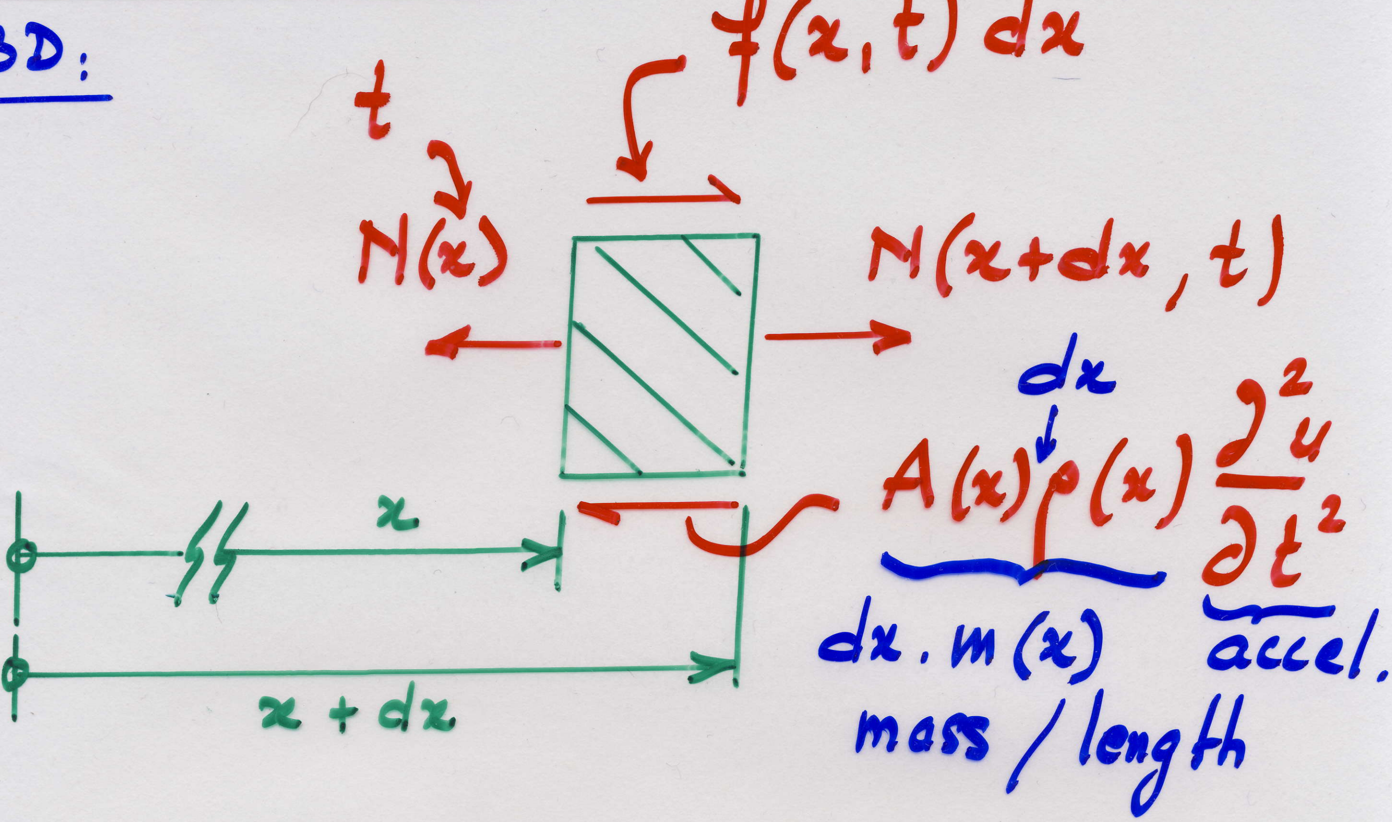
Mtg 28: Fri, 31 Oct 08. EML 4500 — 28-1

HW5: Literature search for composite materials w/ $E(x)$ (varying Young's modulus) (doping, etc.)

FBD:



$$\Sigma F_x = 0 = -N(x,t) + N(x+dx,t) + f(x,t)\,dx \; \cancel{+} - \; m(x)\,\ddot{u}\cdot dx$$

$$= \frac{\partial N(x,t)}{\partial x}\cancel{dx} + \underbrace{h.o.t.}_{\text{higher order terms}}$$

$$+ f(x,t)\,\cancel{dx} \; \cancel{+} - \; m(x)\,\ddot{u}\cdot\cancel{dx} \qquad (1)$$

Neglect h.o.t.

Recall Taylor series exp.:

$$f(x+dx) = f(x) + \frac{df}{dx}(x)\,dx + \underbrace{\frac{1}{2}\frac{d^2f}{dx^2}(x)\,dx^2 + \dots}_{\text{h.o.t.}}$$

Eq. (1) $\Rightarrow$

$$\boxed{\frac{\partial N}{\partial x} + f = m\ddot{u}} \quad (2)$$

$\uparrow$ Eq. of Motion (EOM)

$$\boxed{N(x,t) = A(x)\,\underbrace{\sigma(x,t)}_{E(x)\,\underbrace{\varepsilon(x,t)}_{\frac{\partial u}{\partial x}(x,t)}}} \quad (3)$$

$\uparrow$ Constitutive rel.

(3) in (2) yields:

$$\boxed{\frac{\partial}{\partial x}\left[A(x)E(x)\frac{\partial u}{\partial x}\right] + f(x,t) = m(x)\underbrace{\ddot{u}}_{\frac{\partial^2 u}{\partial t^2}}} \quad (4)$$

PDE of Motion.

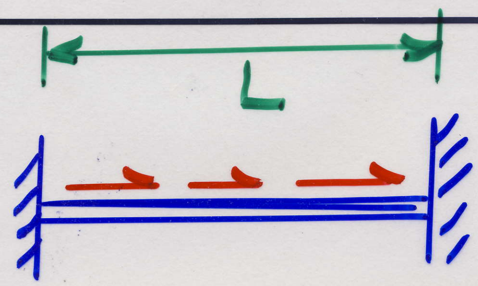
Need 2 b.c.'s (2nd order deriv. wrt x)

2 init. cond. (2nd deriv. wrt t)

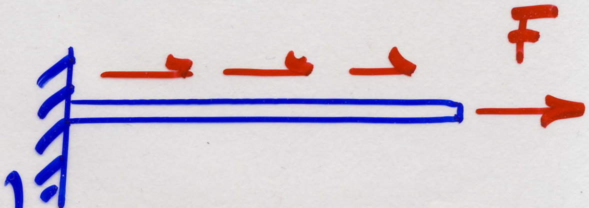
(init. disp, init. veloc.)

(Sam)

Bound. cond.



$$u(0,t) = 0 = u(L,t)$$



1) $\boxed{u(0,t)=0}$

2) $\underbrace{N(L,t)}_{A(L)\,\underbrace{\sigma(L,t)}_{E(L)\,\underbrace{\varepsilon(L,t)}_{\frac{\partial u}{\partial x}(L,t)}}} = F(t)$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x}(L,t) = \frac{F(t)}{A(L)\,E(L)}}$$

Init. cond. At $t=0$, prescribe

$$\boxed{\begin{aligned} u(x,t=0) &= \bar{u}(x) && \text{known func. disp.} \\ \frac{\partial u}{\partial t}(x,t=0) &= \dot{u}(x,t=0) = \bar{v}(x) && \text{known func. veloc.} \end{aligned}}$$

- Comments on HW4: Mechanisms, eigenvect.
  HW6: redo rect. truss.

PVW (continuous) of dyn. of elastic bar

PDE:

$$\boxed{\frac{\partial}{\partial x}\left[(EA)\frac{\partial u}{\partial x}\right] + f = m\ddot{u}} \quad (1)$$

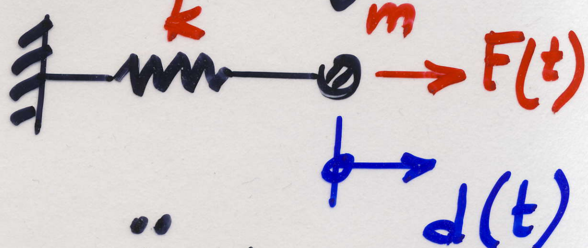
Discrete EOM

$$-\underline{K}\,\underline{d} + \underline{F} = \underline{M}\,\ddot{\underline{d}}$$

$$\Rightarrow \boxed{\underline{M}\,\ddot{\underline{d}} + \underline{K}\,\underline{d} = \underline{F}} \quad (2)$$

Mult. Deg. of Freedom (MDOF)

SDOF = Single DOF



$$m\ddot{d} + kd = F$$

Derive (2) from (1):

$$\int_{x=0}^{x=L} W(x)\left\{\frac{\partial}{\partial x}\left[EA\frac{\partial u}{\partial x}\right] + f - m\ddot{u}\right\}dx$$

$= 0$ for all possible $W(x)$ ← Weighting func. (3)

(1) $\Rightarrow$ (3) trivial.

