Linear Equations

Young Won Lim
06/25/2012

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Linear Equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad\qquad\; \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ \vdots \\ x_n
\end{pmatrix}
\begin{matrix} = \\ = \\ \\ = \end{matrix}
\begin{pmatrix}
b_1 \\ b_2 \\ \vdots \\ b_m
\end{pmatrix}
$$

Linear Equations

$$a_{11} \; x_1 \; + \; \quad \; + \; \cdots \; + \; a_{1n} \; x_n \; = \; b_1$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} b_1 \end{bmatrix}$$

$$\sum_{j=1}^{n} a_{1j} \cdot x_j = b_1$$

row index

col index

mxn Matrix

row index

nx1 Vector

Linear Equations

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$$

$$\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} b_2 \end{bmatrix}$$

$$\sum_{j=1}^{n} a_{2j} \cdot x_j = b_2$$

row index

col index

mxn Matrix

row index

nx1 Vector

Linear Equations

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m$$

$$\begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b_m$$

$$\sum_{j=1}^{n} a_{mj} \cdot x_j = b_m$$

row index
col index
mxn Matrix

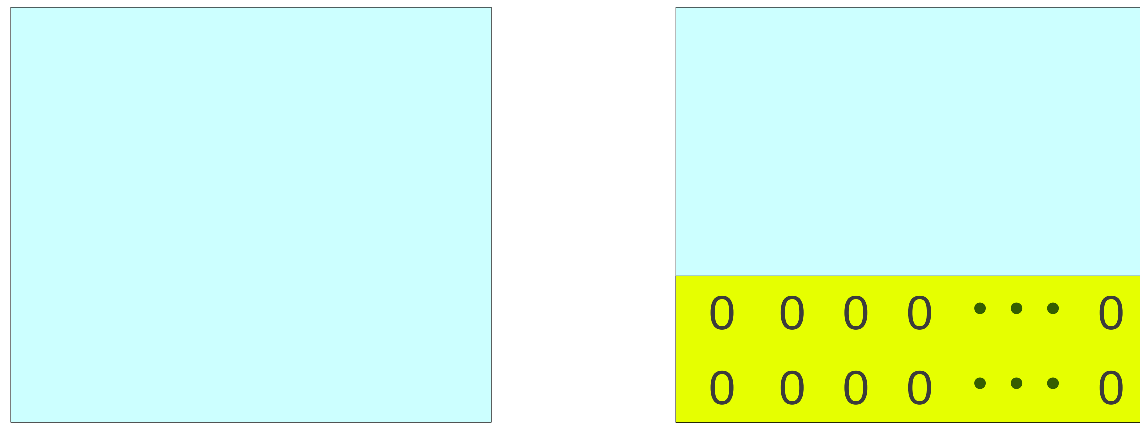
row index
nx1 Vector

Echelon Forms (1)

zero rows	➡	Should be grouped at the bottom
non-zero row	➡	A leading one The 1st non-zero element should be one
Any successive non-zero rows	➡	The leading one of the lower row should be farther to the right than the leading one of the higher row

Echelon Forms (2)

