Minimum Phase (3A)

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8/3/12

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Properties of a Minimum Phase System

Lowest Time Delay

- Group Delay
- Energy Compaction

Invertible

Min Phase Filter	flat response correct phase response
Equalizer	flat response incorrect phase response

Minimum Phase System

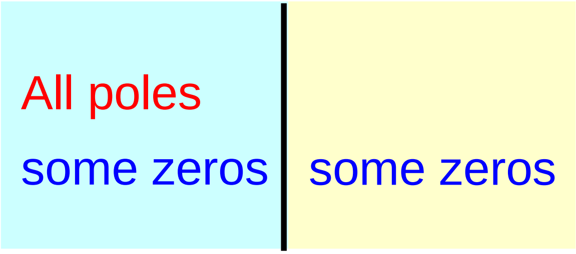
Stable Causal System

All its poles are in the left half of the s plane

Minimum Phase System

Maximum Phase System

Mixed Phase System



Minimum Phase System Properties (1)

Minimum Phase System

If an amplitude response is known

$$A(\omega) = |H(j\omega)| \qquad 0 \le \omega < \infty$$

⬇

the minimum phase response can be determined uniquely

$$\Phi_{min}(\omega) = arg\{H(j\omega)\}$$

Non-Minimum Phase System

some / all zeros in the right half s plane

With the same amplitude response

$$A(\omega) = |H(j\omega)| \qquad 0 \le \omega < \infty$$

The non-minimum phase response is always greater

$$\Phi(\omega) \ge \Phi_{min}(\omega)$$

Minimum Phase System Properties (2)

Minimum Phase System

Phase Response $\Phi(\omega)$ can be unambiguously determined from the amplitude response $A(\omega)$

→

Non-Minimum Phase System

Not valid

Verification of a Minimum Phase System

Check the progression of $\Phi(\omega)$ and $A(\omega)$ at high frequency

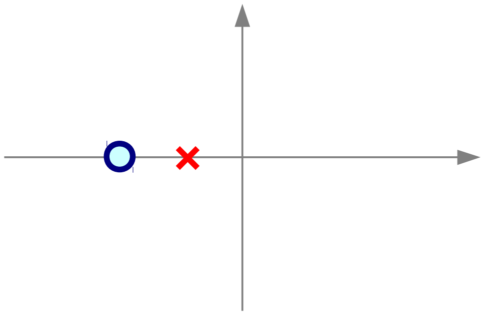
$$H(\omega) = \frac{\overbrace{N(s)}^{\text{order } m}}{\underbrace{D(s)}_{\text{order } n}}$$

	Minimum Phase System	Non-Minimum Phase System
Phase	$\Phi_{min}(\infty) = -90^\circ(n-m)$	$\lvert\Phi(\infty)\rvert \geq \lvert\Phi_{min}(\infty)\rvert$
Slope	$-20(n-m)dB\ /\ decade$	$-20(n-m)dB\ /\ decade$

Example

Minimum Phase System

$$H(s) = \frac{1+2s}{1+4s} \quad \begin{matrix} \longrightarrow -0.5 \\ \longrightarrow -0.25 \end{matrix}$$



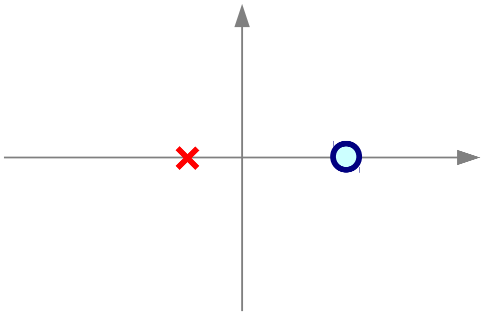
$$\frac{1+j2\omega}{1+j4\omega} = \frac{1+j2\omega}{1+j4\omega} \cdot \frac{1-j4\omega}{1-j4\omega}$$

$$= \frac{(1+8\omega^2) - j2\omega}{1+16\omega^2}$$

$$\Phi(\omega) = -\tan^{-1}\left(\frac{2\omega}{1+8\omega^2}\right)$$

Non-Minimum Phase System

$$H(s) = \frac{1-2s}{1+4s} \quad \begin{matrix} \longrightarrow +0.5 \\ \longrightarrow -0.25 \end{matrix}$$



$$\frac{1-j2\omega}{1+j4\omega} = \frac{1-j2\omega}{1+j4\omega} \cdot \frac{1-j4\omega}{1-j4\omega}$$