(3) $\Rightarrow$ (1) not trivial.

(3) rewritten as : $\int w(x)\, g(x)\, dx = 0$ for all $w(x)$

Since (3) holds for all $w(x)$, select $w(x) = g(x)$, then (3) becomes:

$$\int \underbrace{g^2}_{\geq 0} dx = 0 \Rightarrow g(x) = 0$$

$\equiv$

Int. by parts: $r(x)$, $s(x)$

$$(rs)' = r's + rs'$$

$$r' = \frac{dr}{dx}, \quad s' = \frac{ds}{dx}$$

$$\underbrace{\int (rs)'}_{(rs)} = \int r's + \int rs'$$

$$\Rightarrow \boxed{\int r's = rs - \int rs'}$$

Recall cont. PVW Eq. (3) p. 29-1

1st term: $r(x) = (EA)\dfrac{\partial u}{\partial x}$, $s(x) = W(x)$

By int. by parts:

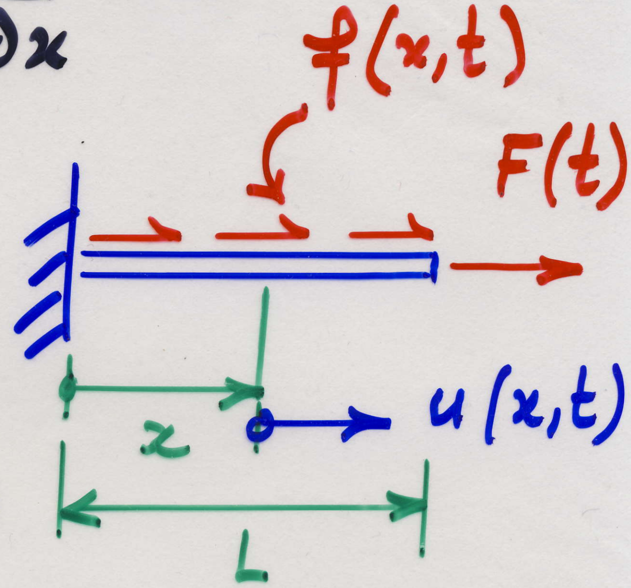
$$\int_{x=0}^{x=L} \underbrace{W(x)}_{s} \frac{\partial}{\partial x}\Big[\underbrace{(EA)\frac{\partial u}{\partial x}}_{r}\Big] dx = \left[W\,(EA)\frac{\partial u}{\partial x}\right]_{x=0}^{x=L} - \int_0^L \frac{dW}{dx}(EA)\frac{\partial u}{\partial x}\,dx$$

$$= \underbrace{W(L)\,(EA)(L)\,\frac{\partial u(L,t)}{\partial x}}_{N(L,t)\,=\,F}$$

$$- \underbrace{W(0)\,(EA)(0)\,\frac{\partial u(0,t)}{\partial x}}_{N(0,t)}$$

$$- \int_0^L \frac{dW}{dx}\,(EA)\,\frac{\partial u}{\partial x}\,dx$$

Now consider model pb :



p.28-3: 2 b.c.'s

At $x=0$, select $W(x)$

st $\boxed{W(0)=0}$ (i.e., kinematically admissible)

Motivation: Discrete PVW applied to Eq. on p. 10-1

$$\underset{6\times1}{\underline{W}} \cdot \Bigg(\underbrace{\underset{6\times2}{[\quad]} \underset{2\times1}{\begin{Bmatrix} d_3 \\ d_4 \end{Bmatrix}}}_{6\times1} - \underset{6\times1}{\underline{F}} \Bigg) = \underset{1\times1}{0} \quad \text{for all } \underline{W}$$

$$\underline{F}^T = \lfloor F_1 \quad F_2 \quad \overset{\text{known}}{\overbrace{F_3 \quad F_4}} \quad F_5 \quad F_6 \rfloor$$

F_1, F_2, F_5, F_6 : unknown reactions

Since $\underline{W}$ can be selected arbitrarily, select $\underline{W}$ st $W_1 = W_2 = W_5 = W_6 = 0$ so to eliminate eqs involving unknown reactions. $\Rightarrow$ elim. rows 1, 2, 5, 6;

$$\underset{2\times2}{\bar{\underline{K}}}\ \underset{2\times1}{\bar{\underline{d}}} = \underset{2\times1}{\bar{\underline{F}}} \qquad (1)$$

$$\bar{\underline{d}} = \begin{Bmatrix} d_3 \\ d_4 \end{Bmatrix} \qquad \bar{\underline{F}} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

Note: $\bar{\underline{W}} \cdot (\bar{\underline{K}}\,\bar{\underline{d}} - \bar{\underline{F}}) = 0$ for all $\bar{\underline{W}}$

$$\bar{\underline{W}} = \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix}$$

step before Eq. (1).

(HW5) ≡

Back to cont. PVW (end motivation)

Unknown reaction $N(0,t) = (EA)(0)\dfrac{\partial u}{\partial x}(0,t)$

Cont. PVW: $W(L)\,F(t)$

$$- \int_0^L \frac{dW}{dx}\,(EA)\,\frac{\partial u}{\partial x}\,dx$$

$$+ \int_0^L W(x)\,\left[f - m\ddot{u} \right] dx = 0$$

for all $W(x)$ st $W(0)=0$.

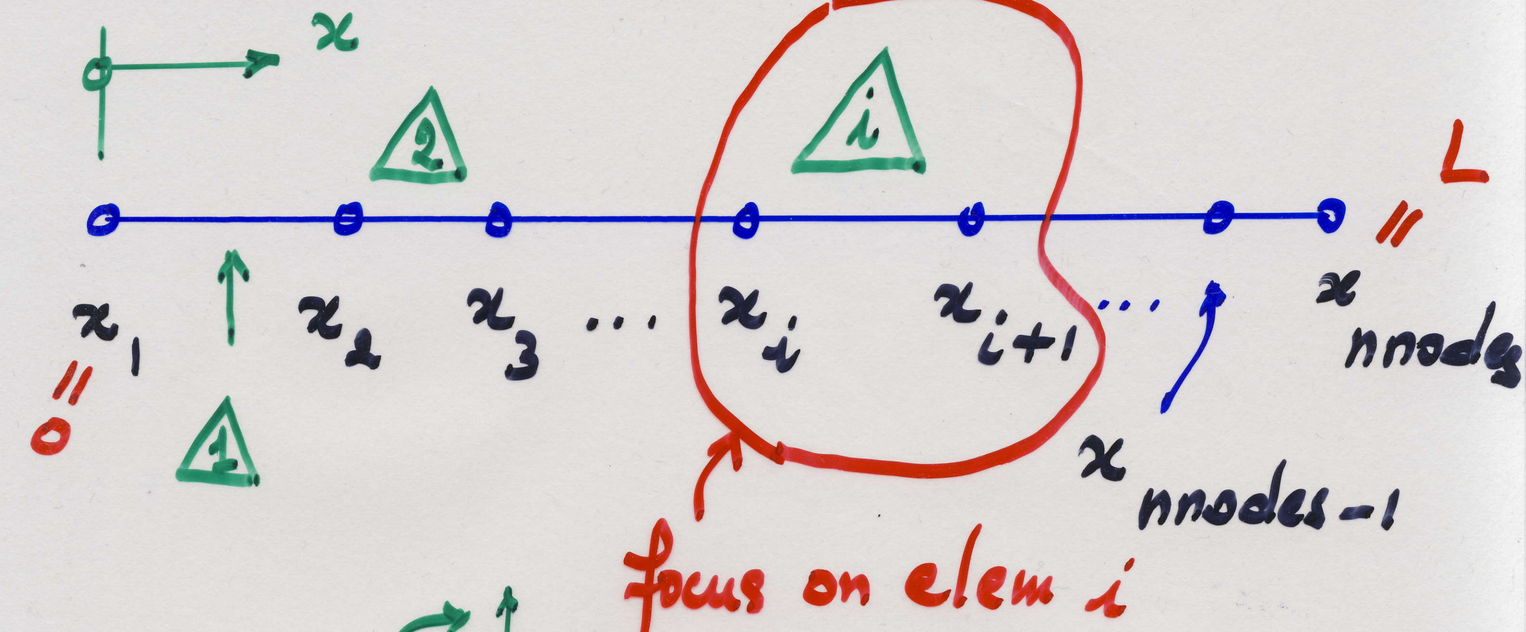
$$\Rightarrow \boxed{\int_0^L W\,(m\,\ddot{u})\,dx + \int_0^L \frac{dW}{dx}\,(EA)\,\frac{\partial u}{\partial x}\,dx = W(L)\,F(t) + \int_0^L W\,f\,dx}$$

