

# CTFS (1B)

---

- Continuous Time Fourier Series

Copyright (c) 2009 - 2011 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

Please send corrections (or suggestions) to [youngwlim@hotmail.com](mailto:youngwlim@hotmail.com).

This document was produced by using OpenOffice and Octave.

# Fourier Series

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx dx$$

$$k = 1, 2, 3, \dots$$



**one-sided spectrum**  
only positive frequencies

# Trigonometric Identities

$$\cos \theta \cos \phi = \frac{1}{2} (\cos(\theta - \phi) + \cos(\theta + \phi))$$

$$\sin \theta \sin \phi = \frac{1}{2} (\cos(\theta - \phi) - \cos(\theta + \phi))$$

$$\sin \theta \cos \phi = \frac{1}{2} (\sin(\theta + \phi) + \sin(\theta - \phi))$$

$$\cos \theta \sin \phi = \frac{1}{2} (\sin(\theta + \phi) - \sin(\theta - \phi))$$

$$\frac{1}{2} (1 + \cos(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\frac{1}{2} (1 - \cos(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\frac{1}{2} (\sin(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\frac{1}{2} (\sin(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\int_{-\pi}^{+\pi} \cos nx \cos mx dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \sin mx dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \cos mx dx = 0$$

$$\int_{-\pi}^{+\pi} \cos nx \sin mx dx = 0$$

$$\int_{-\pi}^{+\pi} \cos nx \cos mx dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin nx \sin mx dx = \pi \quad (n = m)$$

# Trigonometric Orthogonality

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} \underline{f(x) \cos kx} dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} \underline{f(x) \sin kx} dx$$

$k = 1, 2, 3, \dots$

$$\int_{-\pi}^{+\pi} \cos nx \cos mx dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \sin mx dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \cos mx dx = 0$$

$$\int_{-\pi}^{+\pi} \cos nx \sin mx dx = 0$$

$$\int_{-\pi}^{+\pi} \underline{\cos nx \cos mx} dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \underline{\sin nx \sin mx} dx = \pi \quad (n = m)$$

$$a_k \leftarrow \underline{f(x) \cdot \cos kx} = a_0 \cdot \cos kx + \sum_{m=1}^{\infty} (a_m \underline{\cos mx \cdot \cos kx} + b_m \sin mx \cdot \cos kx)$$

$$b_k \leftarrow \underline{f(x) \cdot \sin kx} = a_0 \cdot \sin kx + \sum_{m=1}^{\infty} (a_m \cos mx \cdot \sin kx + b_m \underline{\sin mx \cdot \sin kx})$$

# Any Period $p = 2L$

$$g(v) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kv + b_k \sin kv)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} g(v) dv$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \cos kv dv$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \sin kv dv$$

$k = 1, 2, \dots$

$$v: [-\pi, +\pi]$$

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos \frac{k\pi}{L} x + b_k \sin \frac{k\pi}{L} x \right)$$

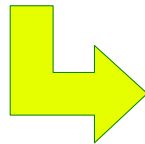
$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x) dx$$

$$a_k = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{k\pi x}{L} dx$$

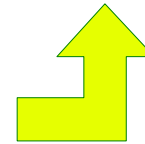
$$b_k = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{k\pi x}{L} dx$$

$k = 1, 2, 3, \dots$

$$x: [-L, +L]$$



$$v = \frac{\pi}{L} x$$
$$dv = \frac{\pi}{L} dx$$



# Time and Frequency

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos \frac{k\pi}{L} x + b_k \sin \frac{k\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x) dx$$

$$a_k = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{k\pi x}{L} dx$$

$$b_k = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{k\pi x}{L} dx$$

$k = 1, 2, 3, \dots$

$$x: [-L, +L]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos \frac{2\pi k}{T} t + b_k \sin \frac{2\pi k}{T} t \right)$$

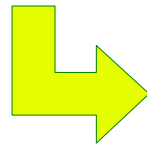
$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos \frac{2\pi k t}{T} dt$$

