

Divergence and Curl (3B)

- Divergence
- Curl
- Green's Theorem

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2-D Vector Field

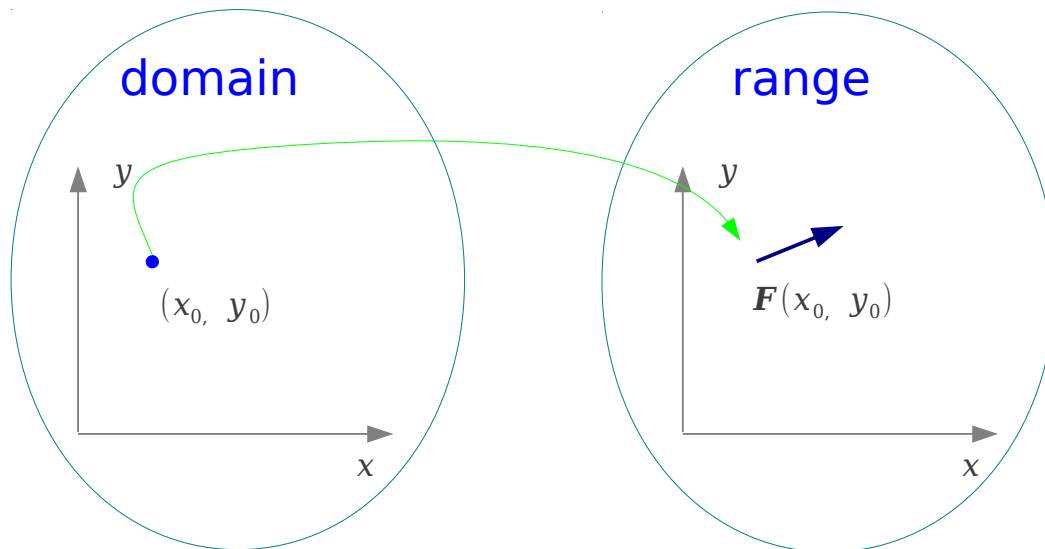
a given point in a 2-d space



A vector

$$(x_0, y_0)$$

$$\langle M(x_0, y_0), N(x_0, y_0) \rangle$$



2 functions

$$(x_0, y_0) \longrightarrow M(x_0, y_0)$$

$$(x_0, y_0) \longrightarrow N(x_0, y_0)$$

$$(x_0, y_0) \longrightarrow \mathbf{F}(x_0, y_0) = M(x_0, y_0)\mathbf{i} + N(x_0, y_0)\mathbf{j}$$