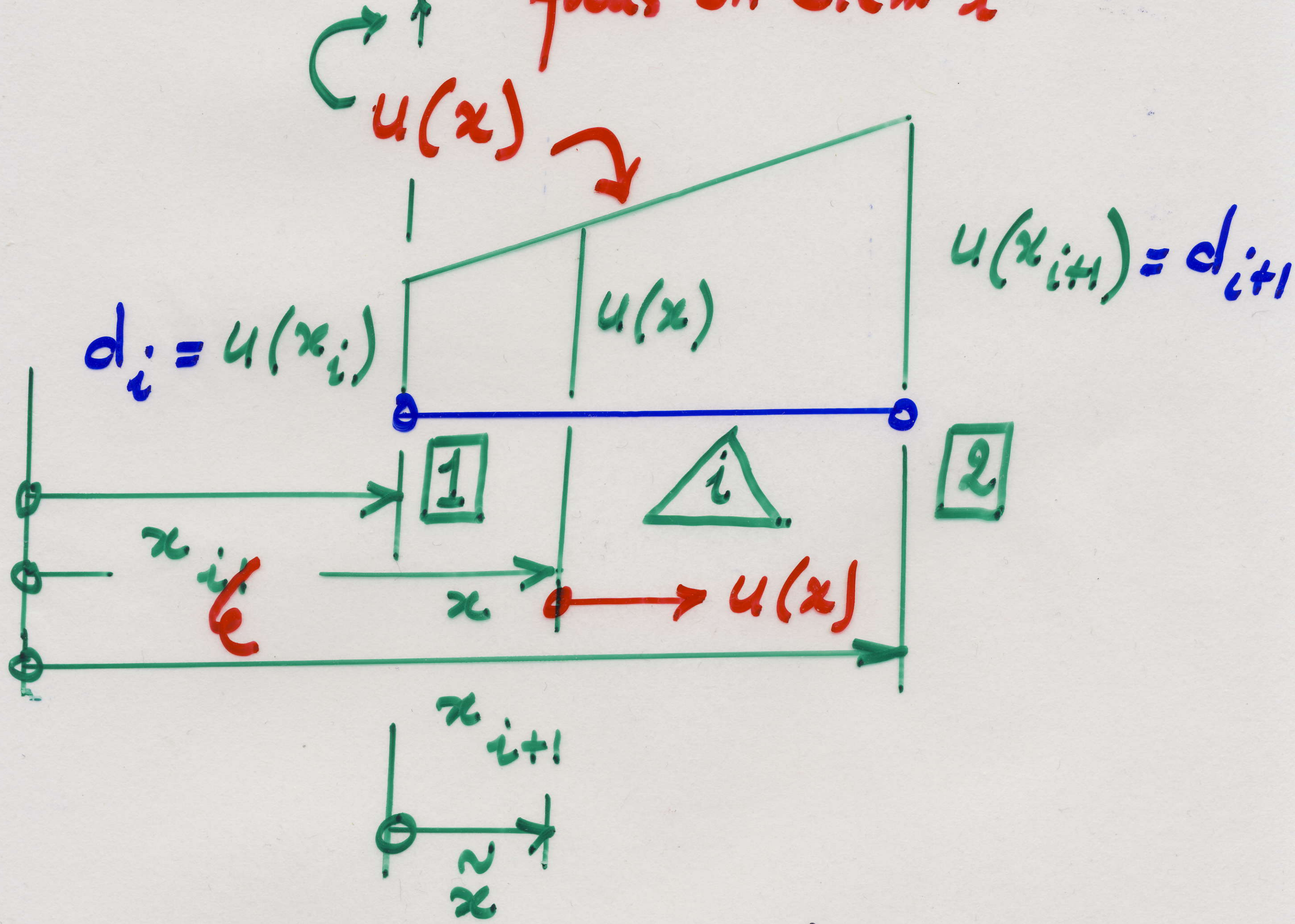
for all $W(x)$ st $W(0)=0$

(Cont. PVW ("Weak form")

Cont. setting (PVW)	Discrete setting (PVW)	
Inertia: $\alpha = \int_0^L w\, m\, \ddot{u}\, dx$	$\bar{\underline{w}} \cdot (\bar{\underline{M}}\, \ddot{\bar{\underline{d}}})$	(1)
Stiffness: $\beta = \int_0^L \frac{dw}{dx} (EA) \frac{\partial u}{\partial x} dx$	$\bar{\underline{w}} \cdot (\bar{\underline{K}}\, \bar{\underline{d}})$	(2)
Applied force: $w(L) F(t) + \int_0^L w f\, dx = \gamma$	$\bar{\underline{w}} \cdot \bar{\underline{F}}$	(3)
for all $w(x)$ st $\underbrace{w(0) = 0}_{\text{b.c.}}$.	for all $\bar{\underline{w}}$ (b.c.'s eliminated)	

Stiffness term:





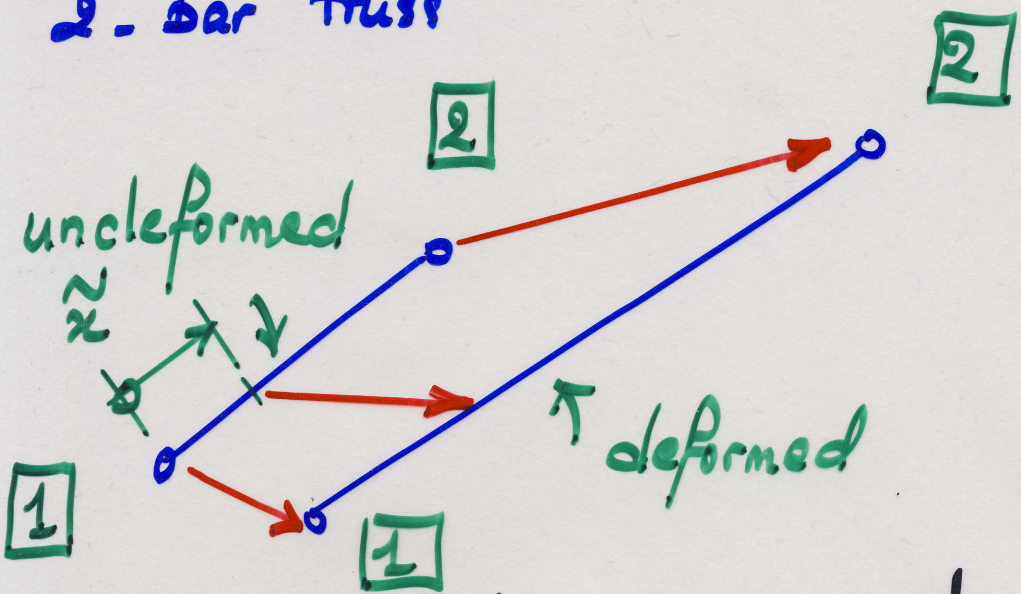
Assume disp $u(x)$ for $x_i \leq x \leq x_{i+1}$

(i.e., $x \in [x_i, x_{i+1}]$)

↑ "belongs to"

Motivation for linear interpolation of $u(x)$

2-bar truss



Deformed shape is a straight line, i.e., there was an implicit assump. of linear interp. of disp. betw. 2 nodes.

Consider the case where there are only axial disp (i.e., zero transv. disp.)

i.e., p. 31-2.

