DFT Analysis (5B)

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Frequency View of a DFT Matrix

$\frac{N}{2}-1$ rows (row 1 to row $(\frac{N}{2}-1)$); $\frac{N}{2}-1$ rows (row $(\frac{N}{2}+1)$ to row $N-1$)

$row\ \ 0$	$f = 0$	$(0\,cycle)$	0
$row\ \ 1$	$f = -1 * f_o$	$(-1\,cycle)$	$\frac{-1}{N}$
$row\ \ 2$	$f = -2 * f_o$	$(-2\,cycles)$	$\frac{-2}{N}$
$row\ \ (\frac{N}{2}-1)$	$f = -(\frac{N}{2}-1) * f_o$	$(-(\frac{N}{2}-1)\,cycles)$	$\frac{-1}{2}+\frac{1}{N}$
$row\ \ (\frac{N}{2})$	$f = -(\frac{N}{2}) * f_o$	$(\frac{N}{2}\ \ cycles)$	$\frac{-1}{2}$
$row\ \ (\frac{N}{2}+1)$	$f = +(\frac{N}{2}-1) * f_o$	$(\frac{N}{2}-1\,cycles)$	$\frac{1}{2}-\frac{1}{N}$
$row\ \ N-2$	$f = +2 * f_o$	$(2\,cycles)$	$\frac{2}{N}$
$row\ \ N-1$	$f = +1 * f_o$	$(1\,cycle)$	$\frac{1}{N}$

Normalized Frequency

$$f_o = \frac{f_s}{N}$$

Frequency View of a X[i] Vector

				Normalized Frequency	
	$X[0]$	$f = 0$	0	0	$(0\,cycle)$
	$X[1]$	$f = +1 * f_o$	$\frac{1}{N} f_s$	$\frac{1}{N}$	$(1\,cycle)$
	$X[2]$	$f = +2 * f_o$	$\frac{2}{N} f_s$	$\frac{2}{N}$	$(2\,cycles)$
$\frac{N}{2} - 1$					
	$X[\frac{N}{2}-1]$	$f = +(\frac{N}{2} - 1) * f_o$	$\frac{(\frac{N}{2}-1)}{N} f_s$	$\frac{1}{2} - \frac{1}{N}$	$(\frac{N}{2}-1\,cycles)$
	$X[\frac{N}{2}]$	$f = +(\frac{N}{2}) * f_o$	$\frac{1}{2} f_s$	$\frac{1}{2}$	$(\frac{N}{2}\quad cycles)$
	$X[\frac{N}{2}+1]$	$f = -(\frac{N}{2} - 1) * f_o$	$-\frac{(\frac{N}{2}-1)}{N} f_s$	$\frac{-1}{2} + \frac{1}{N}$	$(-(\frac{N}{2}-1)\,cycles)$
$\frac{N}{2} - 1$					
	$X[N-2]$	$f = -2 * f_o$	$-\frac{2}{N} f_s$	$\frac{-2}{N}$	$(-2\,cycles)$
	$X[N-1]$	$f = -1 * f_o$	$-\frac{1}{N} f_s$	$\frac{-1}{N}$	$(-1\,cycle)$

Frequency and Time Interval (1)

