

Derivatives (2A)

- Partial Derivative
- Directional Derivative
- Tangent Planes and Normal Lines

Copyright (c) 2012 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Partial Derivatives

Function of one variable $y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Function of two variable $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

treating y as a constant

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

treating x as a constant

Partial Derivatives Notations

Function of one variable $y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Function of two variable $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = z_x = f_x$$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

treating y as a constant

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = z_y = f_y$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

treating x as a constant

Higher-Order & Mixed Partial Derivatives

Second-order Partial Derivatives

$$\frac{\partial^2 z}{\partial \mathbf{x}^2} = \frac{\partial}{\partial \mathbf{x}} \left(\frac{\partial z}{\partial \mathbf{x}} \right)$$

$$\frac{\partial^2 z}{\partial \mathbf{y}^2} = \frac{\partial}{\partial \mathbf{y}} \left(\frac{\partial z}{\partial \mathbf{y}} \right)$$

Third-order Partial Derivatives

$$\frac{\partial^3 z}{\partial \mathbf{x}^3} = \frac{\partial}{\partial \mathbf{x}} \left(\frac{\partial^2 z}{\partial \mathbf{x}^2} \right)$$

$$\frac{\partial^3 z}{\partial \mathbf{y}^3} = \frac{\partial}{\partial \mathbf{y}} \left(\frac{\partial^2 z}{\partial \mathbf{y}^2} \right)$$

Third-order Partial Derivatives

$$\frac{\partial^2 z}{\partial \mathbf{x} \partial \mathbf{y}} = \frac{\partial}{\partial \mathbf{x}} \left(\frac{\partial z}{\partial \mathbf{y}} \right) = \frac{\partial^2 z}{\partial \mathbf{y} \partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{y}} \left(\frac{\partial z}{\partial \mathbf{x}} \right)$$

Chain Rule (1)

Function of two variable

$$z = f(u, v)$$

$$u = g(x, y)$$

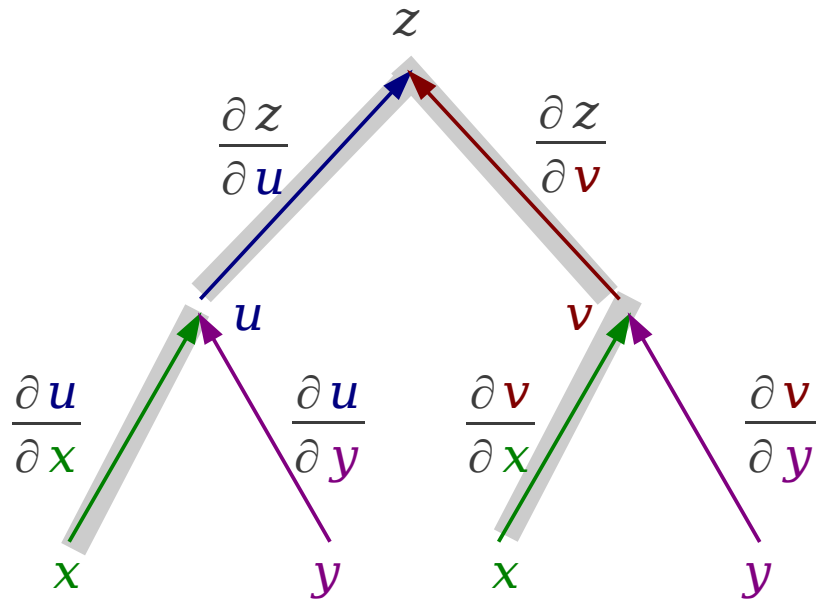
$$v = h(x, y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}$$

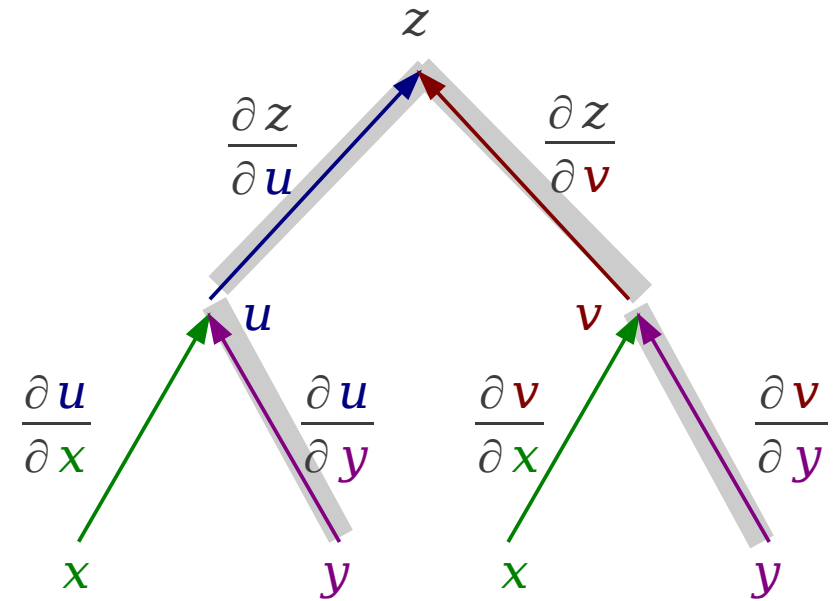
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$$

Chain Rule

Function of two variable $z = f(u, v)$



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}$$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$$

Directional Derivatives

Function of two variable $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

→ value

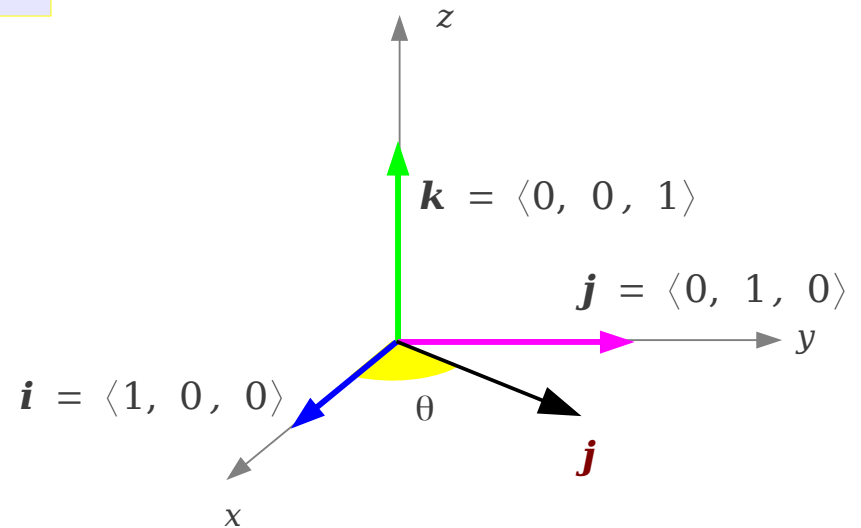
Rate of change of **f** in the **x** direction

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Rate of change of **f** in the **y** direction

Rate of change of **f** in the **u** direction

→ value ?



