

Group Velocity and Phase Velocity (2A)

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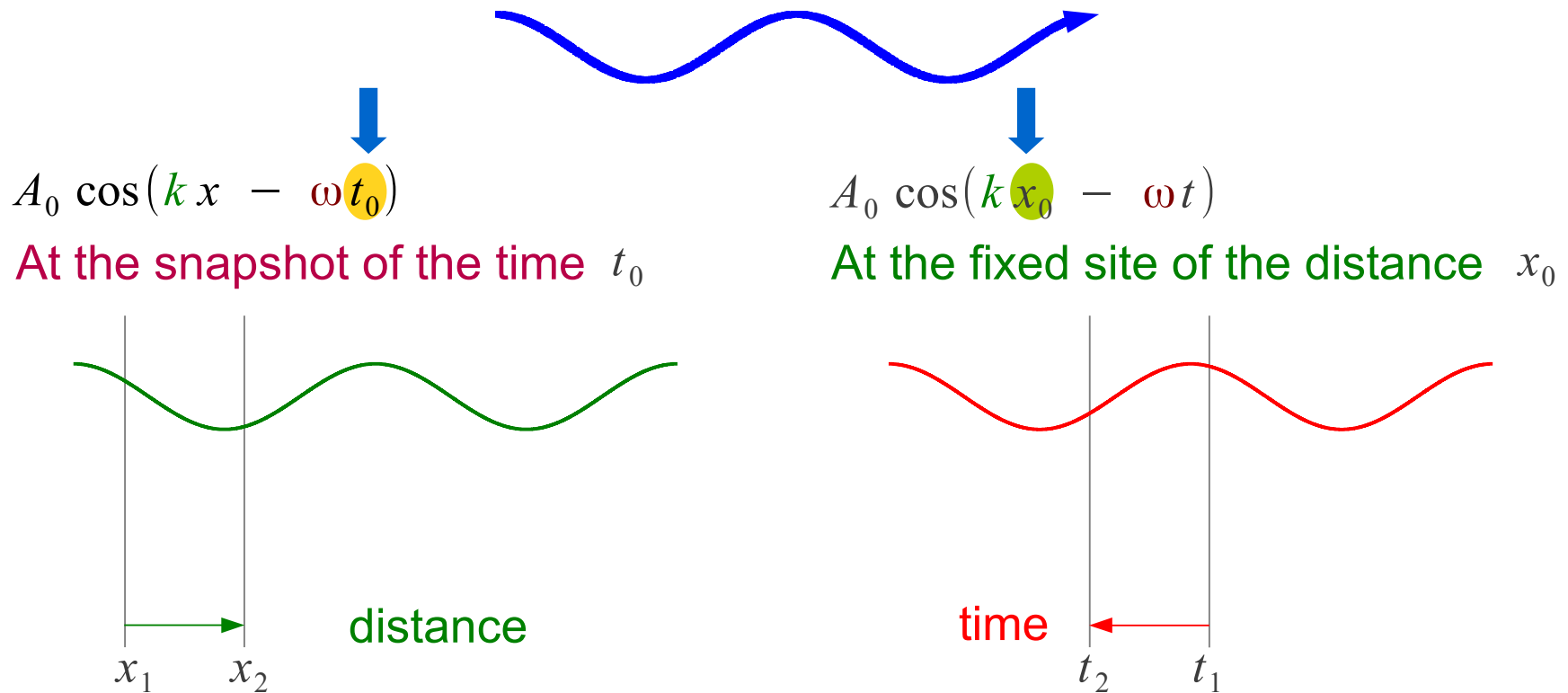
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Wave Equation