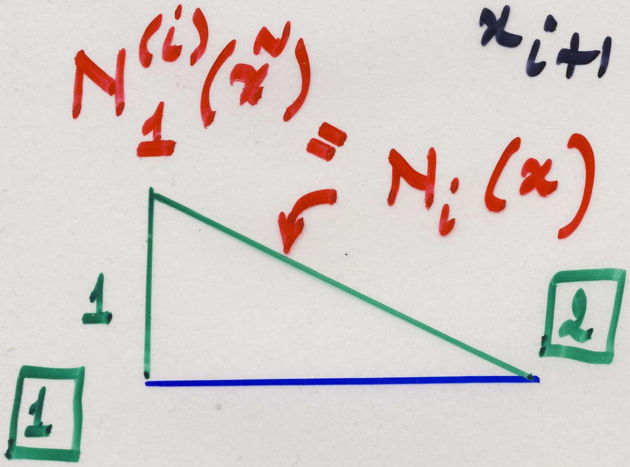
Q: Express $u(x)$ in terms of $d_i = u(x_i)$ and $d_{i+1} = u(x_{i+1})$ as a linear func. in x (i.e, linear interp.)

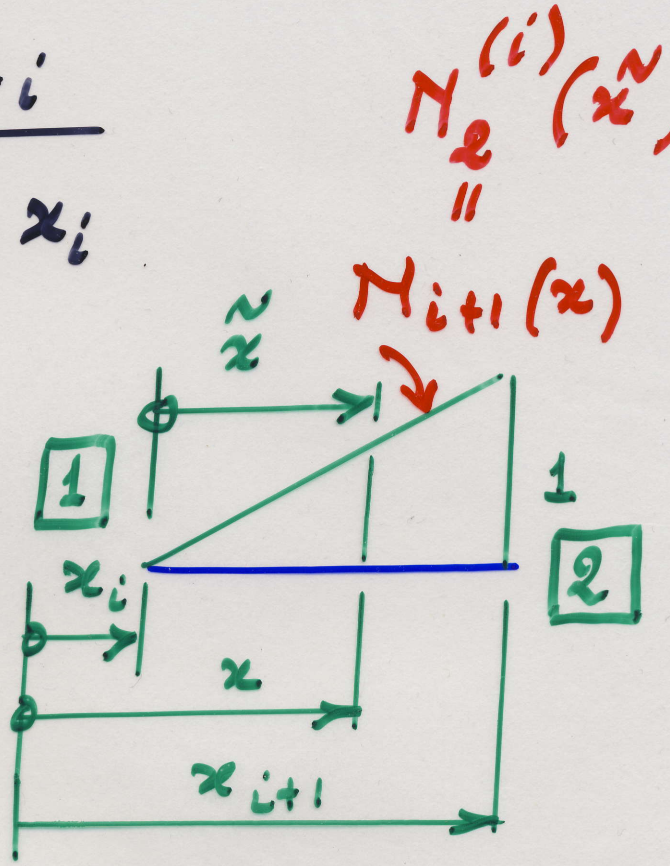
$$\boxed{u(x) = N_i(x)\, d_i + N_{i+1}(x)\, d_{i+1}}$$

where $N_i(x)$ and $N_{i+1}(x)$ are linear func. in x.

$$N_i(x) = \text{HW}$$

$$N_{i+1}(x) = \frac{x - x_i}{x_{i+1} - x_i}$$





- Rube-Goldberg device

  Walter Benjamin
  V Y

  Volkswagen
  F V

- "Honesty, imagination, ethics"

  Narrative in HW reports

Cont. PVW to Discrete PVW (Cont'd)

Lagrangian interp.

Motivation for form of $N_i(x)$ and $N_{i+1}(x)$

1) $N_i(x)$ and $N_{i+1}(x)$ are linear (straight lines), thus any linear combo. of N_i and N_{i+1} is also linear, and in particular expr. for $u(x)$ on p. 31-3.

$$N_i(x) = \alpha_i + \beta_i x \text{ , with } (\alpha_i, \beta_i) \text{ numbers}$$

$N_{i+1}(x) = \alpha_{i+1} + \beta_{i+1} x$,

Linear combo of N_i and $(\alpha_{i+1}, \beta_{i+1})$ numbers

N_{i+1} :

$$N_i d_i + N_{i+1} d_{i+1} = (\alpha_i + \beta_i x) d_i + (\alpha_{i+1} + \beta_{i+1} x) d_{i+1}$$

$$= (\alpha_i d_i + \alpha_{i+1} d_{i+1}) + (\beta_i d_i + \beta_{i+1} d_{i+1}) x$$

is clearly a linear func. in x.

2) Recall eq. for $u(x)$ (interp. of $u(x)$) on p. 31-3 :

$$u(x_i) = \underbrace{N_i(x_i)}_{=1} d_i + \underbrace{N_{i+1}(x_i)}_{=0} d_{i+1}$$

$$= d_i$$

FEM via PVW (cont'd)

$u(x_{i+1}) = d_{i+1}$ HW6

p. 31-3: interp. for $u(x)$

Apply same interp. for $w(x)$, i.e.,

$$\boxed{w(x) = N_i(x)\, w_i + N_{i+1}(x)\, w_{i+1}}$$

Elem stiff. mat. for elem i:

$$(\beta) = \int_{x_i}^{x_{i+1}} [\underbrace{N_i' w_i + N_{i+1}' w_{i+1}}_{w'(x)}] (EA) [\underbrace{N_i' d_i + N_{i+1}' d_{i+1}}_{u'(x)}] dx$$

$$N_i' := \frac{dN_i(x)}{dx}$$

Likewise for N_{i+1}'.

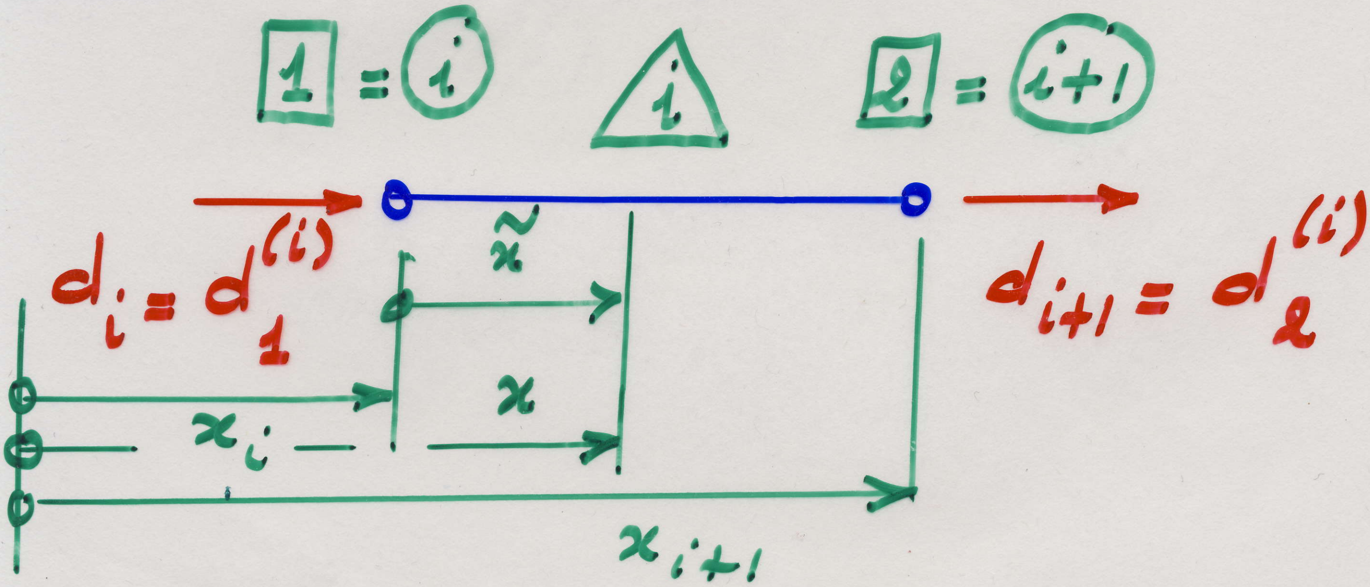
Note: $u(x) = \underbrace{\lfloor N_i(x) \quad N_{i+1}(x) \rfloor}_{\underset{\sim}{N}(x)_{1\times 2}} \begin{Bmatrix} d_i \\ d_{i+1} \end{Bmatrix}_{2\times 1}$

$$\frac{du(x)}{dx} = \underbrace{\lfloor N_i'(x) \quad N_{i+1}'(x) \rfloor}_{\underline{B}(x)}{}_{1\times 2} \begin{Bmatrix} d_i \\ d_{i+1} \end{Bmatrix}_{2\times 1}$$