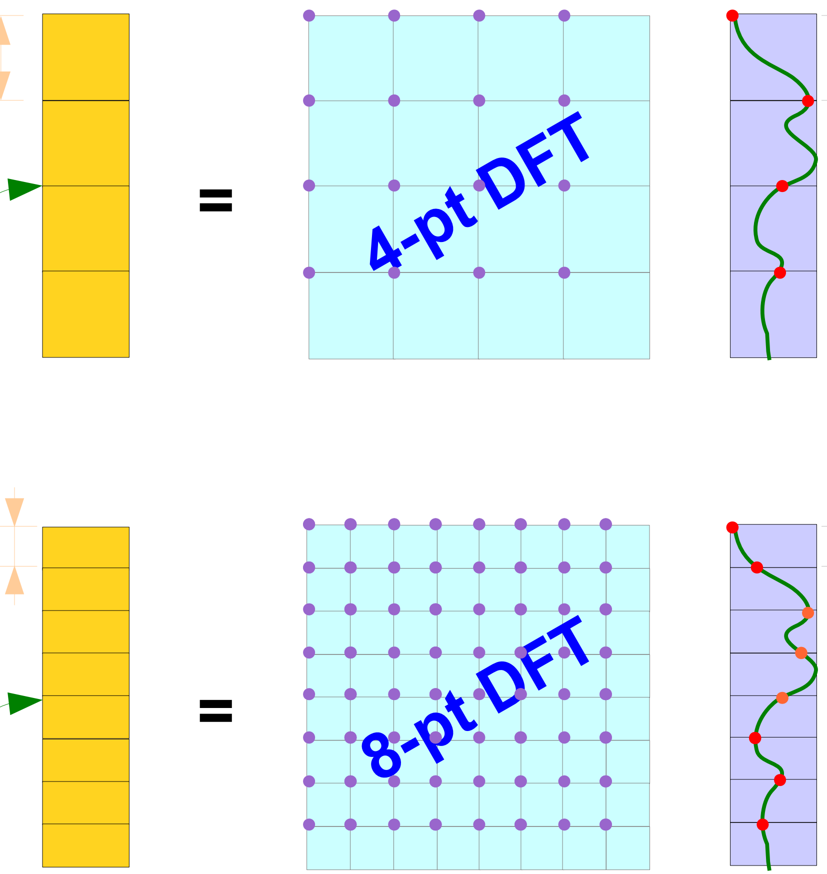
Freq Domain

$$\Delta f_1 = \frac{f_{s1}}{4} = \frac{1}{4\tau}$$

$$f_{h1} = \frac{f_{s1}}{2} = \frac{1}{2\tau}$$

$$\Delta f_2 = \frac{f_{s2}}{8} = \frac{1}{4\tau}$$

$$f_{h2} = \frac{f_{s2}}{2} = \frac{1}{\tau}$$



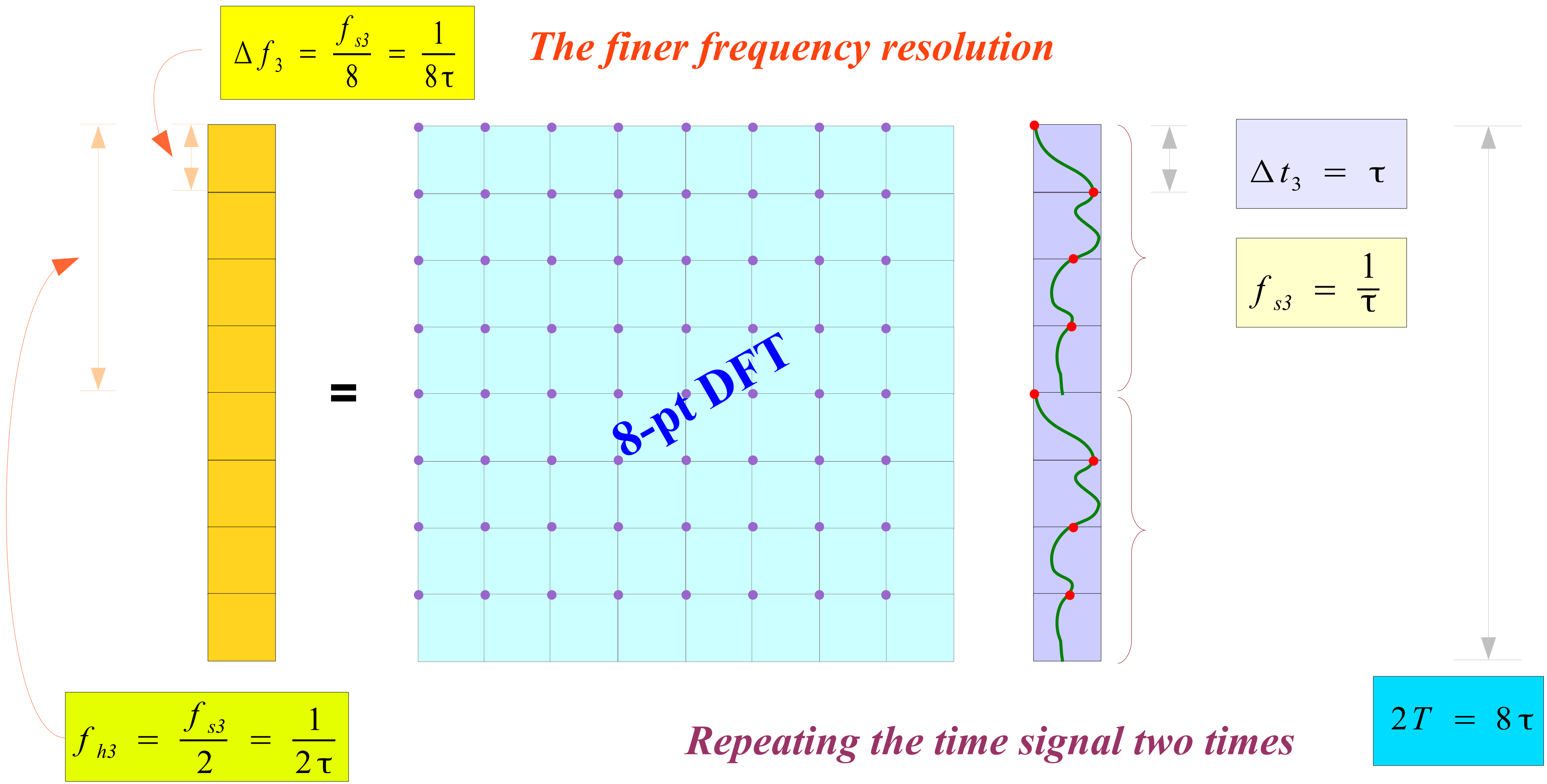
