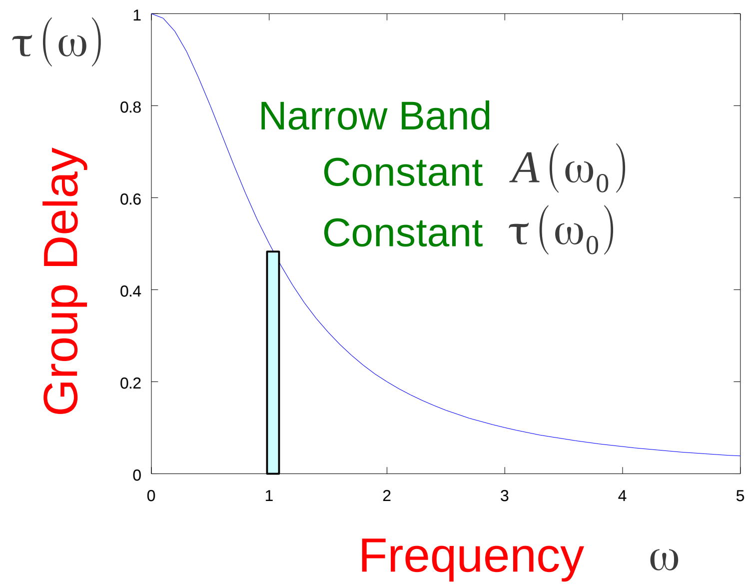
When focusing Narrow Band

Output

Time delayed by $\tau(\omega_0)$

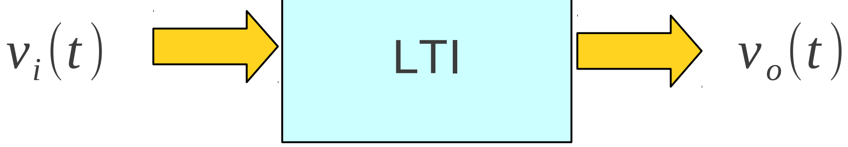
Amplitude scaled by $A(\omega_0)$

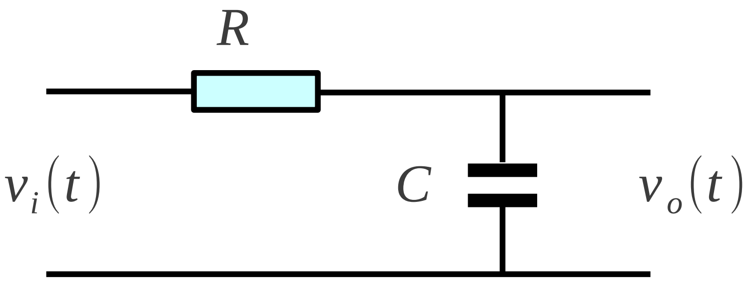
Phase shifted by $\phi(\omega_0)$



Simple Low Pass Filter (4)

Frequency Response





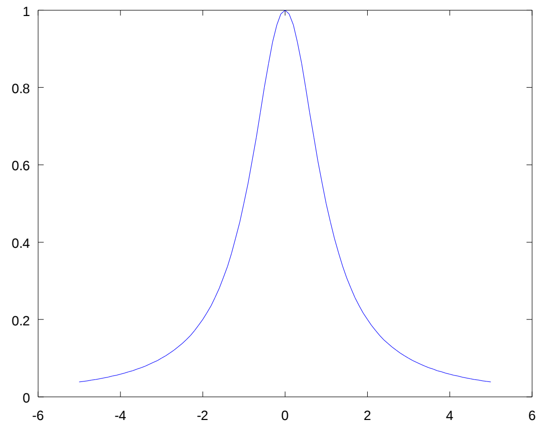
$$H(j\omega) = \frac{1}{1 + j\omega/\omega_0} \qquad \omega_0 = \frac{1}{RC}$$

