

Sampler Spectra (8B)

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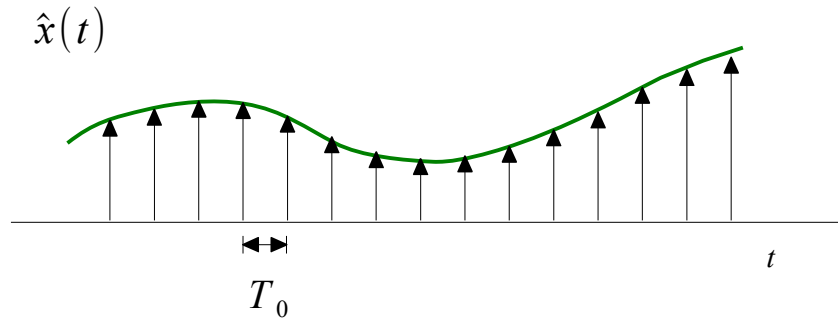
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Sampler

Ideal Sampling

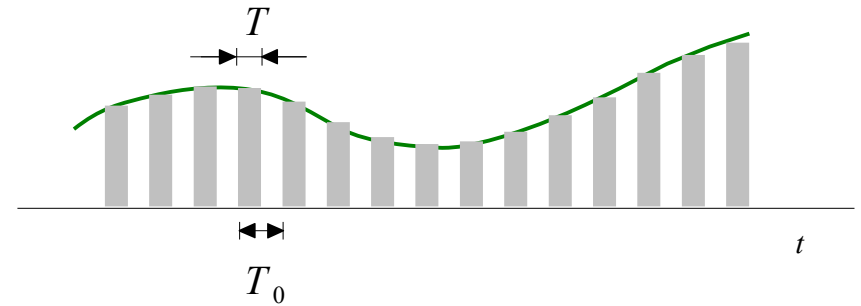


$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0) \delta(t - nT_0)$$

↓ CTFT



Practical Sampling



$$\hat{x}(t) \approx \sum_{n=-\infty}^{+\infty} x(nT_0) p(t - nT_0)$$

↓ CTFT



Square Wave CTFS

$$C_k = \frac{1}{T_0} \cdot \frac{\sin(k \omega_0 T/2)}{k \omega_0/2} = \frac{1}{T_0} \cdot \frac{\sin(T k \omega_0/2)}{k \omega_0/2}$$

$$C_k = 0 \quad \rightarrow \quad \sin(k \omega_0 T/2) = 0$$

$$\sin\left(k \frac{2\pi T}{T_0} \frac{T}{2}\right) = 0 \quad \rightarrow \quad \sin(\pm n \pi) = 0$$

$$k = \pm n \frac{T_0}{T} \quad \rightarrow \quad \omega = \pm n \frac{T_0}{T} \omega_0$$

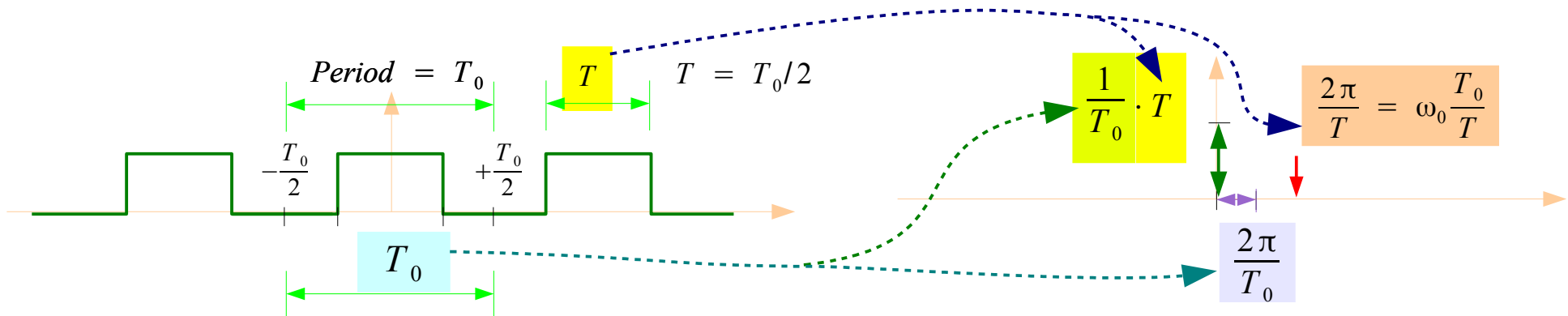
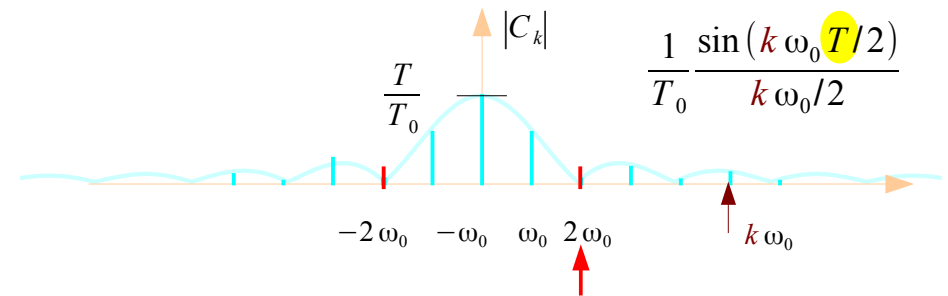
$$C_0 = \lim_{k \rightarrow 0} \frac{1}{T_0} \cdot \frac{\sin(T k \omega_0/2)}{k \omega_0/2}$$

$$C_0 = \lim_{k \rightarrow 0} \frac{1}{T_0} \cdot \frac{(T \omega_0/2) \cos(T k \omega_0/2)}{\omega_0/2} = \frac{T}{T_0}$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_0 \frac{T_0}{T} = \frac{2\pi}{T}$$

$$\omega_0 T = 2\pi \frac{T}{T_0}$$



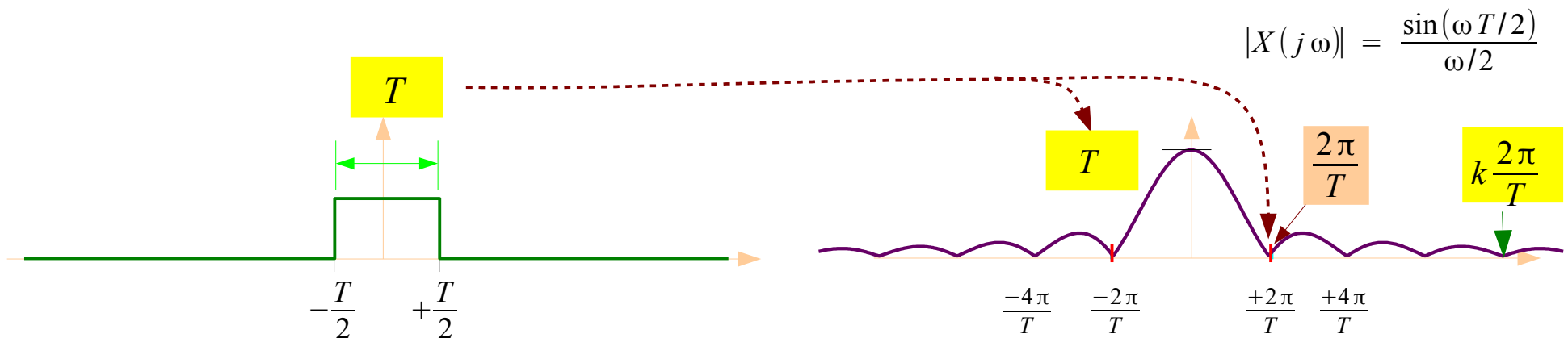
CTFT of a Rect(t/T) function

Continuous Time Fourier Transform

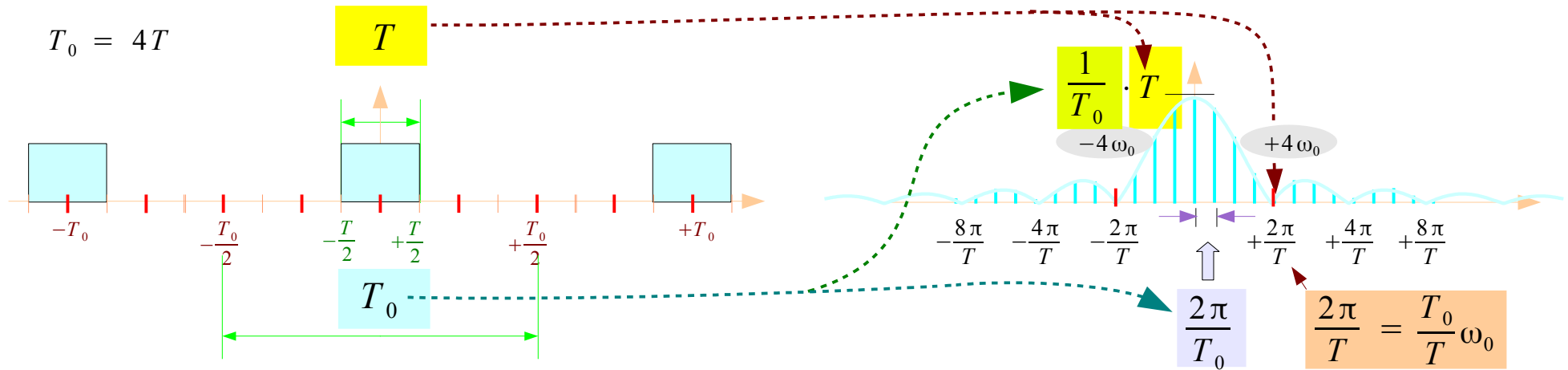
Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

$$X(j\omega) = \int_{-T/2}^{+T/2} e^{-j\omega t} dt = \frac{\sin(\omega T/2)}{\omega/2}$$