Time Domain

$$\Delta t_1 = \tau \qquad f_{s1} = \frac{1}{\tau} \qquad T = 4\tau$$

$$\Delta t_2 = \tau/2 \qquad f_{s2} = \frac{2}{\tau} \qquad T = 4\tau$$

The same frequency resolution

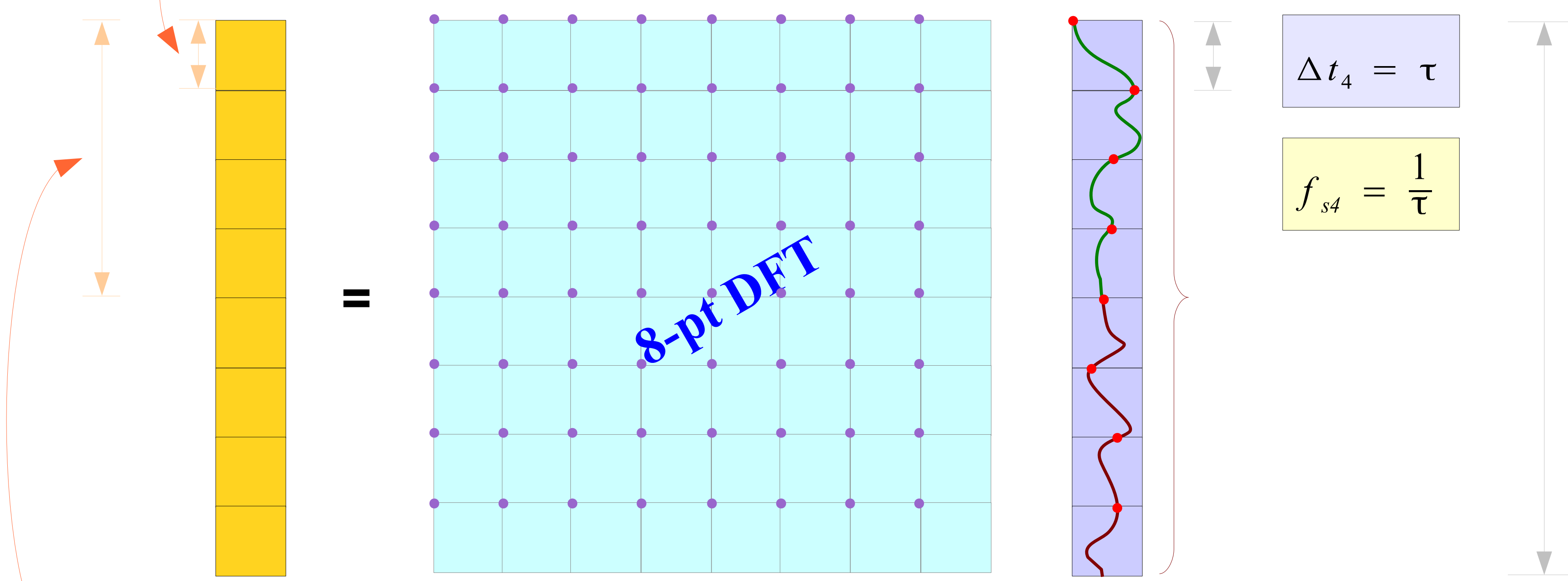
Frequency and Time Interval (2)



Frequency and Time Interval (3)

