Divergence and Curl (3A)

- Divergence
- Curl
- Green's Theorem

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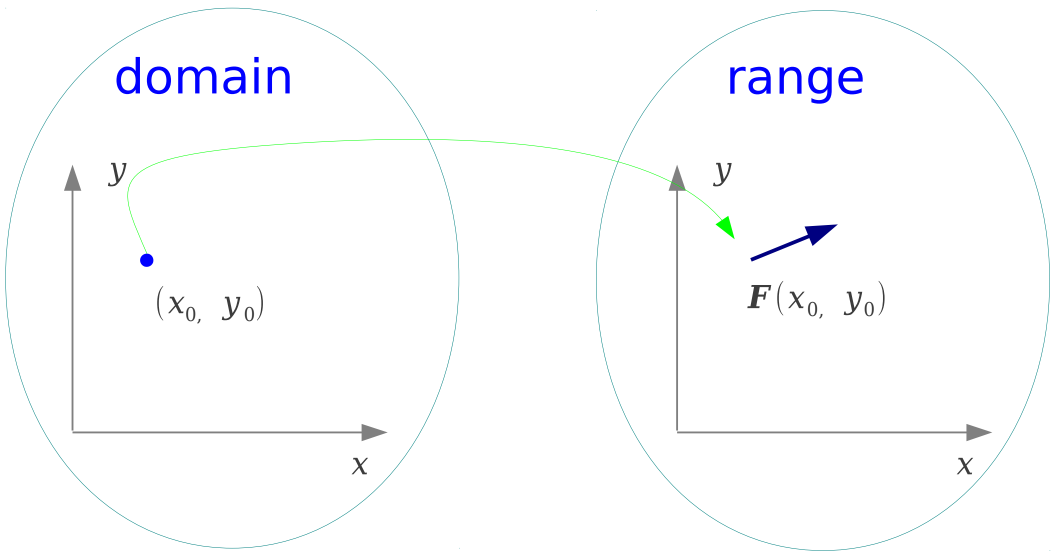
2-D Vector Field

a given point in a 2-d space ➡ A vector

$(x_{0,}\ \ y_0)$ $\langle\, M(x_{0,} y_0),\ \ N(x_{0,} y_0)\,\rangle$



2 functions

$(x_{0,}\ \ y_0) \longrightarrow M(x_{0,} y_0)$

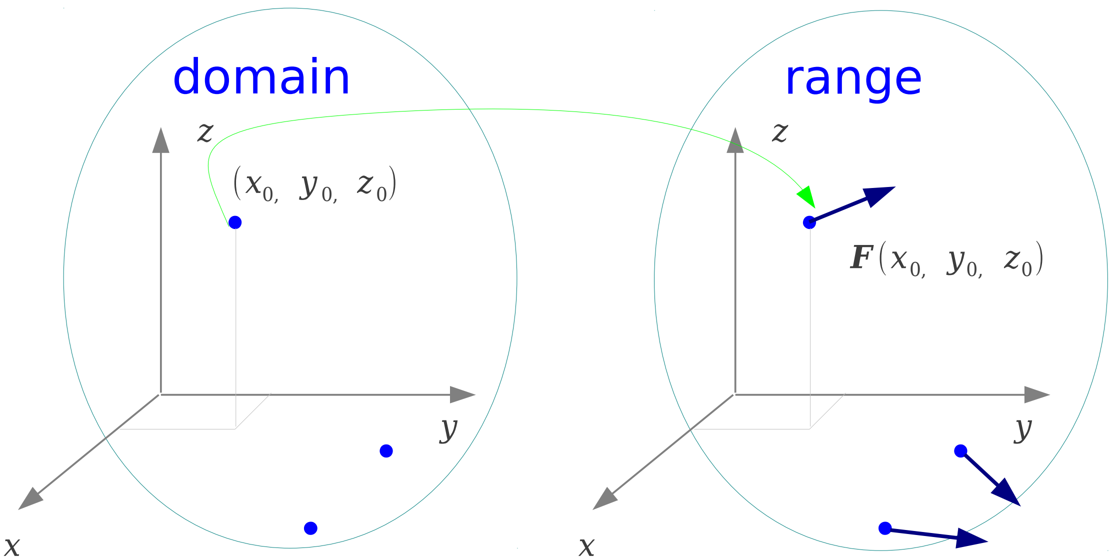
$(x_{0,}\ \ y_0) \longrightarrow N(x_{0,} y_0)$

$(x_{0,}\ \ y_0) \longrightarrow \boldsymbol{F}(x_{0,}\ \ y_0) = M(x_{0,} y_0)\boldsymbol{i} + N(x_{0,} y_0)\boldsymbol{j}$

3-D Vector Field

A given point in a 3-d space ➡ A vector

$(x_{0,}\ \ y_{0,}\ \ z_0)$ $\langle\, M(x_0, y_0, z_0),\ \ N(x_0, y_0, z_0),\ \ P(x_0, y_0, z_0)\,\rangle$



3 functions

$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow M(x_{0,} y_{0,} Z_0)$

$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow N(x_{0,} y_{0,} z_0)$

$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow P(x_{0,} y_{0,} z_0)$

$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow \boldsymbol{F}(x_{0,}\ \ y_{0,}\ \ z_0) \ = \ M(x_{0,} y_{0,} z_0)\boldsymbol{i} \ + \ N(x_{0,} y_{0,} z_0)\boldsymbol{j} \ + \ P(x_{0,} y_{0,} z_0)\boldsymbol{k}$

2-Divergence

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y + \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x \qquad = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y$$

Flux density

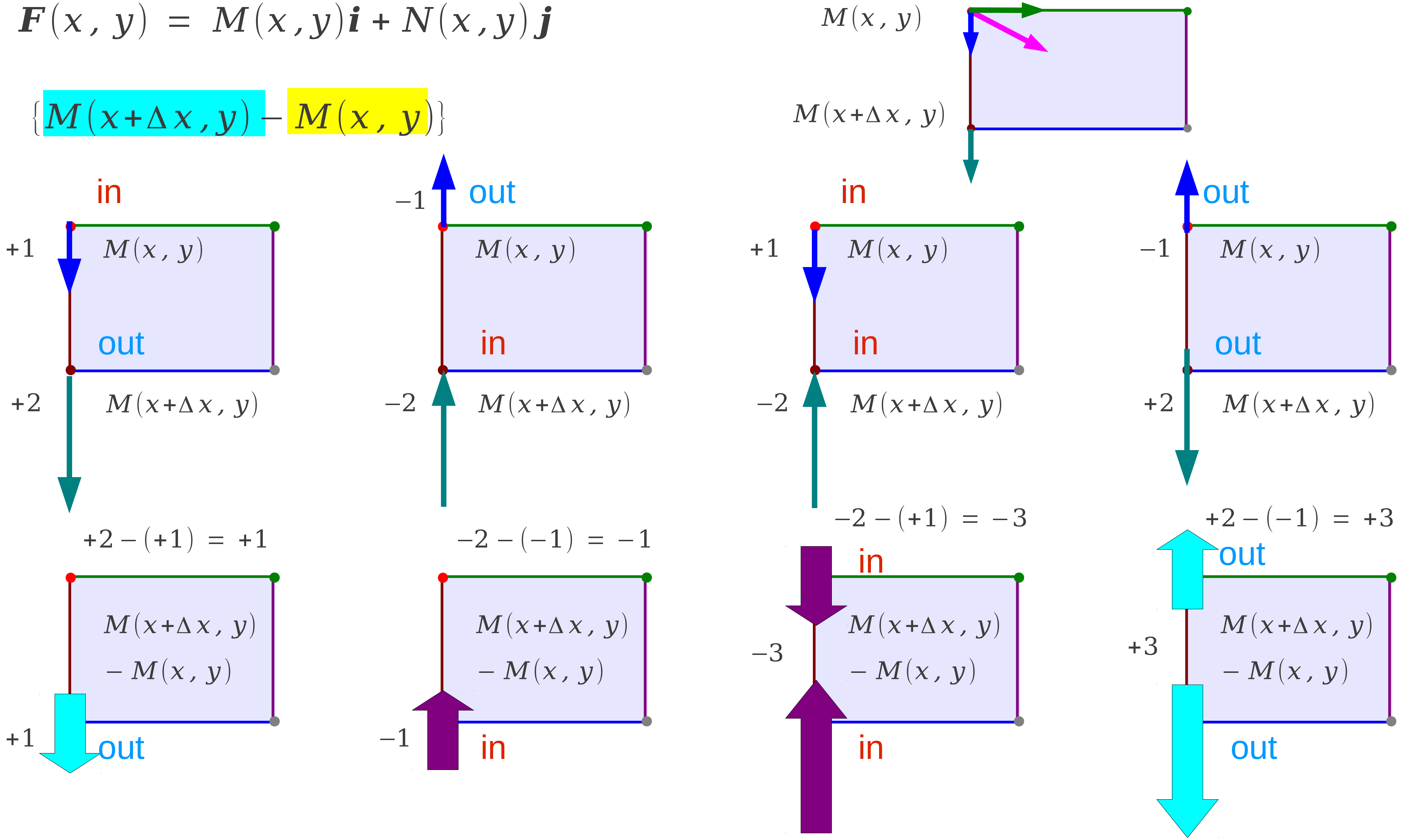
$$= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