Gradient of a 2 Variable Function

Function of two variable $f(x, y)$

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$



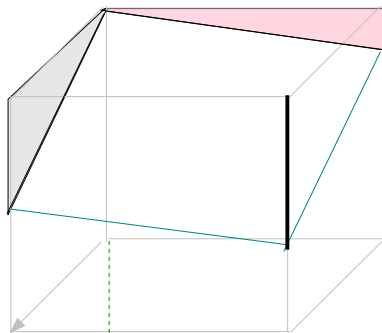
vector

Rate of change of f in the x direction

Rate of change of f in the y direction

Slope in the x direction

$$\frac{\partial f}{\partial x} = -2$$

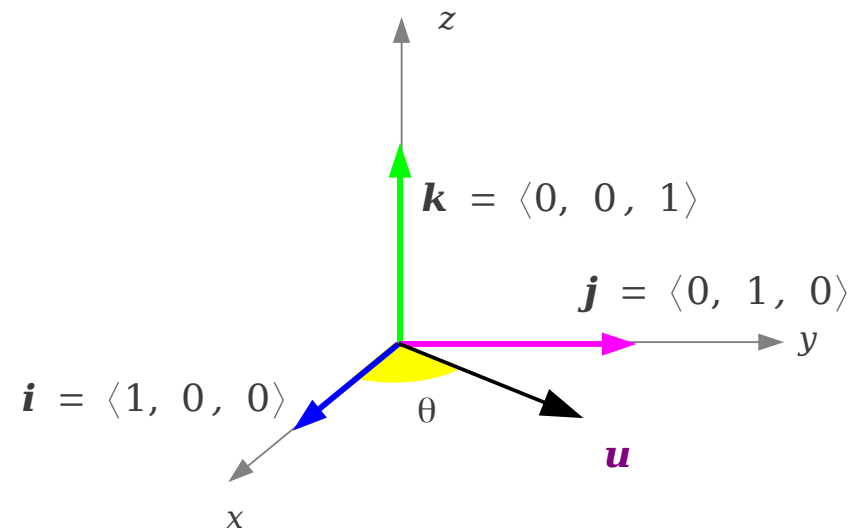


Slope in the y direction

$$\frac{\partial f}{\partial y} = -1$$

$$\frac{\partial f}{\partial y} \mathbf{j} = -1 \mathbf{j}$$

$$\frac{\partial f}{\partial x} \mathbf{i} = -2 \mathbf{i}$$



Gradient of a 3 Variable Function

Function of two variable $f(x, y)$

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$



vector

Rate of change of **f** in the **x** direction
Rate of change of **f** in the **y** direction

Function of three variable $F(x, y, z)$

$$\nabla F(x, y, z) = \frac{\partial F}{\partial x} \mathbf{i} + \frac{\partial F}{\partial y} \mathbf{j} + \frac{\partial F}{\partial z} \mathbf{k}$$



vector

Rate of change of **f** in the **x** direction
Rate of change of **f** in the **y** direction
Rate of change of **f** in the **z** direction

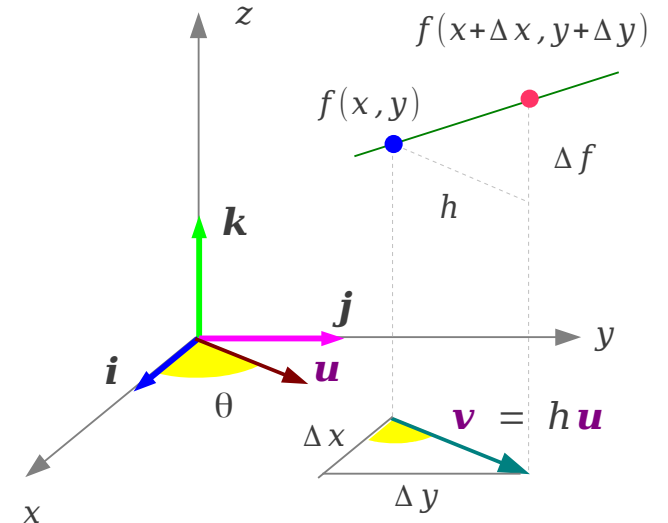
General Partial Differentiation

Function of two variable $f(x, y)$

$$\mathbf{u} = \cos\theta \mathbf{i} + \sin\theta \mathbf{j}$$

$$\begin{aligned} (x, y, 0) &\xrightarrow{\mathbf{v} = h\mathbf{u}} (x+\Delta x, y+\Delta y, 0) \\ \Delta x &= h\cos\theta \\ \Delta y &= h\sin\theta \\ h &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \end{aligned}$$

$$\begin{aligned} &\frac{f(x+\Delta x, y+\Delta y) - f(x, y)}{h} \\ &= \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x, y)}{h} \end{aligned}$$