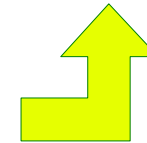
$$b_k = \frac{2}{T} \int_0^T x(t) \sin \frac{2\pi k t}{T} dt$$

$k = 1, 2, \dots$

$$t: [0, T]$$



$$2L = T$$



*Continuous Time Periodic Signal*  $x(t)$

# Harmonic Frequency

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos \frac{2\pi k}{T} t + b_k \sin \frac{2\pi k}{T} t \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos \frac{2\pi k t}{T} dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin \frac{2\pi k t}{T} dt$$

$k = 1, 2, \dots$

$$t: [0, T]$$

resolution frequency

n-th harmonic frequency

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos(2\pi k f_0 t) + b_k \sin(2\pi k f_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(2\pi k f_0 t) dt \quad k = 1, 2, \dots$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(2\pi k f_0 t) dt \quad k = 1, 2, \dots$$

$$t: [0, T]$$

$$f_0 = \frac{1}{T}$$

$$f_n = n f_0 = n \frac{1}{T}$$



# Radial Frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(k 2\pi f_0 t) + b_n \sin(k 2\pi f_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k 2\pi f_0 t) dt \quad k = 1, 2, \dots$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k 2\pi f_0 t) dt \quad k = 1, 2, \dots$$

$$t: [0, T]$$

linear frequency

angular (radial) frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(\mathbf{k} \omega_0 t) + b_n \sin(\mathbf{k} \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(\mathbf{k} \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(\mathbf{k} \omega_0 t) dt$$
$$k = 1, 2, \dots$$

$$t: [0, T]$$

$f$

$$\omega = 2\pi f$$

# Complex Fourier Series Coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k \omega_0 t) dt$$

$$k = 1, 2, \dots$$

$$t: [0, T]$$

**Real coefficients**

$$a_0, a_k, b_k, k = 1, 2, \dots$$

**Complex coefficients**

$$A_0, A_k, B_k, k = 1, 2, \dots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{jk \omega_0 t} + B_k e^{-jk \omega_0 t})$$

$$A_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$A_k = \frac{1}{T} \int_0^T x(t) e^{-jk \omega_0 t} dt$$

$$B_k = \frac{1}{T} \int_0^T x(t) e^{+jk \omega_0 t} dt$$

$$t: [0, T]$$

**one-sided spectrum**

only positive frequencies

**two-sided spectrum**

Both pos and neg frequencies

# Euler Equation (1)

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k \omega_0 t) dt$$

$k = 1, 2, \dots$

$$e^{+j\omega t} = \cos \omega t + j \sin \omega t$$

$$e^{-j\omega t} = \cos \omega t - j \sin \omega t$$

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$\begin{aligned} & a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t) \\ &= a_k \frac{1}{2} (e^{jk\omega_0 t} + e^{-jk\omega_0 t}) + b_k \frac{1}{2j} (e^{jk\omega_0 t} - e^{-jk\omega_0 t}) \\ &= \frac{(a_k - jb_k)}{2} e^{jk\omega_0 t} + \frac{(a_k + jb_k)}{2} e^{-jk\omega_0 t} \\ &= A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \end{aligned}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

zero freq	→	$A_0$	=	$a_0$	}	only positive frequencies
pos freq	→	$A_k$	=	$\frac{1}{2} (a_k - jb_k)$		
neg freq	→	$B_k$	=	$\frac{1}{2} (a_k + jb_k)$		

# Euler Equation (2)

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k \omega_0 t) dt$$

$$k = 1, 2, \dots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

zero freq	→	$A_0 = a_0$	} only positive frequencies
pos freq	→	$A_k = \frac{1}{2} (a_k - jb_k)$	
neg freq	→	$B_k = \frac{1}{2} (a_k + jb_k)$	

$$A_k = \frac{1}{T} \int_0^T x(t) (\cos(k \omega_0 t) - j \sin(k \omega_0 t)) dt$$

$$B_k = \frac{1}{T} \int_0^T x(t) (\cos(k \omega_0 t) + j \sin(k \omega_0 t)) dt$$



$$A_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$B_k = \frac{1}{T} \int_0^T x(t) e^{+jk\omega_0 t} dt$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$



$$x(t) = \sum_{k=0}^{\infty} (A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