$$H(j\omega) = A(j\omega)e^{j\phi(j\omega)}$$

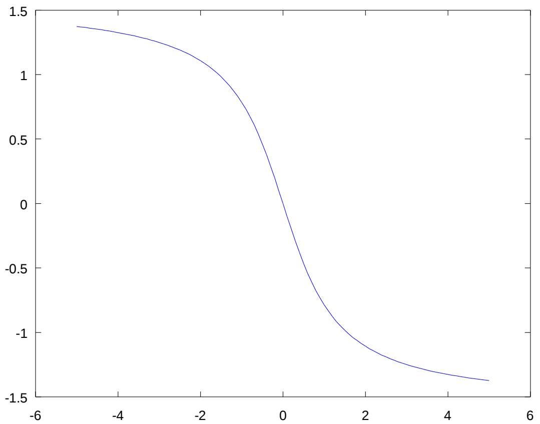
$$A(j\omega) = |H(j\omega)| = \frac{1}{\sqrt{1+\omega^2/\omega_0^2}}$$

$$\phi(j\omega) = arg\{H(j\omega)\} = \tan^{-1}(-\omega/\omega_0)$$

Magnitude Response

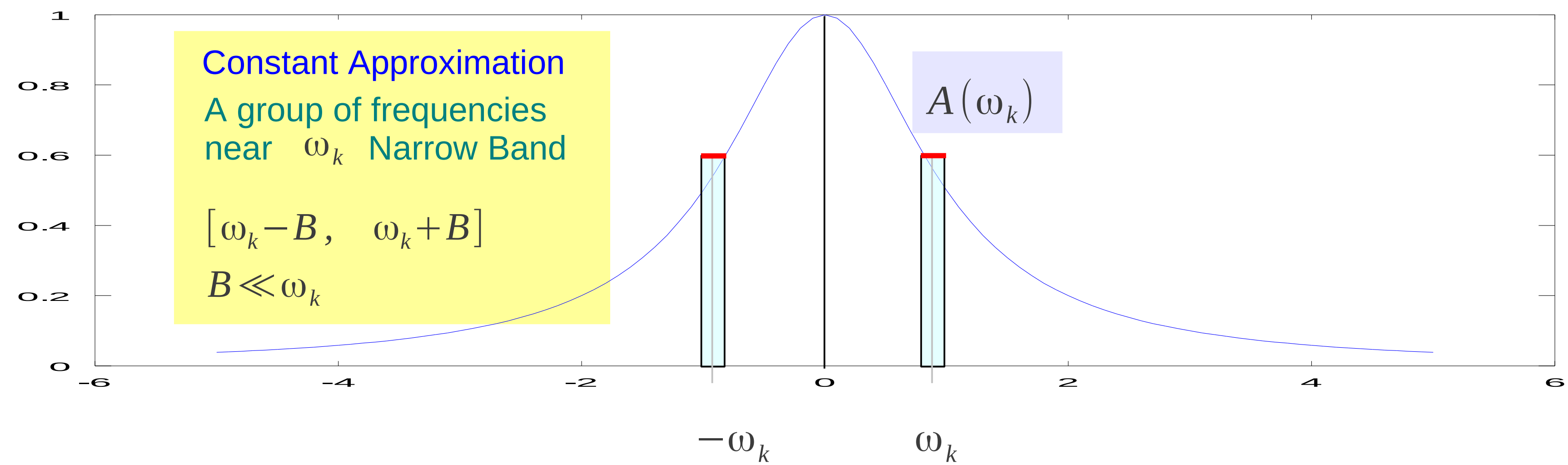


Phase Response

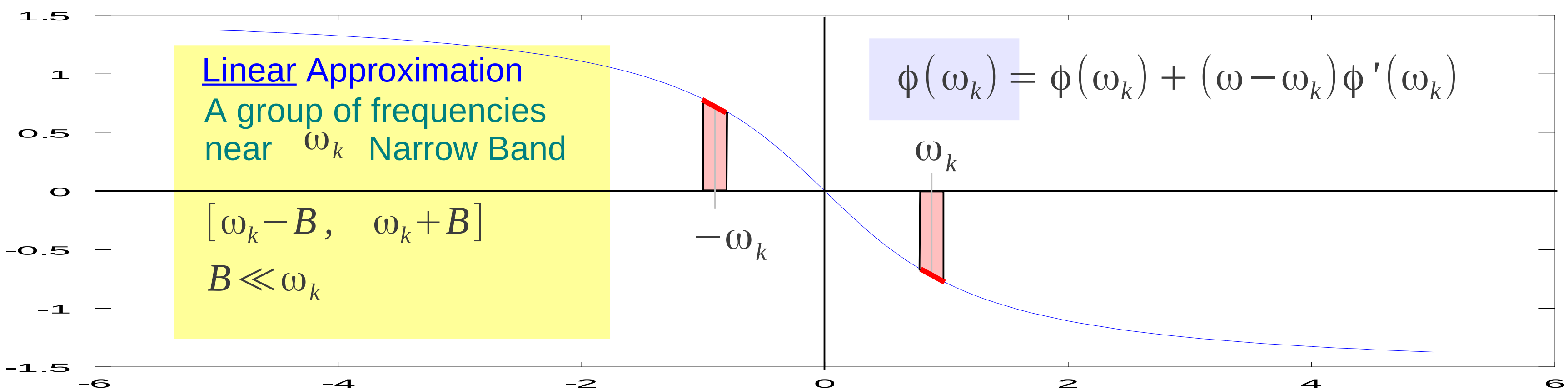


Simple Low Pass Filter (5)

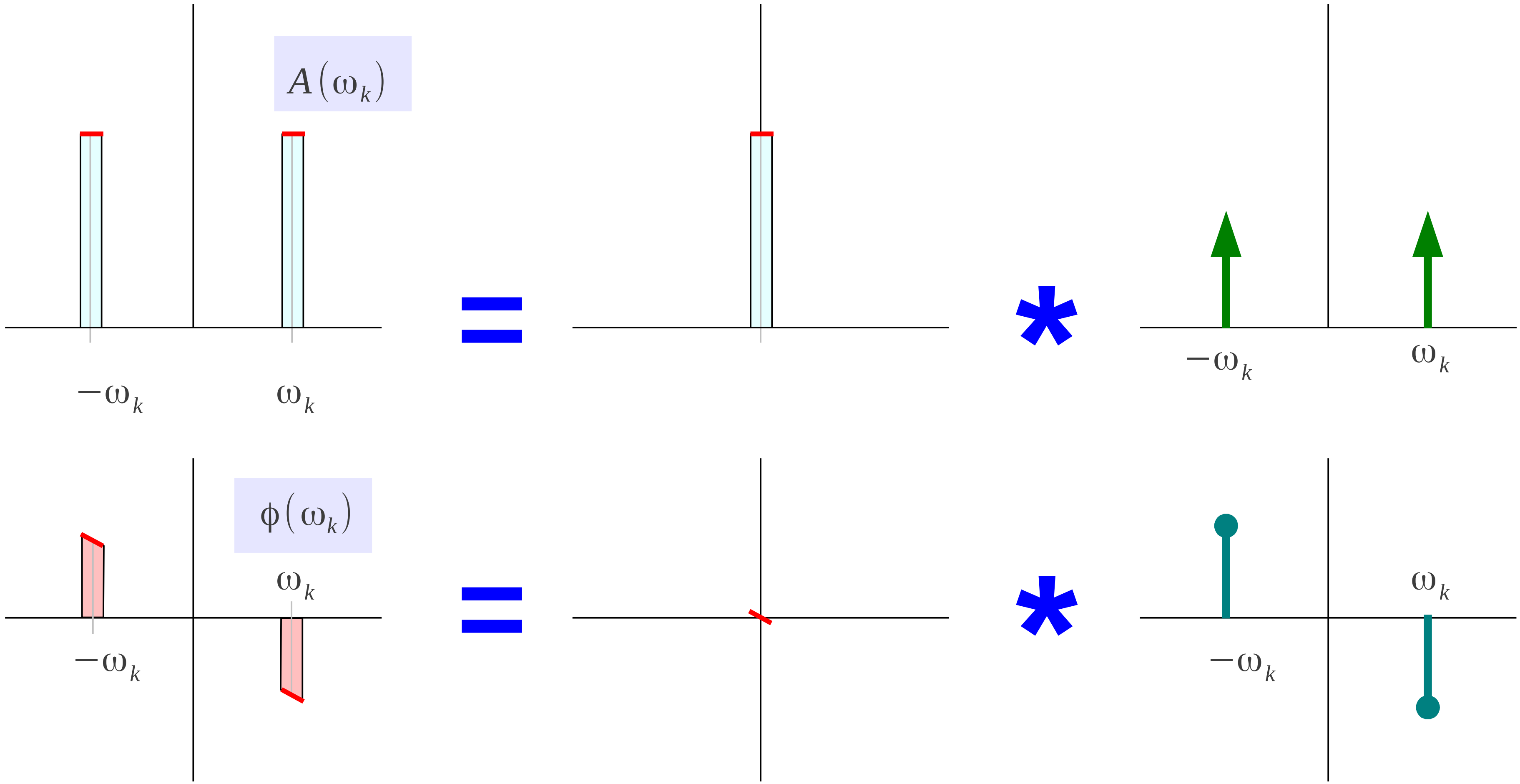
Magnitude Response $A(j\omega) = |H(j\omega)| = 1 \ / \ \sqrt{1+\omega^2/\omega_0^2}$ ➡ $\approx \sum_k A(\omega_k)$