Rate of change of f in the \mathbf{u} direction \rightarrow value

Directional Derivative

Function of two variable $f(x, y)$

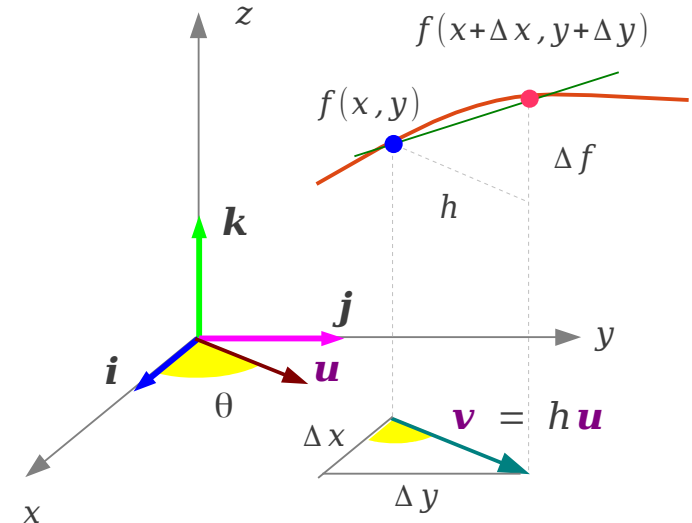
$$\mathbf{u} = \cos\theta \mathbf{i} + \sin\theta \mathbf{j}$$

$$(x, y, 0) \xrightarrow{\mathbf{v} = h\mathbf{u}} (x+\Delta x, y+\Delta y, 0)$$

$$\Delta x = h\cos\theta$$

$$\Delta y = h\sin\theta$$

$$h = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$



Rate of change of f in the \mathbf{u} direction

$$D_{\mathbf{u}} f(x, y) = \lim_{h \rightarrow 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x, y)}{h}$$

→ value

Computing Directional Derivative (1)

Function of two variable $f(x, y)$

$$\mathbf{u} = \cos\theta \mathbf{i} + \sin\theta \mathbf{j}$$

$$D_{\mathbf{u}} f(x, y) = \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h \cos\theta, \mathbf{y} + h \sin\theta) - f(\mathbf{x}, \mathbf{y})}{h}$$

→ value

$$\mathbf{i} = \cos 0 \mathbf{i} + \sin 0 \mathbf{j}$$

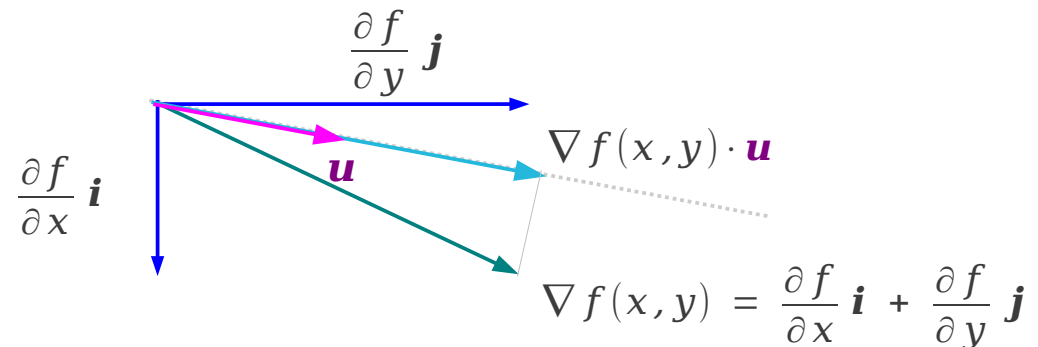
$$D_{\mathbf{i}} f(x, y) = \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h, \mathbf{y}) - f(\mathbf{x}, \mathbf{y})}{h} = \frac{\partial f}{\partial x}$$

$$\mathbf{j} = \cos 90^\circ \mathbf{i} + \sin 90^\circ \mathbf{j}$$

$$D_{\mathbf{j}} f(x, y) = \lim_{h \rightarrow 0} \frac{f(\mathbf{x}, \mathbf{y} + h) - f(\mathbf{x}, \mathbf{y})}{h} = \frac{\partial f}{\partial y}$$

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

$$\mathbf{u} = \cos\theta \mathbf{i} + \sin\theta \mathbf{j}$$



Computing Directional Derivative (2)

Function of two variable $f(x, y)$

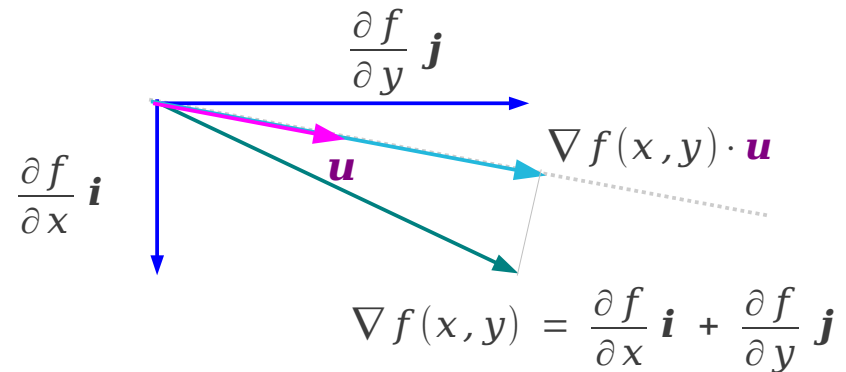
$$\mathbf{u} = \cos\theta \mathbf{i} + \sin\theta \mathbf{j}$$

$$D_{\mathbf{u}} f(x, y) = \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h \cos\theta, \mathbf{y} + h \sin\theta) - f(\mathbf{x}, \mathbf{y})}{h}$$

→ value

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

$$\mathbf{u} = \cos\theta \mathbf{i} + \sin\theta \mathbf{j}$$



$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

→ value

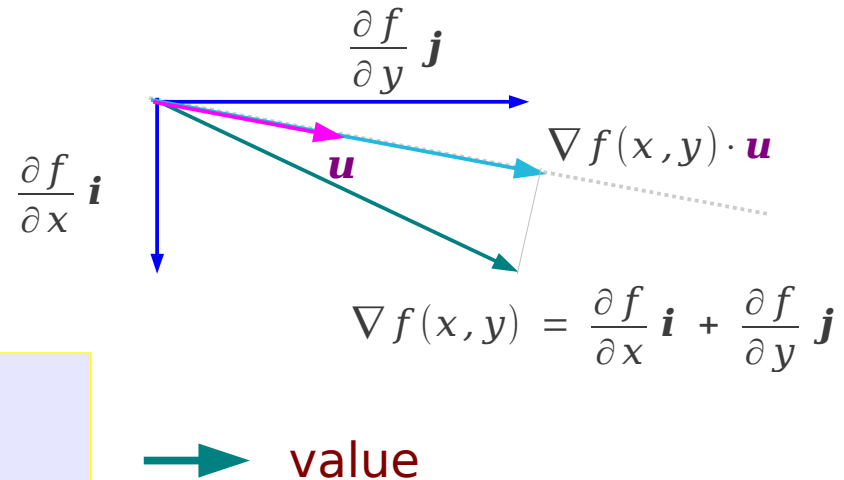
Rate of change of f in the \mathbf{u} direction