CTFT of a Rect(t/T) function



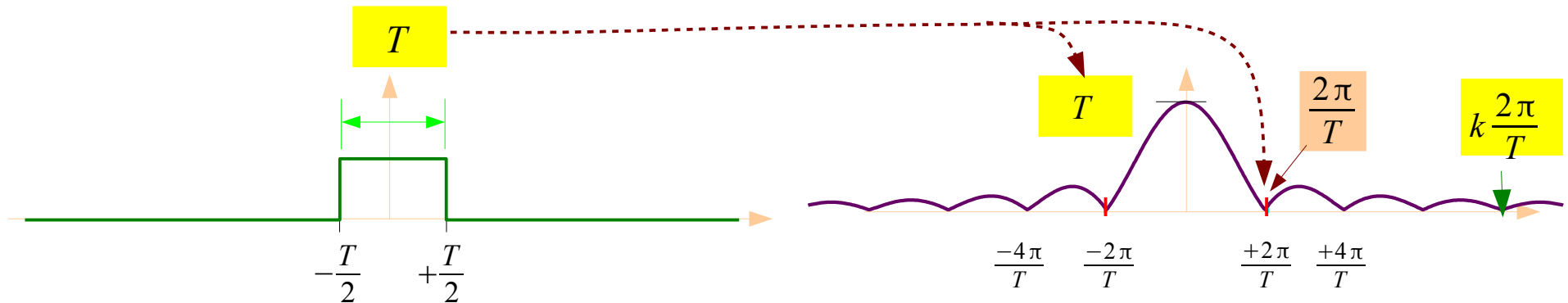
$$C_k T_0 = \frac{\sin(k \omega_0 T/2)}{k \omega_0/2}$$

$$C_k = \frac{1}{T_0} \frac{\sin(k \omega_0 T/2)}{k \omega_0/2}$$

$$X(j\omega) = \lim_{k \omega_0 \rightarrow \omega} \frac{\sin(k \omega_0 T/2)}{k \omega_0/2} = \frac{\sin(\omega T/2)}{\omega/2}$$

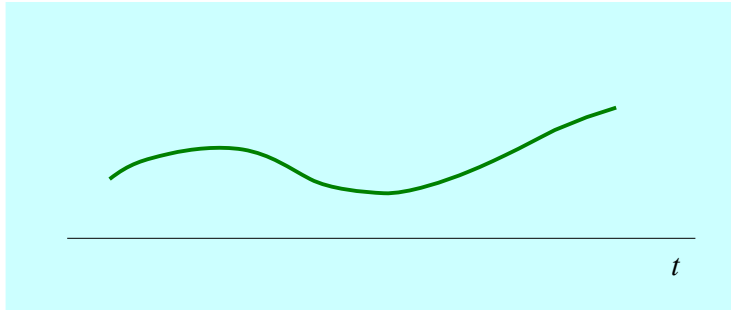
$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

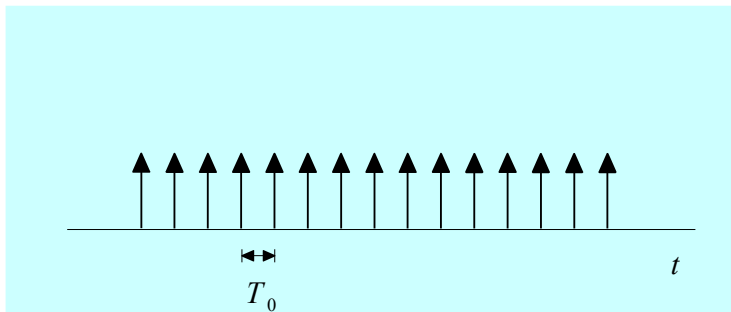


Sampling (1)

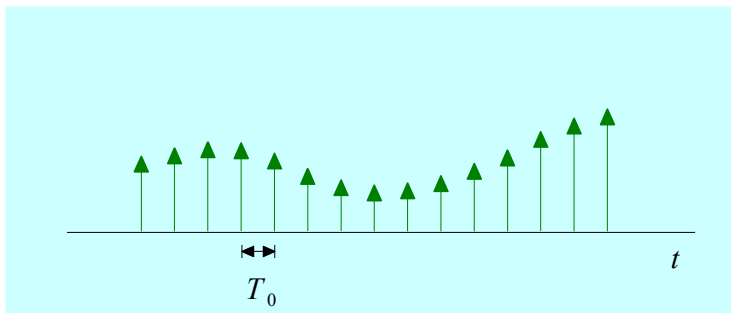
Ideal Sampling



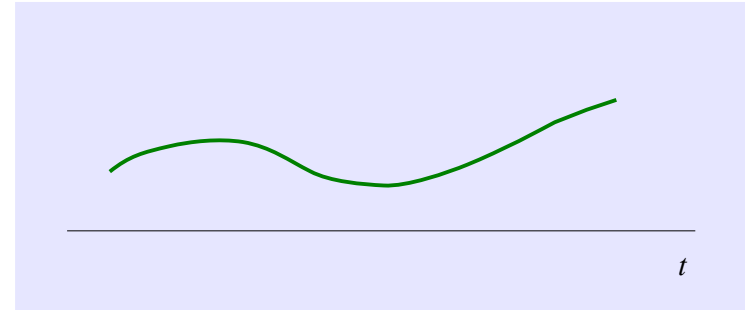
X



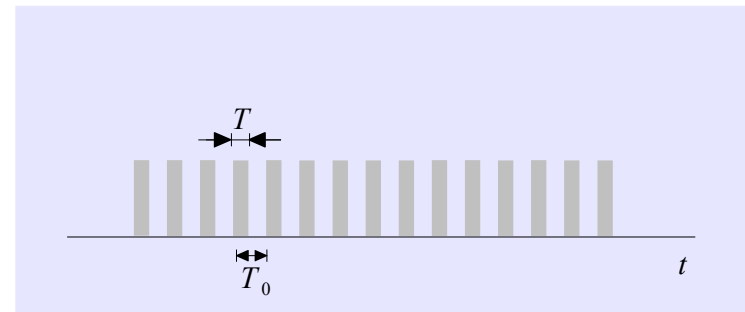
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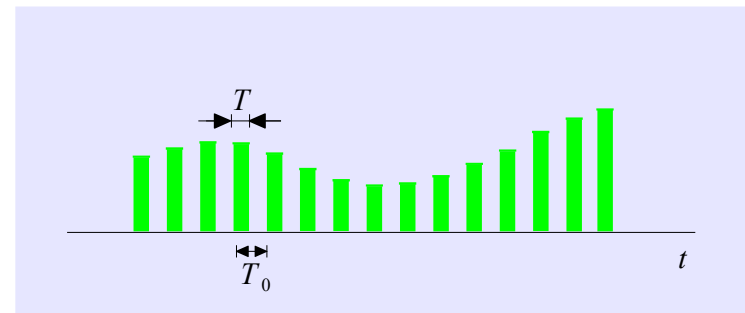
Practical Sampling