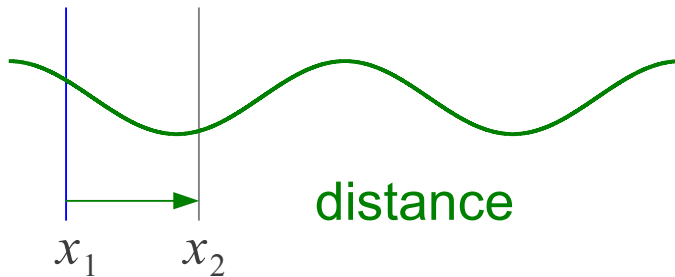
$$A(t, x) = A_0 \cos(kx - \omega t)$$



Wavelength, Frequency

$$A_0 \cos(kx - \omega t_0)$$

At the snapshot of the time t_0

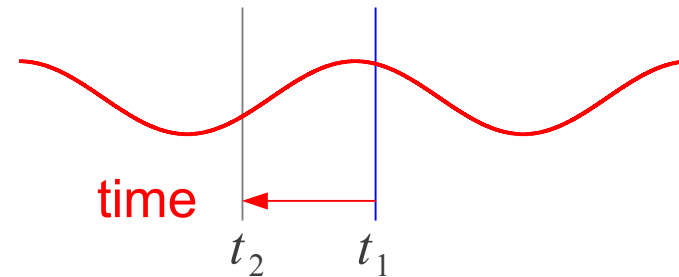


wavelength $\lambda = \frac{2\pi}{k}$

wave number $k = \frac{2\pi}{\lambda}$

$$A_0 \cos(kx_0 - \omega t)$$

At the fixed site of the distance x_0



frequency $f = \frac{\omega}{2\pi}$

period $T = \frac{2\pi}{\omega}$

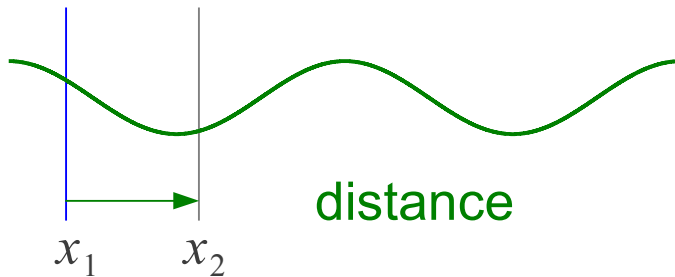
angular frequency $\omega = 2\pi f$

angular frequency $\omega = \frac{2\pi}{T}$

Wave Number, Angular Frequency

$$A_0 \cos(kx - \omega t_0)$$

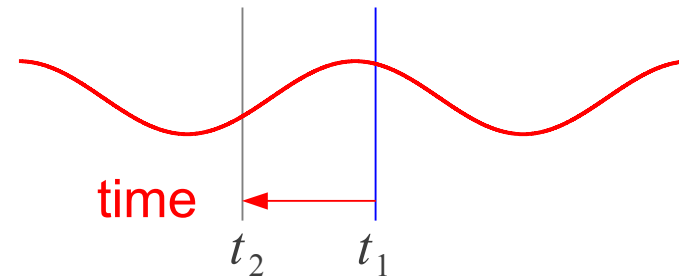
At the snapshot of the time t_0



wave number $k = \frac{2\pi}{\lambda}$
radians per unit distance

$$A_0 \cos(kx_0 - \omega t)$$

At the fixed site of the distance x_0

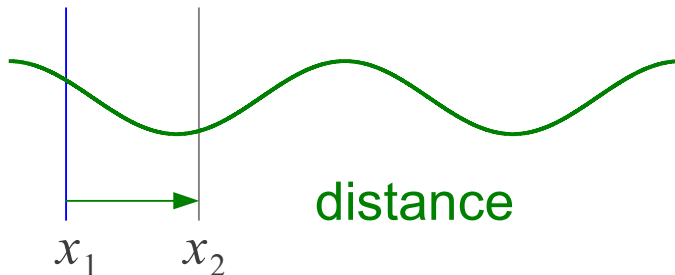


angular frequency $\omega = \frac{2\pi}{T}$
radians per unit time

Phase Velocity (1)