Phase Response $\phi(j\omega) = arg\{H(j\omega)\} = \tan^{-1}(-\omega/\omega_0)$ ➡ $\approx \sum_k \phi(\omega_k)$



Simple Low Pass Filter (6)

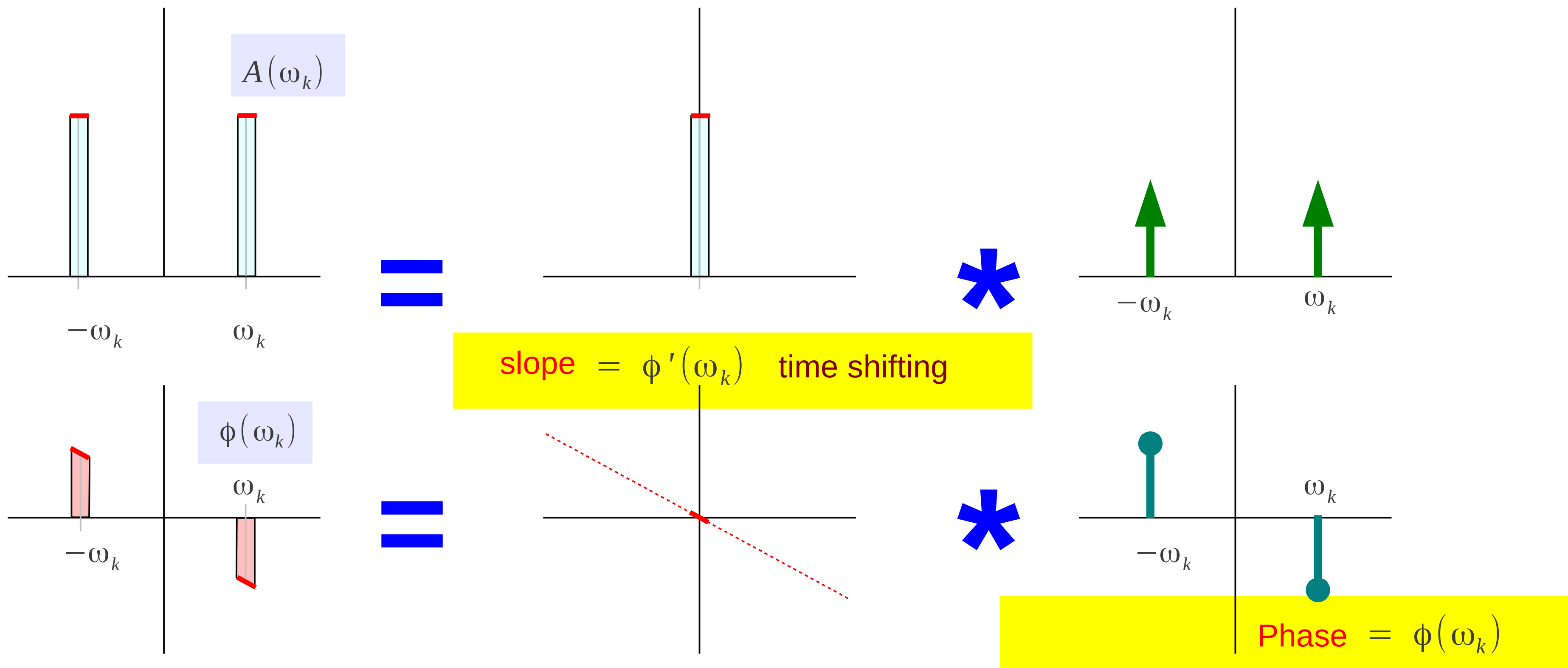


$$2A(\omega_k)B\frac{\sin[B(t+\phi'(\omega_k))]}{\pi B(t+\phi'(\omega_k))}$$

$$\begin{cases} A(\omega_k) & (-B \le \omega \le +B) \\ 0 & else \end{cases}$$

$$\pi[e^{+j\phi(\omega_k)}\delta(\omega-\omega_k) + e^{-j\phi(\omega_k)}\delta(\omega+\omega_k)]$$

Simple Low Pass Filter (6)



Frequency Domain

$$\begin{cases} A(\omega_k) & (-B \le \omega \le +B) \\ 0 & else \end{cases} \quad * \quad \pi[e^{+j\phi(\omega_k)}\delta(\omega-\omega_k) + e^{-j\phi(\omega_k)}\delta(\omega+\omega_k)]$$