Divergence of **F** Flux Density

Inward & Outward Flow – Top, Bottom

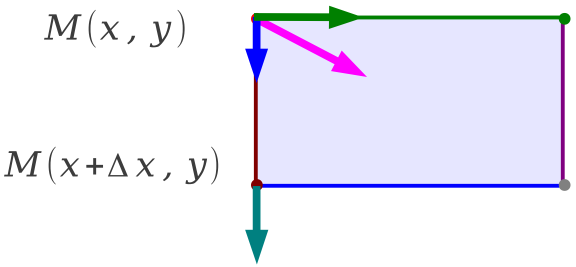
$$\boldsymbol{F}(x, y) = M(x, y)\,\boldsymbol{i} + N(x, y)\,\boldsymbol{j}$$

$$\{M(x+\Delta x, y) - M(x, y)\}$$



Inward & Outward Net Flow – Top, Bottom

$$\boldsymbol{F}(x, y) = M(x, y)\,\boldsymbol{i} + N(x, y)\,\boldsymbol{j}$$



$$\frac{\{M(x+\Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x} > 0$$

- **Positive Slope** of a tangent line parallel to the x axis
- **Outward** bound net flow along the x axis

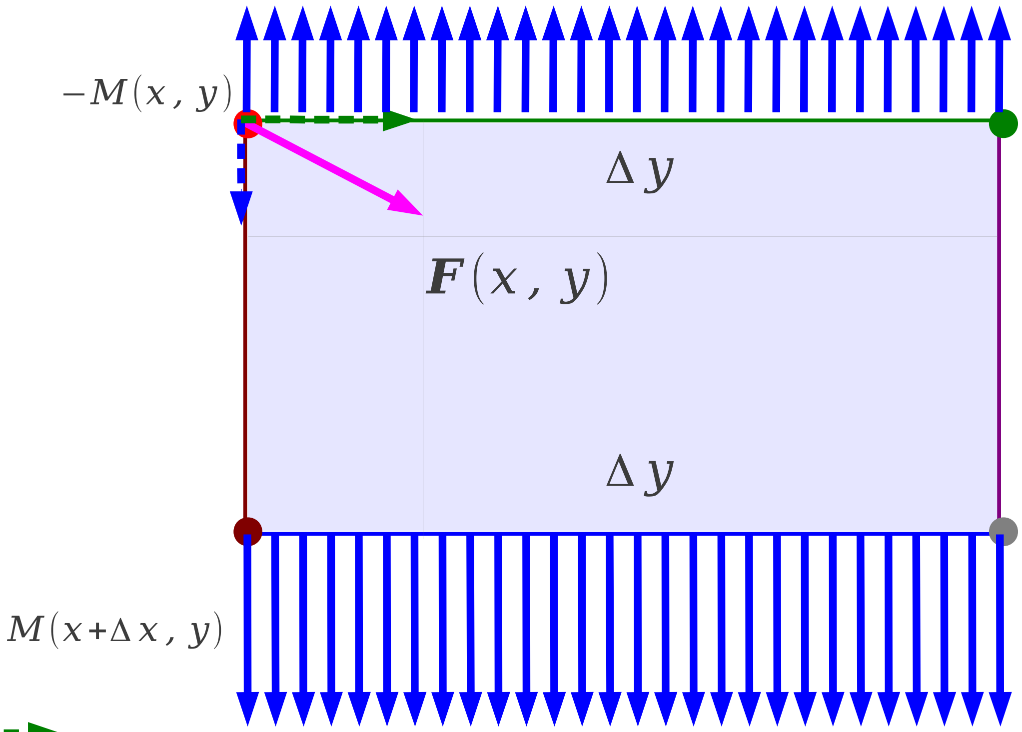
$$\frac{\{M(x+\Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x} < 0$$

- **Negative Slope** of a tangent line parallel to the x axis
- **Inward** bound net flow along the x axis

Net Flow – Top, Bottom

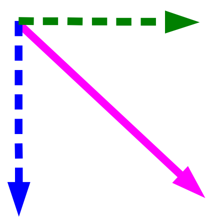
$$\boldsymbol{F}(x, y) = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$$

$$\boldsymbol{F}(x, y)\cdot(-\boldsymbol{i})\Delta y = -M(x, y)\Delta y$$



$$\frac{\{M(x+\Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x}$$

$$\{M(x+\Delta x, y) - M(x, y)\}\Delta y = \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y$$

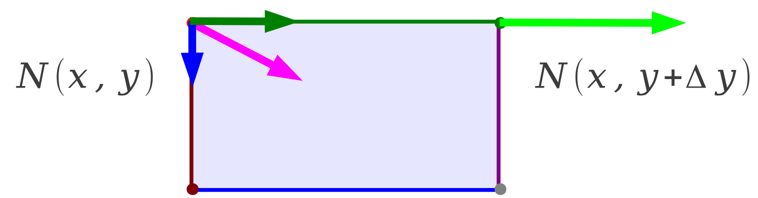


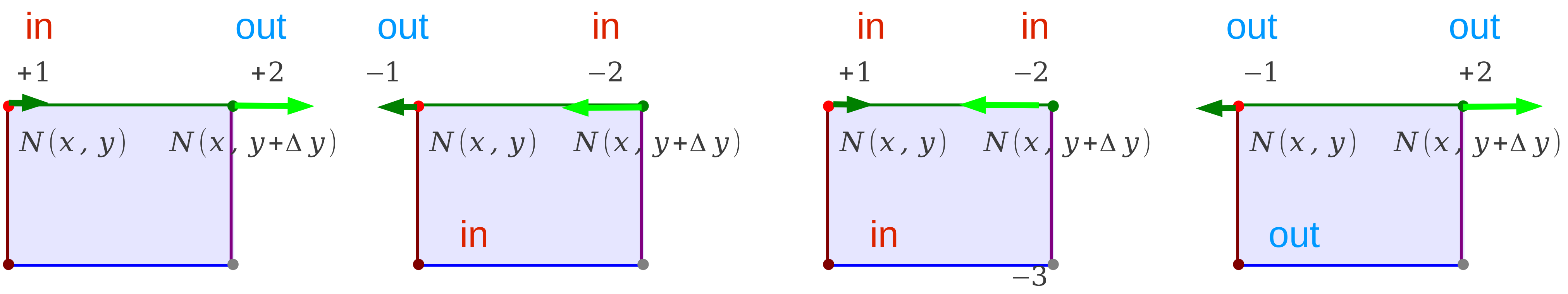
$$\boldsymbol{F}(x+\Delta x, y)\cdot(+\boldsymbol{i})\Delta y = M(x+\Delta x, y)\Delta y$$