$$\Delta f_4 = \frac{f_{s4}}{8} = \frac{1}{8\tau}$$

The finer frequency resolution



$$\Delta t_4 = \tau$$

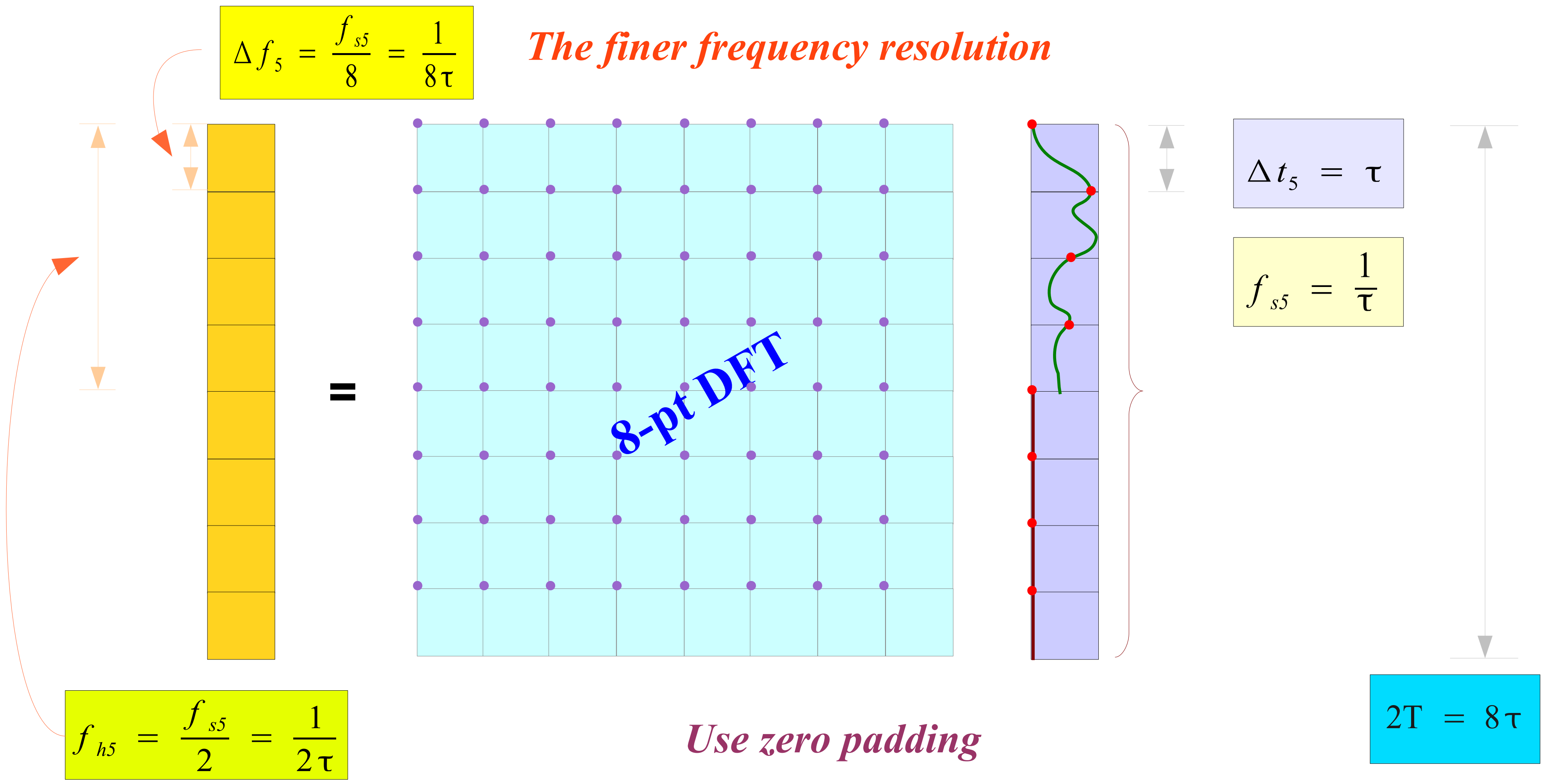
$$f_{s4} = \frac{1}{\tau}$$

$$f_{h4} = \frac{f_{s4}}{2} = \frac{1}{2\tau}$$

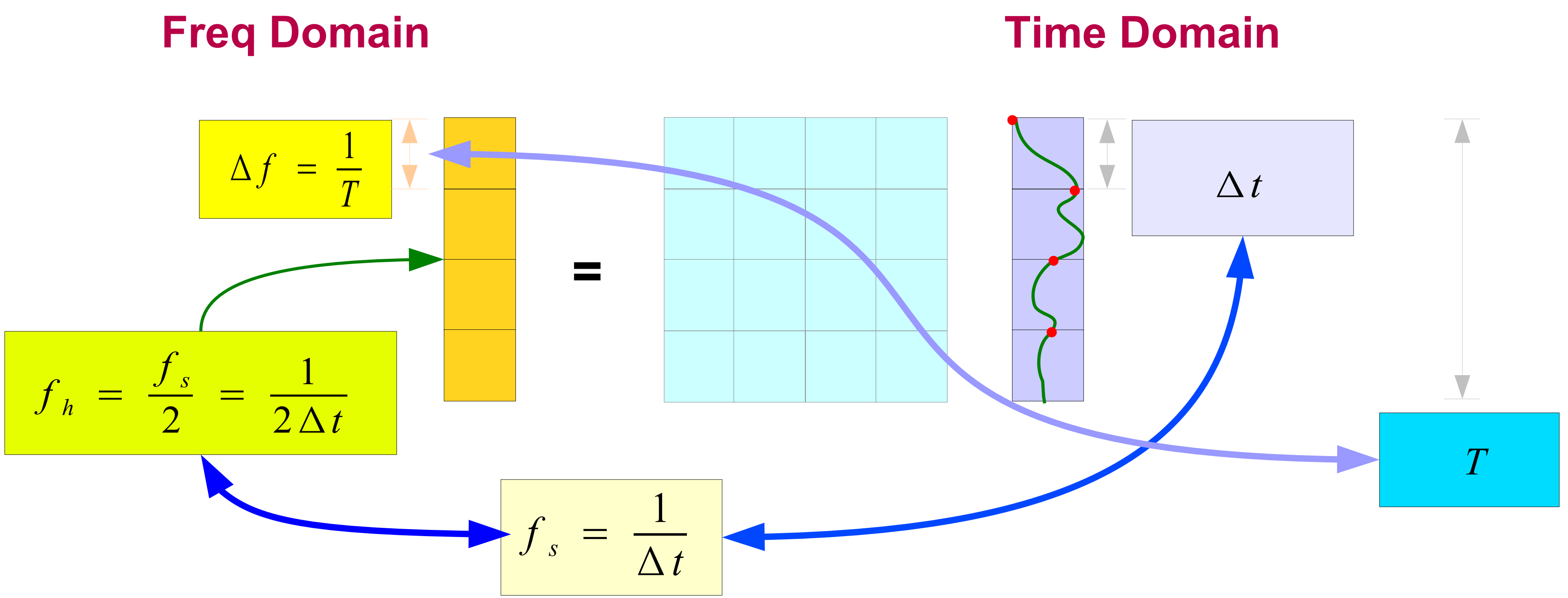
Use two times more samples (2T)

$$2T = 8\tau$$

Frequency and Time Interval (4)



Frequency and Time Interval (5)



Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt$$

$$\boldsymbol{k} = +1, +2, \ldots$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k}\right)$$

$$\boldsymbol{k} = +1, +2, \ldots$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$g_k \cos(k\omega_0 t + \phi_k) = g_k \cos(\phi_k)\cos(k\omega_0 t) - g_k \sin(\phi_k)\sin(k\omega_0 t)$$

$$a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$a_k = g_k \cos(\phi_k)$$

$$-b_k = g_k \sin(\phi_k)$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

Two-Sided Spectrum