Similarly: $$W(x) = \underline{N}(x) \begin{Bmatrix} W_i \\ W_{i+1} \end{Bmatrix}$$

$$\frac{dW(x)}{dx} = \underline{B}(x) \begin{Bmatrix} W_i \\ W_{i+1} \end{Bmatrix}$$

Recall the elem dofs



$$\begin{Bmatrix} d_i \\ d_{i+1} \end{Bmatrix} = \begin{Bmatrix} d_1^{(i)} \\ d_2^{(i)} \end{Bmatrix} = \underline{d}^{(i)}$$

$$\left\{\begin{matrix} w_i \\ w_{i+1} \end{matrix}\right\} = \left\{\begin{matrix} w_1^{(i)} \\ w_2^{(i)} \end{matrix}\right\} = \underline{w}^{(i)}$$

$$\boxed{\begin{aligned} \beta &= \int_{x_i}^{x_{i+1}} \underbrace{\left(\underline{B}\,\underline{w}^{(i)}\right)}_{1\times 1} \underbrace{(EA)}_{1\times 1} \underbrace{\left(\underline{B}\,\underline{d}^{(i)}\right)}_{1\times 1} dx \\ &= \underline{w}^{(i)} \cdot \left(\underline{k}^{(i)}\,\underline{d}^{(i)}\right) \end{aligned}}$$

$$\beta = \int_{x_i}^{x_{i+1}} (EA) \underbrace{\left(\underline{B}\,\underline{w}^{(i)}\right)}_{1\times 1} \cdot \underbrace{\left(\underline{B}\,\underline{d}^{(i)}\right)}_{1\times 1} dx$$

$$- \quad \underbrace{\left(\underline{B}\,\underline{w}^{(i)}\right)^T}_{\begin{matrix} \shortparallel \\ \underbrace{\underline{w}^{(i)T}\,\underline{B}^T}_{\underline{w}^{(i)} \cdot \underline{B}^T} \end{matrix}} \left(\underline{B}\,\underline{d}^{(i)}\right)$$

$$\beta = \underline{w}^{(i)} \cdot \left(\int \underline{B}^T (EA)\, \underline{B}\, dx\right) \underline{d}^{(i)}$$

$$\boxed{\underset{2\times 2}{\underline{k}}^{(i)} = \int_{x_i}^{x_{i+1}} \underbrace{\underline{B}^T(x)}_{2\times 1} \underbrace{\underbrace{(EA)}_{(x)}}_{1\times 1} \underbrace{\underline{B}(x)}_{1\times 2} \, dx}$$

$$\underline{B}(x) = \left\lfloor \text{HW} \quad \frac{1}{L^{(i)}} \right\rfloor$$

$$L^{(i)} = x_{i+1} - x_i \quad (\text{length of elem } i)$$

HW6: Consider $EA = \text{const}$

$$\underline{k}^{(i)} = \frac{EA}{L^{(i)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Transf. of var. (coord) from x to $\tilde{x}$ $\equiv$

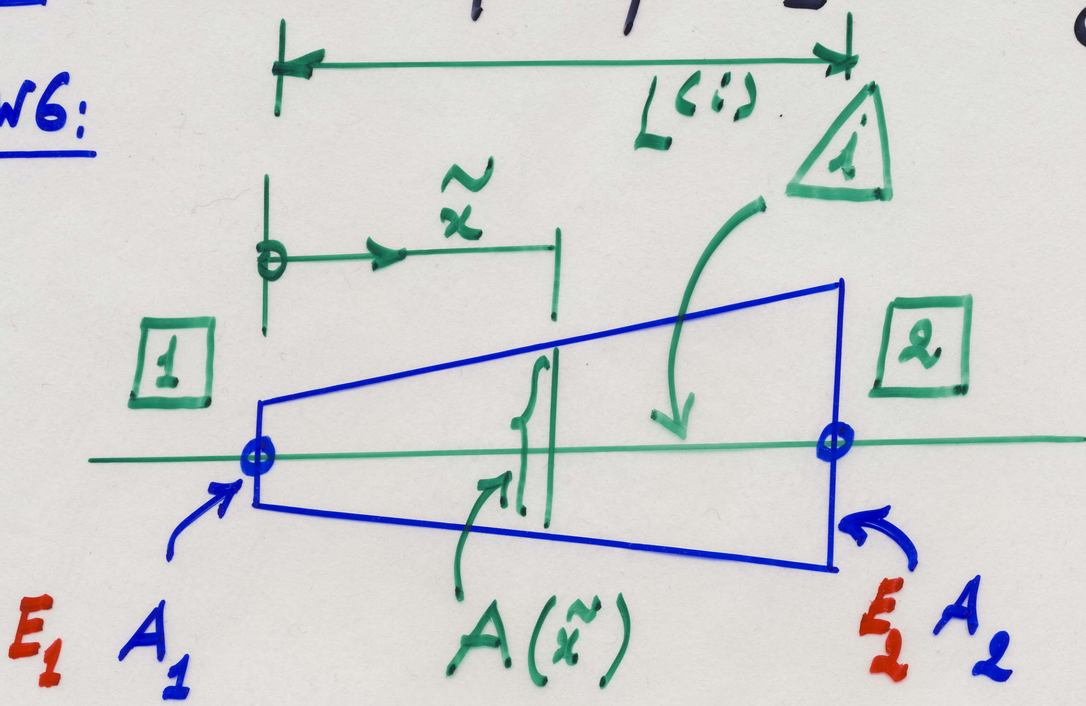
$$\tilde{x} := x - x_i$$

$$d\tilde{x} = dx$$

$$\boxed{\underline{k}^{(i)} = \int_{\tilde{x}=0}^{\tilde{x}=L^{(i)}} \underline{B}^{T}(\tilde{x})\,(EA)(\tilde{x})\,\underline{B}(\tilde{x})\,d\tilde{x}}$$

HW6: Find expr. for $\underline{k}^{(i)}$ using ↺

HW6:



$$A(\tilde{x}) = N_1^{(i)}(\tilde{x})\,A_1 + N_2^{(i)}(\tilde{x})\,A_2$$

$$E(\tilde{x}) = N_1^{(i)}(\tilde{x})\,\cancel{A}_1 \, E_1 + N_2^{(i)}(\tilde{x})\,E_2$$

Find $\underset{2\times 2}{\underline{k}^{(i)}}$ as func of $\{A_1, A_2, E_1, E_2, L^{(i)}\}$

p. 31-4 : $N_1^{(i)}(\tilde{x})$ = HW6

$$N_2^{(i)}(\tilde{x}) = \frac{\tilde{x}}{L^{(i)}} = \begin{cases} 0 & \text{at } \tilde{x} = 0 \\ 1 & \text{at } \tilde{x} = L^{(i)} \end{cases}$$

Shape funcs $N_1^{(i)}$, $N_2^{(i)}$
(basis)

HW6: book p. 159 Set $E_1 = E_2 = E$

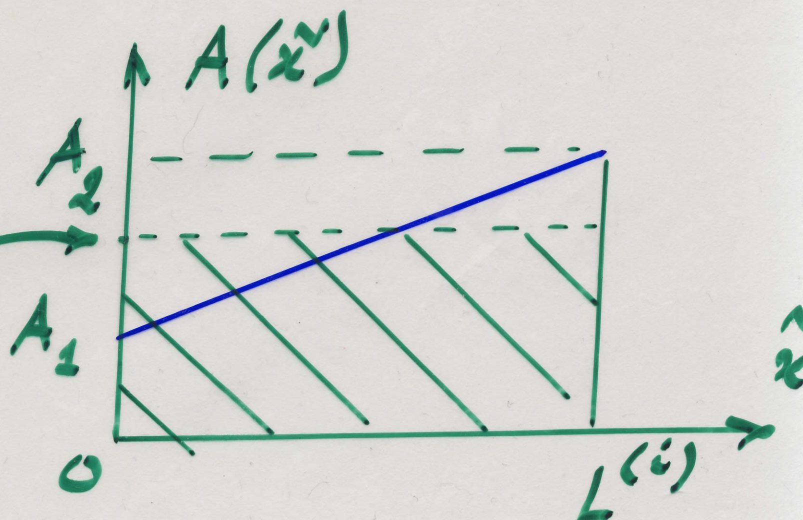
Let $A(\tilde{x})$ be linear as on p. 33-5