Computing Directional Derivative (3)

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

$$\mathbf{u} = \cos\theta \mathbf{i} + \sin\theta \mathbf{j}$$

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$



Rate of change of f in the \mathbf{u} direction

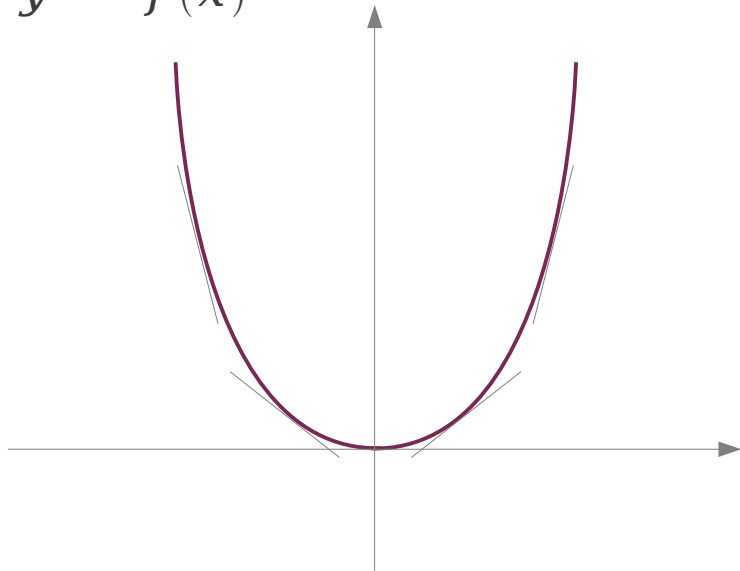
$$D_{\mathbf{u}} f(x, y) = \|\nabla f(x, y)\| \|\mathbf{u}\| \cos\theta = \|\nabla f(x, y)\| \cos\theta$$

$$\min \quad -\|\nabla f(x, y)\| \leq D_{\mathbf{u}} f(x, y) \leq \|\nabla f(x, y)\| \quad \max$$

the direction of $\nabla f(x, y)$ \rightarrow $f(x, y)$ Increases most rapidly

Gradient Example (1)

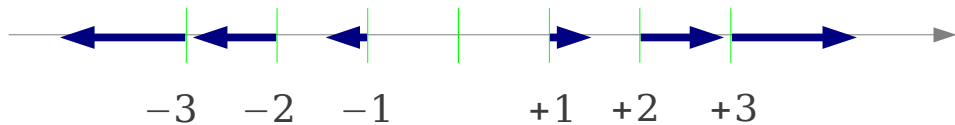
$$y = f(x)$$



$$f(x) = x^2 \quad \frac{df}{dx} = 2x$$

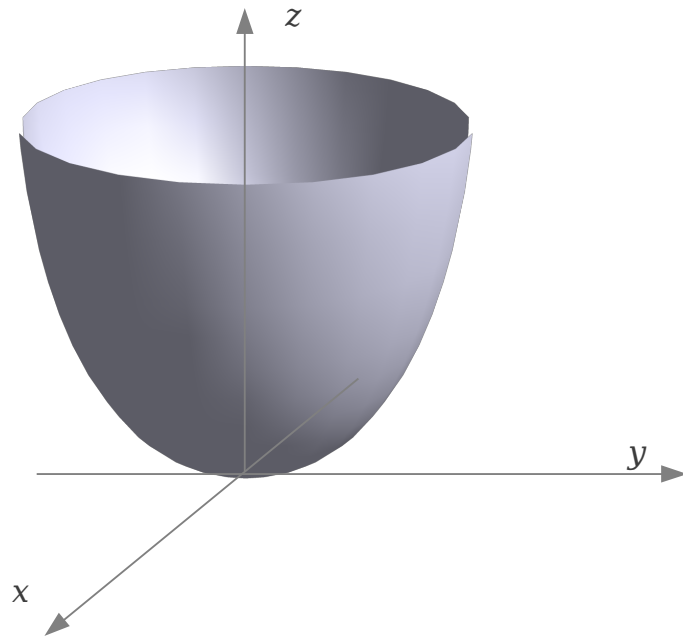
x	-3	-2	-1	0	+1	+2	+3
$\frac{df}{dx}$	-6	-4	-2	0	+2	+4	+6