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} \tan^{-1}(-b_k/a_k) & (k > 0) \\ \tan^{-1}(+b_k/a_k) & (k < 0) \end{cases}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2} g_k \, e^{+j\phi_k} & (k > 0) \\ \frac{1}{2} g_k \, e^{-j\phi_k} & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} +\phi_k & (k > 0) \\ -\phi_k & (k < 0) \end{cases}$$

Power Spectrum *Two-Sided*

$$|C_k|^2 + |C_{-k}|^2 = \frac{1}{2} g_k^2 = \frac{1}{2}(a_k^2 + b_k^2)$$

Periodogram *One-Sided*

$$2 \cdot |C_k| = g_k = \sqrt{a_k^2 + b_k^2}$$

CTFS and DTFS (1)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$ **CTFS**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$\boldsymbol{k} = \ldots,\ -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$x(t) \approx \sum_{k=-M}^{+M} C_k e^{+jk\omega_0 t}$$ $N = 2M + 1$

Truncate coefficeints

$$\boldsymbol{k} = -M,\ \ldots,\ -1,\ 0,\ +1,\ \ldots,\ +M$$

N Time Samples: N equations

$$t \rightarrow \boldsymbol{n}Ts = \boldsymbol{n}\left(\tfrac{T}{N}\right)$$

$$x[\boldsymbol{n}] = \sum_{k=-M}^{+M} \gamma_k\, e^{+jk\omega_0 n\left(\frac{T}{N}\right)}$$