Obtain $\underline{k}^{(i)}$ from previous pb

and Comp. to expr. given in book

$$\frac{E}{L^{(i)}} \underbrace{\frac{(A_1 + A_2)}{2}}_{\text{Ave. area}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \underline{k}^{(i)}$$

(Aranus) $\frac{A_1 + A_2}{2}$



Next, compare the general $\underline{k}^{(i)}$ on p. 33-5 to stiff. mat. obtained by using $\frac{1}{2}(A_1 + A_2)$ and $\frac{1}{2}(E_1 + E_2)$

Note: $E_1 \neq E_2$.

$$\frac{(E_1 + E_2)(A_1 + A_2)}{4L^{(i)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \underline{k}^{(i)}_{ave}$$

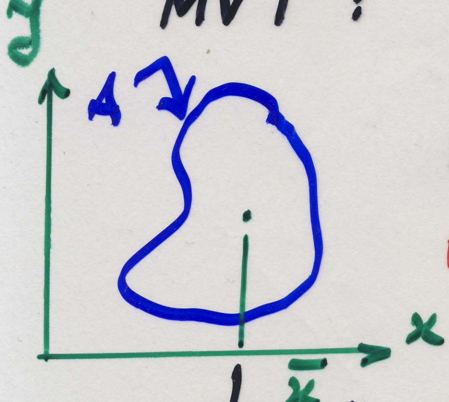
Find $\underline{k}^{(i)} - \underline{k}^{(i)}_{ave}$.

__Rem:__ Recall the Mean Value Thm (MVT) and its rel. to centroid:

MVT: $$\int_{x=a}^{x=b} f(x)\,dx = f(\bar{x})[b-a]$$

for $\bar{x} \in [a, b]$

"belongs to" $a \leq \bar{x} \leq b$



Centroid $$\int_A x\,dA = \bar{x}\int_A dA = \bar{x}A$$

$$\int_{x=a}^{x=b} f(x)\, g(x)\, dx = f(\bar{x})\, g(\bar{x})\, [b-a]$$

$$a \leq \bar{x} \leq b$$

But

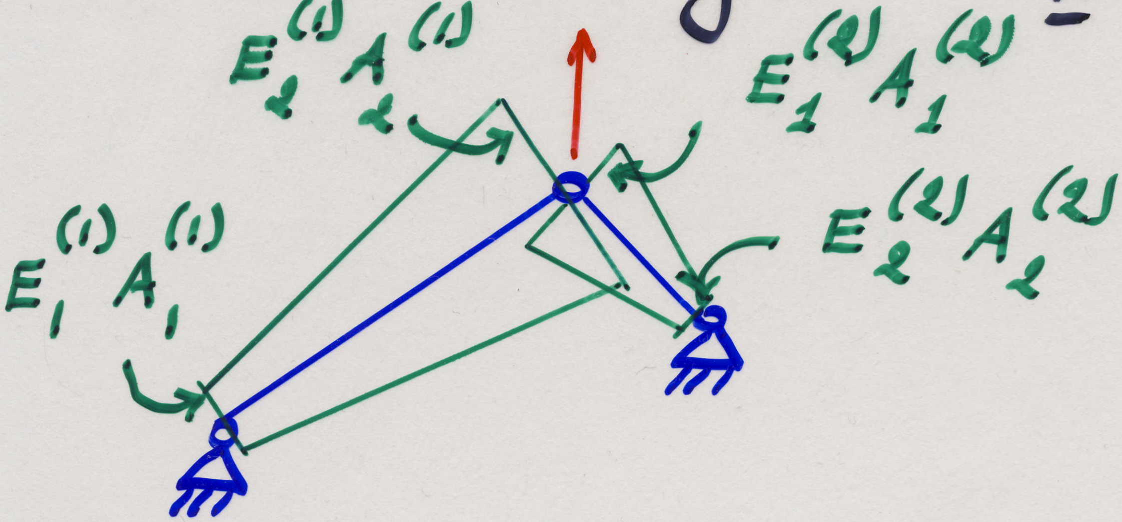
$$f(\bar{x}) \neq \underbrace{\frac{1}{b-a}\int_a^b f(x)\, dx}_{\text{Ave value of } f}$$

In general.

$$g(\bar{x}) \neq \underbrace{\frac{1}{b-a}\int_a^b g(x)\, dx}_{\text{Ave value of } g}$$

HW7:

Modify 2-bar truss code to accommodate general $\underline{k}^{(i)}$ (end Rem on p. 33-5)



Elem 1: $E_1^{(1)} = 2$, $E_2^{(1)} = 4$

$A_1^{(1)} = 0.5$, $A_2^{(1)} = 1.5$

Elem 2: $E_1^{(2)} = 3$, $E_2^{(2)} = 7$

$A_1^{(2)} = 1$, $A_2^{(2)} = 3$

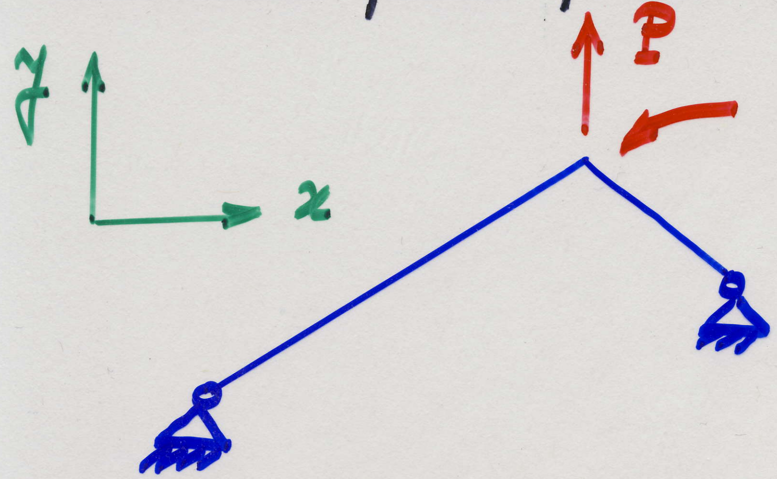
- comp. soln for 2-bar truss w/ tapered elems
- Plot deformed shape, and also deformed shape for previous 2-bar truss w/ ave. E and ave. A.

Frame elem = truss (bar) elem ⊕ beam elem

axial deform.