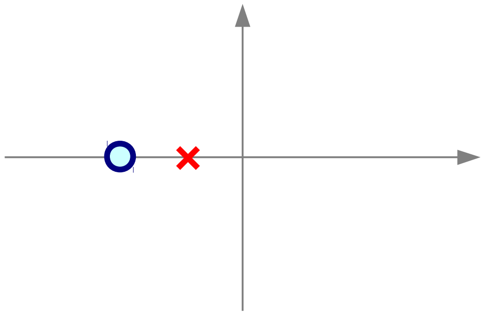
$$= \frac{(1-8\omega^2) - j6\omega}{1+16\omega^2}$$

$$\Phi(\omega) = -\tan^{-1}\left(\frac{6\omega}{1+8\omega^2}\right)$$

Example - Decomposition

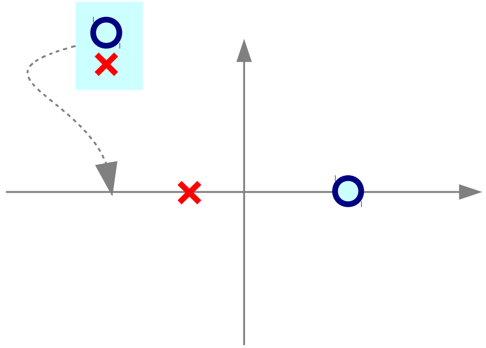
Minimum Phase System

$$H(s) = \frac{1+2s}{1+4s}$$



Non-Minimum Phase System

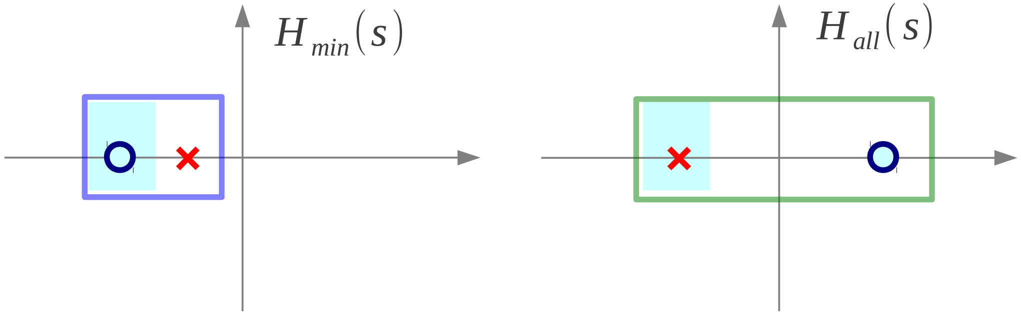
$$H(s) = \frac{1-2s}{1+4s} = \frac{1-2s}{1+4s} \cdot \frac{1+2s}{1+2s}$$



$$H(s) = \frac{1-2s}{1+4s} \cdot \frac{1+2s}{1+2s}$$

$$= \boxed{\frac{1+2s}{1+4s}} \cdot \boxed{\frac{1-2s}{1+2s}}$$

$$= H_{min}(s) \cdot H_{all}(s)$$



A non-minimum phase system can always be decomposed into $H_{min}(s) \cdot H_{all}(s)$

Example - All Pass Filter (1)

$$H_{all}(s) = \frac{1-2s}{1+2s}$$

Flat Magnitude

$$\left|\frac{1-j2\omega}{1+j2\omega}\right| = \frac{|\ 1-j2\omega\ |}{|\ 1+j2\omega\ |}$$

$$= \frac{\sqrt{1+4\omega^2}}{\sqrt{1+4\omega^2}} = 1$$

$$|H_{all}(j\omega)| = \frac{\sqrt{1+4\omega^2}}{\sqrt{1+4\omega^2}} = 1$$

A Pure Phase Shifter

$$\frac{1-j2\omega}{1+j2\omega} = \frac{1-j2\omega}{1+j2\omega}\cdot\frac{1-j2\omega}{1-j2\omega}$$

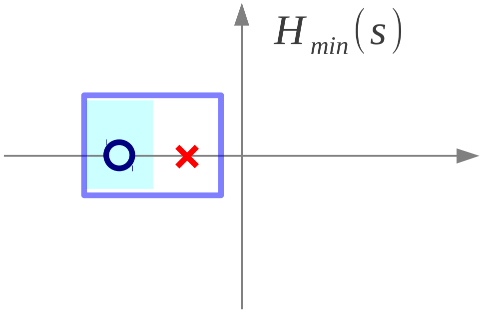
$$= \frac{(1-4\omega^2)-j4\omega}{1+4\omega^2}$$