3-D Vector Field

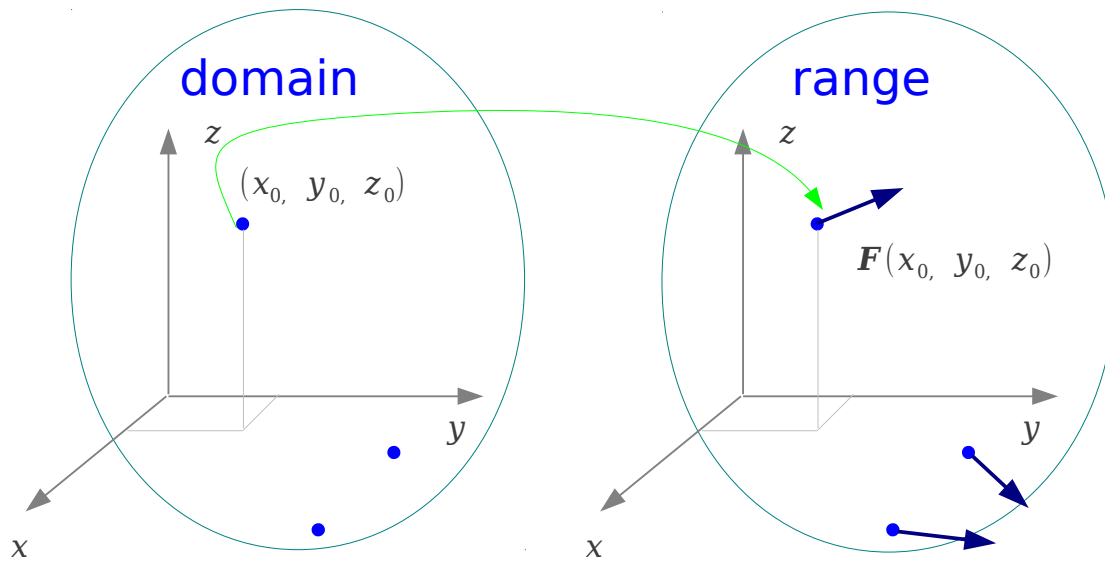
A given point in a 3-d space

$$(x_0, y_0, z_0)$$



A vector

$$\langle M(x_0, y_0, z_0), N(x_0, y_0, z_0), P(x_0, y_0, z_0) \rangle$$



3 functions

$$(x_0, y_0, z_0) \longrightarrow M(x_0, y_0, z_0)$$

$$(x_0, y_0, z_0) \longrightarrow N(x_0, y_0, z_0)$$

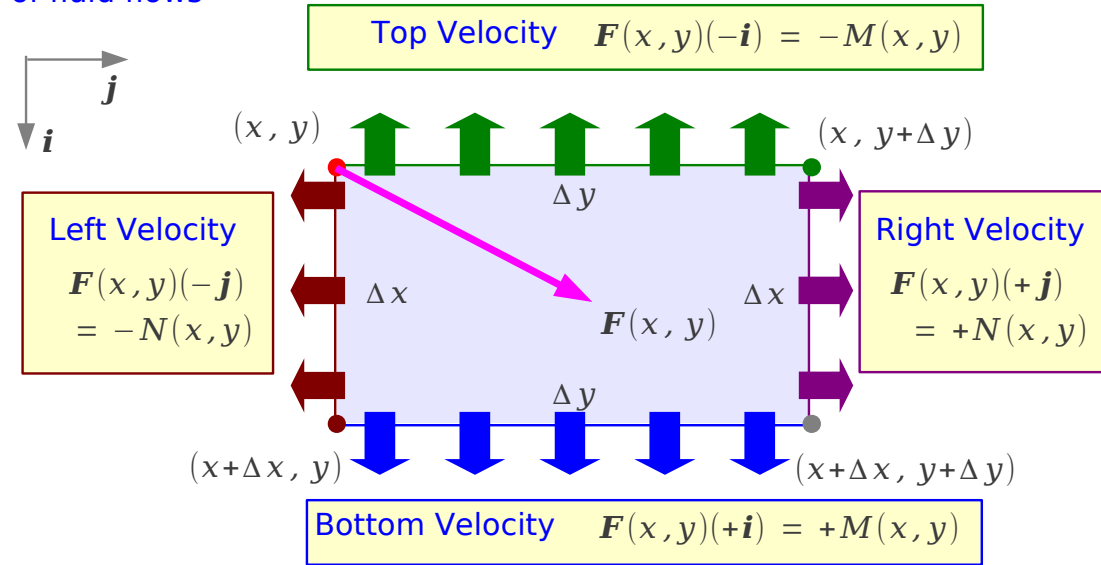
$$(x_0, y_0, z_0) \longrightarrow P(x_0, y_0, z_0)$$

$$(x_0, y_0, z_0) \longrightarrow \mathbf{F}(x_0, y_0, z_0) = M(x_0, y_0, z_0)\mathbf{i} + N(x_0, y_0, z_0)\mathbf{j} + P(x_0, y_0, z_0)\mathbf{k}$$

Inward & Outward Bound

2-D Divergence (5)

Velocity Fields of fluid flows



$$\{M(x+\Delta x, y) - M(x, y)\} \Delta y = \left(\frac{\partial M}{\partial x} \Delta x \right) \Delta y$$

$$\{N(x, y+\Delta y) - N(x, y)\} \Delta x = \left(\frac{\partial N}{\partial y} \Delta y \right) \Delta x$$

Flow rate of outward bound fluid

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x} \Delta x \right) \Delta y + \left(\frac{\partial N}{\partial y} \Delta y \right) \Delta x = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta x \Delta y$$

Flux density

$$= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)$$

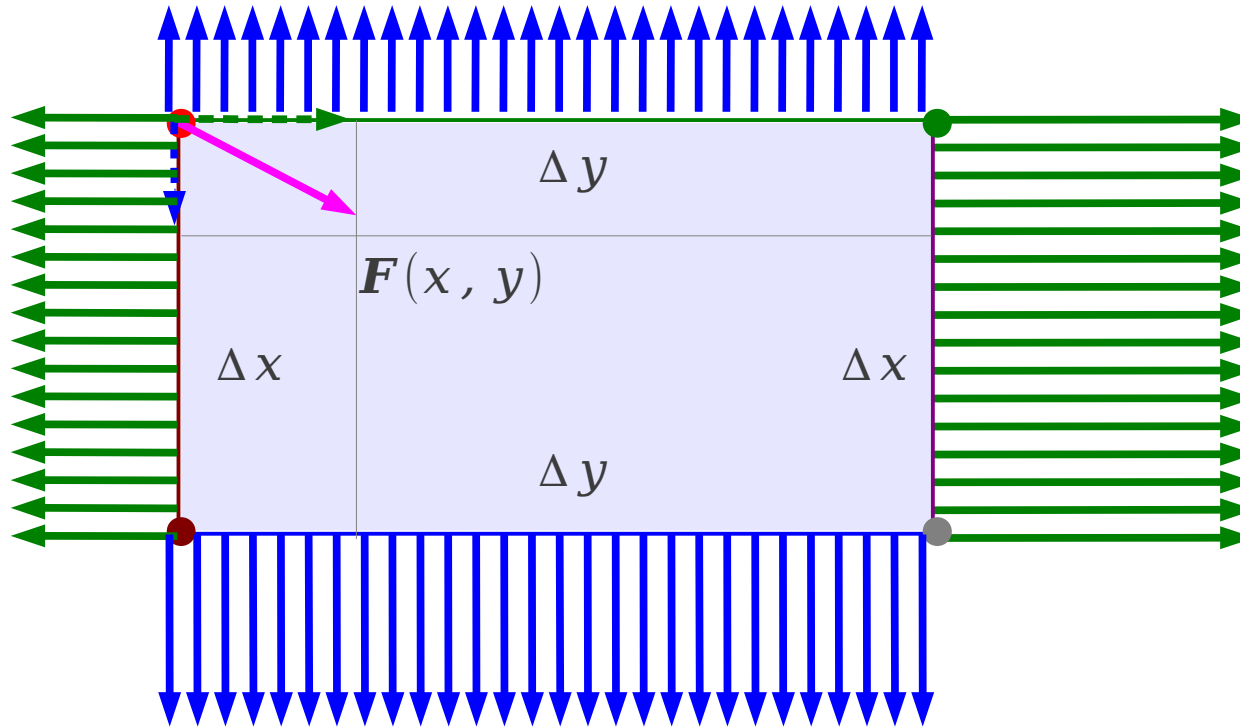
Divergence of \mathbf{F}

Flux Density

2-D Divergence (d)