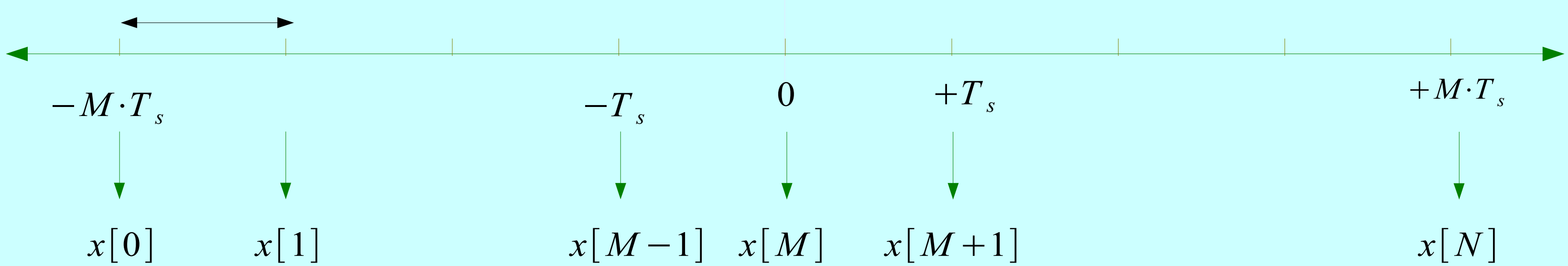
$$jk\omega_0 t \rightarrow \boldsymbol{k}\left(\frac{2\pi}{T}\right)\boldsymbol{n}\left(\frac{T}{N}\right) = \left(\frac{2\pi}{T}\right)\boldsymbol{n}k$$

$$x[\boldsymbol{n}] = \sum_{k=-M}^{+M} \gamma_k\, e^{+j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{n} = 0,\ 1,\ 2,\ \ldots,\ N-1,$$

$$T_s = \frac{T}{N}$$

Time Samples



$0 \le n < L$ $T_s \rightarrow 0$

CTFS and DTFS (2)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$ **CTFS**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots,\ -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$x[n] = \sum_{k=-M}^{+M} \gamma_k \, e^{+j\left(\frac{2\pi}{N}\right)nk}$$

$$n = 0,\ 1,\ 2,\ \ldots,\ N-1,$$

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)nk}$$

$$k = -M,\ \ldots,\ 0,\ \ldots,\ +M$$

Approximate Continuous Signal
With the truncated coefficients

$$x_{FS}(t) = \sum_{k=-M}^{+M} \gamma_k \, e^{+jk\omega_0 t}$$ **DTFS**

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$ **Approximated Fourier Coefficients**

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$ **DFT**

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$ **IDFT**

$0 \le n < L$ $\quad T_s \to 0$

CTFS and DTFS (3)

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$x[\boldsymbol{n}] = \sum_{\boldsymbol{k}=-M}^{+M} \gamma_{\boldsymbol{k}}\, e^{+j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{n} = 0,\ 1,\ 2,\ \ldots\ ,\ N-1,$$

$$\gamma_{\boldsymbol{k}} = \frac{1}{N} \sum_{\boldsymbol{n}=0}^{N-1} x[\boldsymbol{n}]\, e^{-j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{k} = -M,\ \ldots\ ,\ 0,\ \ldots\ ,\ +M$$

$$x(t) = \sum_{\boldsymbol{k}=-\infty}^{+\infty} C_{\boldsymbol{k}}\, e^{+j\boldsymbol{k}\omega_0 t}$$ **CTFS**

Truncate Fourier Coefficients

$$x_{FS}(t) = \sum_{\boldsymbol{k}=-M}^{+M} \gamma_{\boldsymbol{k}}\, e^{+j\boldsymbol{k}\omega_0 t}$$ **DTFS**

$$C_k \approx \gamma_{\boldsymbol{k}} = \frac{X[k]}{N}$$

Approximated Fourier Coefficients

Power Spectrum using FFT

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

`X = fft(x)`

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

`x = ifft(X)`

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

CTFS

Approximated
Fourier Series Coefficients

$$x_{FS}(t) = \sum_{k=-M}^{+M} \gamma_k\, e^{+jk\omega_0 t}$$

DTFS

`fc = fft(x)/N     = X/N`

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$

Approximated
Fourier Coefficients

`x = ifft(fc)*N`

$$|C_k|^2 \approx \frac{|X[k]|^2}{N^2}$$

Approximated
Power Spectrum

Periodogram using FFT

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$ **Approximated Fourier Coefficients**

$$|C_k|^2 \approx \frac{|X[k]|^2}{N^2}$$ **Approximated Power Spectrum**

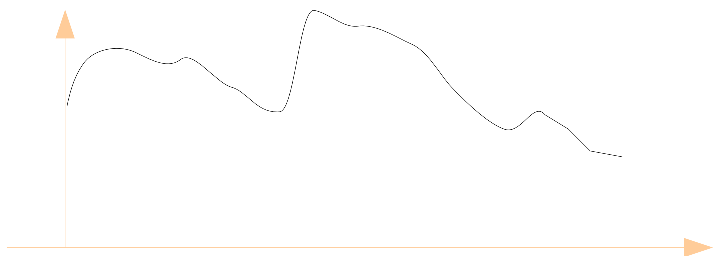
$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2$$ **Average Power**

$$\Rightarrow \left(\sqrt{\frac{\sum_{k=0}^{N-1} \frac{|X[k]|^2}{N}}{N}}\right)^2$$ **RMS of Periodogram**