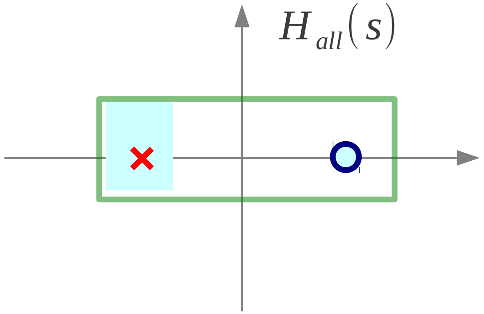
$$arg\{H_{all}(j\omega)\} = -\tan^{-1}\left(\frac{4\omega}{1-4\omega^2}\right)$$

$$H(s) = \frac{1-2s}{1+4s}\cdot\frac{1+2s}{1+2s}$$

$$= \boxed{\frac{1+2s}{1+4s}}\cdot\boxed{\frac{1-2s}{1+2s}}$$

$$= H_{min}(s)\ \cdot\ H_{all}(s)$$





A non-minimum phase system can always be decomposed into $H_{min}(s)\cdot H_{all}(s)$

Example - All Pass Filter (2)

$$H_{all}(s) = \frac{s - 0.5}{s + 0.5}$$

$$= \frac{s + 0.5 - 1}{s + 0.5}$$

$$H(s) = 1 - \frac{2}{(s + 0.5)}$$

⬇ Inverse Laplace Transform

$$h(t) = \delta(t) - e^{-0.5t}$$

$$H_{all}(j\omega) = \frac{j\omega - 0.5}{j\omega + 0.5}$$

Flat Magnitude

$$|H_{all}(j\omega)| = \frac{\sqrt{\omega^2 + 0.25}}{\sqrt{\omega^2 + 0.25}} = 1$$

Phase Shifter

$$arg\{H_{all}(j\omega)\} = -2\tan^{-1}\left(\frac{\omega}{0.5}\right)$$

Group Delay

$$-\frac{d}{d\omega}\left(arg\{H_{all}(j\omega)\}\right)$$

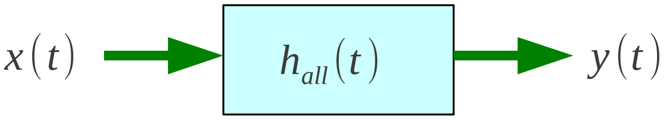
$$= -\frac{d}{d\omega}\left(-2\tan^{-1}\left(\frac{\omega}{0.5}\right)\right)$$

$$= \frac{4}{(1 + \omega^2/0.25)} > 0$$

Properties of a Minimum Phase System

Properties of a Minimum Phase System

All Pass Filter (4)

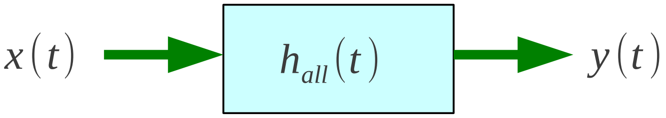


Parseval's Theorem

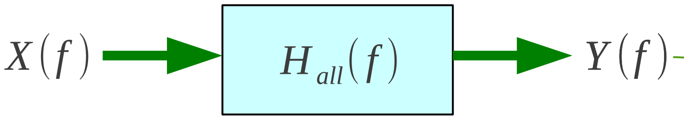
$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |y(t)|^2\, dt$$

Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2\, dt \geq \int_{-\infty}^{t_0} |y(t)|^2\, dt$$



The energy build-up in the input is more **rapid** than in the output



Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |X(f)|^2\, dt$$

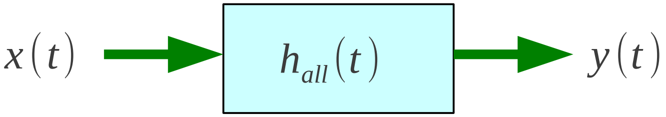
$$\int_{-\infty}^{+\infty} |y(t)|^2\, dt = \int_{-\infty}^{+\infty} |Y(f)|^2\, dt$$

$$Y(f) = H_{all}(f)\, X(f) \Rightarrow = \int_{-\infty}^{+\infty} |H_{all}(f)|^2 |X(f)|^2\, dt$$

$$|H_{all}(f)| = 1 \Rightarrow = \int_{-\infty}^{+\infty} |X(f)|^2\, dt$$

Allpass Filter $\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |y(t)|^2\, dt$