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x} \Delta x \right) \Delta y + \left(\frac{\partial N}{\partial y} \Delta y \right) \Delta x = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta x \Delta y$$



Flux density = $\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)$

Divergence of \mathbf{F}

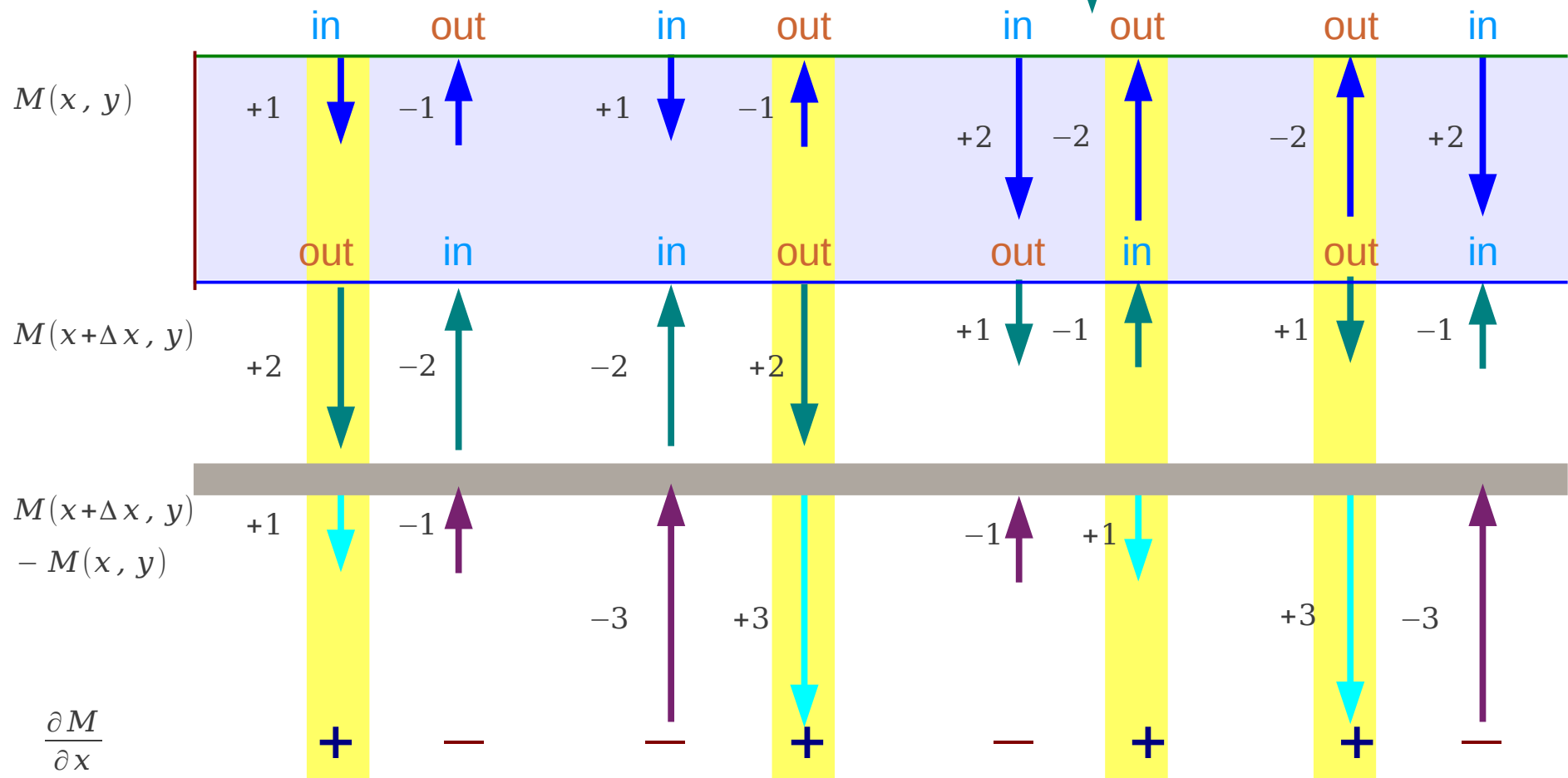
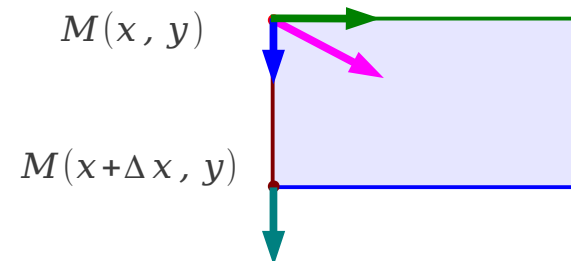
Flux Density

Inward & Outward Bound

Inward & Outward Bound - X (i)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