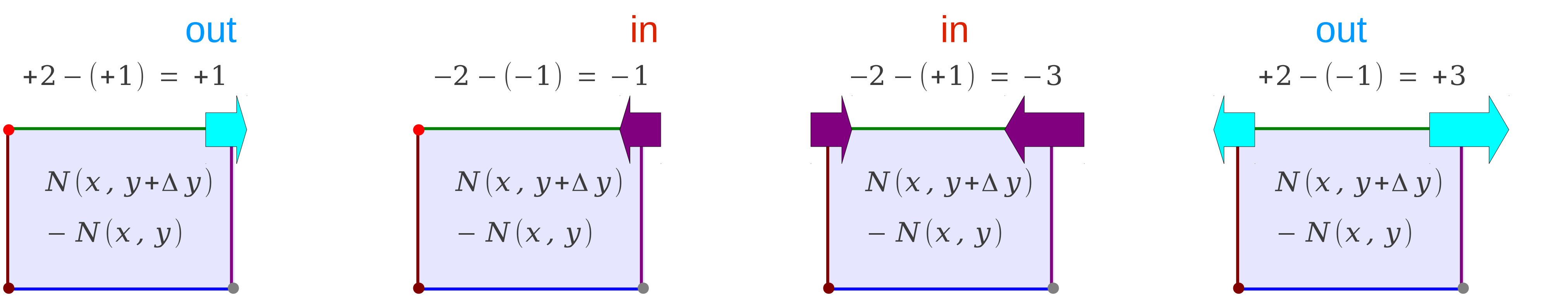
Inward & Outward Flow – Left, Right

$$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$$

$$\{N(x, y+\Delta y) - N(x, y)\}$$

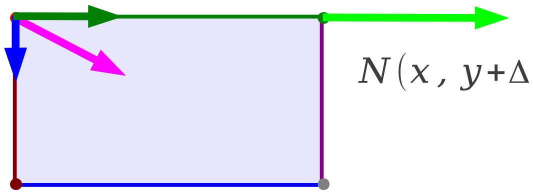






Inward & Outward Net Flow – Left, Right

$$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$$



$$\frac{\{N(x,y+\Delta y) - N(x, y)\}}{\Delta y} \approx \frac{\partial N}{\partial y} > 0$$

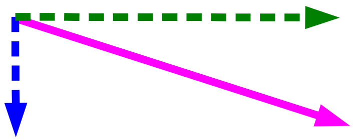
- **Positive Slope** of a tangent line parallel to the y axis
- **Outward** bound net flow along the y axis

$$\frac{\{N(x,y+\Delta y) - N(x, y)\}}{\Delta y} \approx \frac{\partial N}{\partial y} < 0$$

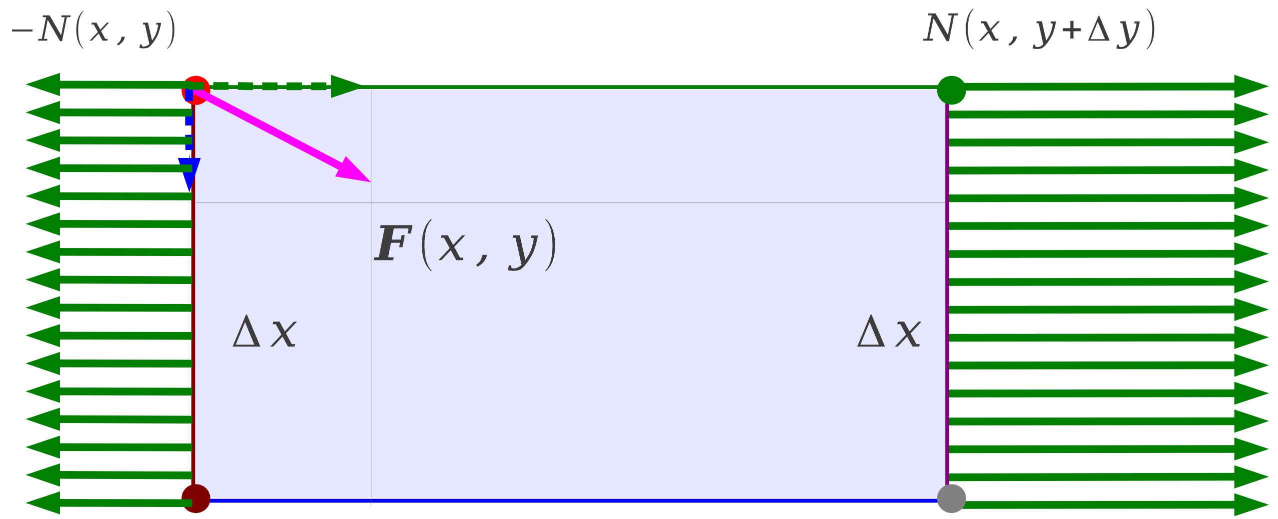
- **Negative Slope** of a tangent line parallel to the y axis
- **Inward** bound net flow along the y axis

Net Flow – Left, Right

$$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$$



$$\boldsymbol{F}(x, y) \cdot (-\boldsymbol{j})\Delta x = -N(x, y)\Delta x$$



$$\boldsymbol{F}(x, y+\Delta y) \cdot (+\boldsymbol{j})\Delta x = N(x, y+\Delta y)\Delta x$$