# Complex Fourier Series

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k \omega_0 t) dt$$
$$k = 1, 2, \dots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

$$A_0 = a_0$$

$$A_k = \frac{1}{2} (a_k - jb_k)$$

$$B_k = \frac{1}{2} (a_k + jb_k)$$

$$k = 1, 2, \dots$$

$$x(t) = \sum_{k=0}^{\infty} (A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

$$A_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$
$$k = 0, 1, 2, \dots$$

$$B_k = \frac{1}{T} \int_0^T x(t) e^{+jk\omega_0 t} dt$$
$$k = 1, 2, \dots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$
$$k = -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} A_0 & (k = 0) \\ A_k & (k > 0) \\ B_k & (k < 0) \end{cases}$$

# Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t))$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k \omega_0 t + \phi_k)$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k \omega_0 t) dt$$
$$k = 1, 2, \dots$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1} \left( -\frac{b_k}{a_k} \right)$$

$$k = 1, 2, \dots$$

$$g_k \cos(k \omega_0 t + \phi_k) = \underline{g_k \cos(\phi_k)} \cos(k \omega_0 t) - \underline{g_k \sin(\phi_k)} \sin(k \omega_0 t)$$

$$\cos(\alpha + \beta) = \underline{\cos(\alpha)} \cos(\beta) - \underline{\sin(\alpha)} \sin(\beta)$$

$$\underline{a_k} \cos(k \omega_0 t) + \underline{b_k} \sin(k \omega_0 t)$$

# Two-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k \omega_0 t) dt$$
$$k = 1, 2, \dots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

$$A_0 = a_0$$

$$A_n = \frac{1}{2} (a_n - j b_n)$$

$$B_n = \frac{1}{2} (a_n + j b_n)$$

$$x(t) = \sum_{n=-\infty}^{+\infty} C_k e^{jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$k = -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} A_0 & (k = 0) \\ A_k & (k > 0) \\ B_k & (k < 0) \end{cases}$$

$$|C_k| = |A_k| = |B_k| \quad (k \neq 0)$$

$$\text{Arg}(C_k) = \begin{cases} +\phi_k = \tan^{-1} \left( -\frac{b_k}{a_k} \right) & (k > 0) \\ -\phi_k = \tan^{-1} \left( +\frac{b_k}{a_k} \right) & (k < 0) \end{cases}$$

# CTFS of Impulse Train (1)

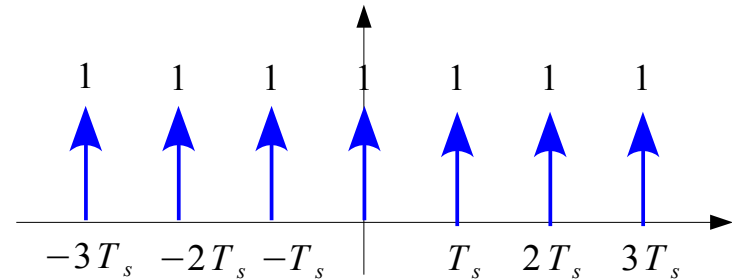
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

## Fourier Series Expansion of Impulse Train

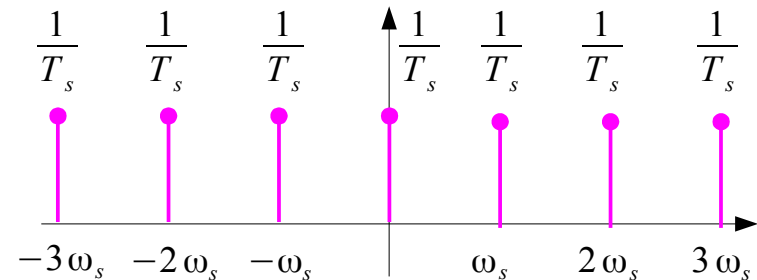
$$p(t) = \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t}$$