Group Delay

Phase Delay

Time Domain

$$2A(\omega_k)B\frac{\sin[B(t+\phi'(\omega_k))]}{\pi B(t+\phi'(\omega_k))} \quad \times \quad \cos(\omega_k t+\phi(\omega_k))$$

Beat Signal

Very similar frequency signals

1.1 Hz $\cos(2\pi * 1.1 * t)$

0.9 Hz $\cos(2\pi * 0.9 * t)$

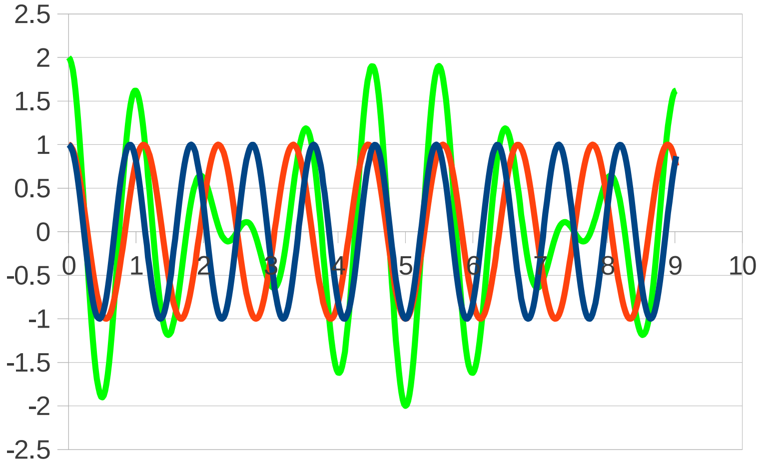
$$\cos(2\pi * 1.1 * t) + \cos(2\pi * 0.9 * t)$$

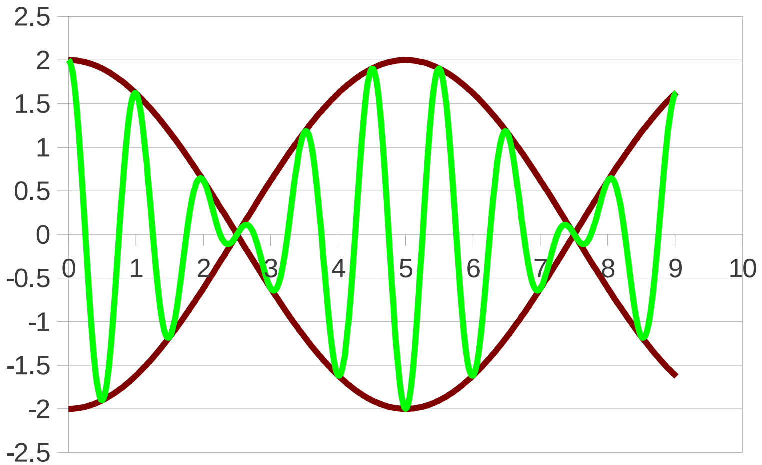
$$= \cos\left(2\pi * \frac{(1.1-0.9)}{2} * t\right) \cdot \cos\left(2\pi * \frac{(1.1+0.9)}{2} * t\right)$$