X

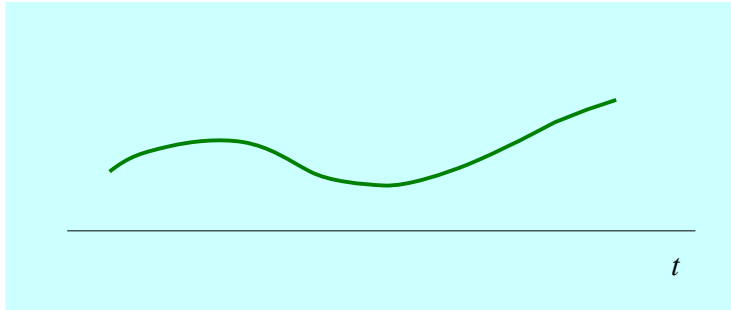


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Sampling (2)

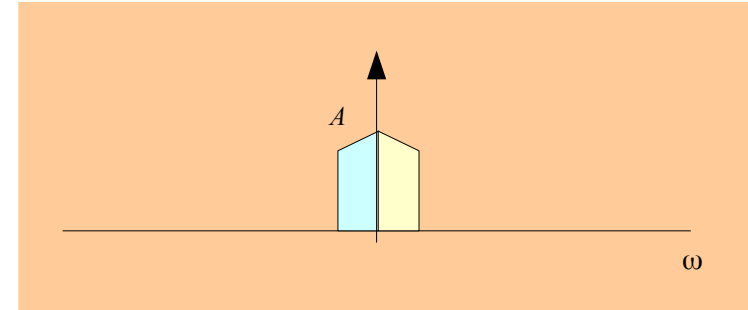
Ideal Sampling



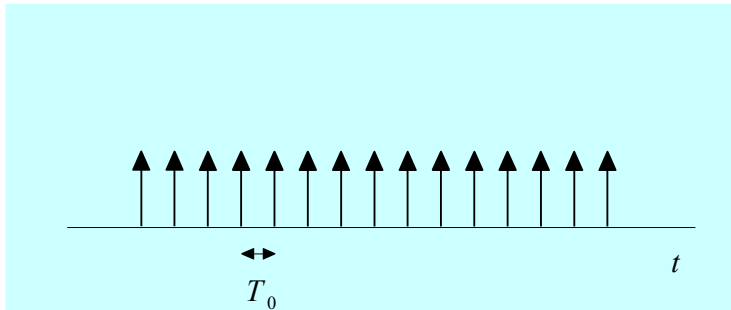
CTFT



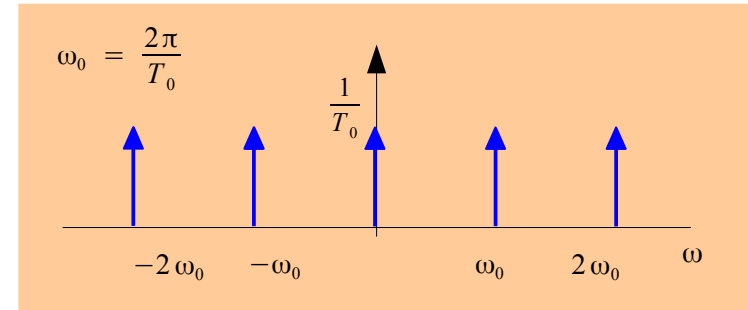
Frequency Domain



X

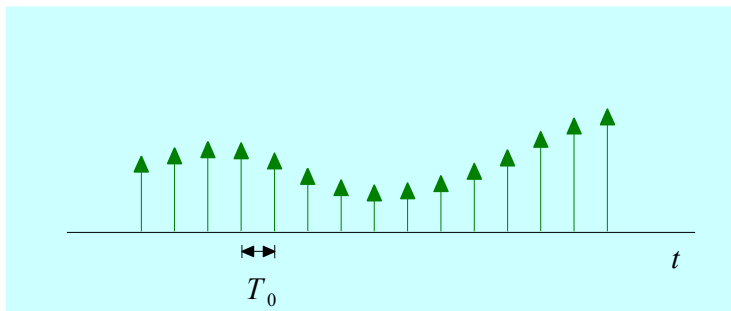


CTFT

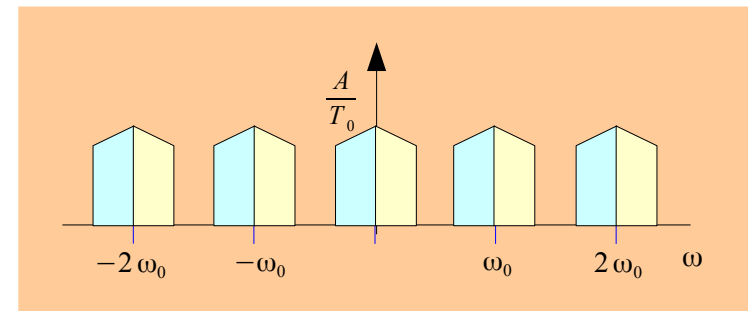


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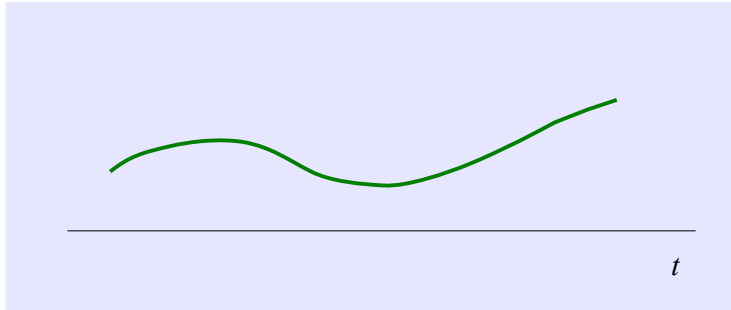
CTFT



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Sampling (3)