## Fourier Series Coefficients

$$\begin{aligned} a_k &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jk\omega_s 0} dt \\ &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) dt = \frac{1}{T_s} \end{aligned}$$



$$\omega_s = \frac{2\pi}{T_s}$$

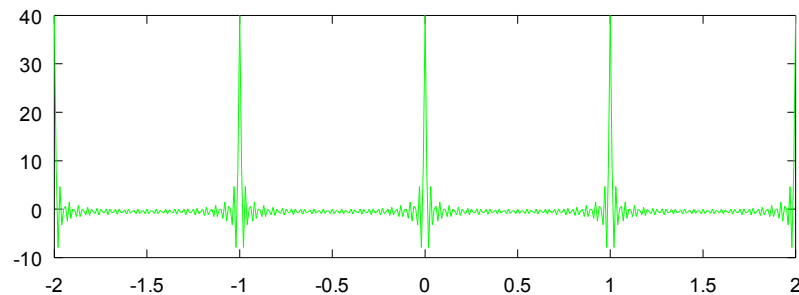
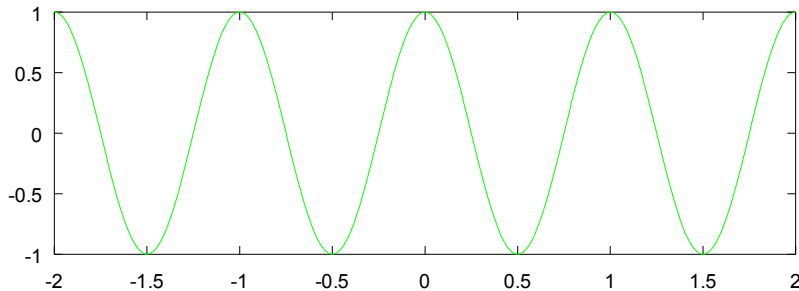




# CTFS of Impulse Train (2)

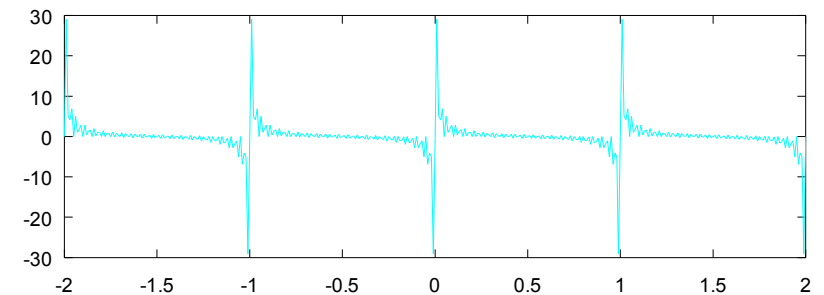
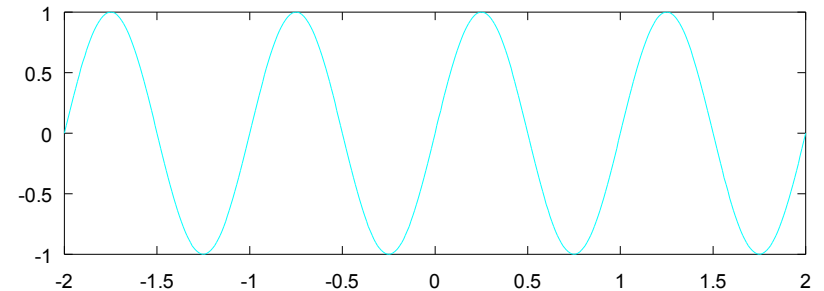
$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} (\cos k\omega_s t - j \sin k\omega_s t)$$