zero rows → Should be grouped at the bottom



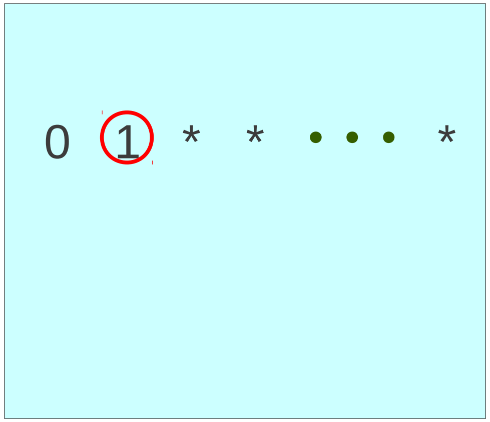
0 0 0 0 • • • 0

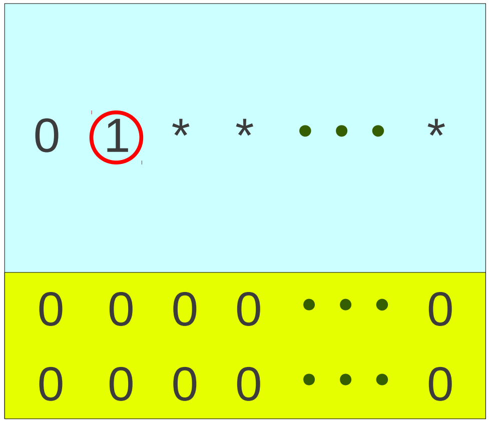
0 0 0 0 • • • 0

Echelon Forms (3)

non-zero row → A leading one

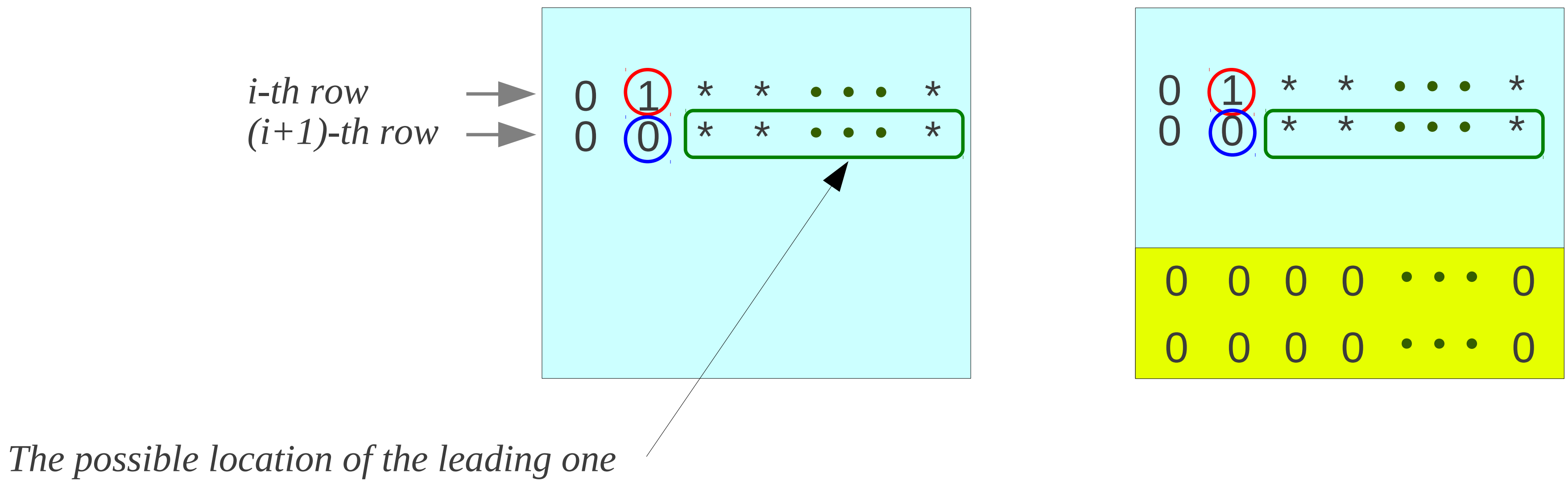
The 1st non-zero element should be one

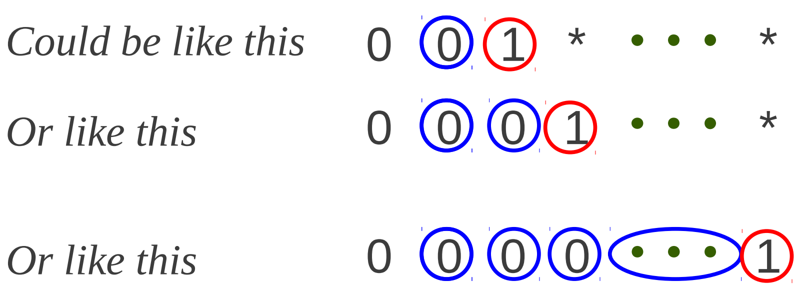




Echelon Forms (3)