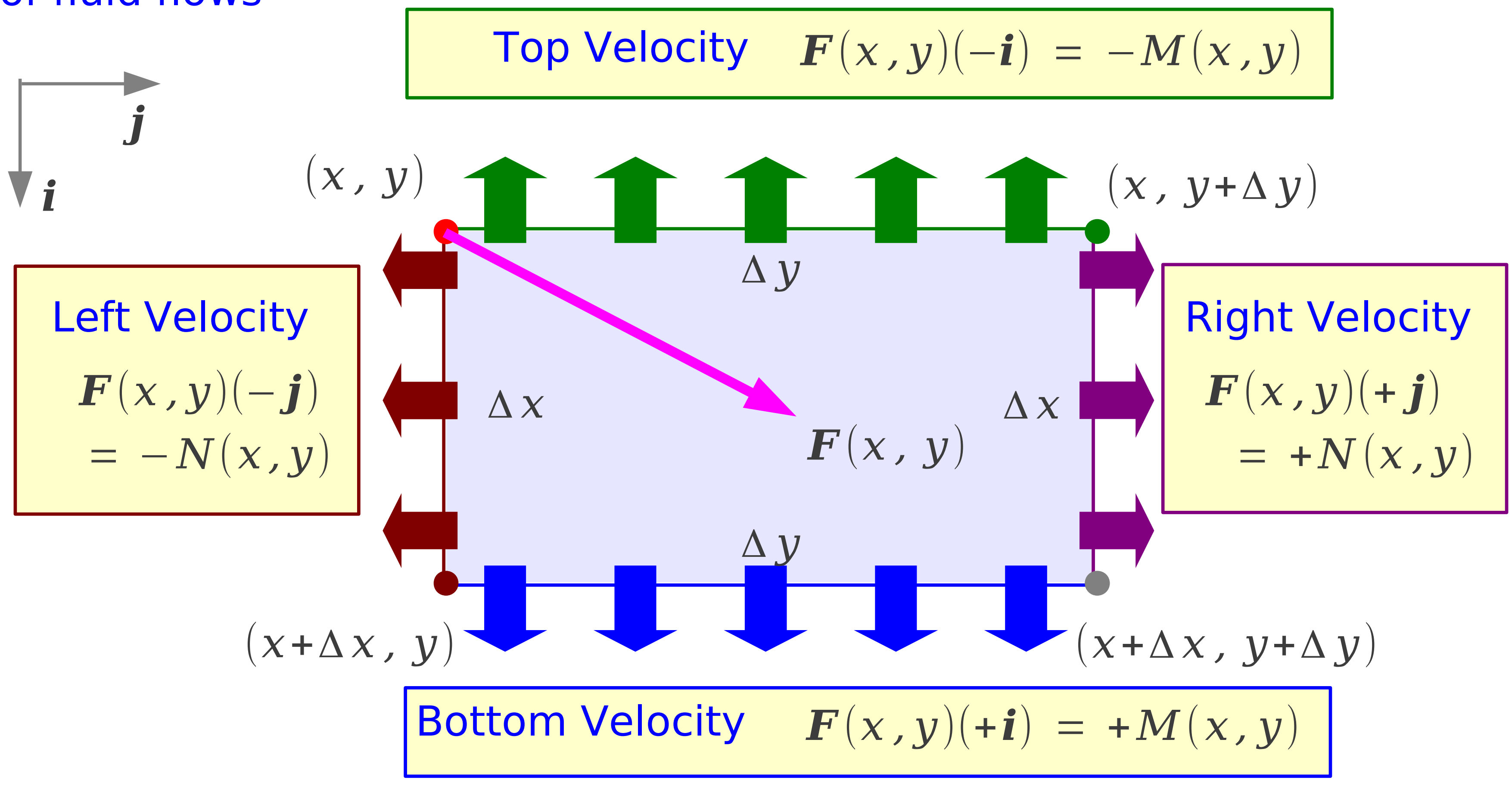
$$\frac{\{N(x, y+\Delta y) - N(x, y)\}}{\Delta y} \approx \frac{\partial N}{\partial y}$$

$$\{N(x, y+\Delta y) - N(x, y)\}\Delta x = \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$$

2-D Divergence (1)

Velocity Fields
of fluid flows

$$\mathbf{F}(x, y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$$



Flow rate of outward bound fluid

2-D Divergence (2)

The rate at which fluid leave the rectangle

Across top $\boldsymbol{F}(x, y) \cdot (-\boldsymbol{i}) \Delta y = -M(x, y) \Delta y$

Across bottom $\boldsymbol{F}(x+\Delta x, y) \cdot (+\boldsymbol{i}) \Delta y = M(x+\Delta x, y) \Delta y$

Across left $\boldsymbol{F}(x, y) \cdot (-\boldsymbol{j}) \Delta x = -N(x, y) \Delta x$

Across right $\boldsymbol{F}(x, y+\Delta y) \cdot (+\boldsymbol{j}) \Delta x = N(x, y+\Delta y) \Delta x$

Across top + bottom $\{M(x+\Delta x, y) - M(x, y)\} \Delta y = \left(\frac{\partial M}{\partial x} \Delta x\right) \Delta y$

Across left + right $\{N(x, y+\Delta y) - N(x, y)\} \Delta x = \left(\frac{\partial N}{\partial y} \Delta y\right) \Delta x$

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x} \Delta x\right) \Delta y + \left(\frac{\partial N}{\partial y} \Delta y\right) \Delta x = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) \Delta x \Delta y$$

Flux density $= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$ Divergence of **F** Flux Density

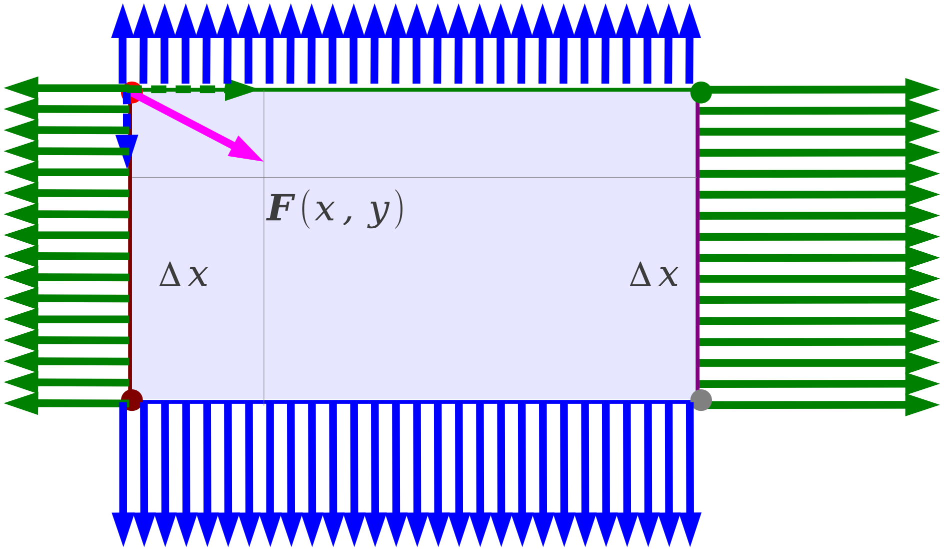
2-D Divergence and Del Operator

2-D Vector Field

$$\boldsymbol{F}(x,\, y) \;=\; M(x,y)\,\boldsymbol{i} + N(x,y)\,\boldsymbol{j}$$

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y \;+\; \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x \qquad = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x\, \Delta y$$



Flux density $= \left(\dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}\right)$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j}\right) \cdot (M\,\boldsymbol{i} + N\,\boldsymbol{j})$$

$$= \nabla \cdot \boldsymbol{F}$$

Divergence of **F**

3-D Divergence and Del Operator

2-D Vector Field $\qquad \boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$

Divergence of **F**

$$= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j}\right) \cdot (M\boldsymbol{i} + N\boldsymbol{j}) = \nabla \cdot \boldsymbol{F}$$

3-D Vector Field $\qquad \boldsymbol{F}(x, y, z) = M(x,y,z)\boldsymbol{i} + N(x,y,z)\boldsymbol{j} + P(x,y,z)\boldsymbol{k}$

Divergence of **F**

$$= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \cdot (M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{k}) = \nabla \cdot \boldsymbol{F}$$

2-Curl (1 – 5)

Circulation around rectangle boundary

$$\approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y \quad - \left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x \qquad = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x \Delta y$$