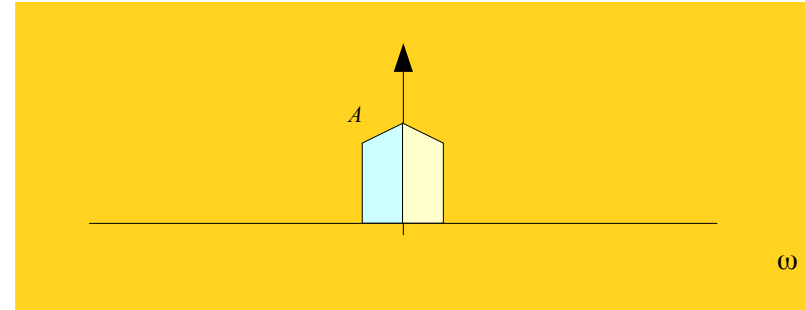
Practical Sampling



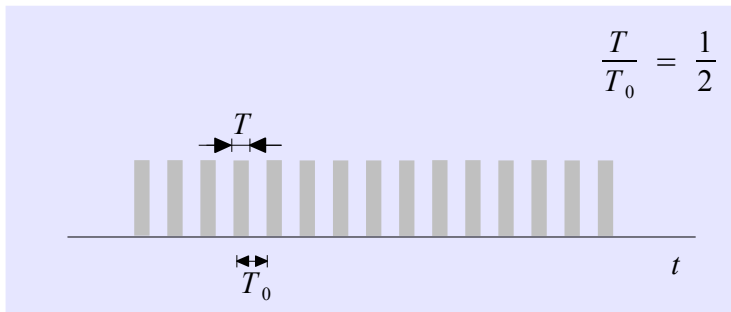
CTFT



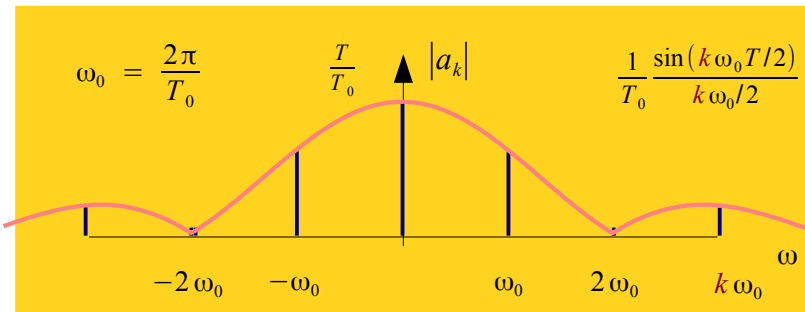
Frequency Domain



X

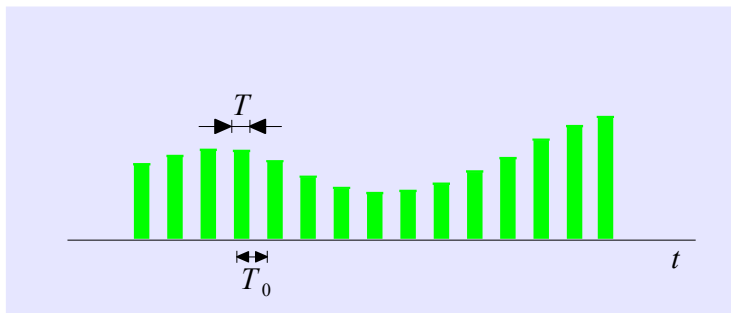


CTFT

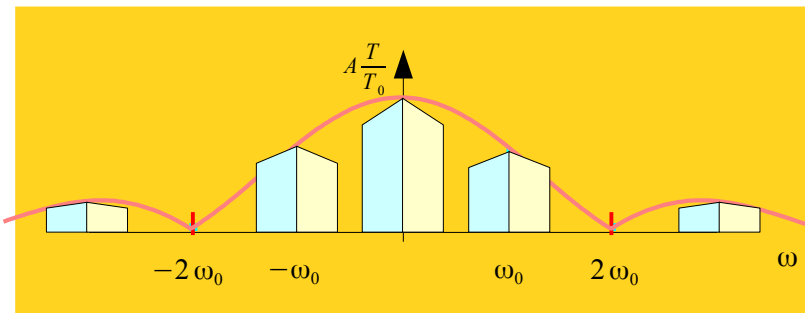


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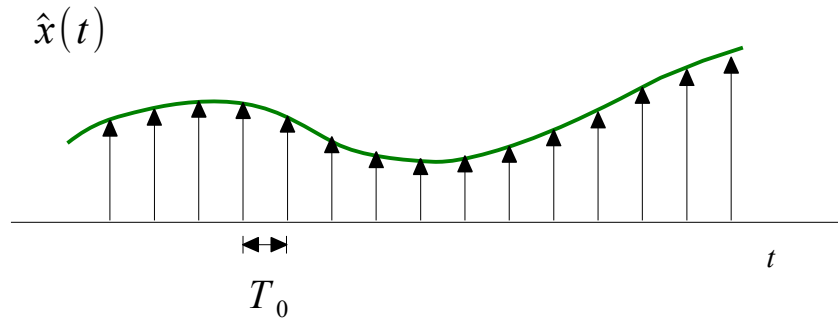
CTFT



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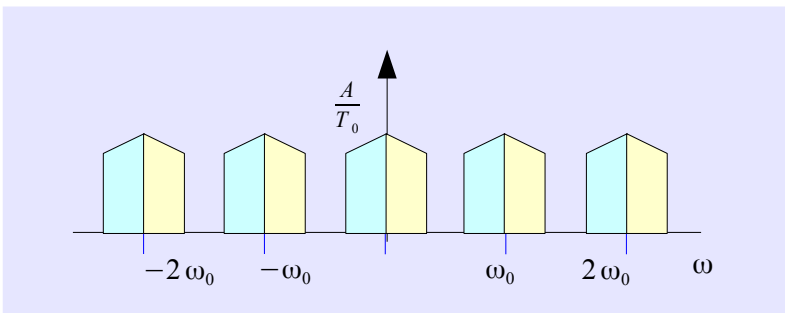
Sampling CTFT

Ideal Sampling

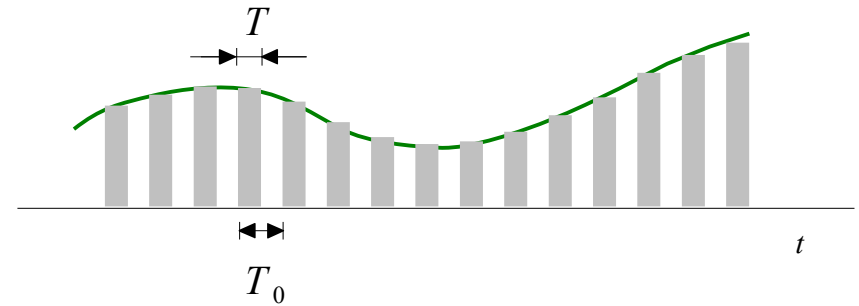


$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0) \delta(t-nT_0)$$

↓ CTFT

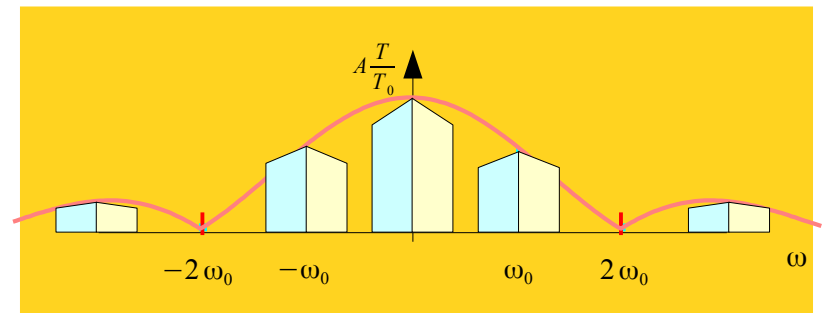


Practical Sampling

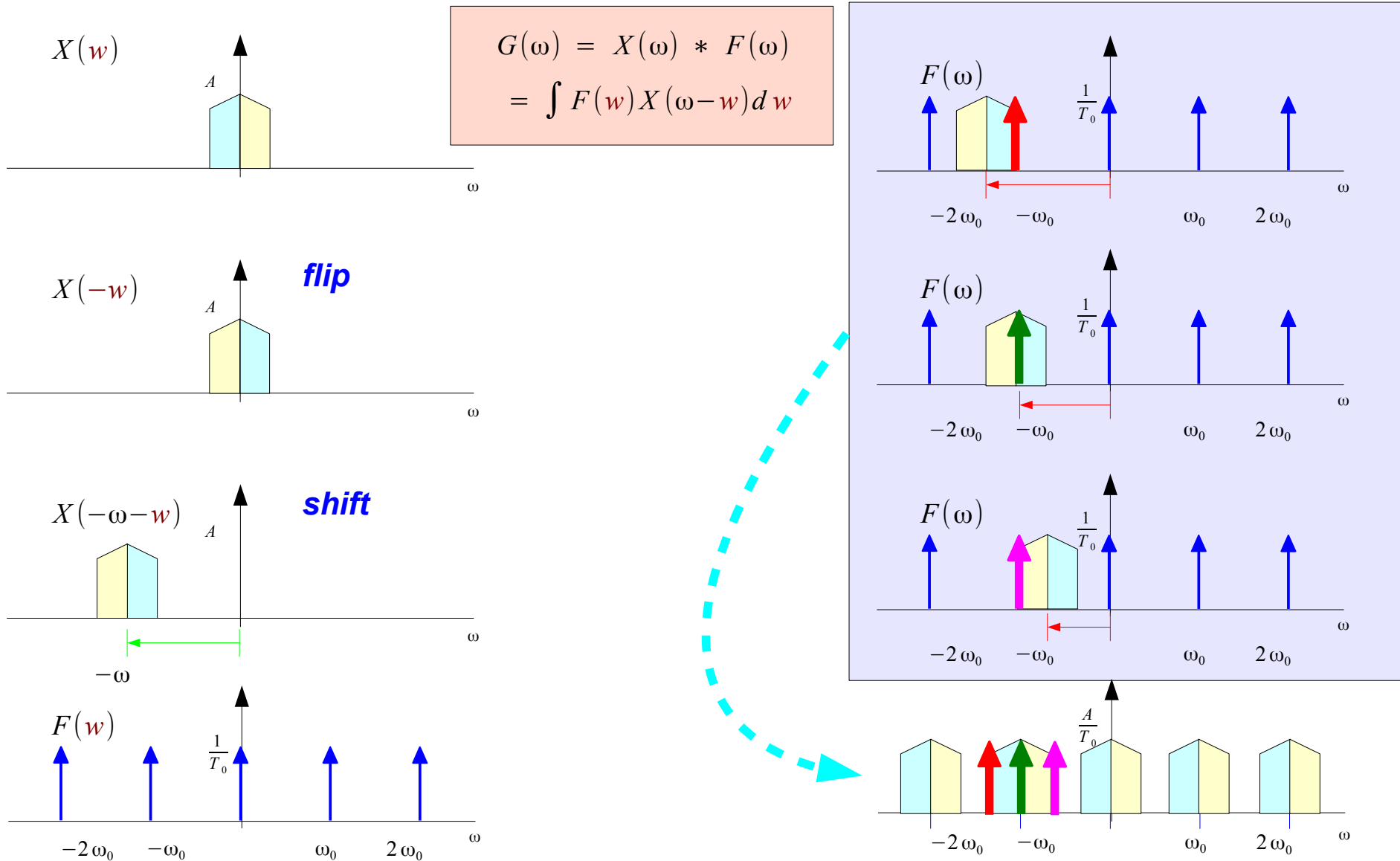


$$\hat{x}(t) \approx \sum_{n=-\infty}^{+\infty} x(nT_0) p(t-nT_0)$$

↓ CTFT

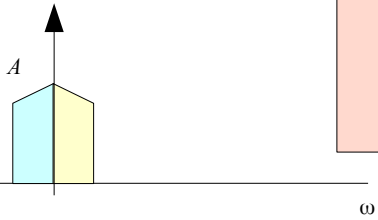


Convolution with Impulse Train



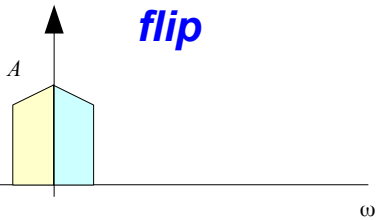
Convolution with Sinc Impulse Train

$X(\omega)$



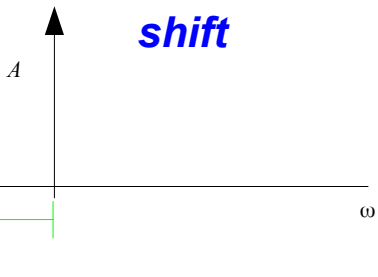
$$G(\omega) = X(\omega) * F(\omega) = \int F(\omega') X(\omega - \omega') d\omega'$$