All Pass Filter (5)



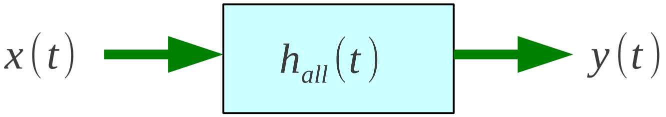
$$x(t) \longrightarrow \boxed{h_{all}(t)} \longrightarrow y(t)$$

Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |y(t)|^2 \, dt$$

Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2 \, dt \geq \int_{-\infty}^{t_0} |y(t)|^2 \, dt$$



$$x(t) \longrightarrow \boxed{h_{all}(t)} \longrightarrow y(t)$$

The <u>energy build-up</u> in the input is more **rapid** than in the output

Truncated input

$$x_1(t) = \begin{cases} x(t) & (t \leq t_0) \\ 0 & (t > t_0) \end{cases}$$

For $t \leq t_0$ $\Rightarrow$ $x_1(t) = x(t)$ $\Rightarrow$

$$y_1(t) = \int_{-\infty}^{t_0} h(t-\tau) x_1(\tau) d\tau = \int_{-\infty}^{t} h(t-\tau) x(\tau) d\tau = y(t)$$

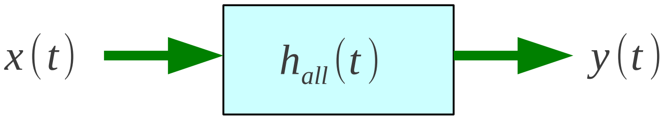
$$\int_{-\infty}^{+\infty} |x_1(t)|^2 \, dt = \int_{-\infty}^{+\infty} |y_1(t)|^2 \, dt$$

For $t > t_0$ $\Rightarrow$ $x_1(t) = 0$ $\Rightarrow$

$$\underline{\int_{-\infty}^{t_0} |x_1(t)|^2 \, dt} = \int_{-\infty}^{+\infty} |y_1(t)|^2 \, dt = \underline{\int_{-\infty}^{t_0} |y_1(t)|^2 \, dt} + \int_{t_0}^{+\infty} |y_1(t)|^2 \, dt$$

$$\int_{-\infty}^{t_0} |x(t)|^2 \, dt \geq \int_{-\infty}^{t_0} |y(t)|^2 \, dt \qquad \textit{For} \quad t \leq t_0$$

All Pass Filter (6)

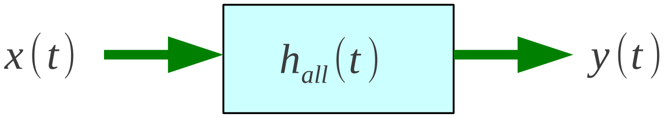


Parseval's Theorem

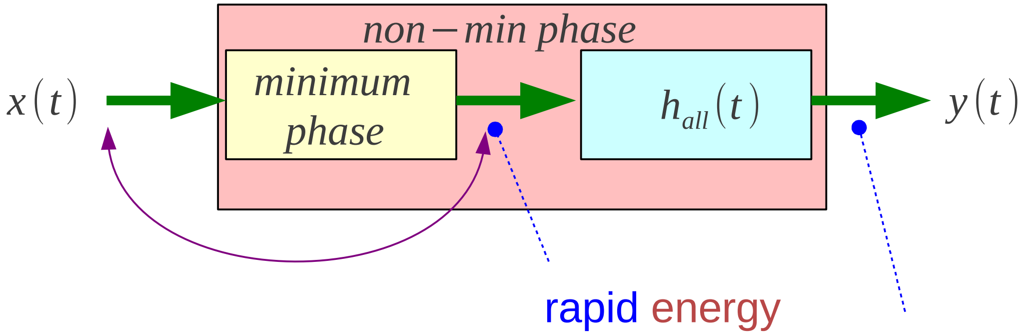
$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |y(t)|^2 \, dt$$

Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2 \, dt \geq \int_{-\infty}^{t_0} |y(t)|^2 \, dt$$