$$A_0 \cos(kx - \omega t_0)$$

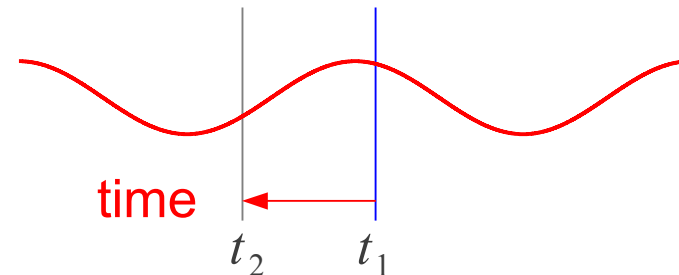
At the snapshot of the time t_0



wave number $k = \frac{2\pi}{\lambda}$
radians per unit distance

$$A_0 \cos(kx_0 - \omega t)$$

At the fixed site of the distance x_0



angular frequency $\omega = \frac{2\pi}{T}$
radians per unit time

Phase Velocity $v_p = \frac{\lambda}{T} = \frac{2\pi/k}{2\pi/\omega} = \frac{\omega}{k}$ $v_p = \frac{\omega}{k}$

Phase Velocity (2)

Phase Velocity $v_p = \frac{\omega}{k}$

$$A \cos(kx - \omega t)$$

Given time t ,  ωt oscillations

Corresponding distance x ,  the same oscillations

$$kx = \omega t$$

$$v_p = \frac{x}{t} = \frac{\omega}{k}$$

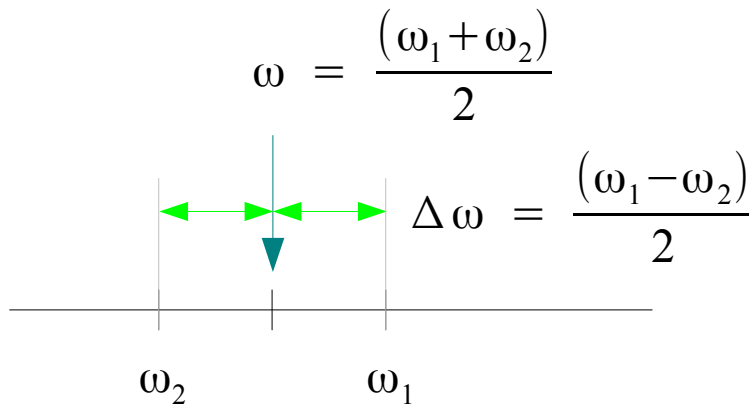
Phase Velocity, Group Velocity

Phase Velocity $v_p = \frac{\omega}{k}$

Group Velocity $v_g = \frac{\partial \omega}{\partial k}$

Group Velocity Explanation (1)