Circulation density

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

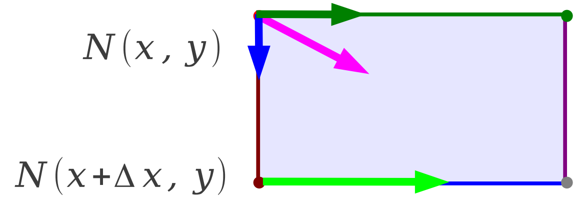
k-component
Curl of **F**

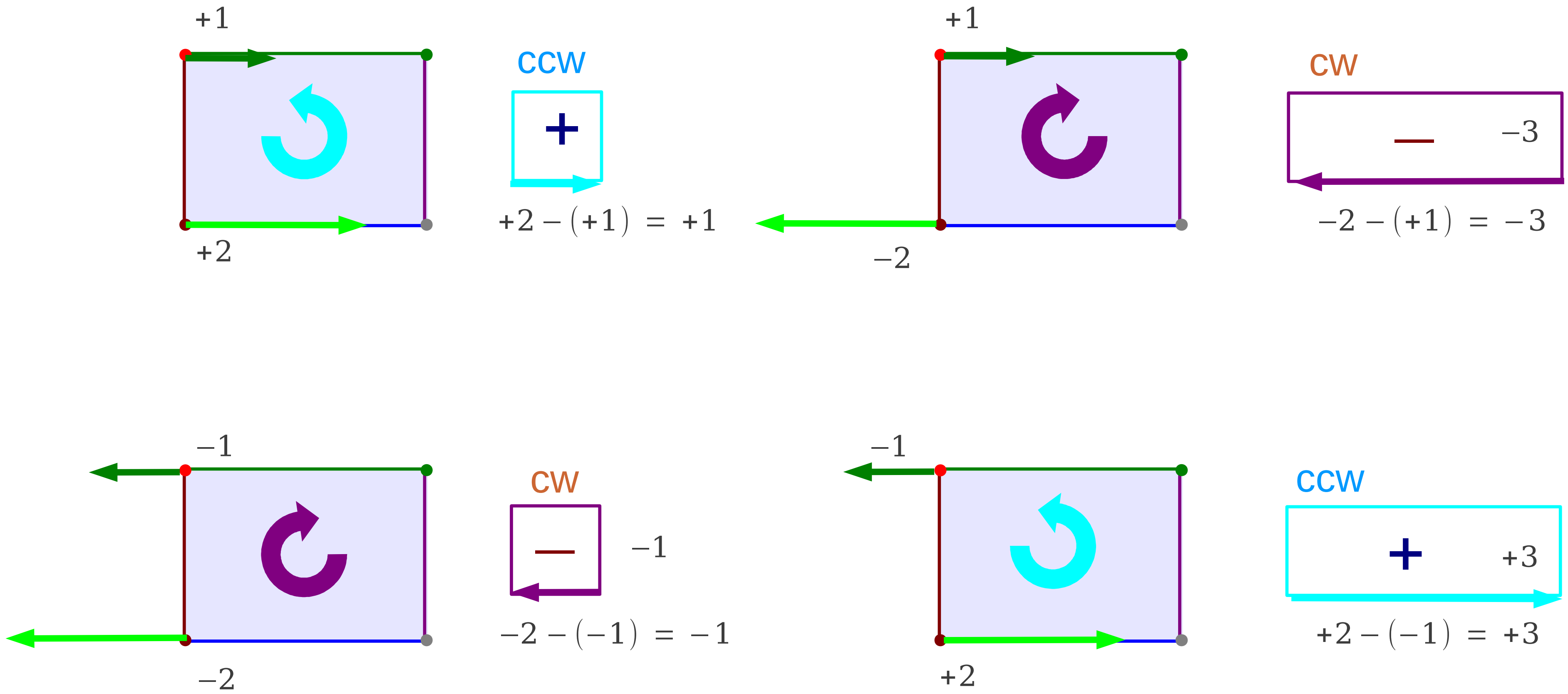
Circulation Density

CW & CCW Spin – Top, Bottom

$$\boldsymbol{F}(x, y) = M(x, y)\,\boldsymbol{i} + N(x, y)\,\boldsymbol{j}$$

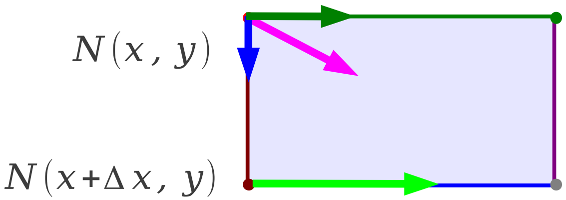
$$\{N(x+\Delta x, y) - N(x, y)\}$$





CW & CCW Net Spin – Top, Bottom

$$\boldsymbol{F}(x, y) = M(x, y)\,\boldsymbol{i} + N(x, y)\,\boldsymbol{j}$$



$$\frac{\{N(x+\Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} > 0$$

- **Positive Slope** of a tangent line parallel to the x axis
- **CCW** bound net flow along the z axis

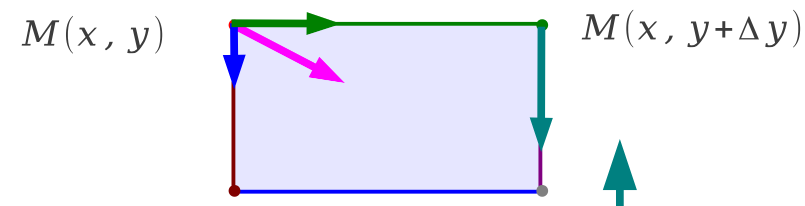
$$\frac{\{N(x+\Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} < 0$$

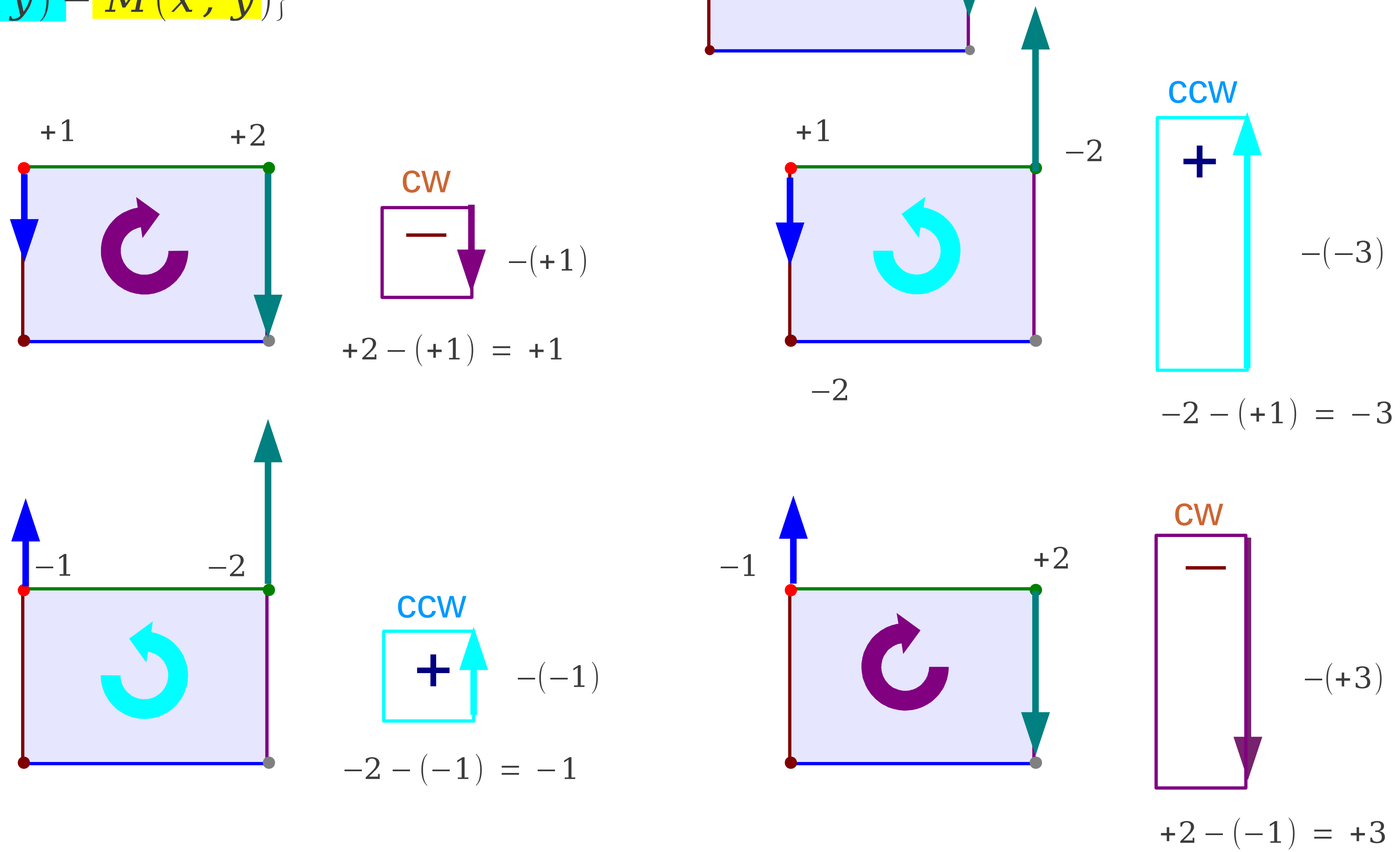
- **Negative Slope** of a tangent line parallel to the x axis
- **CW** bound net flow along the z axis