Any successive non-zero rows → The leading one of the lower row should be farther to the right than the leading one of the higher row



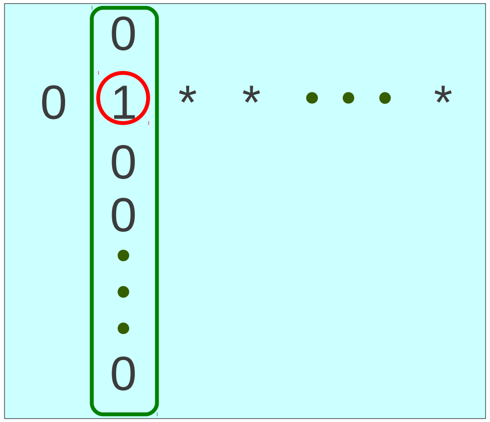


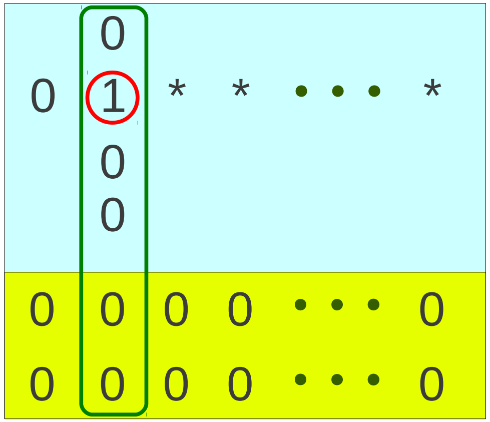
Reduced Echelon Forms

zero rows	→	Should be grouped at the bottom
non-zero row	→	A leading one The 1st non-zero element should be one
Any successive non-zero rows	→	The leading one of the lower row should be farther to the right than the leading one of the higher row
Any column that contains a leading one	→	All other elements except the leading one are all zeros

Reduced Echelon Forms

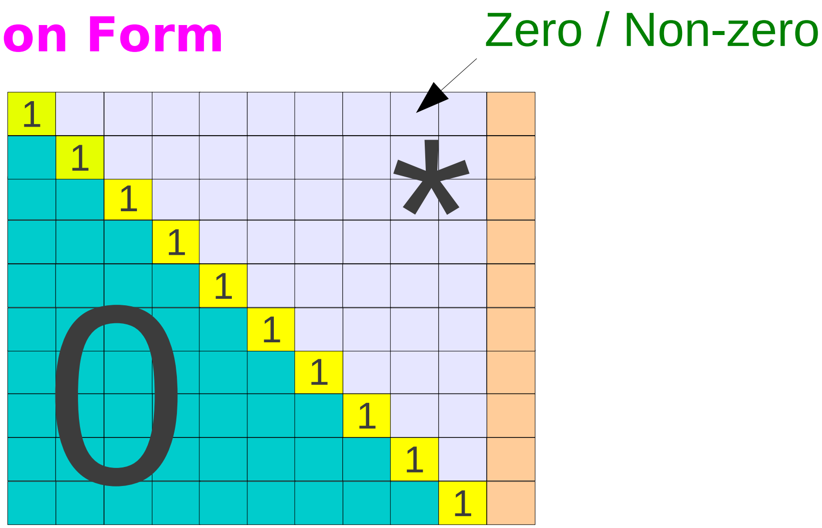
Any column that contains a leading one → All other elements except the leading one are all zeros