The energy build-up in the input
is more **rapid** than in the output



rapid energy
build-up than the output

The signal energy until t_0 of the minimum phase

≥ any other causal signal
with the same magnitude response

Thus minimum phase signals

➡ maximally concentrated toward time 0
when compared against all causal signals
having the same magnitude response

minimum phase signals

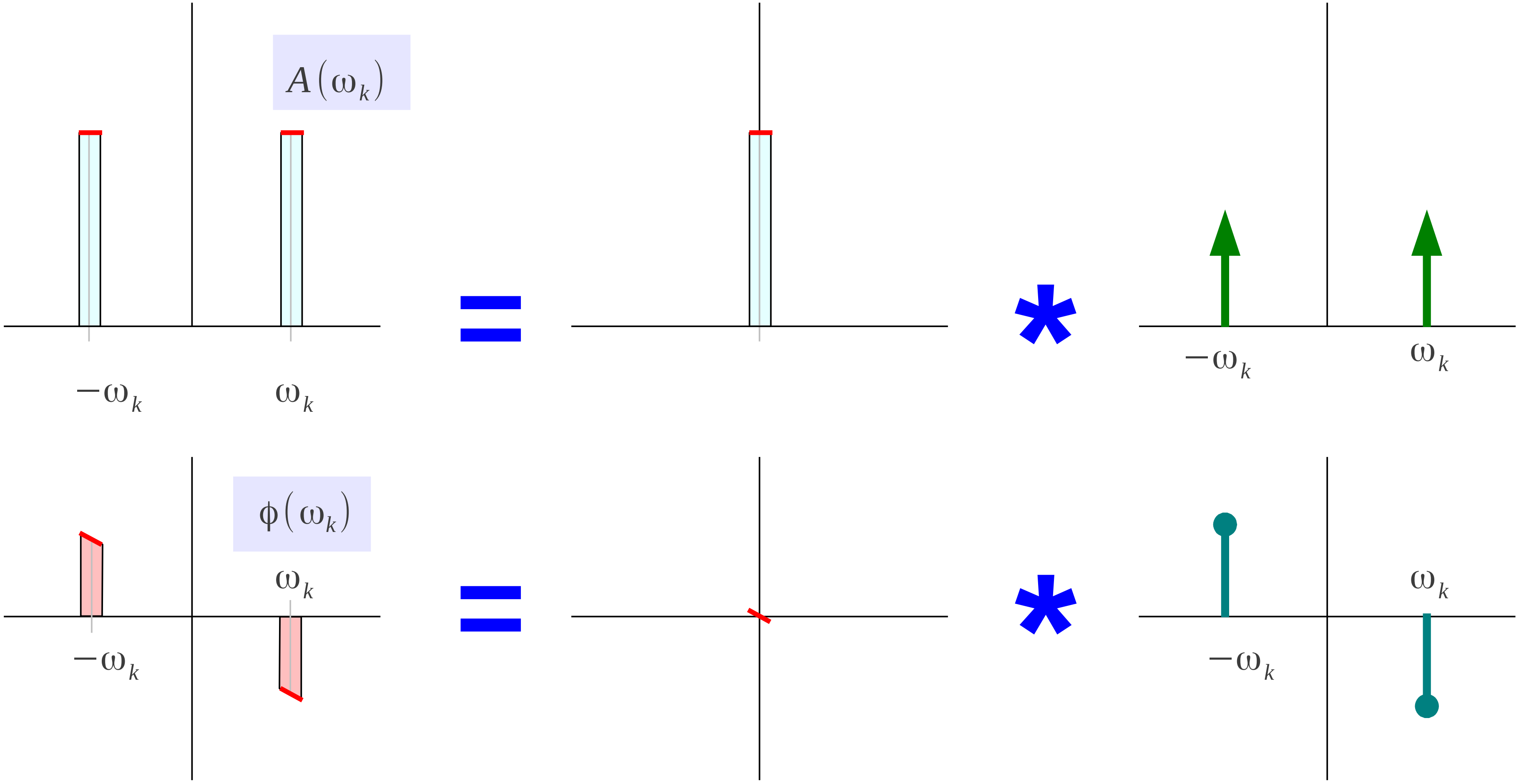
➡ minimum delay signals

Properties of a Minimum Phase System

Properties of a Minimum Phase System

Properties of a Minimum Phase System

Simple LPF – Approximation (2)