$\cos 2\pi \cdot 1 \cdot t$



$$\sum_{k=1}^{40} \cos 2\pi \cdot k \cdot t$$

$\sin 2\pi \cdot 1 \cdot t$

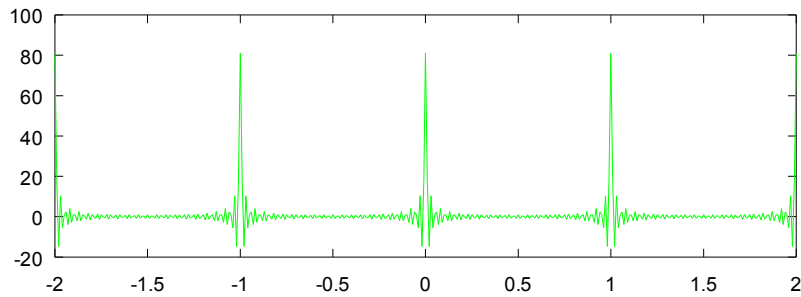
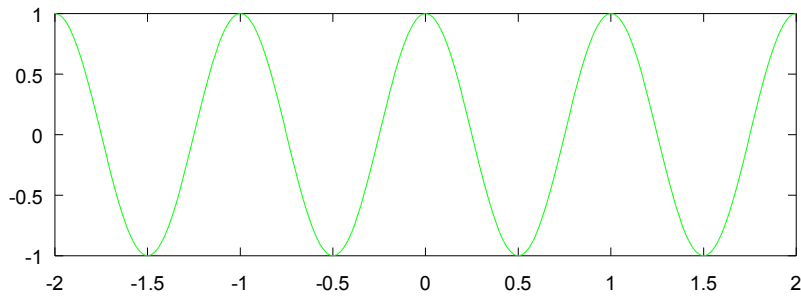


$$\sum_{k=1}^{40} \sin 2\pi \cdot k \cdot t$$

# CTFS of Impulse Train (3)

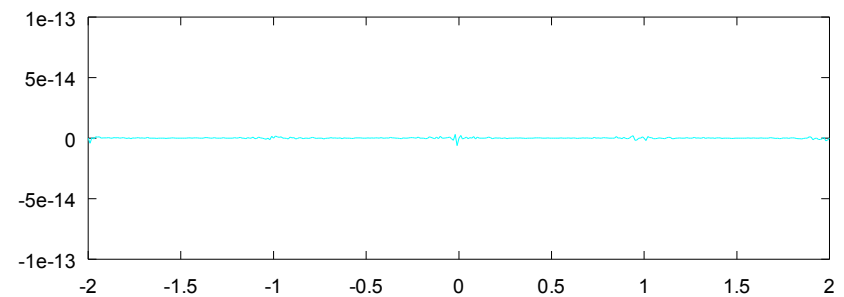
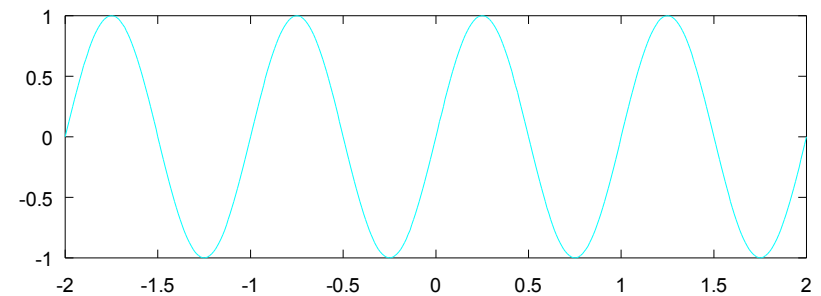
$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} (\cos k\omega_s t - j \sin k\omega_s t)$$

$\cos 2\pi \cdot 1 \cdot t$



$$\sum_{k=-40}^{40} \cos 2\pi \cdot k \cdot t$$

$\sin 2\pi \cdot 1 \cdot t$



$$\sum_{k=-40}^{40} \sin 2\pi \cdot k \cdot t$$

## References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] G. Beale, [http://teal.gmu.edu/~gbeale/ece\\_220/fourier\\_series\\_02.html](http://teal.gmu.edu/~gbeale/ece_220/fourier_series_02.html)
- [4] C. Langton, <http://www.complextoreal.com/chapters/fft1.pdf>