$$\omega_1 > \omega_2$$

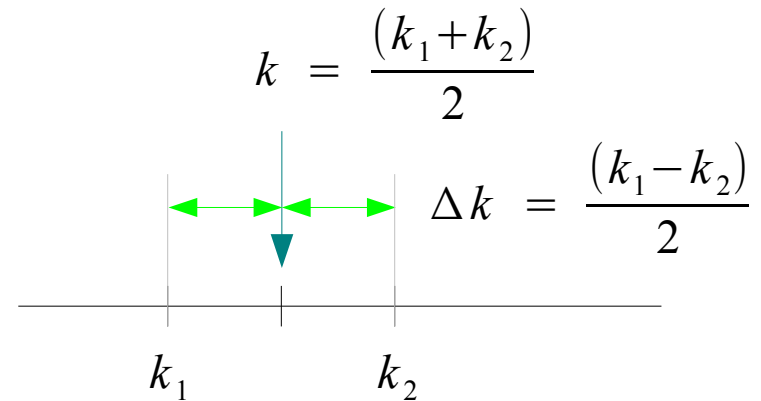


$$\omega_1 = \omega + \Delta\omega$$

$$\omega_2 = \omega - \Delta\omega$$

$$e^{j(k_1x - \omega_1t)}$$

$$k_1 > k_2$$

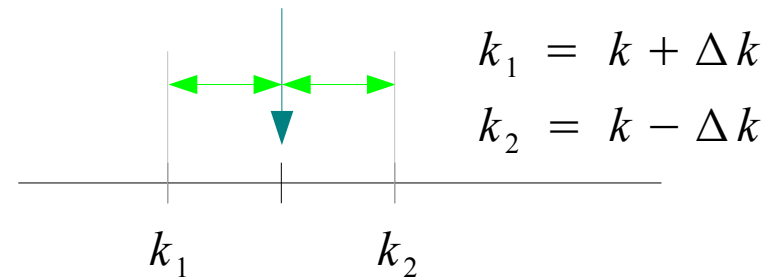
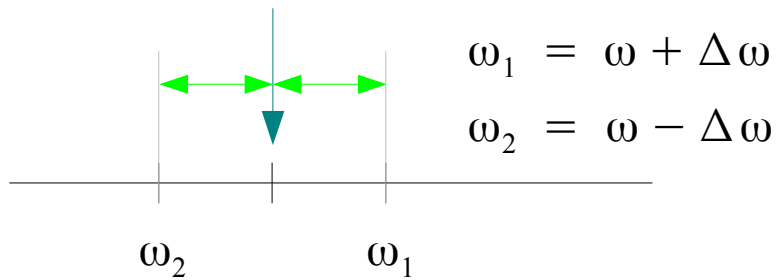


$$k_1 = k + \Delta k$$

$$k_2 = k - \Delta k$$

$$e^{j(k_2x - \omega_2t)}$$

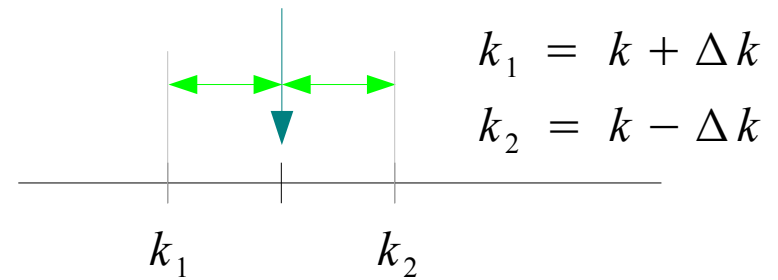
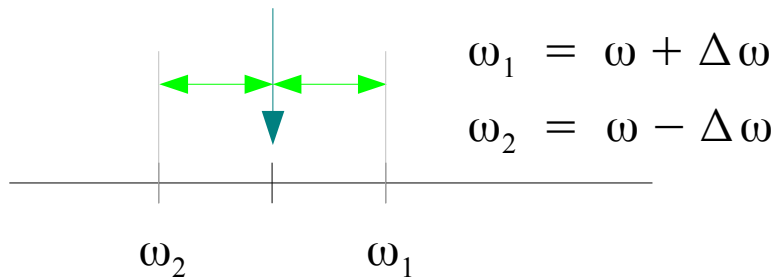
Group Velocity Explanation (2)



$$\begin{aligned}
 & e^{j(k_1 x - \omega_1 t)} + e^{j(k_2 x - \omega_2 t)} \\
 &= e^{j\{(k + \Delta k)x - (\omega + \Delta\omega)t\}} + e^{j\{(k - \Delta k)x - (\omega - \Delta\omega)t\}} \\
 &= e^{j\{(kx - \omega t) + (\Delta kx - \Delta\omega t)\}} + e^{j\{(kx - \omega t) - (\Delta kx - \Delta\omega t)\}} \\
 &= e^{j(kx - \omega t)} \{ e^{j(\Delta kx - \Delta\omega t)} + e^{-j(\Delta kx - \Delta\omega t)} \} \\
 &= \underline{2 \cos(\Delta kx - \Delta\omega t)} e^{j(kx - \omega t)}
 \end{aligned}$$