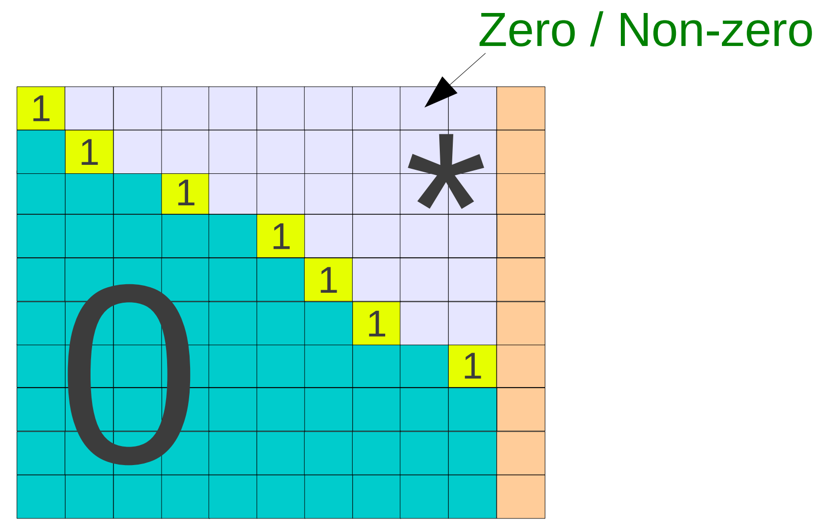




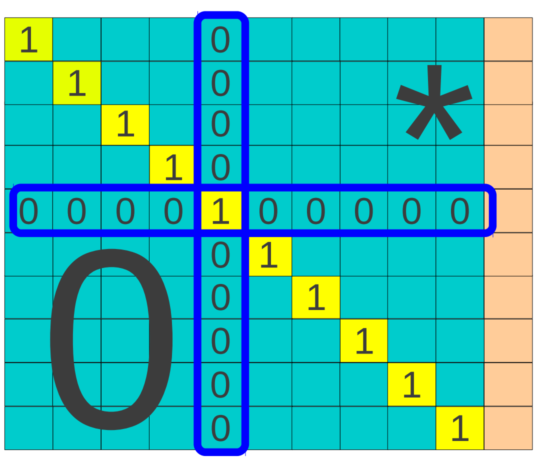
Examples

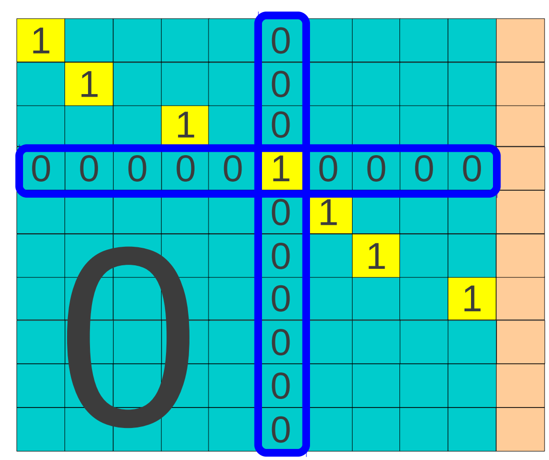
Echelon Form





Reduced Echelon Form





Example

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

$$
\begin{aligned}
2x_1 + 1x_2 - 1x_3 &= +8 \\
-3x_1 - 1x_2 + 2x_3 &= -11 \\
-2x_1 + 1x_2 + 2x_3 &= -3
\end{aligned}
$$

$$
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
=
\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}
$$

$$
\begin{pmatrix} +2 & +1 & -1 \\ -3 & -1 & +2 \\ -2 & +1 & +2 \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix}
=
\begin{pmatrix} +8 \\ -11 \\ -3 \end{pmatrix}
$$

Gauss-Jordan Elimination

$$\begin{pmatrix} +2 & +1 & -1 \\ -3 & -1 & +2 \\ -2 & +1 & +2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} +8 \\ -11 \\ -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$

Gauss-Jordan Elimination – Step 1

$$+2x_1 + x_2 - x_3 = 8 \qquad (L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right]$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = 4 \qquad \left(\tfrac{1}{2} \times L_1\right)$$

+2/2 +1/2 –1/2 +8/2

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = 4 \qquad \left(\tfrac{1}{2} \times L_1\right)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right]$$