$X(-\omega)$



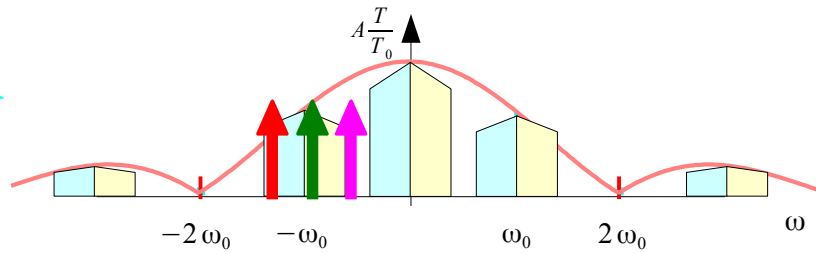
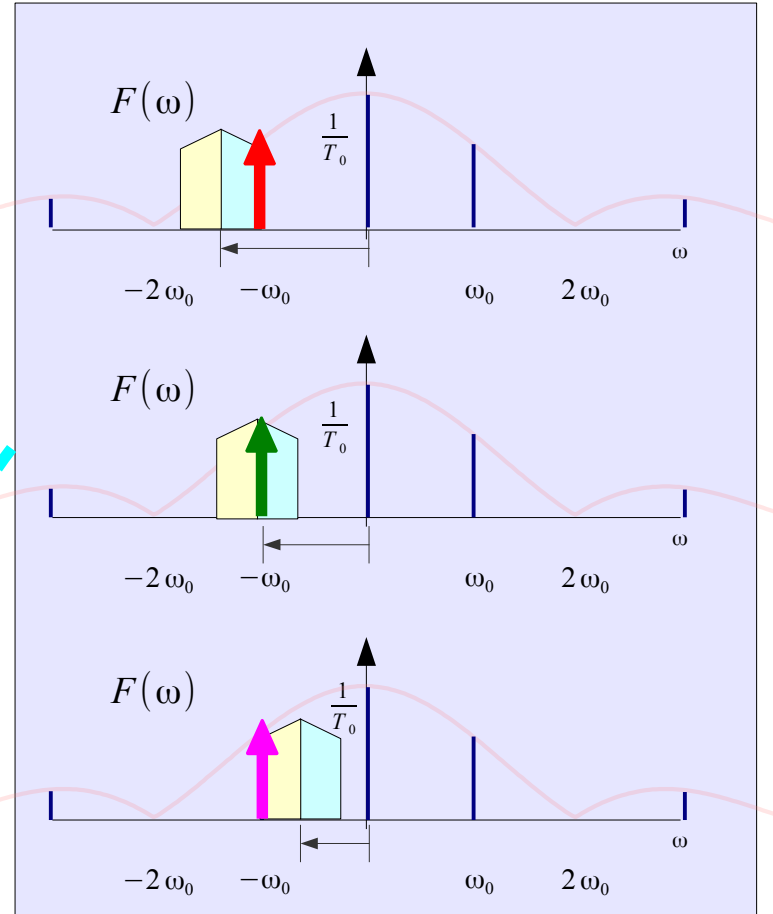
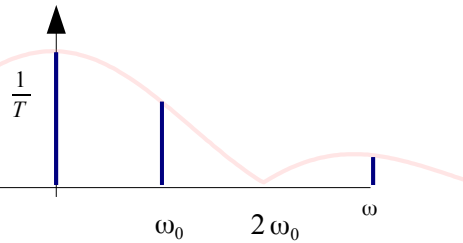
flip

$X(-\omega - \omega')$

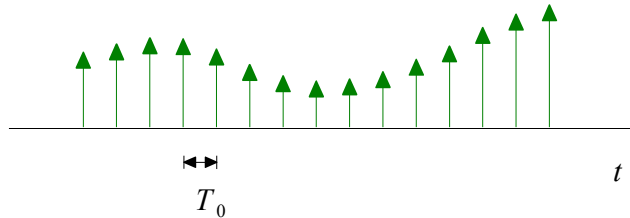


shift

$F(\omega)$

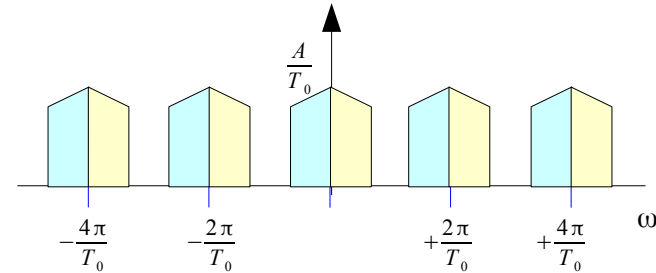


CTFT of Sampled Signal



$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0) \delta(t-nT_0)$$

CTFT



$$\begin{aligned} \hat{X}(f) &= \int_{-\infty}^{+\infty} \hat{x}(t) e^{-j2\pi f t} dt \\ &= \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} x(nT_0) \delta(t-nT_0) e^{-j2\pi f t} dt \\ &= \sum_{n=-\infty}^{+\infty} x(nT_0) e^{-j2\pi f T_0 n} \end{aligned}$$

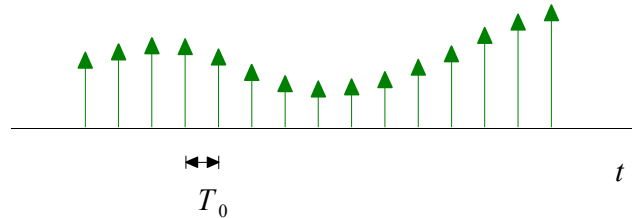
$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0) \delta(t-nT_0)$$