Envelope

Group Velocity Explanation (3)



$$e^{j(k_1 x - \omega_1 t)} + e^{j(k_2 x - \omega_2 t)}$$

$$= \underline{2 \cos(\Delta k x - \Delta \omega t)} e^{j(k x - \omega t)}$$

Envelope

$\Delta k \ll k_1, k_2 \rightarrow$ *Small Wave number* \rightarrow *Long Wavelength*

$\Delta \omega \ll \omega_1, \omega_2 \rightarrow$ *Small Frequency* \rightarrow *Long Period*

Envelope Velocity

$$v_g = \frac{\Delta \omega}{\Delta k}$$

Group Velocity

$$v_g = \frac{d \omega}{d k}$$

Group Velocity & Fourier Transform (1)

A periodic function

$$f(\theta) = \sum_k a_k \sin(k\theta) + b_k \cos(k\theta)$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(\theta) \sin(k\theta) d\theta \quad b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(\theta) \cos(k\theta) d\theta$$

A non-periodic function

$$f(x) = \int_{-\infty}^{+\infty} F(k) e^{jkx} dk$$

$$F(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{-jkx} dx$$

Group Velocity & Fourier Transform (2)

A non-periodic function

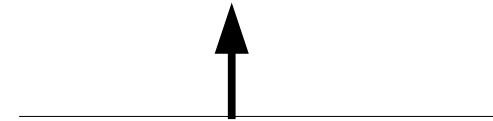
$$f(x) = \int_{-\infty}^{+\infty} F(k) e^{jkx} dk$$

$$F(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{-jkx} dx$$

Infinite number of sine waves
Well-defined wavelength
Well-defined k (wave number)

Lots of sine waves of diff wavelengths
Short wave packet

A long wave packet



A small spread in k
Sharp peak

Group Velocity & Fourier Transform (3)

A non-periodic function

$$f(x) = \int_{-\infty}^{+\infty} F(k) e^{jkx} dk \quad F(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{-jkx} dx$$

A long wave packet

*A small spread in k
Sharp peak at k_0*

At some initial time $t = t_0$

$$f(x, 0) = \int_{-\infty}^{+\infty} F(x) e^{jkx} dk$$

Taylor expansion to 1st order

$$\omega(k) = \omega_0 + \frac{d\omega}{dk}(k - k_0)$$

After time t

$$f(x, t) = \int_{-\infty}^{+\infty} F(x) e^{j(kx - \omega(k)t)} dk$$

$f(x, t)$

$$= \int_{-\infty}^{+\infty} F(x) e^{j(kx - (\omega_0 + \frac{d\omega}{dk}(k - k_0))t)} dk$$

$\omega(k)$ *different wavelength components
have different frequencies*

Group Velocity & Fourier Transform (3)