CW & CCW Spin – Left, Right

$$\boldsymbol{F}(x,y) = M(x,y)\,\boldsymbol{i} + N(x,y)\,\boldsymbol{j}$$

$$-\{M(x,y+\Delta y) - M(x,y)\}$$

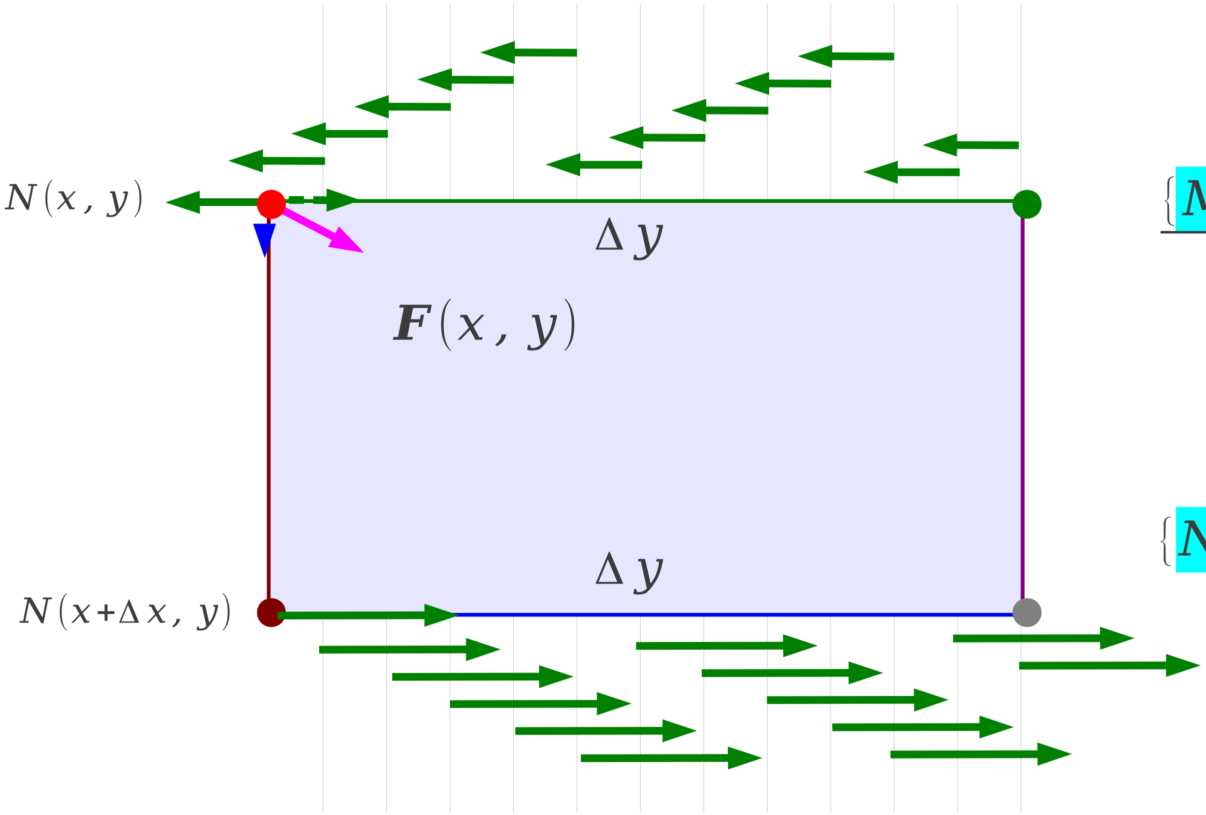




Net Spin – Top, Bottom

$$\boldsymbol{F}(x, y) = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$$

$$\boldsymbol{F}(x, y)\cdot(-\boldsymbol{j})\Delta y = -N(x, y)\Delta y$$



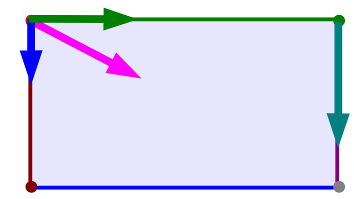
$$\frac{\{M(x+\Delta x, y) - M(x, y)\}}{\Delta x} = \left(\frac{\partial N}{\partial x}\right)$$

$$\{N(x+\Delta x, y) - N(x, y)\}\Delta y = \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y$$

$$\boldsymbol{F}(x+\Delta x, y)\cdot(+\boldsymbol{j})\Delta y = N(x+\Delta x, y)\Delta y$$

CW & CCW Net Spin – Left, Right

$$\boldsymbol{F}(x, y) = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$$



$$-\frac{\{M(x, y+\Delta y) - M(x, y)\}}{\Delta y} \approx -\frac{\partial M}{\partial y} > 0$$

- **Positive Slope** of a tangent line parallel to the y axis
- **CCW** bound net flow along the z axis

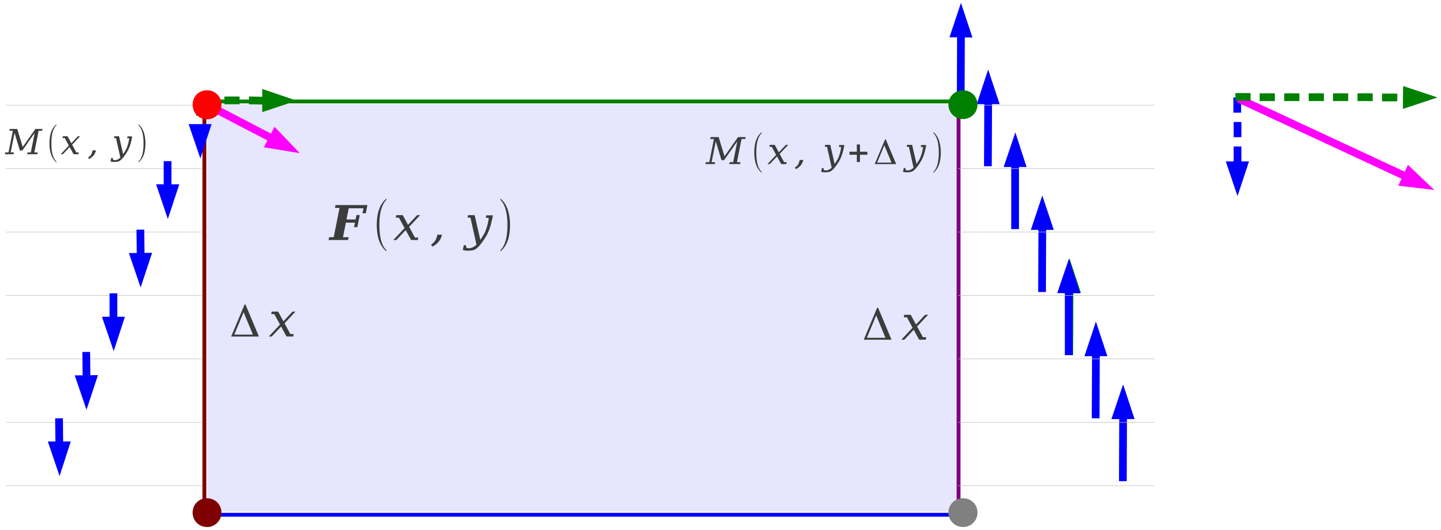
$$-\frac{\{M(x, y+\Delta y) - M(x, y)\}}{\Delta y} \approx -\frac{\partial M}{\partial y} < 0$$