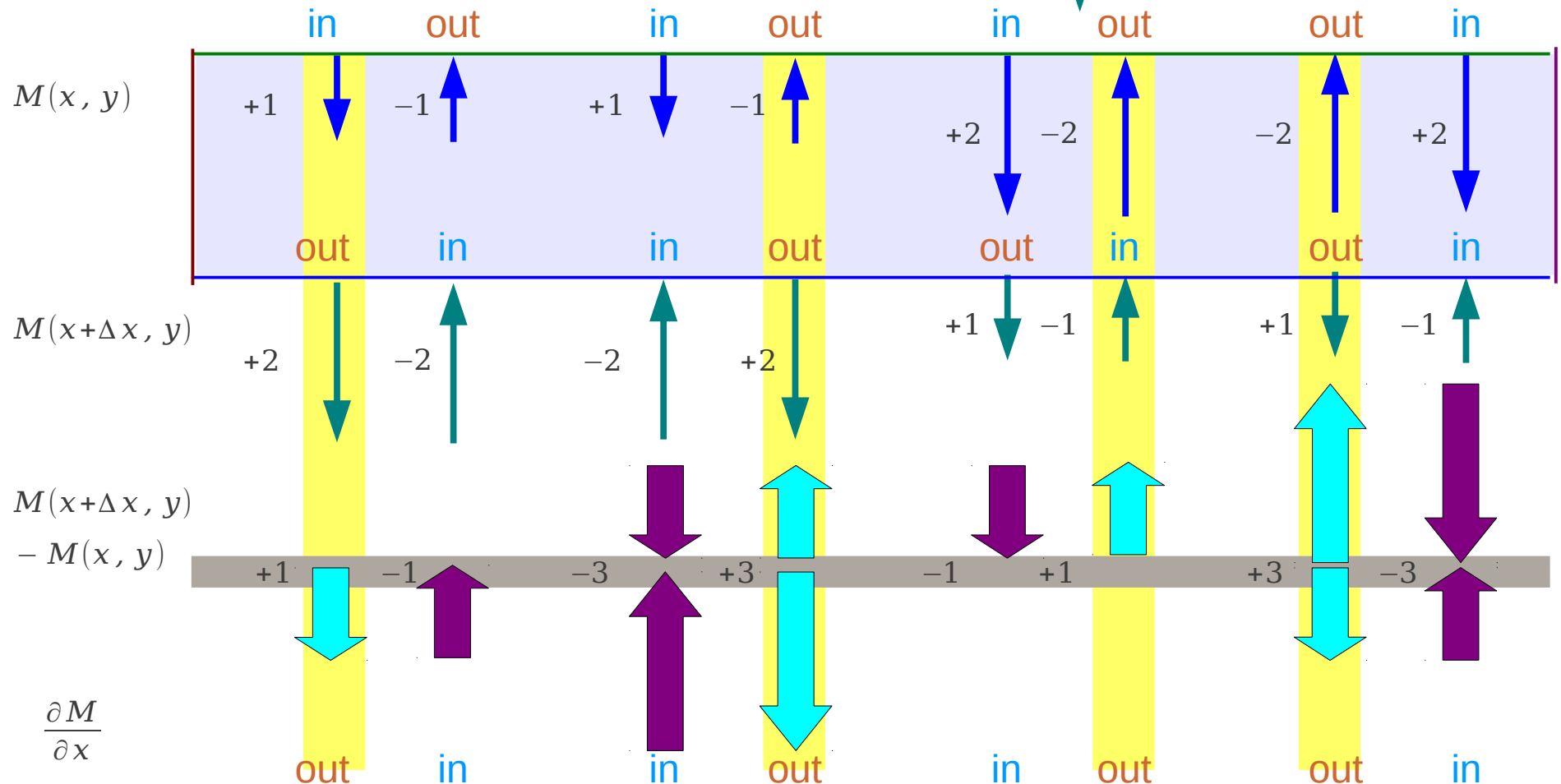
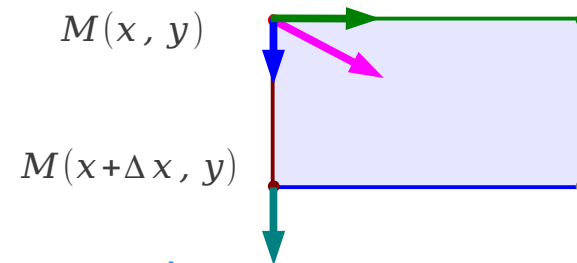
$$\{M(x+\Delta x, y) - M(x, y)\}$$



Inward & Outward Bound - X (ii)

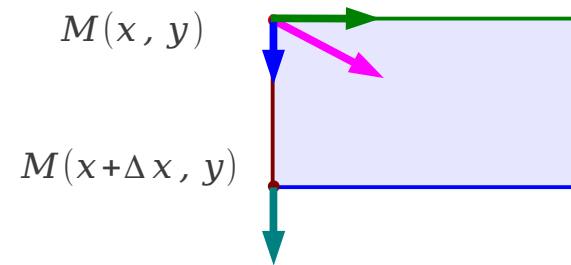
$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

$$\{M(x+\Delta x, y) - M(x, y)\}$$



Inward & Outward Bound - X (iii)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