A long wave packet

*A small spread in k
Sharp peak at k_0*

After time t

$$f(x, t) = \int_{-\infty}^{+\infty} F(k) e^{j(kx - \omega(k)t)} dk$$

Taylor expansion to 1st order

$$\omega(k) = \omega_0 + \frac{d\omega}{dk}(k - k_0)$$

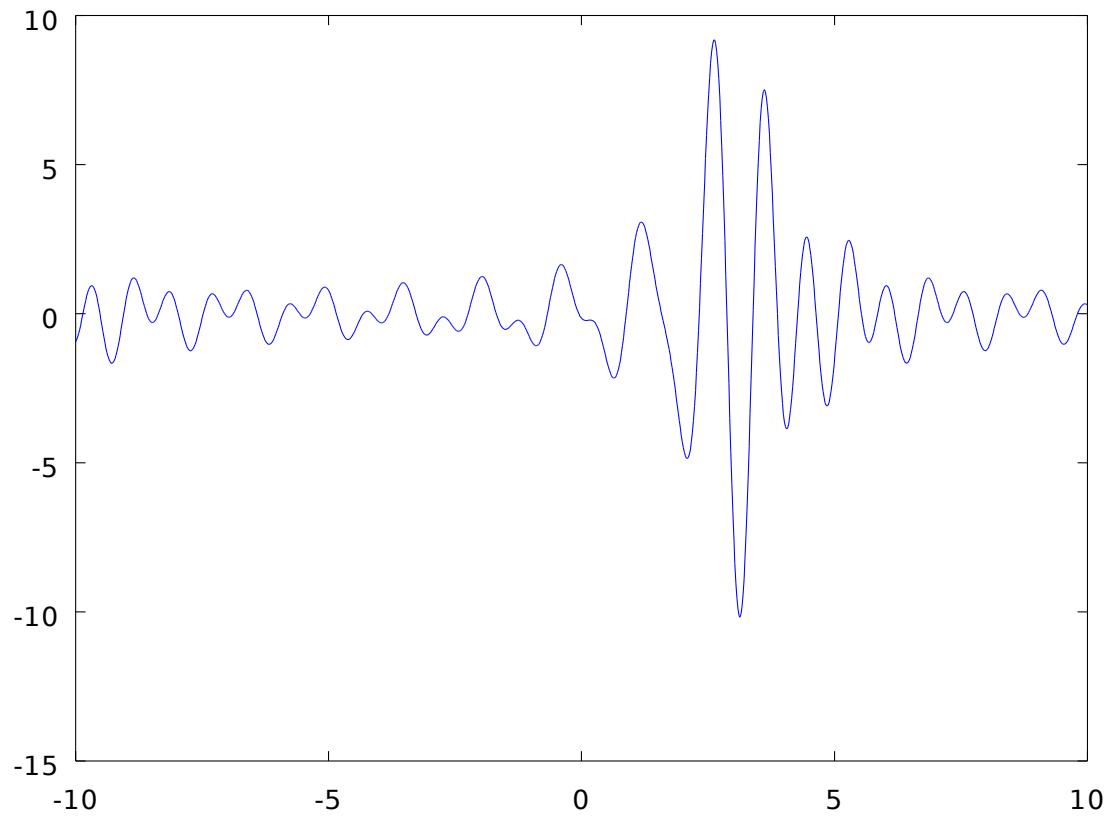
$$\begin{aligned} f(x, t) &= \int_{-\infty}^{+\infty} F(k) e^{j(kx - (\omega_0 + \frac{d\omega}{dk}(k - k_0))t)} dk \\ &= \int_{-\infty}^{+\infty} F(k) e^{j(k_0x + kx - k_0x - (\omega_0 + \frac{d\omega}{dk}(k - k_0))t)} dk \\ &= e^{j(k_0x - \omega_0t)} \int_{-\infty}^{+\infty} F(k) e^{j((k - k_0)x - \frac{d\omega}{dk}(k - k_0)t)} dk \\ &= e^{j(k_0x - \omega_0t)} \int_{-\infty}^{+\infty} F(k) e^{j(k - k_0)\left(x - \frac{d\omega}{dk}t\right)} dk \end{aligned}$$

Group Velocity

$$v_g = \frac{d\omega}{dk}$$



$$\left(x - \frac{d\omega}{dk}t\right)$$



```
x = linspace(-10, +10, 1000);
```

```
y = zeros(1, 1000);
```

```
for k= 1.0:0.1:2.0
```

```
    y = y + cos(4*k*(x-k));
```

```
end
```

```
plot(x, y);
```


References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] <http://www.mathpages.com/>, Phase, Group, and Signal Velocity
- [4] R. Barlow, www.hep.man.ac.uk/u/roger/PHYS10302/lecture15.pdf
- [5] P. Hofmann, www.philiphofmann.net/book_material/notes/groupphasevelocity.pdf