CTFT



$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT_0) e^{-j2\pi f T_0 n}$$

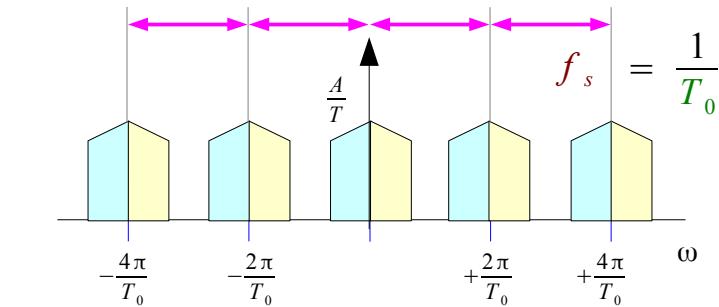
Periodicity in Frequency



$$f_s = \frac{1}{T_0} \quad 2\pi f_s = \frac{2\pi}{T_0} = \omega_0$$

$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0) \delta(t - nT_0)$$

CTFT



$$2\pi f = \omega$$

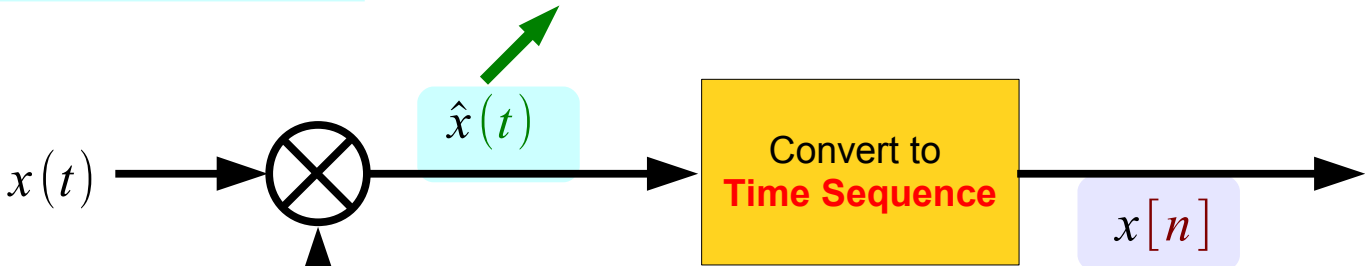
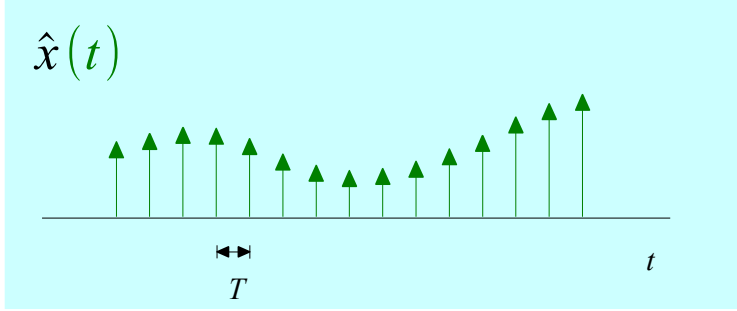
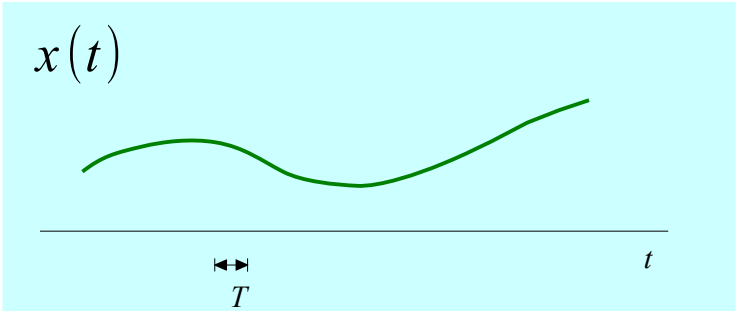
$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT_0) e^{-j2\pi f T_0 n}$$

$$e^{-j2\pi(f+f_s)T_0 m} = e^{-j2\pi(f)T_0 n} \leftarrow f_s T_0 = 1$$

1 / Period = Sampling Frequency f_s

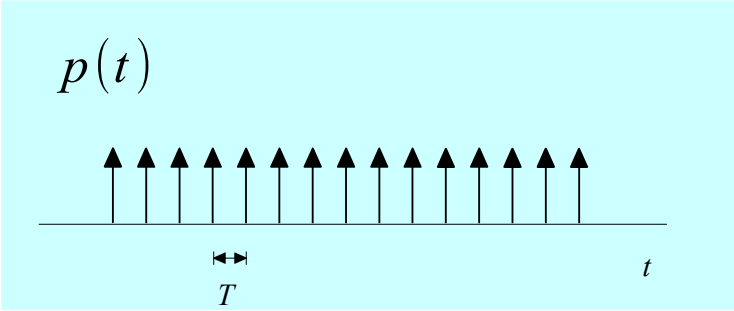
$$\hat{X}(f) = \hat{X}(f + f_s)$$

Time Sequence

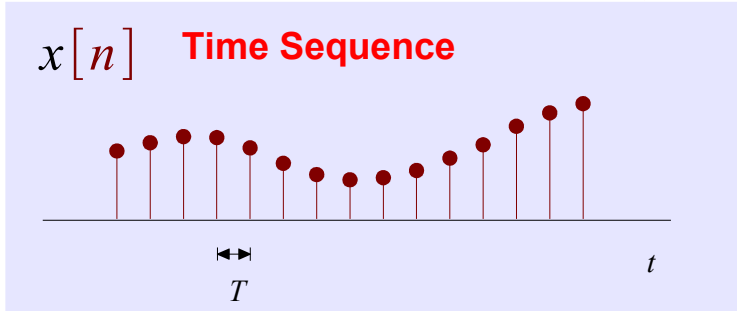


Ideal Sampling

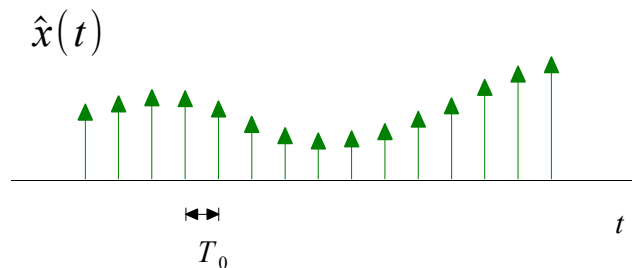
$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$



T Sampling Period

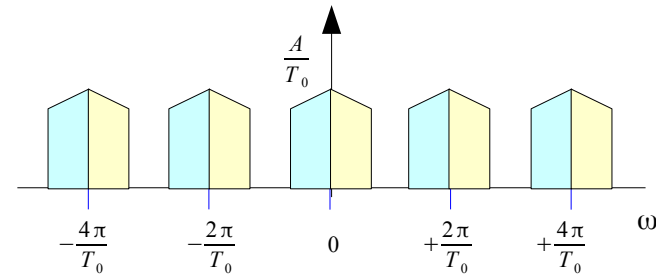


DTFT of a Time Sequence

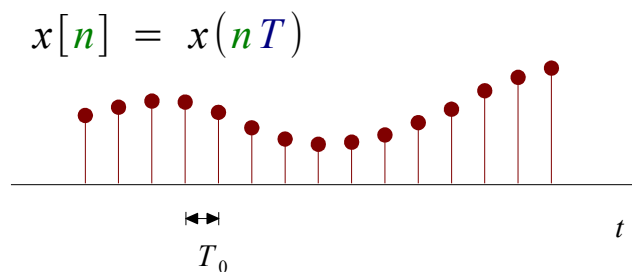


$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0) \delta(t - nT_0)$$

CTFT

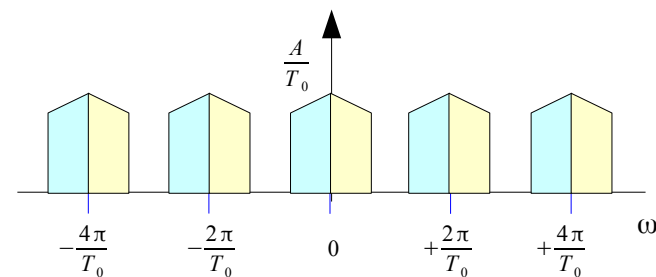


$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT_0) e^{-j2\pi f T_0 n}$$



$$x[n], \text{ Sampling Period } T_0$$

DTFT

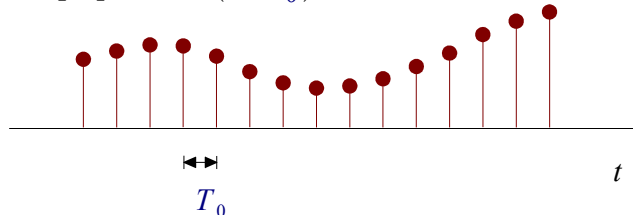


$$X(f) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi f T_0 n}$$

Here, $X(f)$ does not denote the CTFT of $x(t)$

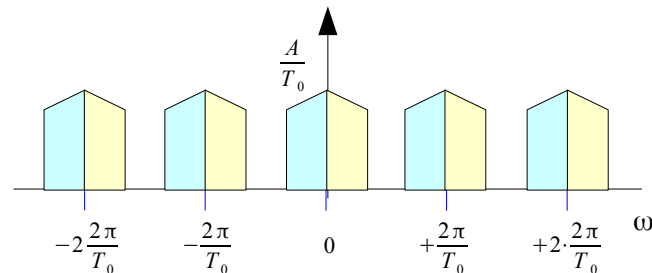
Discrete Time Fourier Transform (1)

$$x[n] = x(nT_0)$$



$x[n]$, Sampling Period T_0

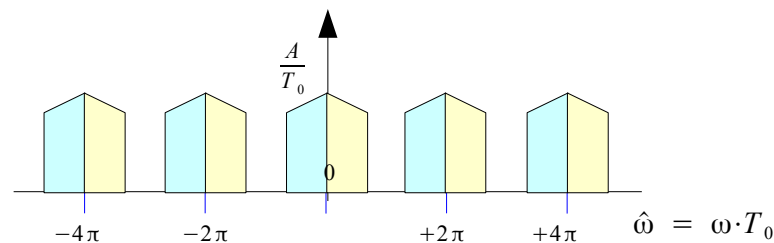
DTFT



$$X(f) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi f T_0 n}$$

Normalized Angular Frequency

$$2\pi f T_0 = \frac{2\pi f}{1/T_0} = 2\pi \frac{f}{f_s} = \hat{\omega}$$



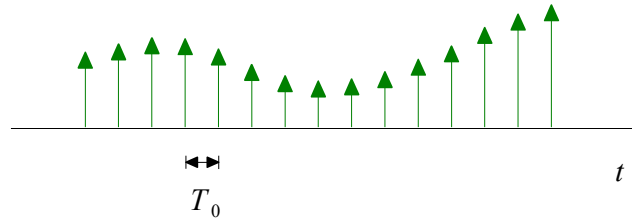
$x[n]$, Sampling Period T_0

DTFT



$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega} n}$$

Discrete Time Fourier Transform (2)