Scaled arrows

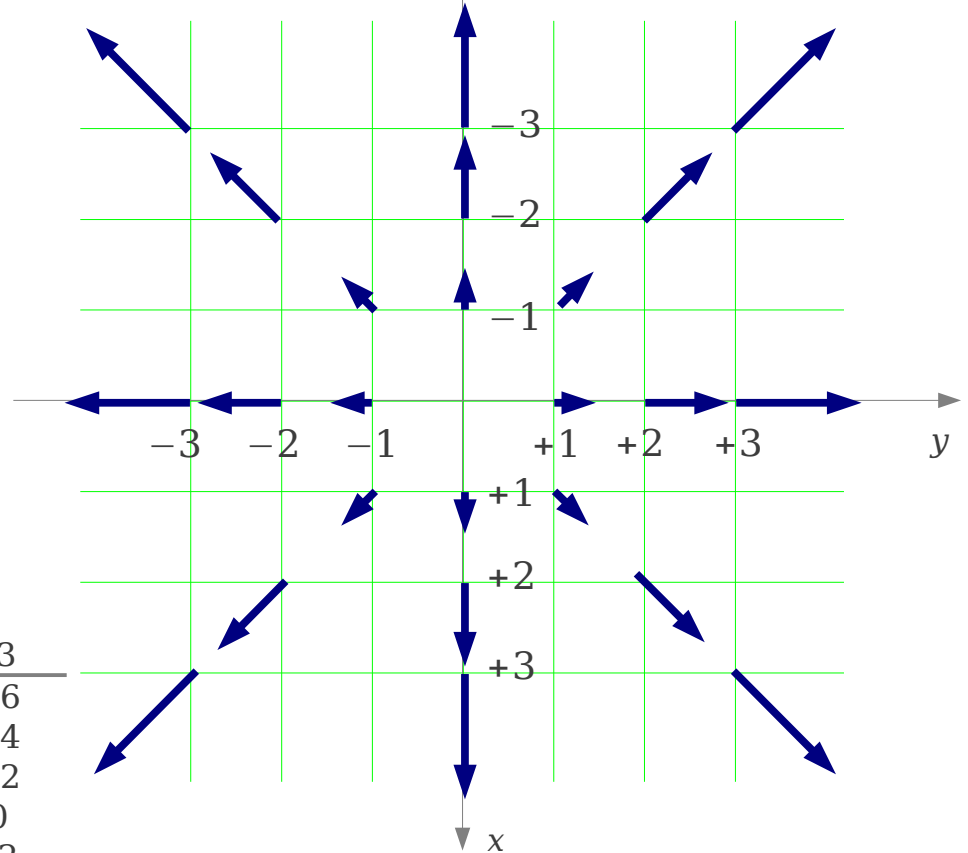


Gradient Example (2)

$$z = f(x, y)$$



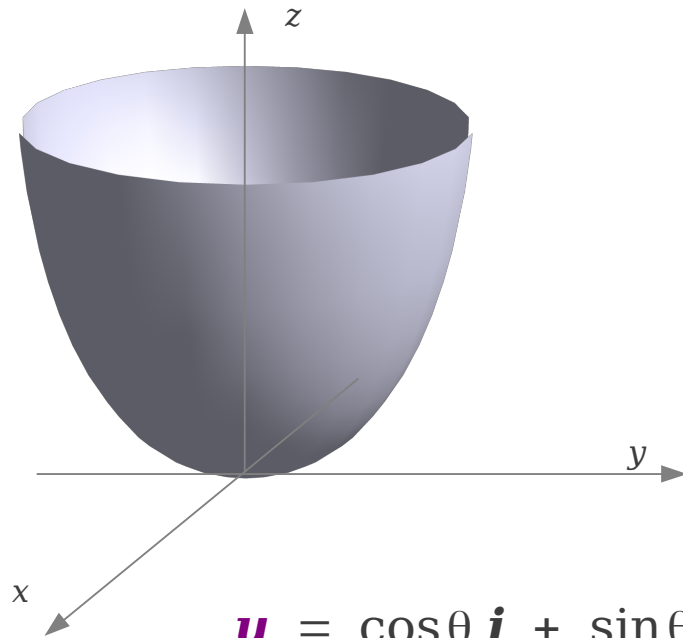
$$f(x, y) = x^2 + y^2 \quad \frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y$$



	$x=-3$	$x=-2$	$x=-1$	$x=0$	$x=+1$	$x=+2$	$x=+3$
$y=-3$	-6,-6	-4,-6	-2,-6	0,-6	+2,-6	+4,-6	+6,-6
$y=-2$	-6,-4	-4,-4	-2,-4	0,-4	+2,-4	+4,-4	+6,-4
$y=-1$	-6,-2	-4,-2	-2,-2	0,-2	+2,-2	+4,-2	+6,-2
$y=0$	-6,0	-4,0	-2,0	0,0	+2,0	+4,0	+6,0
$y=+1$	-6,+2	-4,+2	-2,+2	0,+2	+2,+2	+4,+2	+6,+2
$y=+2$	-6,+4	-4,+4	-2,+4	0,+4	+2,+4	+4,+4	+6,+4
$y=+3$	-6,+6	-4,+6	-2,+6	0,+6	+2,+6	+4,+6	+6,+6

Gradient Example (2)

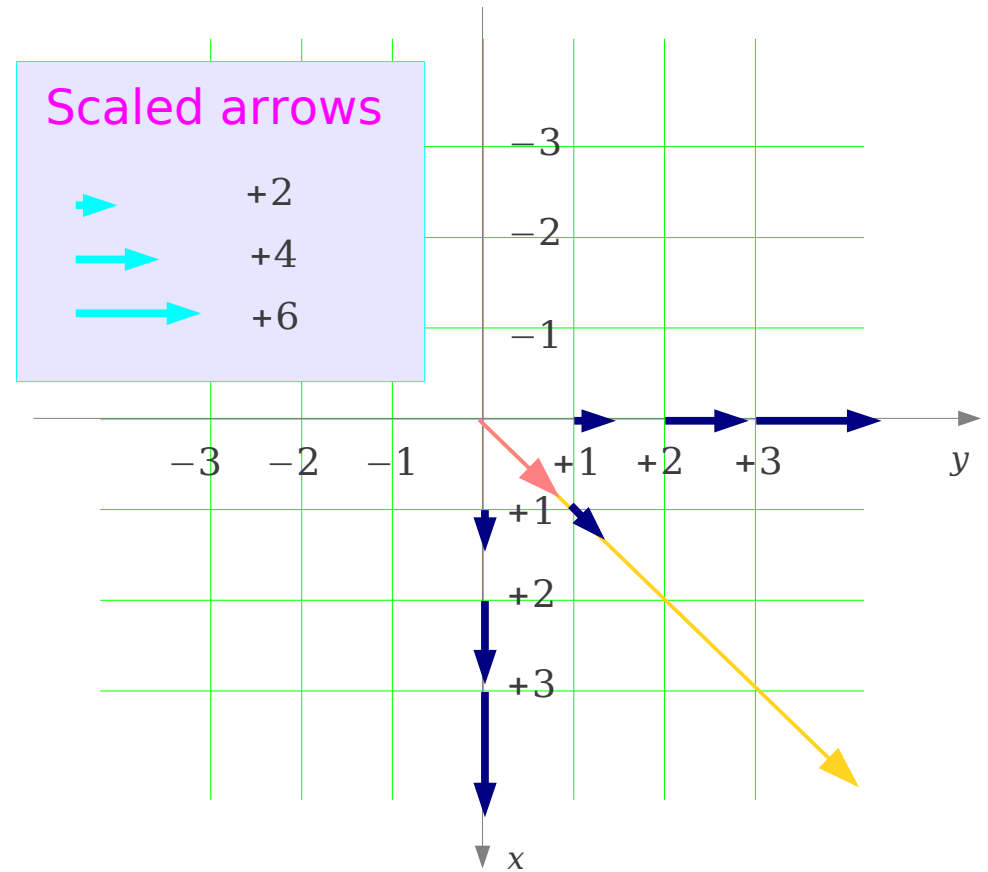
$$y = f(x, y)$$



$$f(x, y) = x^2 + y^2 \quad \frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y$$