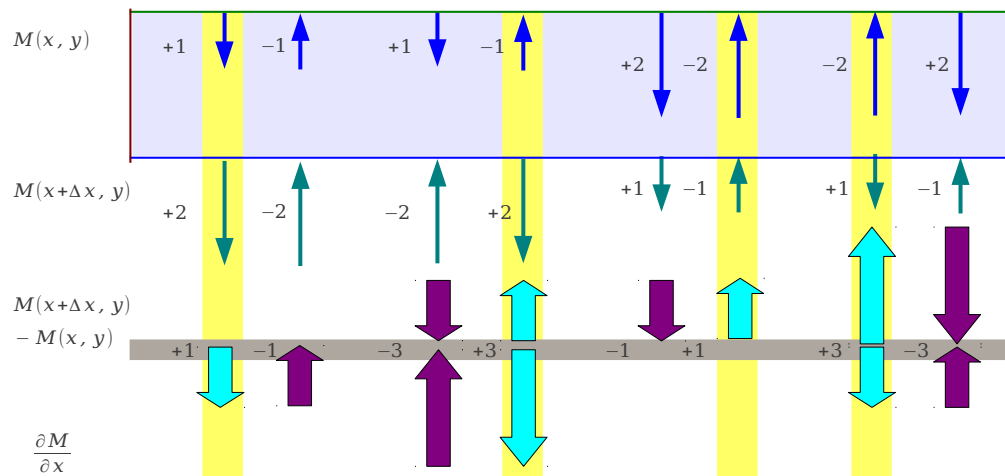


$$\{M(x + \Delta x, y) - M(x, y)\} > 0 \quad \Delta x > 0$$

Outward bound net flow

$$\{M(x + \Delta x, y) - M(x, y)\} < 0 \quad \Delta x > 0$$

Inward bound net flow



$$\frac{\{M(x + \Delta x, y) - M(x, y)\}}{\Delta x}$$

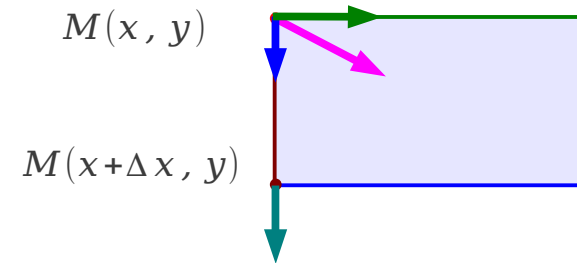
$$\approx \frac{\partial M}{\partial x} > 0 \quad \text{Outward}$$

$$\frac{\{M(x + \Delta x, y) - M(x, y)\}}{\Delta x}$$

$$\approx \frac{\partial M}{\partial x} < 0 \quad \text{Inward}$$

Inward & Outward Bound - X (iv)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$



$$\frac{\{M(x + \Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x} > 0$$

- **Positive Slope** of a tangent line parallel to the x axis
- **Outward** bound net flow along the x axis

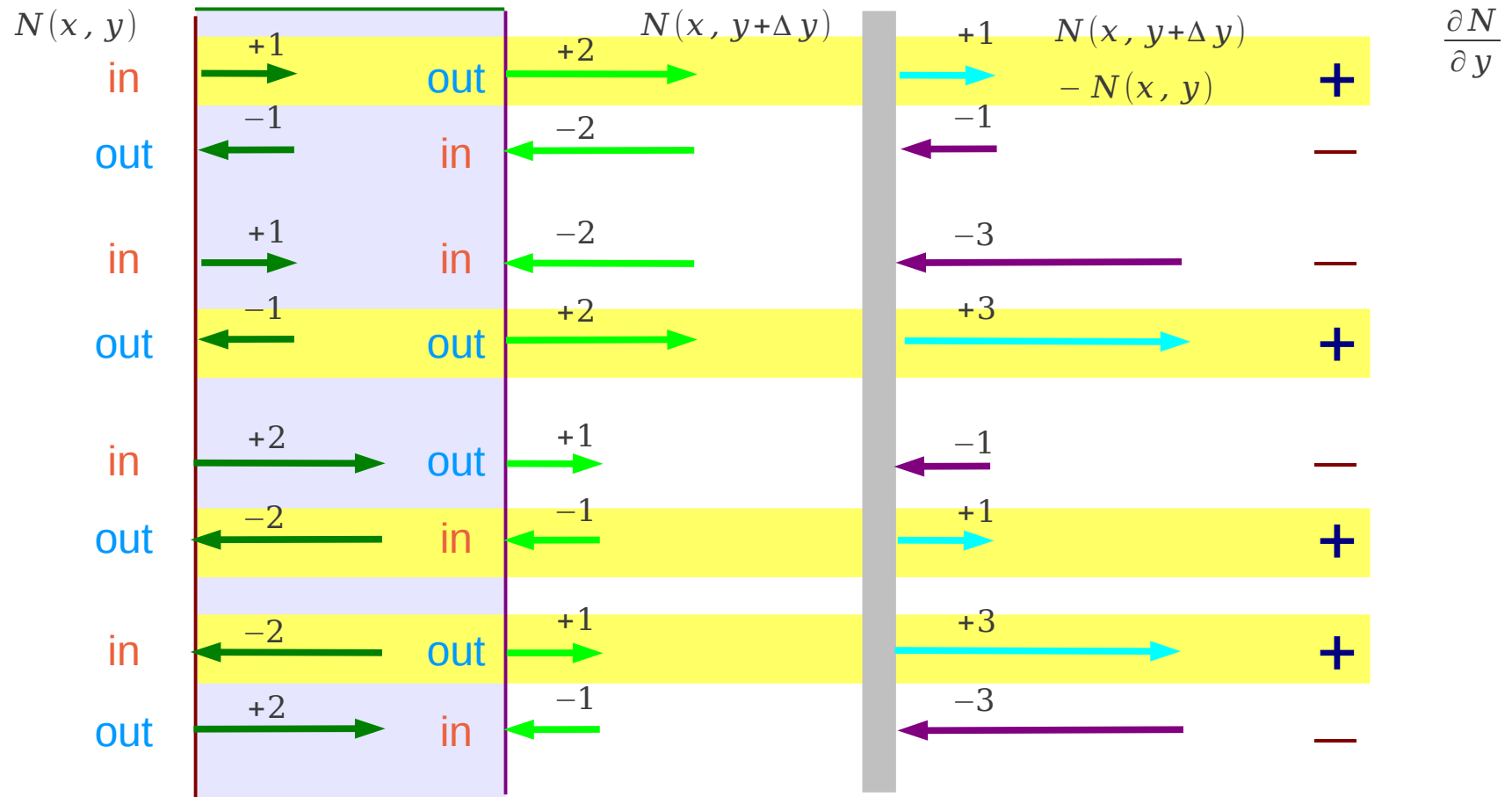
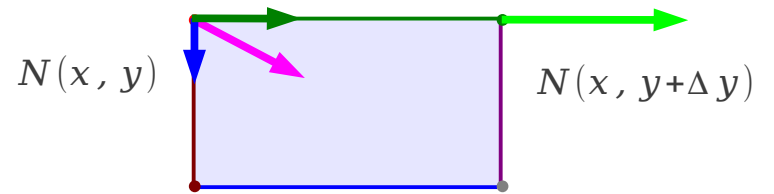
$$\frac{\{M(x + \Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x} < 0$$