$$= \cos(2\pi * 0.1 * t) \cdot \cos(2\pi * 1.0 * t)$$

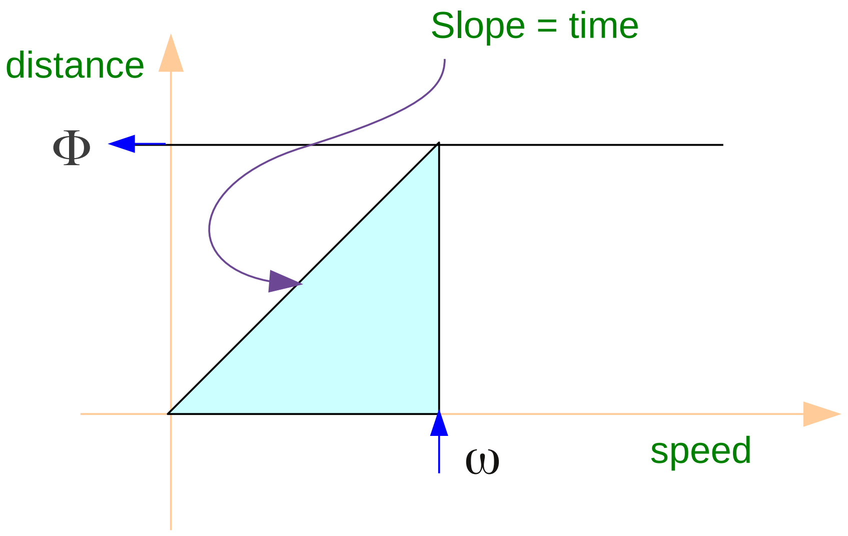
Slow moving envelop

Fast moving carrier



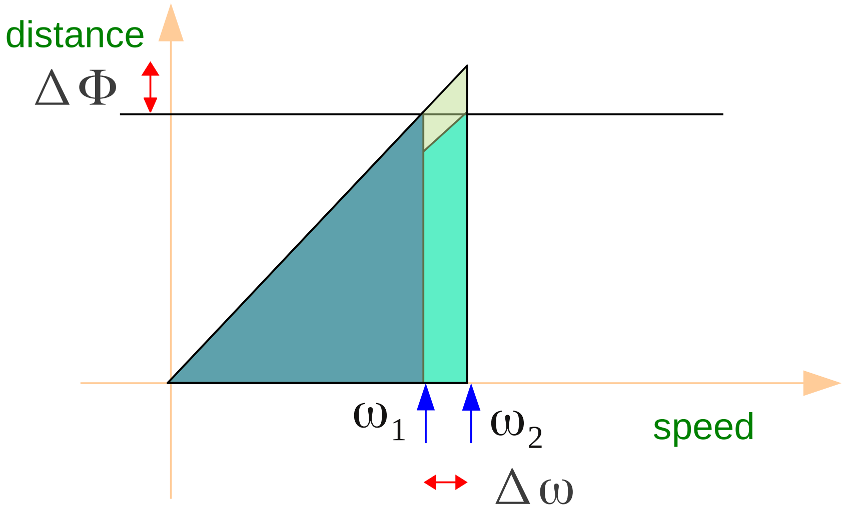


Angle and Angular Speed



$$\Phi = \omega \cdot t$$

$$t = \frac{\Phi}{\omega}$$



$$\Delta\Phi = \Delta\omega \cdot \Delta t$$

$$\Delta t = \frac{\Delta\Phi}{\Delta\omega}$$

Group Delay

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] http://www.libinst.com/tpfd.htm