Gauss-Jordan Elimination – Step 2

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right)$$

$$+3x_1 + \tfrac{3}{2}x_2 - \tfrac{3}{2}x_3 = +12 \qquad (3 \times L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$

+3	+3/2	–3/2	+12
–3	–1	+2	–11

$$+2x_1 + \tfrac{2}{2}x_2 - \tfrac{2}{2}x_3 = +8 \qquad (2 \times L_1)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

+2	+2/2	–2/2	+8
–2	+1	+2	–3

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = +1 \qquad (3 \times L_1 + L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (2 \times L_1 + L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right)$$

Gauss-Jordan Elimination – Step 3

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = +1 \qquad (L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right]$$

$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (2 \times L_2)$$

0	+1	+1	+2

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (2 \times L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right]$$

Gauss-Jordan Elimination – Step 4

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{+2} & +1 & +5 \end{array}\right)$$

$$0x_1 - 2x_2 - 2x_3 = -4 \qquad \boxed{(-2 \times L_2)}$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

0	–2	–2	–4
0	+2	+1	+5

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 - 1x_3 = +1 \qquad \boxed{(-2 \times L_2)} + L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{0} & -1 & +1 \end{array}\right)$$

Gauss-Jordan Elimination – Step 5

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 - 1x_3 = +1 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right]$$

$$0x_1 - 0x_2 + 1x_3 = -1 \qquad (-1 \times L_3)$$

0	0	+1	–1

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \qquad (-1 \times L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right]$$

Forward Phase

$$\left(\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

Forward Phase - Gaussian Elimination

Gauss-Jordan Elimination – Step 6

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \qquad (L_3)$$

+1	+1/2	–1/2	+4
0	+1	+1	+2
0	0	+1	–1

$$0x_1 + 0x_2 + \tfrac{1}{2}x_3 = -\tfrac{1}{2} \qquad (+\tfrac{1}{2} \times L_3)$$
$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$

0	0	+1/2	–1/2
+1	+1/2	–1/2	+4

$$0x_1 + 0x_2 - 1x_3 = +1 \qquad (-1 \times L_3)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$

0	0	–1	+1
0	+1	+1	+2

$$+1x_1 + 0x_2 - 0x_3 = +2 \qquad (+1 \times L_3 + L_1)$$
$$0x_1 + 1x_2 + 0x_3 = +3 \qquad (-1 \times L_3 + L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \qquad (L_3)$$

+1	+1/2	0	+7/2
0	+1	0	+3
0	0	+1	–1

Gauss-Jordan Elimination – Step 7

$$+1\,x_1 + 0\,x_2 - 0\,x_3 = +2 \qquad (L_1)$$
$$0\,x_1 + 1\,x_2 + 0\,x_3 = +3 \qquad (L_2)$$
$$0\,x_1 + 0\,x_2 + 1\,x_3 = -1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & \boxed{+1/2} & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

$$0\,x_1 - \tfrac{1}{2}x_2 + 0\,x_3 = -\tfrac{3}{2} \qquad \boxed{\left(-\tfrac{1}{2} \times L_2\right)}$$
$$+1\,x_1 + 0\,x_2 - 0\,x_3 = +2 \qquad (L_1)$$

$$\begin{array}{cccc} 0 & -1/2 & 0 & -3/2 \\ +1 & +1/2 & 0 & +7/2 \end{array}$$

$$+1\,x_1 + 0\,x_2 - 0\,x_3 = +2 \qquad (+1 \times L_3 + L_1)$$
$$0\,x_1 + 1\,x_2 + 0\,x_3 = +3 \qquad (-1 \times L_3 + L_2)$$
$$0\,x_1 + 0\,x_2 + 1\,x_3 = -1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & \boxed{0} & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

Backward Phase

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & 0 & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

Gauss-Jordan Elimination

Forward Phase - Gaussian Elimination

$$\left[\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right] \rightarrow$$

Backward Phase

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} +1 & 0 & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right] \rightarrow$$

Storing Magnetic Energy

Dissipate Magnetic Energy

Pulse

Pulse

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003