$$\frac{|X[k]|^2}{N} \quad k = 0, 1, \dots, N-1$$ **Approximated Periodogram**

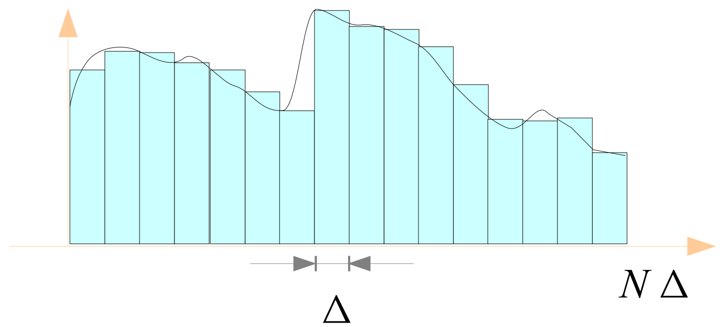
$$\frac{|X[k]|}{\sqrt{N}} \quad k = 0, 1, \dots, N-1$$

RMS in continuous time



$$\frac{1}{T}\int_0^T g^2(t)\, dt$$

RMS in discrete time



$$\frac{1}{N\Delta}\sum_{k=0}^{N-1} |g[k]|^2 \Delta = \frac{1}{N}\sum_{k=0}^{N-1} |g[k]|^2$$

	Periodic Signals	Aperiodic Signals	Random Signals	
Frequency Spacing	$\Delta f = \frac{1}{N \Delta t}$	$\Delta f = \frac{1}{N \Delta t}$	$\sum S \Delta f = \frac{1}{N \Delta t} \sum S$	$\frac{1}{N \Delta t} \sum x^2 \Delta t$
Two Sided	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$	$S(k) = \frac{\Delta t}{N} \lvert X(k) \rvert^2$	$P = \sum_{k=0}^{N-1} S(k) \Delta f$
One Sided				
$k = 0,\ \frac{N}{2}$	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$	$S_1(k) = 2\mathrm{S}(k)$	$P = \sum_{k=0}^{N/2} S_1(k) \Delta f$
$k = 1, \cdots, \frac{N}{2} - 1$	$\frac{2}{N} X(k)$	$\frac{2 \Delta t}{N} X(k)$	$S_1(k) = S(k)$	
Frequency Scale	$k \Delta f$	$k \Delta f$	$k \Delta f$	