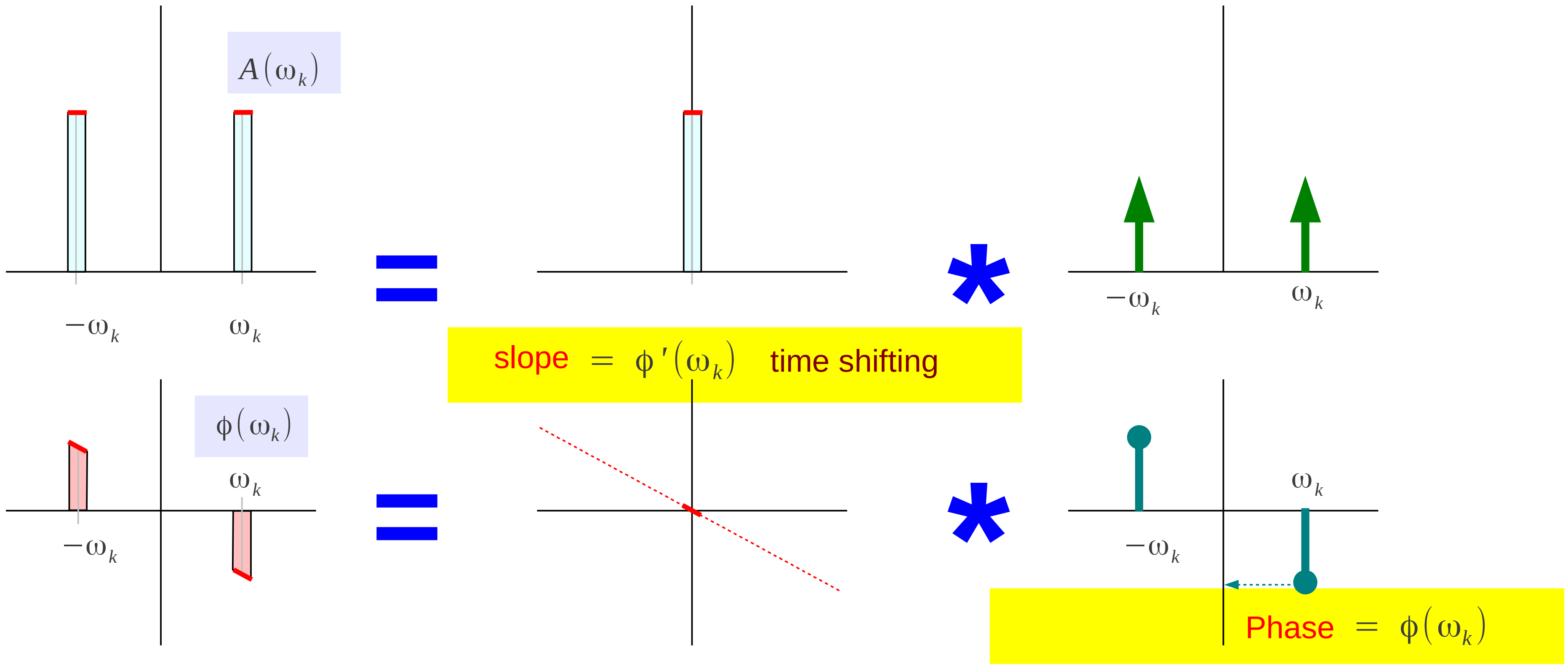


$$2A(\omega_k)B\frac{\sin[B(t+\phi'(\omega_k))]}{\pi B(t+\phi'(\omega_k))}$$

$$\begin{cases} A(\omega_k) & (-B \le \omega \le +B) \\ 0 & else \end{cases}$$

$$\pi[e^{+j\phi(\omega_k)}\delta(\omega-\omega_k) + e^{-j\phi(\omega_k)}\delta(\omega+\omega_k)]$$

Simple LPF – Approximation (3)



Frequency Domain

$$\begin{cases} A(\omega_k) & (-B \le \omega \le +B) \\ 0 & else \end{cases} \quad * \quad \pi[e^{+j\phi(\omega_k)}\delta(\omega-\omega_k) + e^{-j\phi(\omega_k)}\delta(\omega+\omega_k)]$$

Group Delay

Phase Delay

Time Domain

$$2A(\omega_k)B\frac{\sin[B(t+\phi'(\omega_k))]}{\pi B(t+\phi'(\omega_k))} \quad \times \quad \cos(\omega_k t+\phi(\omega_k))$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] http://www.libinst.com/tpfd.htm
[4] K.Shin, J.K. Hammond, “Fundamentals of Signal Processing for Sound and Vibration Engineers”
[5] www.radiolab.com.au/DesignFile/DN004.pdf