EigenSpaces (5A)

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EigenValues and EigenVectors

n x n

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

eigenvalue

eigenvector

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$$

$$\begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

characteristic Equation

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$$

Characteristic Equation

n x n

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

characteristic Equation

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$$

$$\begin{pmatrix} \lambda - a_{22} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$$

characteristic Equation

$$det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$$

$$\left.\begin{aligned} det(\boldsymbol{A} - \lambda \boldsymbol{I}) &= 0 \\ det(\lambda \boldsymbol{I} - \boldsymbol{A}) &= 0 \end{aligned}\right\} \qquad \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n = 0$$

Triangular Matrix

n x n

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Upper Triangular

n x n

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Lower Triangular

n x n

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Diagonal

$\boldsymbol{A} - \lambda \boldsymbol{I}$

$$\begin{bmatrix} a_{11} - \lambda & & & \\ & a_{22} - \lambda & & \\ & & & \\ & & & a_{nn} - \lambda \end{bmatrix}$$

characteristic Equation

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0 \quad det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$$

$$(\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}) = 0$$

$$\lambda = a_{11}, \quad \lambda = a_{22}, \quad \cdots, \quad \lambda = a_{nn}$$

Powers of Matrix

n x n	A
eigenvalue	λ
eigenvector	x

⇒

n x n	A^k
eigenvalue	λ^k
eigenvector	x

$$A^2 x = A(Ax) = A(\lambda I)x = \lambda A x = \lambda^2 x$$

$$A^2 x = \lambda^2 x$$

EigenValue 0

n x n

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

characteristic Equation

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$$

$$det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n$$

$$c_n = 0 \Leftrightarrow \lambda = 0 \Leftrightarrow det(-\boldsymbol{A}) = c_n \Leftrightarrow \text{Non-invertible } \boldsymbol{A}$$

$$(-1)^n det(\boldsymbol{A}) = c_n$$

$$det(\boldsymbol{A}) = 0$$

A nxn Matrix A (1)

1. **A** is invertible
2. **Ax** = **0** has only the trivial solution
3. The RREF(**A**) = $\mathbf{I}_n$
4. **A** can be written as a product of elementary matrix
5. **Ax** = **b** is consistent for every n x 1 **b**
6. **Ax** = **b** has exactly one solution for every n x 1 **b**
7. det(**A**) $\neq$ 0
8. The column vectors are linearly independent
9. The row vectors are linearly independent
10. The column vectors span R^n
11. The row vectors span R^n
12. The column vectors form a basis for R^n
13. The row vectors form a basis for R^n
14. rank(**A**) = n
15. nullity(**A**) = 0
16. The orthogonal complement of the null space is R^n
17. The orthogonal complement of the row space is {**0**}

A nxn Matrix A (2)

18. The range of T_A is R^n
19. T_A is one-to-one
20. $\lambda=0$ is not the eigenvalue of **A**

Diagonalizable

n x n $\qquad$ **n x n**

$$\boldsymbol{A} \Rightarrow \boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$$

$$det(\boldsymbol{B}) = det(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}) = det(\boldsymbol{P}^{-1})det(\boldsymbol{A})det(\boldsymbol{P})$$

$$= \frac{1}{det(\boldsymbol{P})}det(\boldsymbol{A})det(\boldsymbol{P}) = det(\boldsymbol{A})$$

$$rank(\boldsymbol{B}) = rank(\boldsymbol{A})$$

$$nullity(\boldsymbol{B}) = nullity(\boldsymbol{A})$$

$$(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0 \qquad (\lambda\boldsymbol{I} - \boldsymbol{B}) = 0$$

Similarity Transform

n x n → n x n

$A \Rightarrow D = P^{-1} A P$: **Diagonal Matrix**

A: diagonalizable ⇔ **n linear independent eigenvectors**

A: diagonalizable → $D = P^{-1} A P$ $\quad PD = AP$

$$P = [\, p_1 \quad p_2 \quad \cdots \quad p_n \,]$$

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 p_1 & \lambda_2 p_2 & \cdots & \lambda_n p_n \end{pmatrix}$$

References

[1] http://en.wikipedia.org/
[2] Anton, et al., Elementary Linear Algebra, 10th ed, Wiley, 2011
[3] Anton, et al., Contemporary Linear Algebra,