Periodic Signals

Frequency Spacing

$\Delta f = \frac{1}{N \Delta t}$

Two Sided Fourier Series Coefficient

$\frac{1}{N} X(k)$

One Sided Fourier Series Coefficient

$\frac{1}{N} X(k) \qquad k = 0, \ \frac{N}{2}$

$\frac{2}{N} X(k) \qquad k = 1, \cdots, \frac{N}{2} - 1$

Frequency Scale

$k \Delta f$

Aperiodic Signals

$\Delta f = \frac{1}{N \Delta t}$

Two Sided Fourier Series Coefficient

$\frac{\Delta t}{N} X(k)$

One Sided Fourier Series Coefficient

$\frac{\Delta t}{N} X(k) \qquad k = 0, \ \frac{N}{2}$

$\frac{2 \Delta t}{N} X(k) \qquad k = 1, \cdots, \frac{N}{2} - 1$

$k \Delta f$

Random Signals

One-sided Power Spectral Density

$$P = \sum_{k=0}^{N-1} S(k)\,\Delta f$$

One-sided Power Spectral Density

$$P = \sum_{k=0}^{N/2} S_1(k)\,\Delta f$$

$$S_1(k) = 2\mathrm{S}(k) \qquad k = 1, \ldots, \tfrac{N}{2} - 1$$

$$S_1(k) = S(k) \qquad k = 0, \tfrac{N}{2}$$

Two Sided Fourier Series Coefficient

$$\frac{1}{N\,\Delta t} \sum x^2\,\Delta t$$

$$\sum S\,\Delta f = \frac{1}{N\,\Delta t} \sum S$$

$$S(k) = \frac{\Delta t}{N} |X(k)|^2$$

$$k\,\Delta f$$

Amplitude Spectrum

$$A_k = \tfrac{1}{N}|X(k)| = \tfrac{1}{N}\sqrt{\Re^2(X(k))+\Im^2(X(k))}$$

$$k = 0,1,2,\cdots,N-1$$

Power Spectrum

$$P_k = \tfrac{1}{N^2}|X(k)|^2 = \tfrac{1}{N^2}\{\Re^2(X(k))+\Im^2(X(k))\}$$

$$k = 0,1,2,\cdots,N-1$$

One Sided Amplitude Spectrum

$$\bar{A}_k = \tfrac{1}{N}|X(0)| \qquad k=0$$

$$\bar{A}_k = \tfrac{2}{N}|X(0)| \qquad k=1,2,\cdots,N/2$$

One Sided Power Spectrum

$$\bar{P}_k = \tfrac{1}{N^2}|X(0)|^2 \qquad k=0$$

$$\bar{P}_k = \tfrac{2}{N^2}|X(0)|^2 \qquad k=1,2,\cdots,N/2$$

Frequency Bin

$$f = \frac{k f_s}{N}$$

Frequency Bin

$$f = \frac{k f_s}{N}$$

Phase Spectrum

$$\phi_k = \tan^{-1}\left(\frac{\Im(X(k))}{\Re(X(k))}\right) \qquad k=0,1,2,\cdots,N-1$$

Data Truncation
Frequency Resolution
Zero Padding
Periodogram
Spectral Plot

Amplitude spectrum in quantity peak
Phase spectrum in radians
Amplitude spectrum in volts rms
Phase spectrum in degrees
Power spectrum

Signals without discontinuity
Signals with discontinuity

Sampling frequency is not an integer multiple of the FFT length

Leakage

$$[0, \quad \frac{f_s}{2}]$$

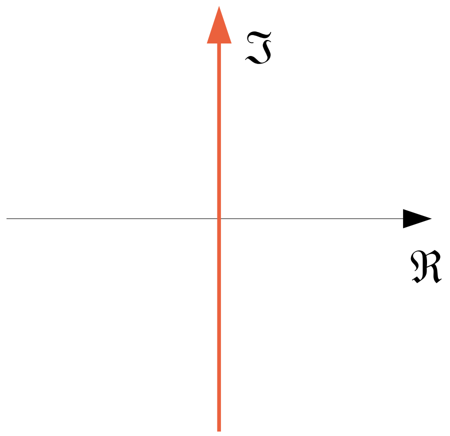
Fourier Transform

$f(t)$ A continuous sum of weighted exponential functions :

$$f(t)\, e^{-j\omega t}$$

$$-\infty \;<\; \omega \;<\; +\infty$$

Not so useful in transient analysis

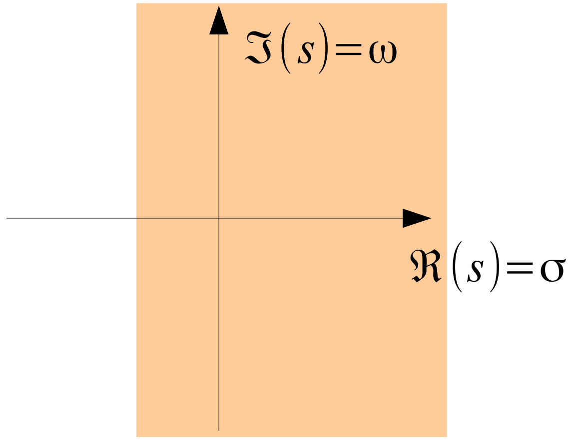


Laplace Transform

$$f(t)\, e^{-st} \;=\; f(t) e^{-(\sigma + j\omega)t}$$

Linear Time Domain Analysis

Initial Condition



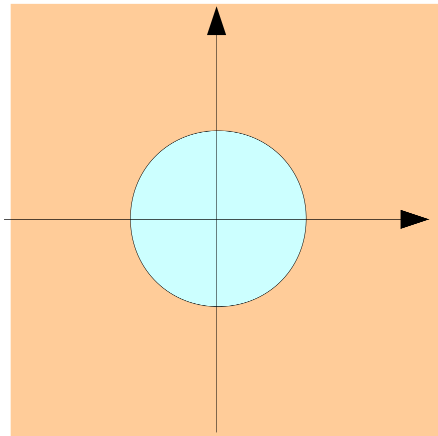
z Transform

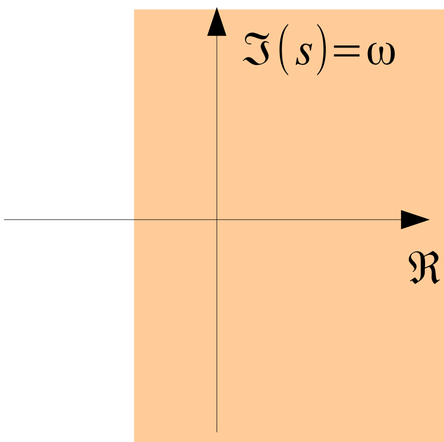
$$f[n]\, z^{-n}$$

Discrete Time System

Difference Equation

$$z \;=\; e^{sT} \;=\; e^{\sigma T}\, e^{j\omega T}$$





References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A "graphical interpretation" of the DFT and FFT, by Steve Mann