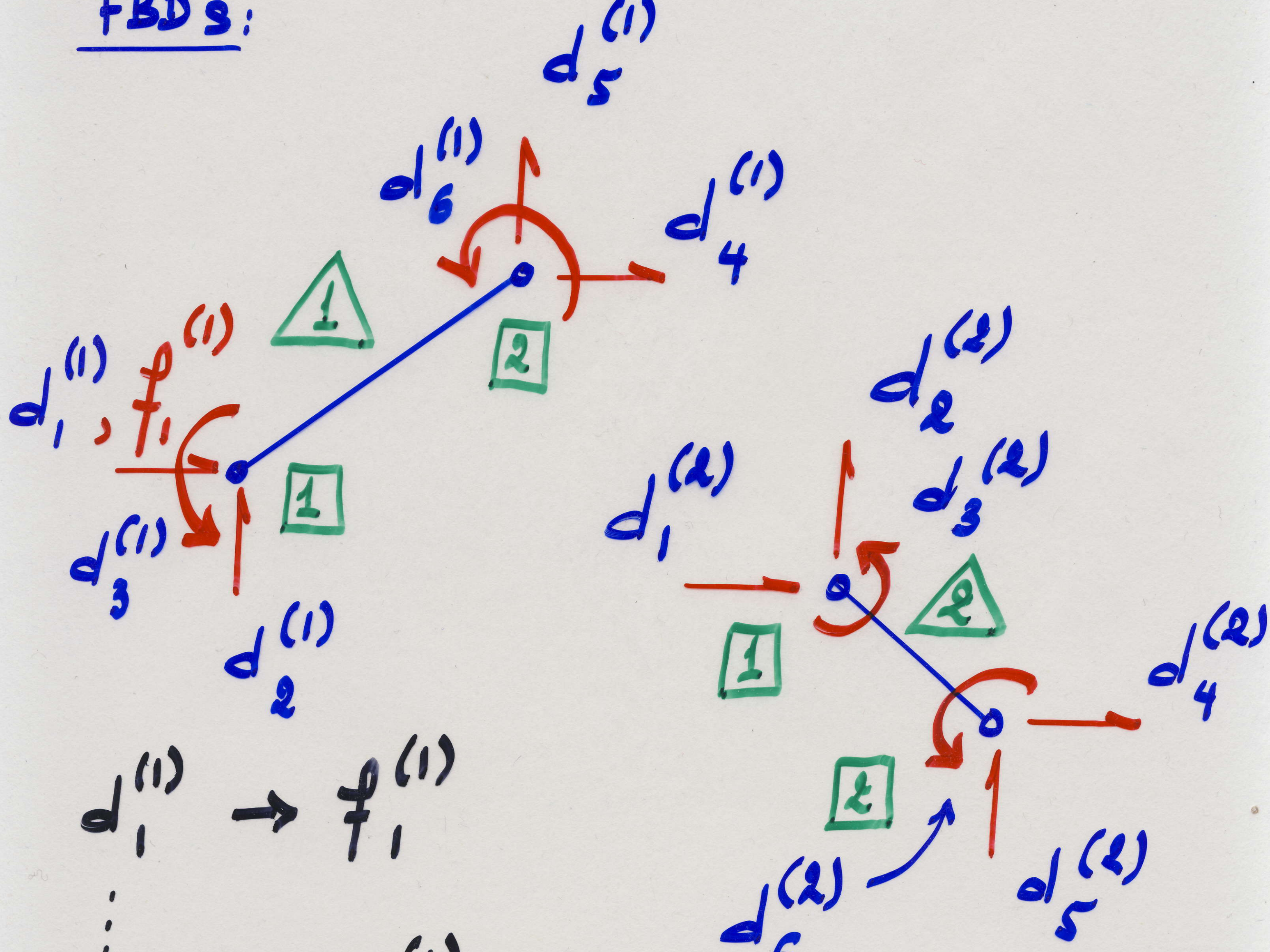
transv. deform.

Model frame pb w/ 2 elems:



rigid connection (angle betw. 2 elems const. after deform.)

FBDs:



$$d_1^{(1)} \rightarrow f_1^{(1)}$$

$$\vdots$$

$$d_6^{(1)} \rightarrow f_6^{(1)}$$

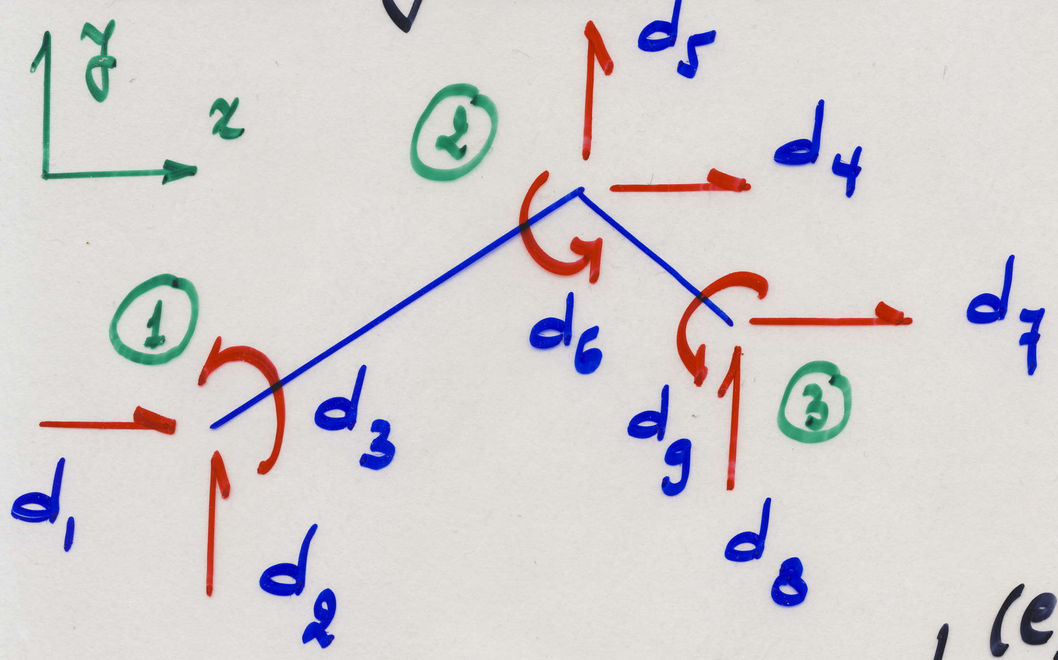
In general, $\underbrace{d_i^{(e)}}_{\text{gen. disp.}} \longrightarrow \underbrace{f_i^{(e)}}_{\text{gen. forces}}$

$e = 1, 2$

$i = 1, \dots, 6$

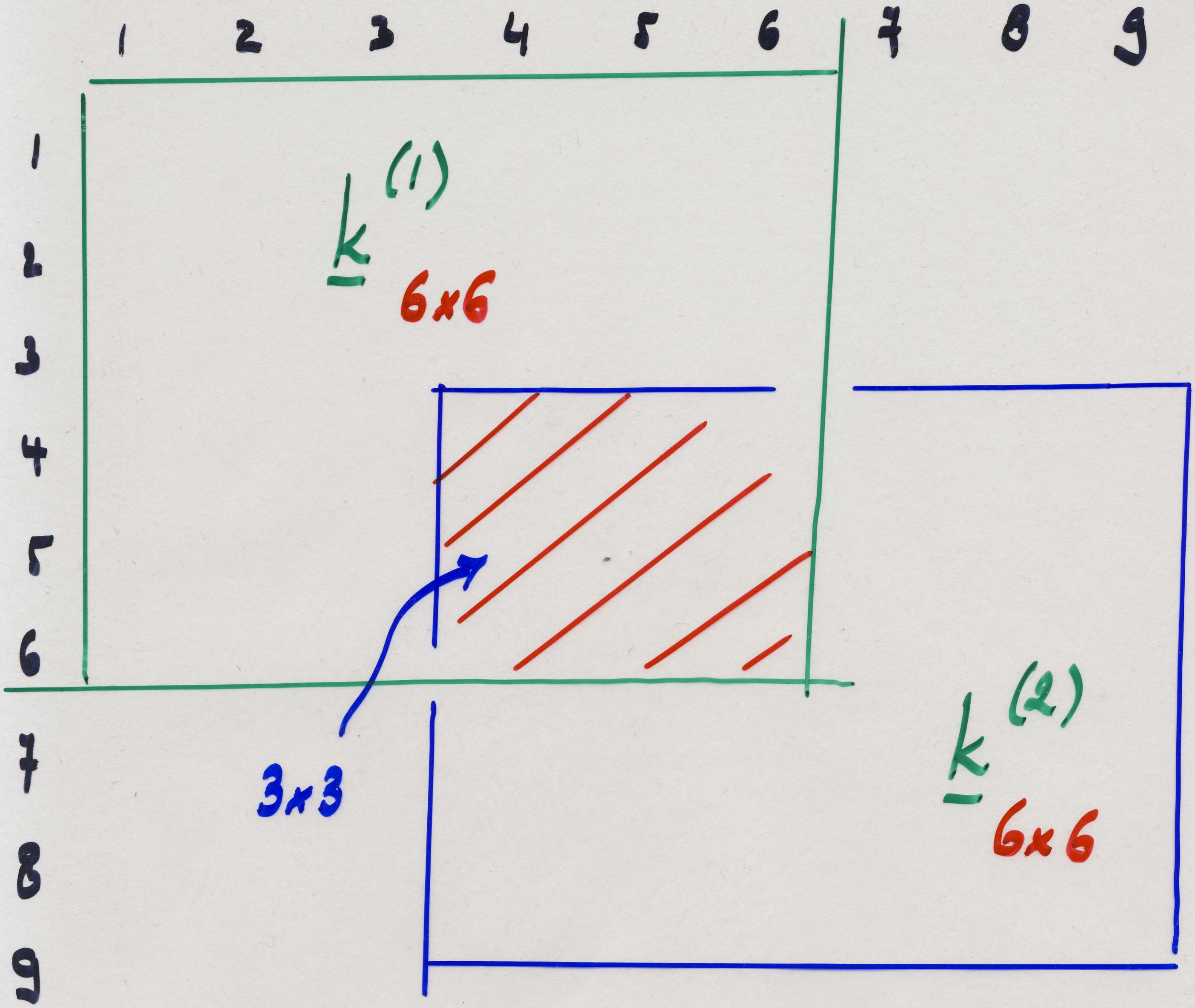
$$\left.\begin{array}{l} d_3^{(e)} \\ d_6^{(e)} \end{array}\right\} \text{rot. dofs} \longrightarrow \left.\begin{array}{l} f_3^{(e)} \\ f_6^{(e)} \end{array}\right\} \text{bend. mom.}$$