- **Negative Slope** of a tangent line parallel to the x axis
- **Inward** bound net flow along the x axis

Inward & Outward Bound - Y (i)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

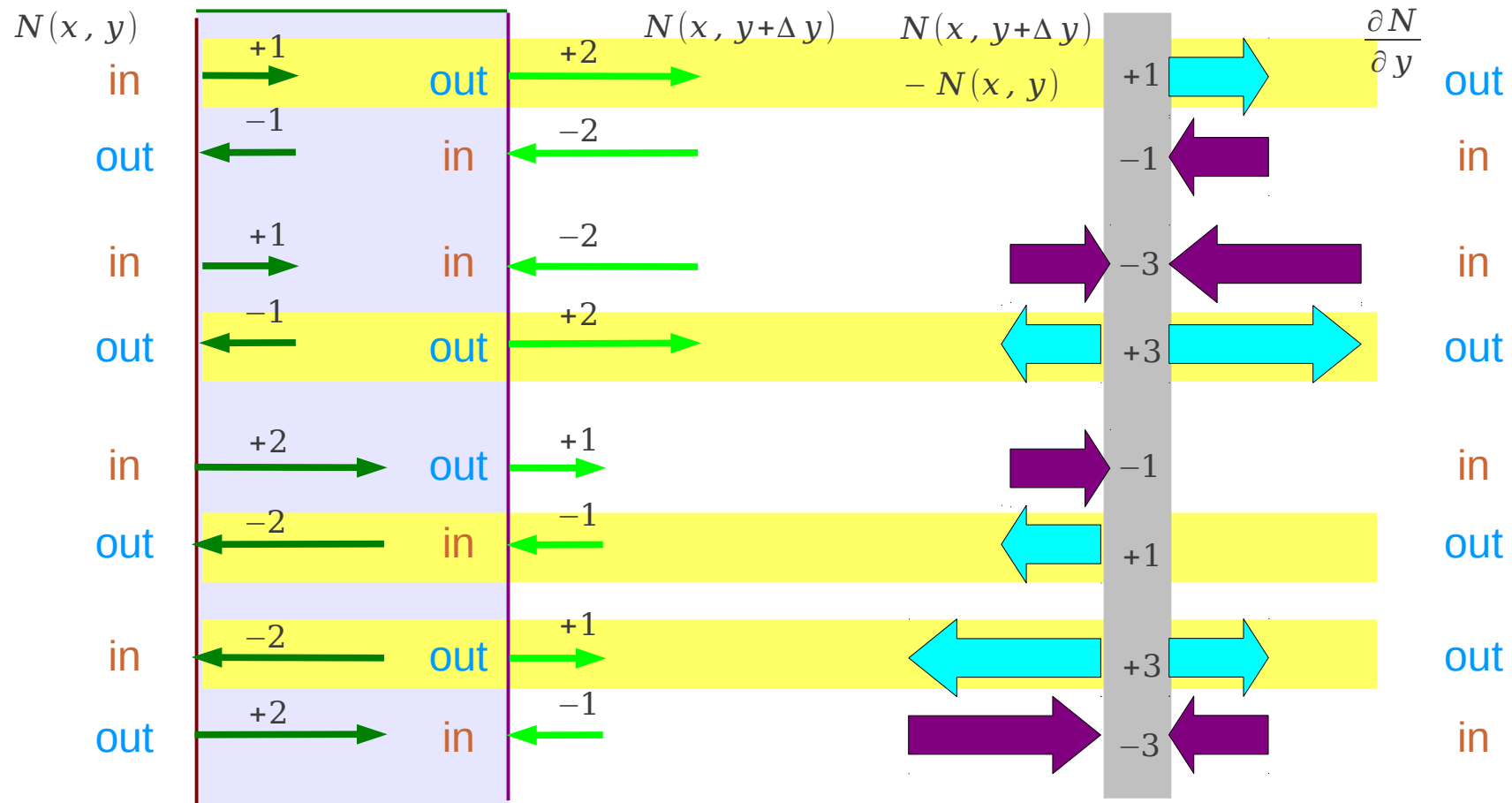
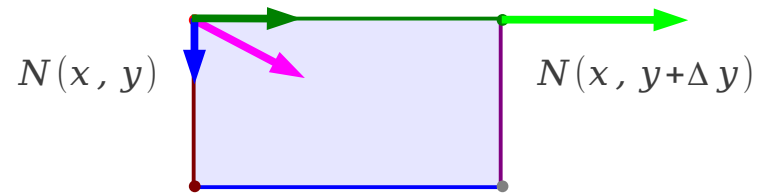
$$\{M(x+\Delta x, y) - M(x, y)\}$$



Inward & Outward Bound - Y (ii)