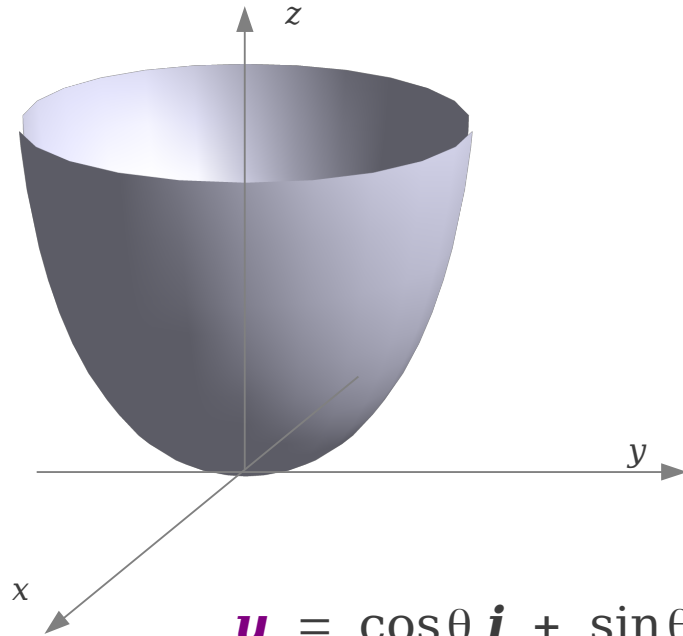
$$\begin{aligned} \mathbf{u} &= \cos\theta \mathbf{i} + \sin\theta \mathbf{j} \\ &= \frac{1}{\sqrt{2}} \mathbf{i} + \frac{1}{\sqrt{2}} \mathbf{j} \end{aligned}$$

$$\begin{aligned} D_{\mathbf{u}} f(1, 1) &= \nabla f(1, 1) \cdot \mathbf{u} \\ &= (2, 2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\ &= 2\sqrt{2} \end{aligned}$$



Gradient Example (3)

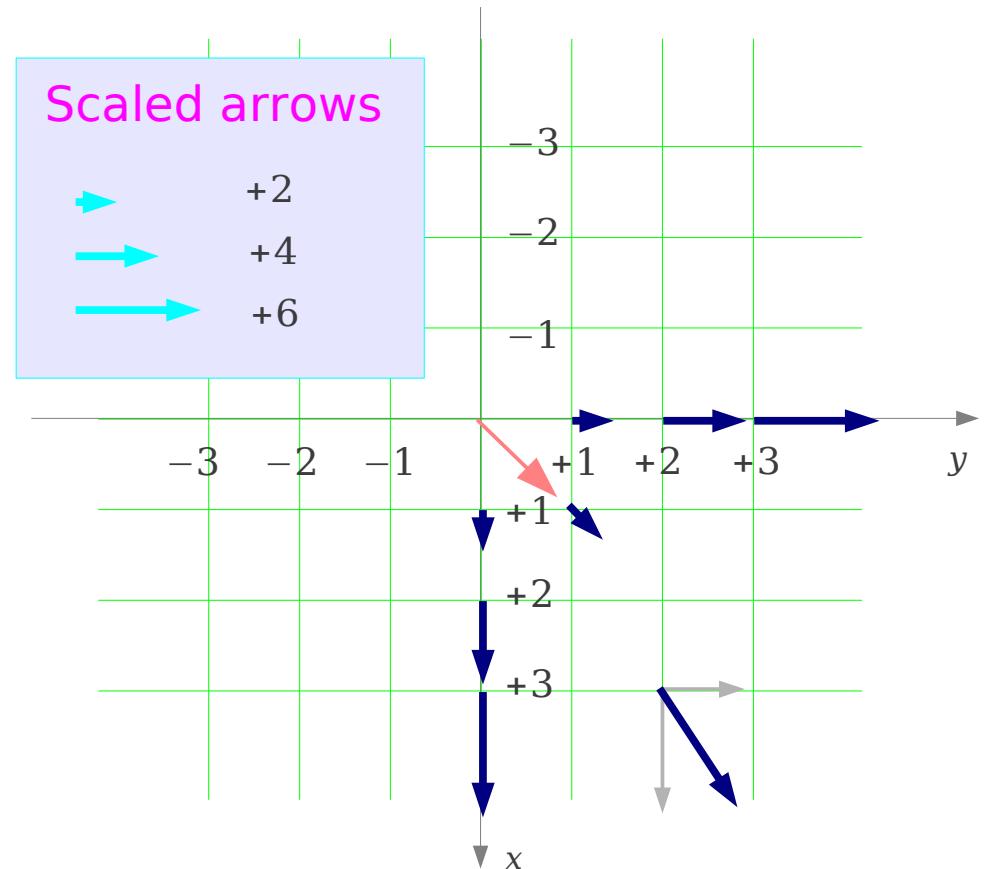
$$y = f(x, y)$$



$$f(x, y) = x^2 + y^2 \quad \frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y$$

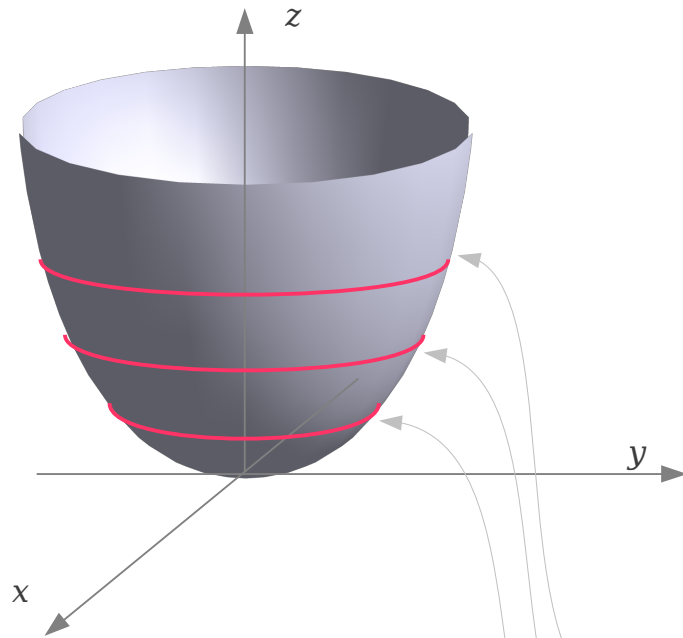
$$\begin{aligned} \mathbf{u} &= \cos\theta \mathbf{i} + \sin\theta \mathbf{j} \\ &= \frac{1}{\sqrt{2}} \mathbf{i} + \frac{1}{\sqrt{2}} \mathbf{j} \end{aligned}$$

$$\begin{aligned} D_{\mathbf{u}} f(3, 2) &= \nabla f(3, 2) \cdot \mathbf{u} \\ &= (6, 4) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\ &= 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2} \end{aligned}$$