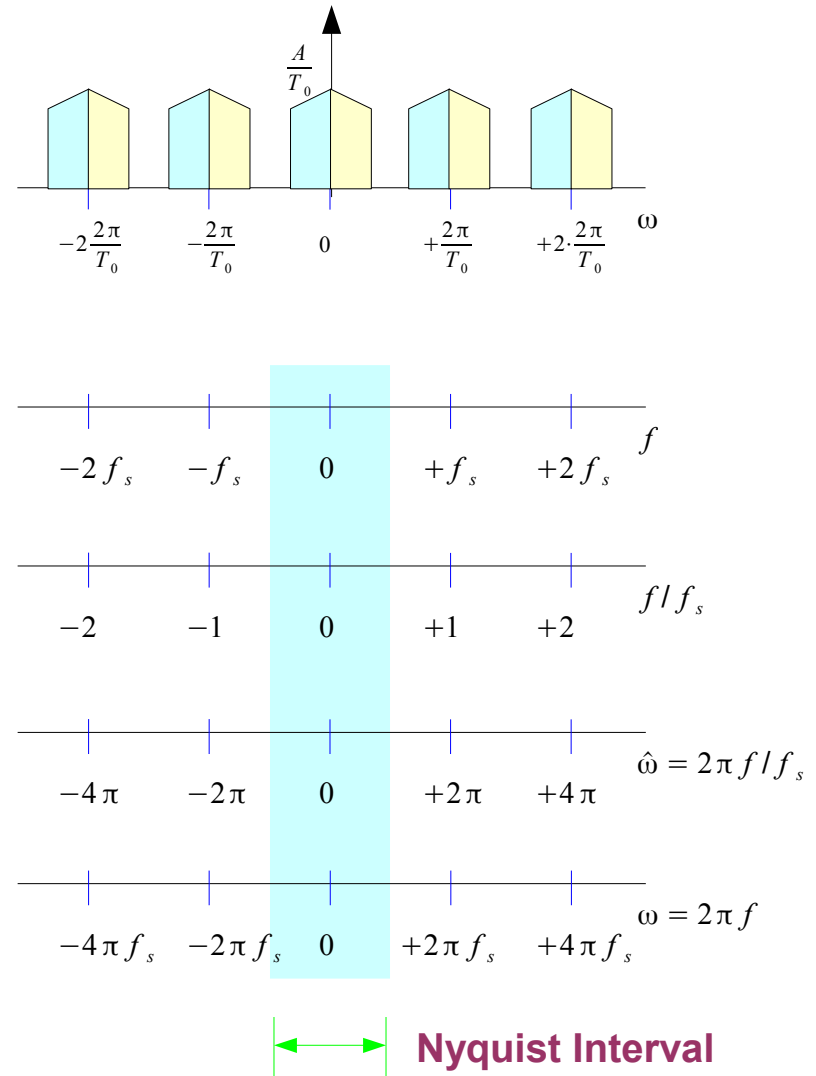
$$f_s = \frac{1}{T} \quad 2\pi f_s = \frac{2\pi}{T} = \omega_0$$

$$X(f) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi f T n}$$

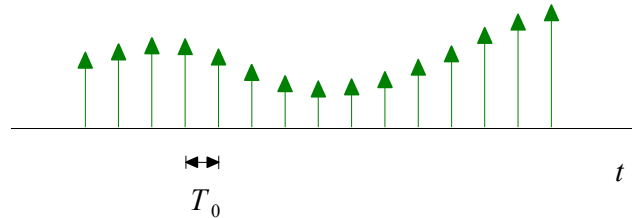
Normalized Angular Frequency

$$2\pi f T = \frac{2\pi f}{1/T} = 2\pi \frac{f}{f_s} = \hat{\omega}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega} n}$$



Discrete Time Fourier Transform (3)



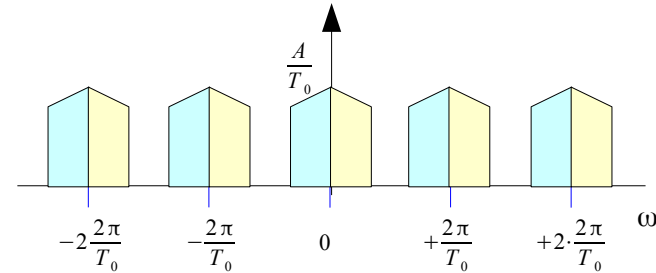
$$f_s = \frac{1}{T_0} \quad 2\pi f_s = \frac{2\pi}{T_0} = \omega_0$$

$$X(f) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi f T_0 n}$$

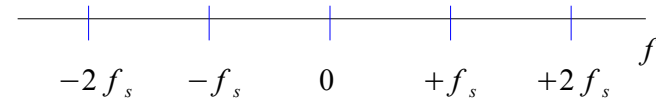
Normalized Angular Frequency

$$2\pi f T_0 = \frac{2\pi f}{1/T_0} = 2\pi \frac{f}{f_s} = \hat{\omega}$$

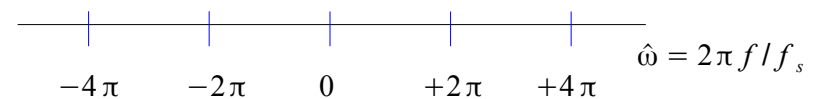
$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega} n}$$



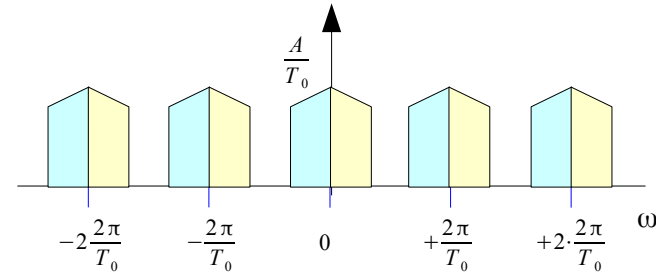
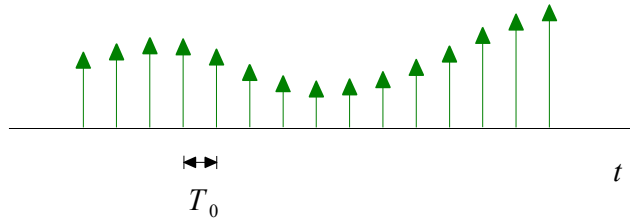
$\hat{X}(f)$ Absolute Frequency



$\hat{X}(e^{j\hat{\omega}})$ Normalized Angular Frequency
unit circle →
emphasize the periodic nature



Fourier Series Interpretation



$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0) \delta(t - nT_0)$$

CTFT



$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT_0) e^{-j2\pi f T_0 n}$$

$$x(nT_0) = \frac{1}{f_s} \int_{-f_s/2}^{+f_s/2} \hat{X}(f) e^{+j2\pi f T_0 n} df$$

CTFS



$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT_0) e^{-j2\pi f T_0 n}$$

$$= \int_{-\pi}^{+\pi} \hat{X}(\omega) e^{+j\omega n} \frac{d\omega}{2\pi}$$

$$\omega = 2\pi f / f_s \quad \frac{df}{f_s} = \frac{d\omega}{2\pi}$$

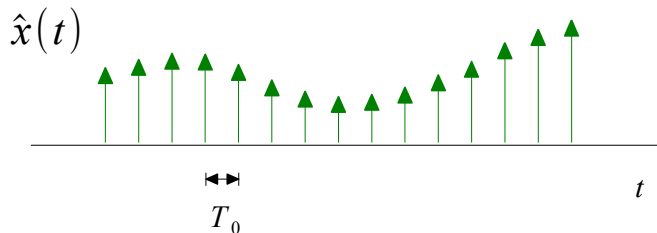
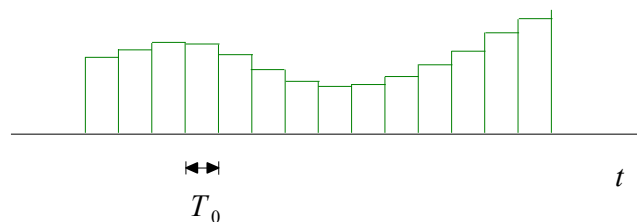
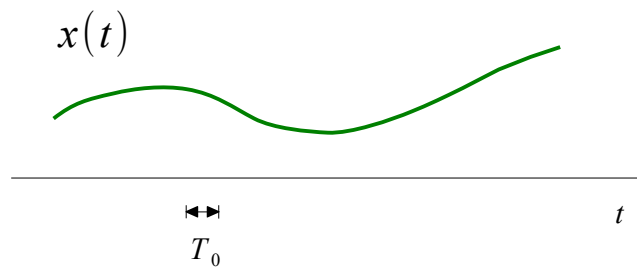
Fourier Series Coefficients $x(nT_0)$