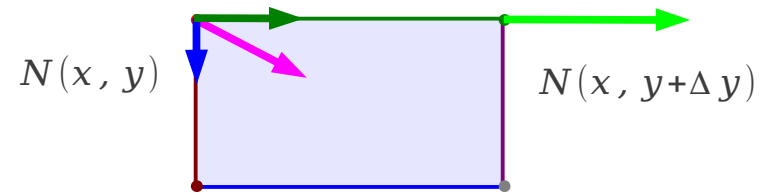
$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

$$\{M(x+\Delta x, y) - M(x, y)\}$$



Inward & Outward Bound - Y (iii)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

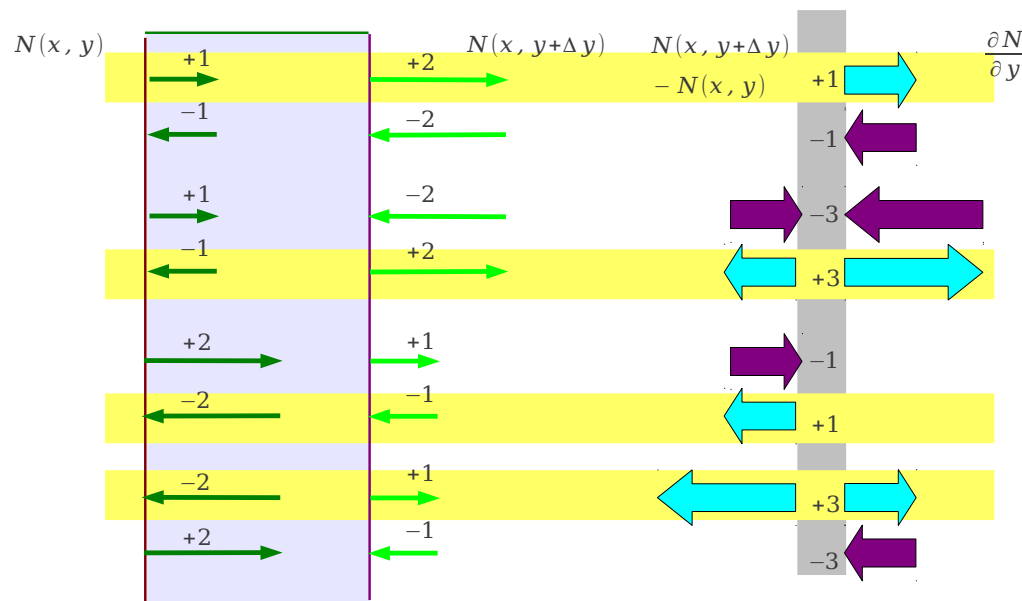


$$\{N(x, y + \Delta y) - N(x, y)\} > 0 \quad \Delta y > 0$$

Outward bound net flow

$$\{N(x, y + \Delta y) - N(x, y)\} < 0 \quad \Delta y > 0$$

Inward bound net flow



$$\{N(x, y + \Delta y) - N(x, y)\}$$

$$\approx \frac{\partial N}{\partial y} \Delta y > 0$$

Outward

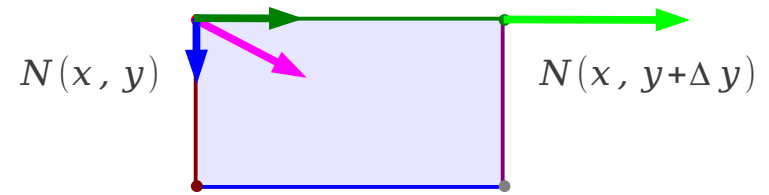
$$\{N(x, y + \Delta y) - N(x, y)\}$$

$$\approx \frac{\partial N}{\partial y} \Delta y < 0$$

Inward

Inward & Outward Bound - Y (iv)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$



$$\frac{\{N(x, y + \Delta y) - N(x, y)\}}{\Delta y} \approx \frac{\partial N}{\partial y} > 0$$

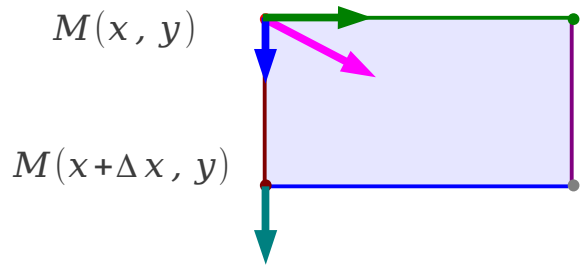
- **Positive Slope** of a tangent line parallel to the y axis
- **Outward** bound net flow along the y axis

$$\frac{\{N(x, y + \Delta y) - N(x, y)\}}{\Delta y} \approx \frac{\partial N}{\partial y} < 0$$

- **Negative Slope** of a tangent line parallel to the y axis
- **Inward** bound net flow along the y axis

Inward & Outward Bound

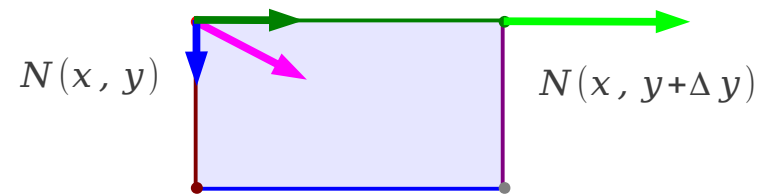
Inward & Outward Bound



$$\frac{\{M(x + \Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x} > 0 \quad \text{Outward}$$

$$\frac{\{M(x + \Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x} < 0 \quad \text{Inward}$$

$$\begin{aligned} & \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \right) \cdot (M \mathbf{i}) \\ &= \nabla \cdot (M \mathbf{i}) \end{aligned}$$



$$\frac{\{N(x, y + \Delta y) - N(x, y)\}}{\Delta y} \approx \frac{\partial N}{\partial y} > 0 \quad \text{Outward}$$

$$\frac{\{N(x, y + \Delta y) - N(x, y)\}}{\Delta y} \approx \frac{\partial N}{\partial y} < 0 \quad \text{Inward}$$

$$\begin{aligned} & \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \right) \cdot (N \mathbf{j}) \\ &= \nabla \cdot (N \mathbf{j}) \end{aligned}$$