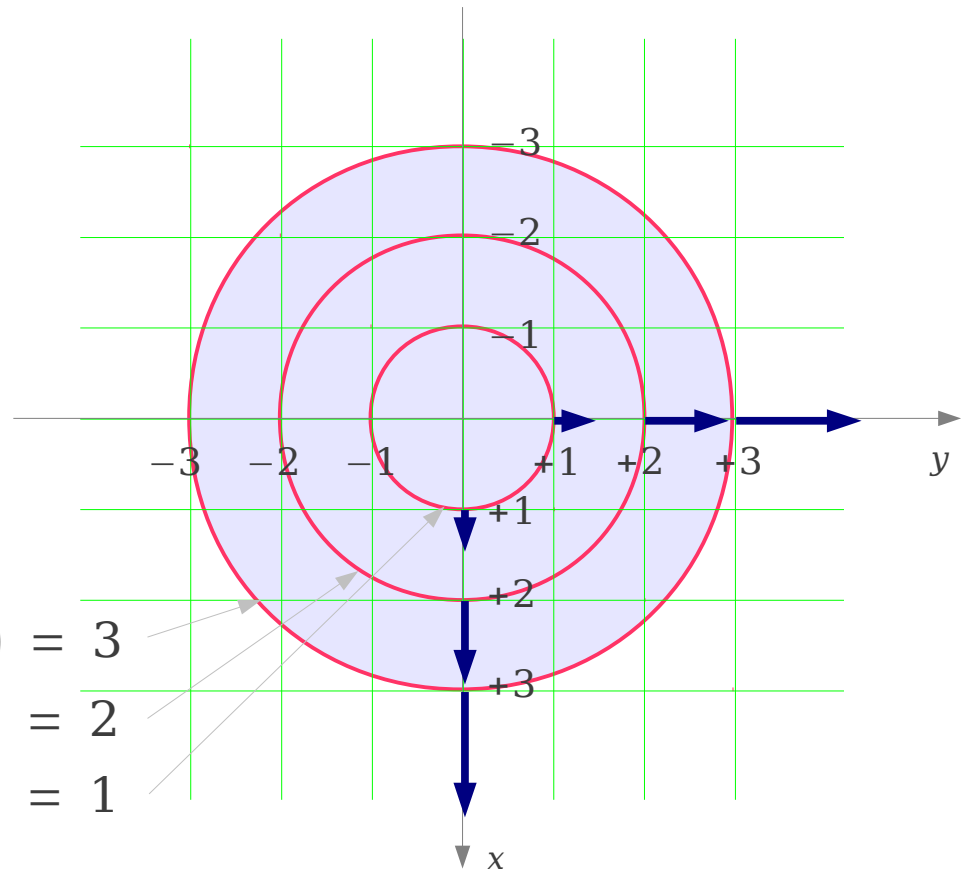


Level Curve (1)

$$z = f(x, y)$$



$$f(x, y) = x^2 + y^2 \quad \frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y$$



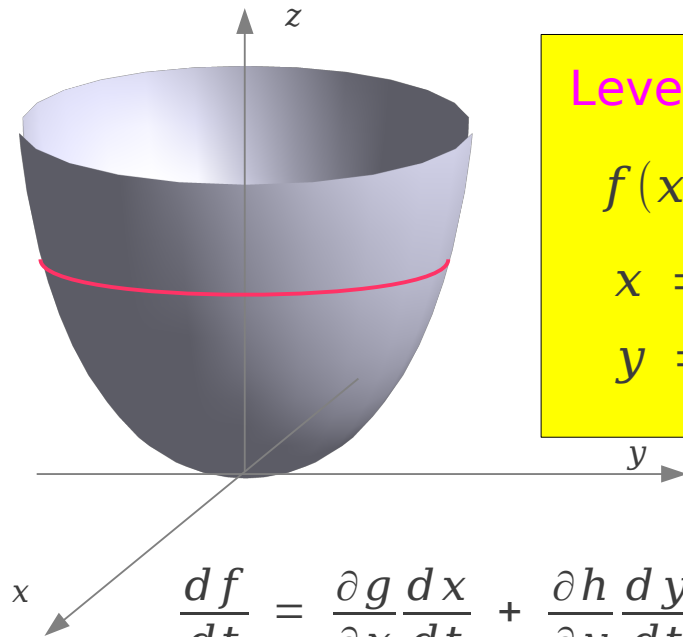
Level Curve

$$f(x, y) = c$$

$$\begin{aligned} f(x, y) &= 3 \\ f(x, y) &= 2 \\ f(x, y) &= 1 \end{aligned}$$

Level Curve (2)

$$y = f(x, y)$$



$$f(x, y) = x^2 + y^2 \quad \frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y$$