- **Negative Slope** of a tangent line parallel to the y axis
- **CW** bound net flow along the z axis

Net Spin – Left, Right

$$\boldsymbol{F}(x,\ y)\ =\ M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$$

$$\boldsymbol{F}(x,\ y)\cdot(+\boldsymbol{i})\Delta x\ =\ M(x,\ y)\Delta x \qquad \boldsymbol{F}(x,\ y+\Delta y)\cdot(-\boldsymbol{i})\Delta x\ =\ -M(x,\ y+\Delta y)\Delta x$$

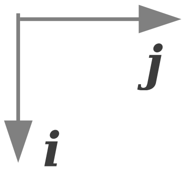


$$\frac{-\{M(x,\ y+\Delta y) - M(x,\ y)\}}{\Delta y}\ =\ -\left(\frac{\partial M}{\partial y}\right)$$

$$-\{M(x,\ y+\Delta y) - M(x,\ y)\}\Delta x\ =\ -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

2-D Curl (1)

Velocity Fields
of fluid flows



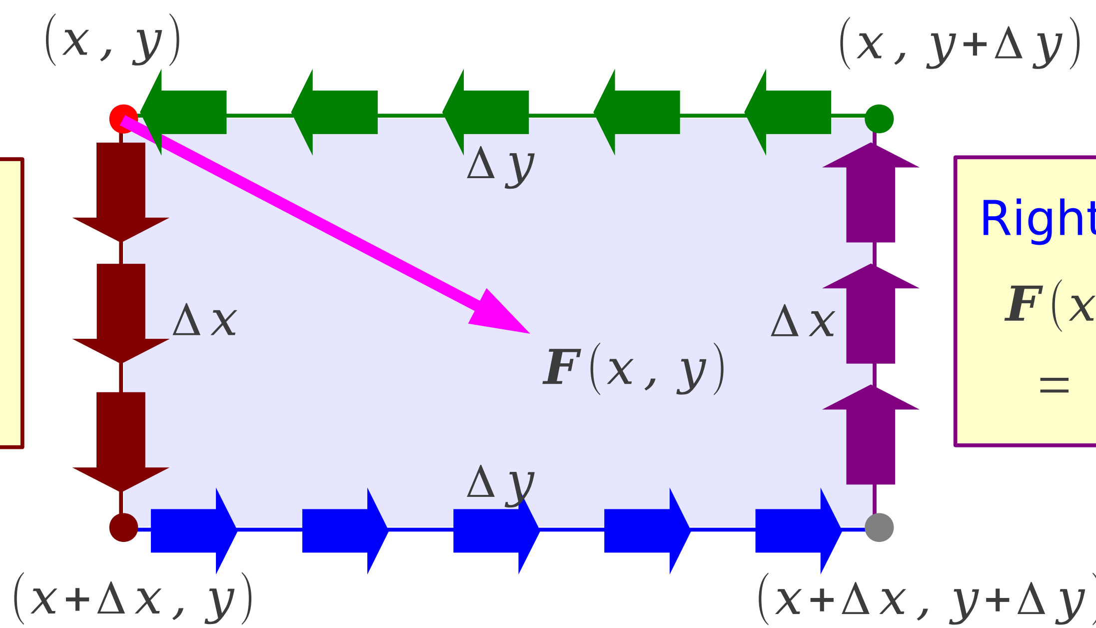
$\boldsymbol{F}(x,\ y) \ = \ M(x,y)\,\boldsymbol{i} + N(x,y)\,\boldsymbol{j}$

Top Velocity $\quad \boldsymbol{F}(x,y)(-\boldsymbol{j}) \ = \ -N(x,y)$

Left Velocity

$\boldsymbol{F}(x,y)(+\boldsymbol{i})$

$= \ +M(x,y)$



Right Velocity

$\boldsymbol{F}(x,y)(-\boldsymbol{i})$

$= \ -M(x,y)$

Bottom Velocity $\quad \boldsymbol{F}(x,y)(+\boldsymbol{j}) \ = \ +N(x,y)$

Flow rate of counter clock wise circulating fluid

2-D Curl (2)

The flow rate of counter clock wise circulation

Along top $\quad \boldsymbol{F}(x, y) \cdot (-\boldsymbol{j}) \Delta y = -N(x, y) \Delta y$

Along bottom $\quad \boldsymbol{F}(x+\Delta x, y) \cdot (+\boldsymbol{j}) \Delta y = N(x+\Delta x, y) \Delta y$

Along left $\quad \boldsymbol{F}(x, y) \cdot (+\boldsymbol{i}) \Delta x = M(x, y) \Delta x$

Along right $\quad \boldsymbol{F}(x, y+\Delta y) \cdot (-\boldsymbol{i}) \Delta x = -M(x, y+\Delta y) \Delta x$

Along top + bottom $\quad \{N(x+\Delta x, y) - N(x, y)\} \Delta y = \left(\frac{\partial N}{\partial x} \Delta x\right) \Delta y$

Along left + right $\quad -\{M(x, y+\Delta y) - M(x, y)\} \Delta x = -\left(\frac{\partial M}{\partial y} \Delta y\right) \Delta x$

Circulation around rectangle boundary

$$\approx \left(\frac{\partial N}{\partial x} \Delta x\right) \Delta y - \left(\frac{\partial M}{\partial y} \Delta y\right) \Delta x = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \Delta x \Delta y$$

Circulation density $\quad = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

k-component Curl of **F**

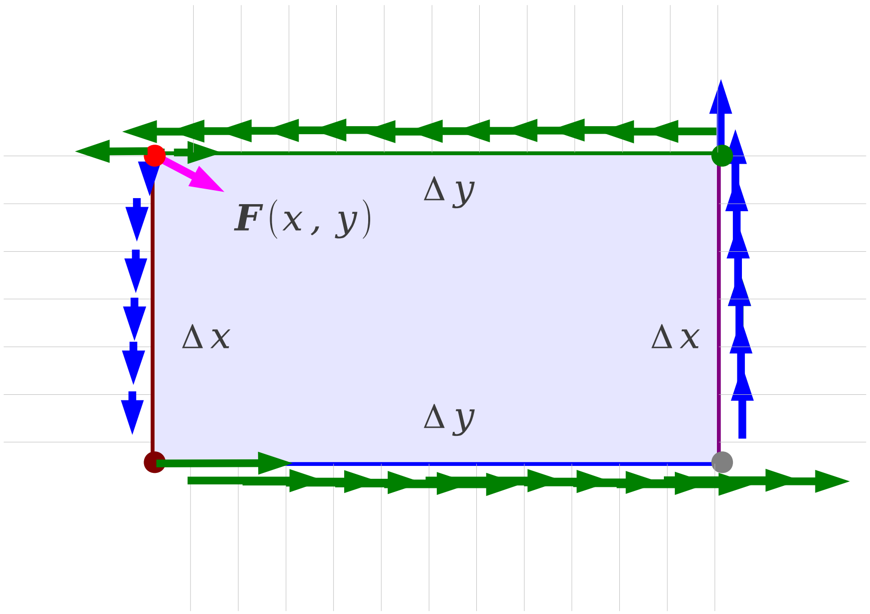
Circulation Density

2-D Curl and Del Operator

2-D Vector Field $\quad \boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$