$\hat{X}(f)$ **Continuous Periodic Function**

View as a Fourier Series Expansion

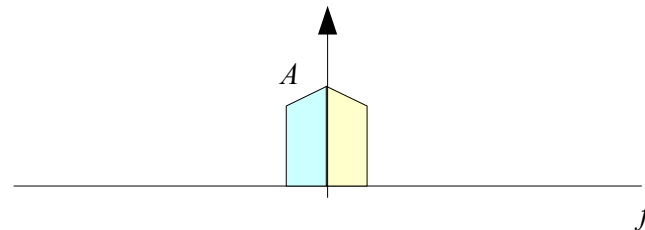
Numerical Approximation

$$X(f) = \lim_{T_0 \rightarrow 0} T_0 \cdot \hat{X}(f)$$



$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0) \delta(t - nT_0)$$

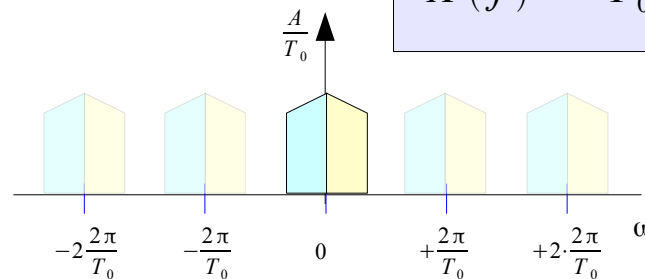
CTFT



$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{+j2\pi f t} dt$$

$$\approx \sum_{n=-\infty}^{+\infty} x(nT_0) e^{-j2\pi f T_0 n} \cdot T_0$$

$$X(f) \approx T_0 \cdot \hat{X}(f)$$



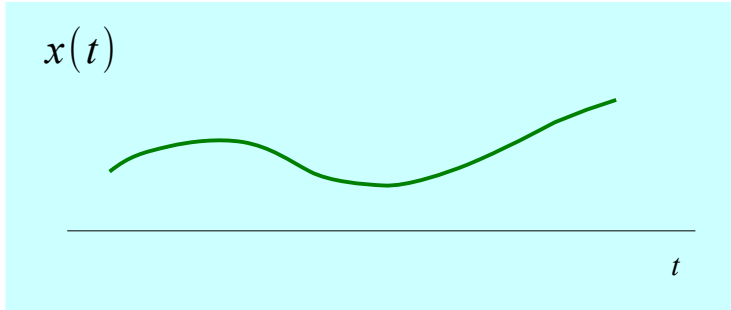
CTFT



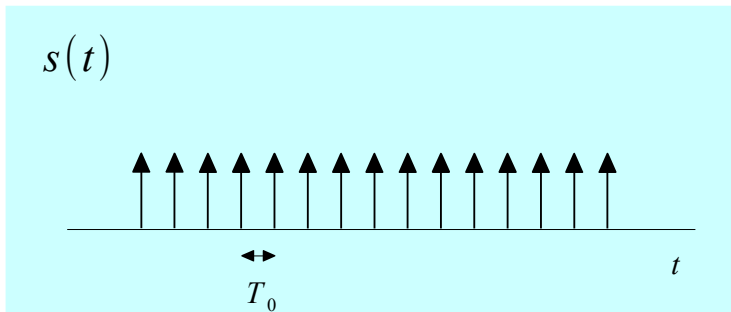
$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT_0) e^{-j2\pi f T_0 n}$$

Spectrum Replication (1)

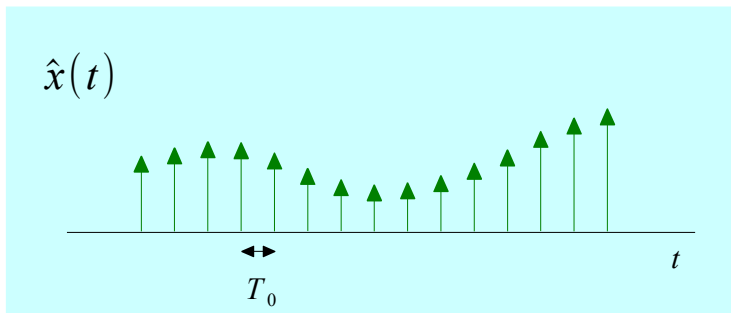
Ideal Sampling



X



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$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0) \delta(t-nT_0)$$

$$\begin{aligned} s(t) &= \sum_{n=-\infty}^{+\infty} \delta(t-nT_0) \\ &= \frac{1}{T_0} \sum_{m=-\infty}^{+\infty} e^{+j2\pi m f_s t} \end{aligned}$$

$$\hat{x}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} x(t) e^{+j2\pi m f_s t}$$

Shift Property



$$\hat{X}(f) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} X(f-m f_s)$$

Spectrum Replication (2)

$$S(f) = \frac{1}{T} \sum_{m=-\infty}^{+\infty} \delta(f - m f_s)$$

Convolution in Frequency

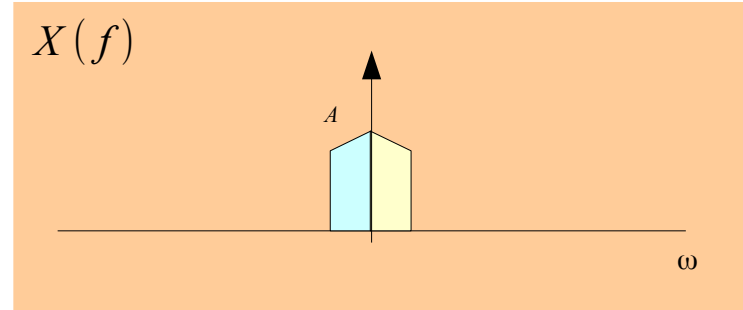
$$\hat{X}(f) = X(f) * S(f)$$

$$= \int_{-\infty}^{+\infty} X(f - f') S(f') d f'$$

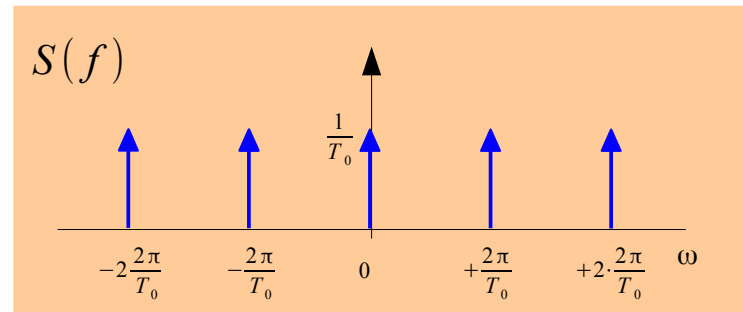
$$= \frac{1}{T_0} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(f - f') \delta(f' - m f_s) d f'$$

$$\hat{X}(f) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} X(f - n f_s)$$

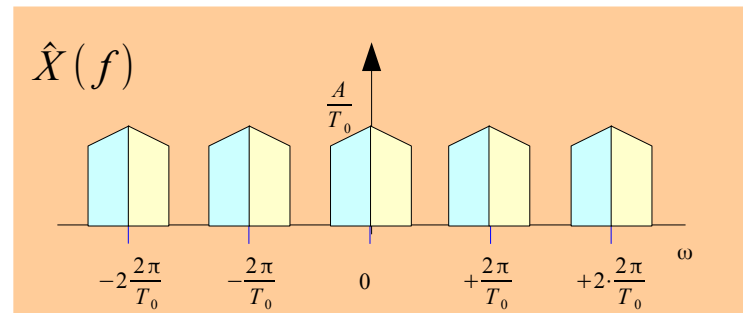
Frequency Domain



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