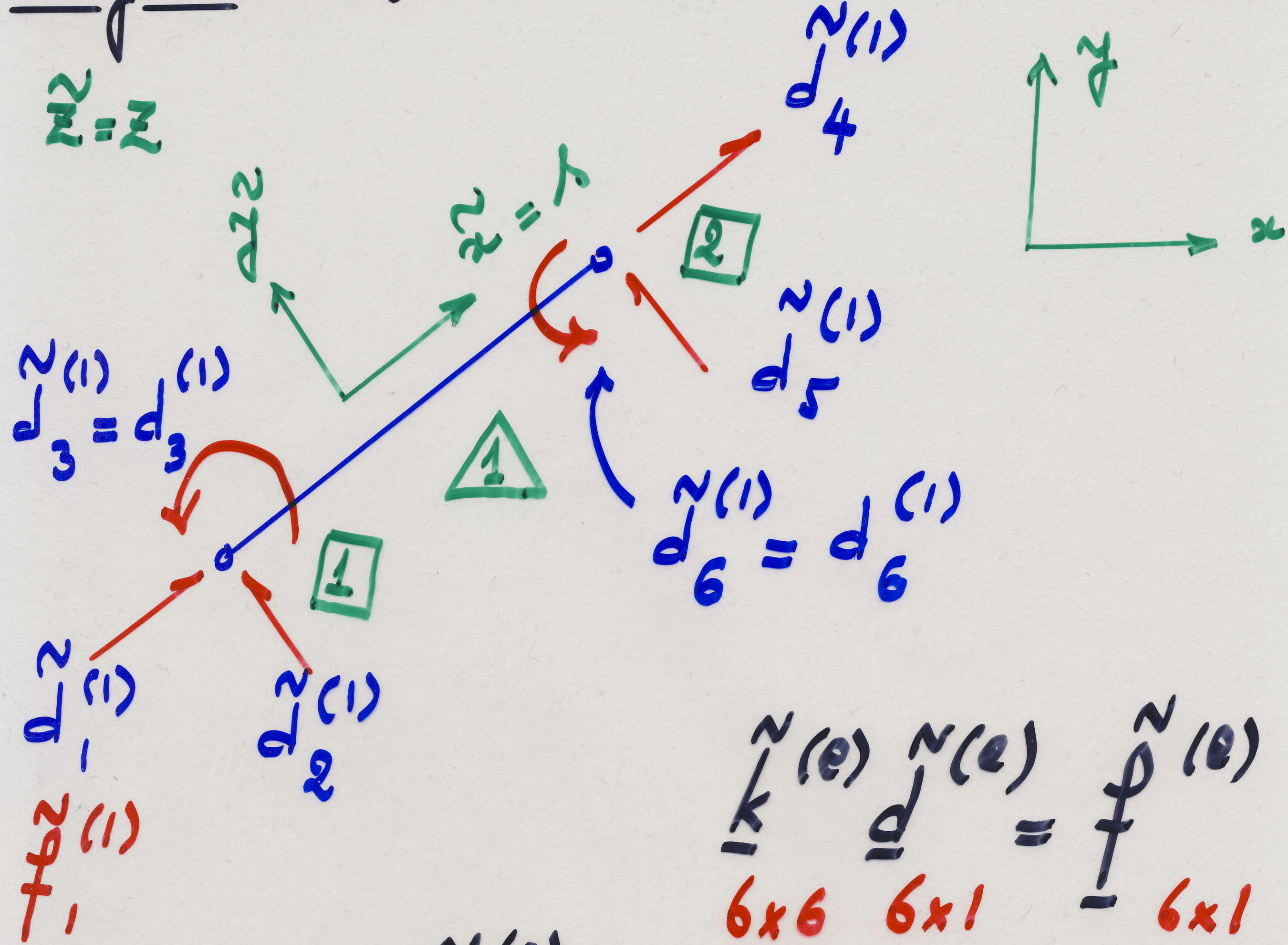
2-D Frame global dofs:



2 elem stiff. mat. $\underset{6\times 6}{\underline{k}^{(e)}}$, $e = 1, 2$

Global stiff. mat. $\underset{9\times 9}{\underline{K}} = \overset{e=2}{\underset{e=1}{\mathsf{A}}} \underset{6\times 6}{\underline{k}^{(e)}}$


1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9
$\underline{k}^{(1)}_{6\times6}$
3x3
$\underline{k}^{(2)}_{6\times6}$



$$\underset{6\times 6}{\underline{\tilde{k}}^{(e)}} \; \underset{6\times 1}{\underline{\tilde{d}}^{(e)}} = \underset{6\times 1}{\underline{\tilde{f}}^{(e)}}$$

$$\underline{\tilde{d}}^{(e)} = \begin{Bmatrix} \tilde{d}_1^{(e)} \\ \vdots \\ \tilde{d}_6^{(e)} \end{Bmatrix}, \quad \underline{\tilde{f}}^{(e)} = \begin{Bmatrix} \tilde{f}_1^{(e)} \\ \vdots \\ \tilde{f}_6^{(e)} \end{Bmatrix}$$

Note: $\tilde{f}_3^{(e)} = f_3^{(e)}$, $\tilde{f}_6^{(e)} = f_6^{(e)}$

Moments (about $\tilde{z} = z$)

$$\underset{6\times 6}{\underline{\tilde{k}}} = \begin{array}{c} \begin{array}{cccccc} \quad\tilde{d}_1 \quad & \quad\tilde{d}_2\quad & \quad\tilde{d}_3\quad & \quad\tilde{d}_4\quad & \quad\tilde{d}_5\quad & \quad\tilde{d}_6\quad \end{array} \\ \left[\begin{array}{cccccc} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ & & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ & & & \frac{EA}{L} & 0 & 0 \\ & \text{sym.} & & & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ & & & & & \frac{4EI}{L} \end{array}\right] \end{array} \begin{array}{c} \\ \tilde{d}_1 \\ \tilde{d}_2 \\ \tilde{d}_3 \\ \tilde{d}_4 \\ \tilde{d}_5 \\ \tilde{d}_6 \end{array}$$

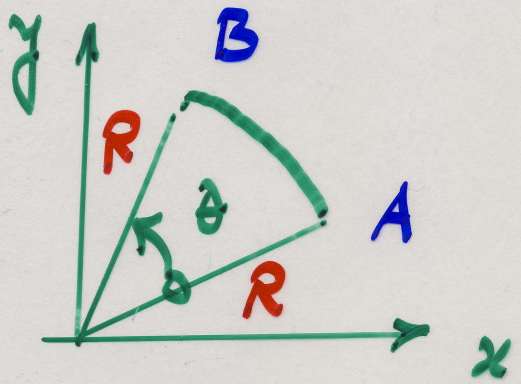
$$[\tilde{d}_1] = L = [\tilde{d}_i] \qquad i = 1, 2, 4, 5$$

"dimen. of" "length" disp.

$$[\tilde{d}_3] = 1 = [\tilde{d}_6]$$

rot.

(no. dimen)



$$\underbrace{\widehat{AB}}_{\text{arc length}} = R \cdot \theta$$

meas. in rad

$$\theta = \frac{\widehat{AB}}{R}$$

$$[\theta] = \frac{[\widehat{AB}]}{[R]} = \frac{L}{L} = 1$$

$$\sigma = E\,\varepsilon \Rightarrow [\sigma] = [E]\,\underbrace{[\varepsilon]}_{= 1}$$

why?