Circulation around rectangle boundary

$$\approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y - \left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x \Delta y$$



Circulation density $= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + 0\boldsymbol{k}\right) \times (M\boldsymbol{i} + N\boldsymbol{j} + 0\boldsymbol{k})$$

Curl of **F** $= \nabla \times \boldsymbol{F}$

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix}$$

3-D Curl and Del Operator

2-D Vector Field $\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$

Curl of **F** $= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\boldsymbol{k}$ $\quad \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix}$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + 0\boldsymbol{k}\right) \times (M\boldsymbol{i} + N\boldsymbol{j} + 0\boldsymbol{k}) = \nabla \times \boldsymbol{F}$$

3-D Vector Field $\boldsymbol{F}(x, y, z) = M(x,y,z)\boldsymbol{i} + N(x,y,z)\boldsymbol{j} + P(x,y,z)\boldsymbol{k}$

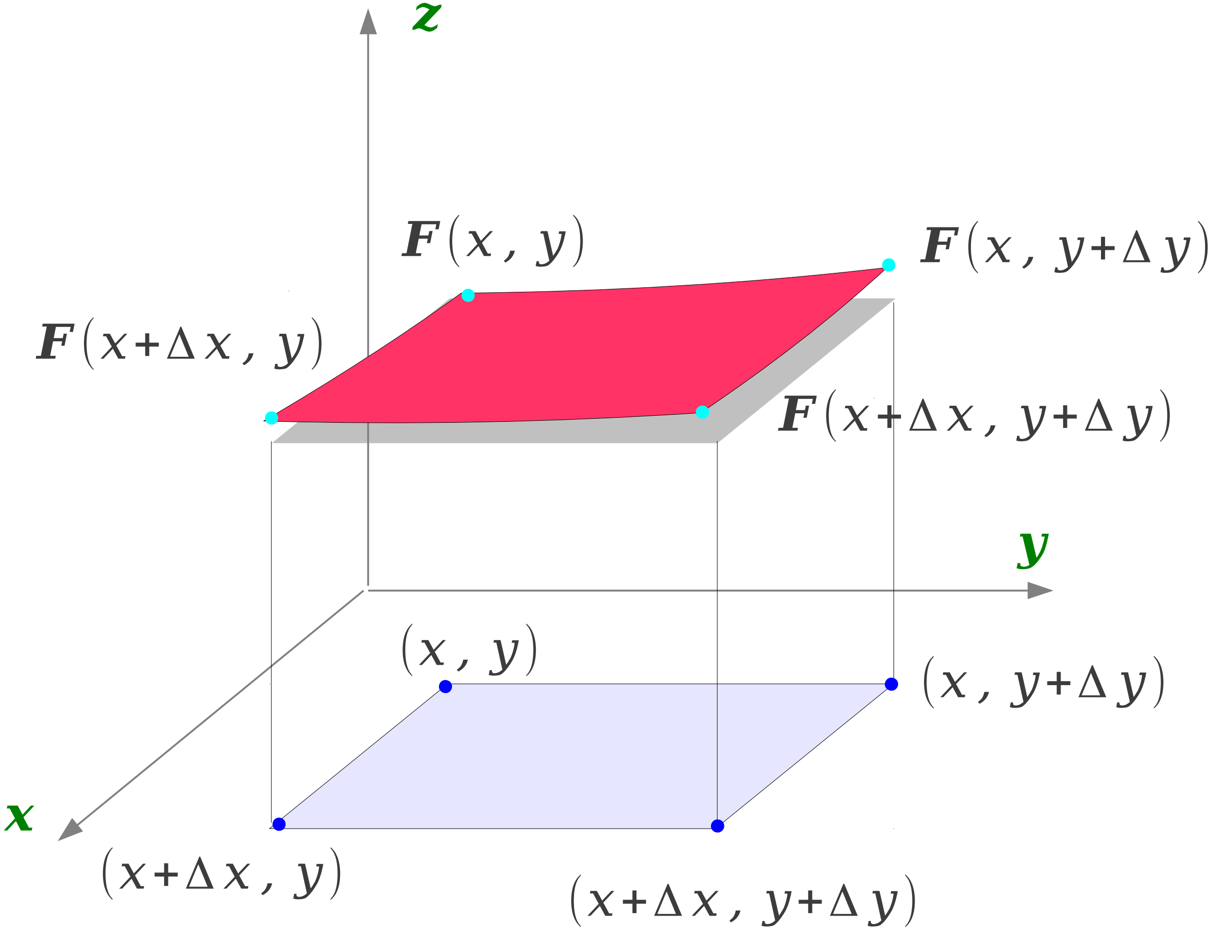
Curl of **F** $= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\boldsymbol{k}$ $\quad \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \times (M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{k}) = \nabla \times \boldsymbol{F}$$

2

i

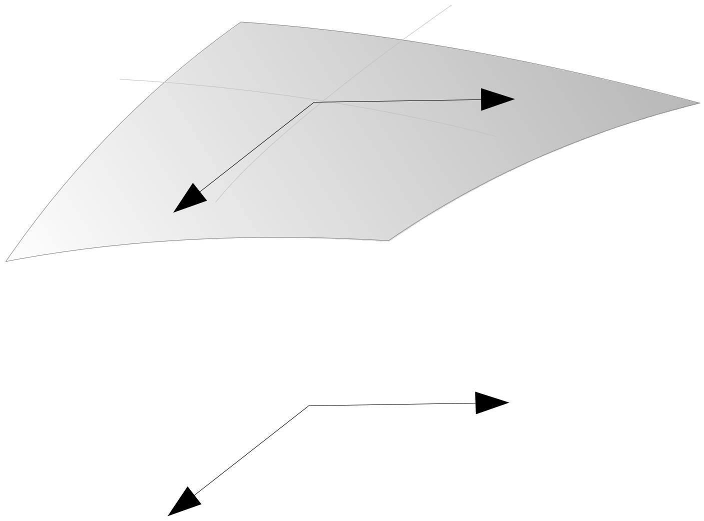
2-D Divergence

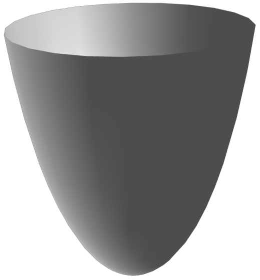


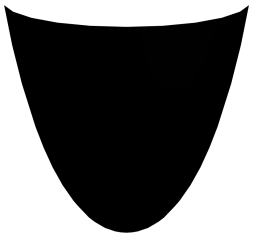
Chain Rule

Function of two variable $y = f(u, v)$

$u = g(x, y)$ $v = h(x, y)$







References

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