Level Curve

$$f(x, y) = 3$$

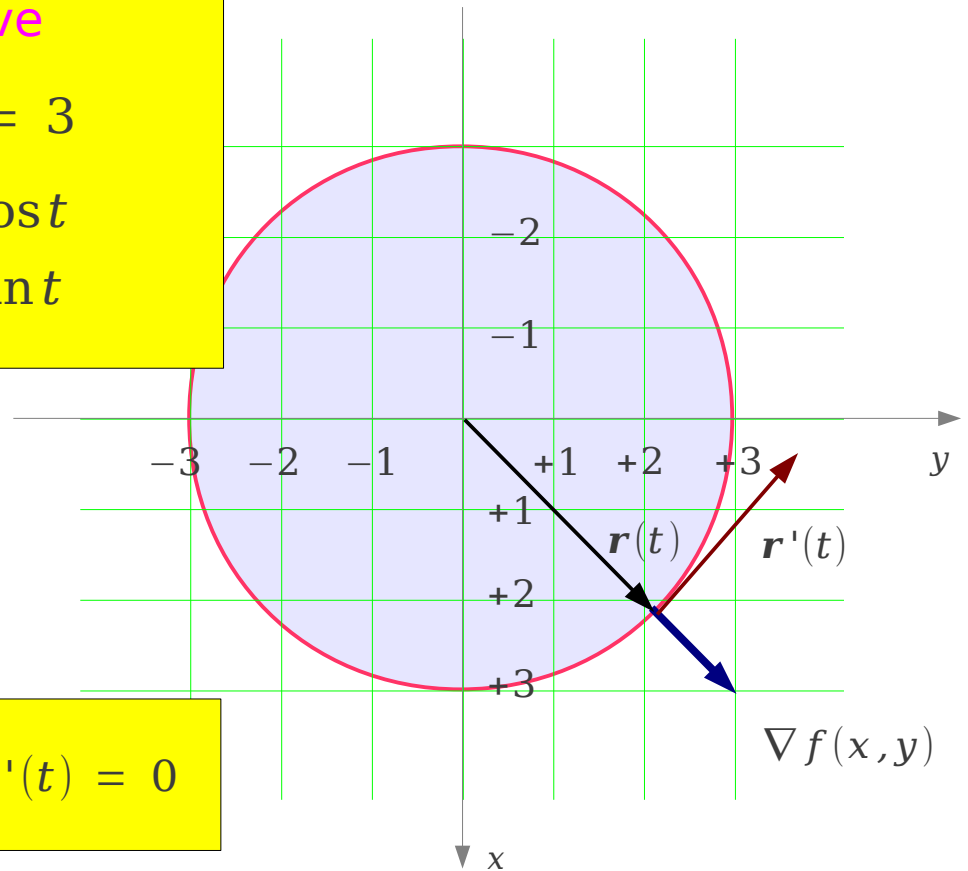
$$x = 3 \cos t$$

$$y = 3 \sin t$$

$$\frac{df}{dt} = \frac{\partial g}{\partial x} \frac{dx}{dt} + \frac{\partial h}{\partial y} \frac{dy}{dt} = 0$$

$$\begin{cases} \nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \\ \mathbf{r}'(t) = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} \end{cases}$$

$$\nabla f(x, y) \cdot \mathbf{r}'(t) = 0$$



Level Surface

$$\begin{array}{ll} \text{Function of two variable} & f(x, y) \\ \text{Level Curve} & f(x, y) = c \end{array}$$

$$\left\{ \begin{array}{l} x=g(t) \\ y=h(t) \end{array} \right. \Rightarrow \frac{df}{dt} = \frac{\partial g}{\partial x} \frac{dx}{dt} + \frac{\partial h}{\partial y} \frac{dy}{dt} = 0 \Rightarrow \left\{ \begin{array}{l} \nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \\ \mathbf{r}'(t) = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} \end{array} \right.$$

∇f orthogonal to the level curve at P

$$\nabla f(x, y) \cdot \mathbf{r}'(t) = 0$$

$$\begin{array}{ll} \text{Function of three variable} & F(x, y, z) \\ \text{Level Surface} & F(x, y, z) = c \end{array}$$

$$\left\{ \begin{array}{l} x=f(t) \\ y=g(t) \\ z=h(t) \end{array} \right. \Rightarrow \frac{dF}{dt} = 0 \Rightarrow \left\{ \begin{array}{l} \nabla F(x, y, z) = \frac{\partial F}{\partial x} \mathbf{i} + \frac{\partial F}{\partial y} \mathbf{j} + \frac{\partial F}{\partial z} \mathbf{k} \\ \mathbf{r}'(t) = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \end{array} \right.$$
$$\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial g}{\partial y} \frac{dy}{dt} + \frac{\partial h}{\partial z} \frac{dz}{dt} = 0$$

∇F normal to the level surface at P

$$\nabla F(x, y, z) \cdot \mathbf{r}'(t) = 0$$

Tangent Plane

Function of three variable $F(x, y, z)$

Level Surface $F(x, y, z) = c$

P_0 $F(x_0, y_0, z_0) = c$

$$\nabla F(x_0, y_0, z_0) \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial F}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

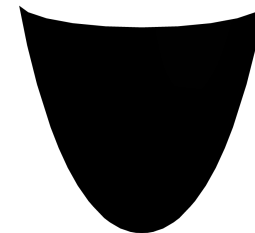
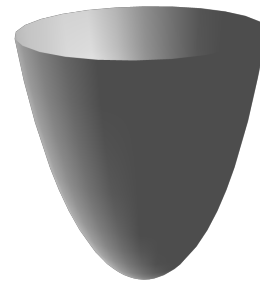
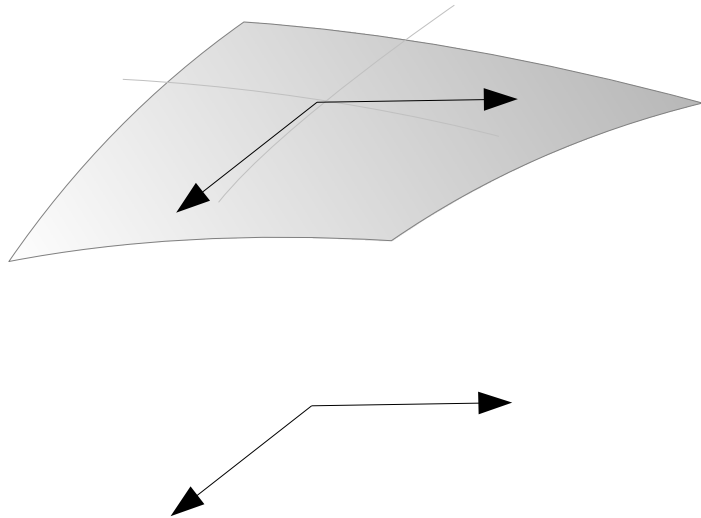
Chain Rule

Function of two variable

$$y = f(u, v)$$

$$u = g(x, y)$$

$$v = h(x, y)$$



References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [4] D.G. Zill, “Advanced Engineering Mathematics”