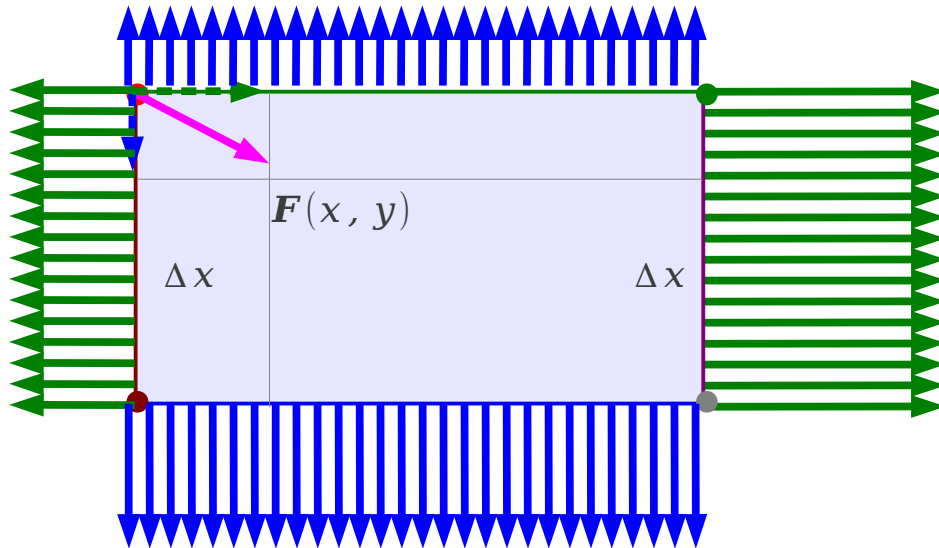
2-D Divergence and Del Operator

2-D Vector Field

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x} \Delta x \right) \Delta y + \left(\frac{\partial N}{\partial y} \Delta y \right) \Delta x = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta x \Delta y$$



$$\text{Flux density} = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)$$

$$= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \right) \cdot (M\mathbf{i} + N\mathbf{j})$$

$$= \nabla \cdot \mathbf{F}$$

Divergence of \mathbf{F}

3-D Divergence and Del Operator

2-D Vector Field

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

$$\begin{aligned}\text{Divergence of } \mathbf{F} &= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \\ &= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \right) \cdot (M\mathbf{i} + N\mathbf{j}) = \nabla \cdot \mathbf{F}\end{aligned}$$

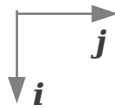
3-D Vector Field

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

$$\begin{aligned}\text{Divergence of } \mathbf{F} &= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) \\ &= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \cdot (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) = \nabla \cdot \mathbf{F}\end{aligned}$$

2-D Curl (4)

Velocity Fields of fluid flows

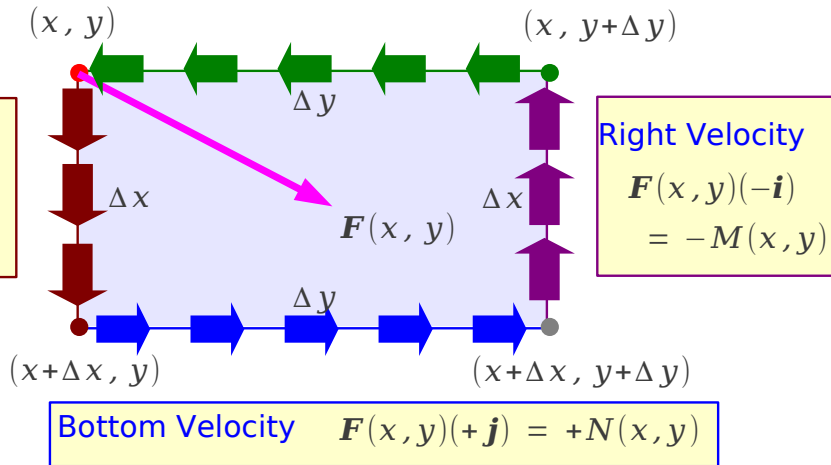


$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

$$\text{Top Velocity } \mathbf{F}(x, y)(-\mathbf{j}) = -N(x, y)$$

Left Velocity

$$\mathbf{F}(x, y)(+\mathbf{i}) = +M(x, y)$$



Right Velocity

$$\mathbf{F}(x, y)(-\mathbf{i}) = -M(x, y)$$

$$\text{Bottom Velocity } \mathbf{F}(x, y)(+\mathbf{j}) = +N(x, y)$$

$$\begin{aligned} & \{N(x+\Delta x, y) - N(x, y)\} \Delta y \\ & = \left(\frac{\partial N}{\partial x} \Delta x \right) \Delta y \\ & - \{M(x, y+\Delta y) - M(x, y)\} \Delta x \\ & = - \left(\frac{\partial M}{\partial y} \Delta y \right) \Delta x \end{aligned}$$

Flow rate of counter clock wise circulating fluid

Circulation around rectangle boundary

$$\approx \left(\frac{\partial N}{\partial x} \Delta x \right) \Delta y - \left(\frac{\partial M}{\partial y} \Delta y \right) \Delta x = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \Delta x \Delta y$$

Circulation density

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

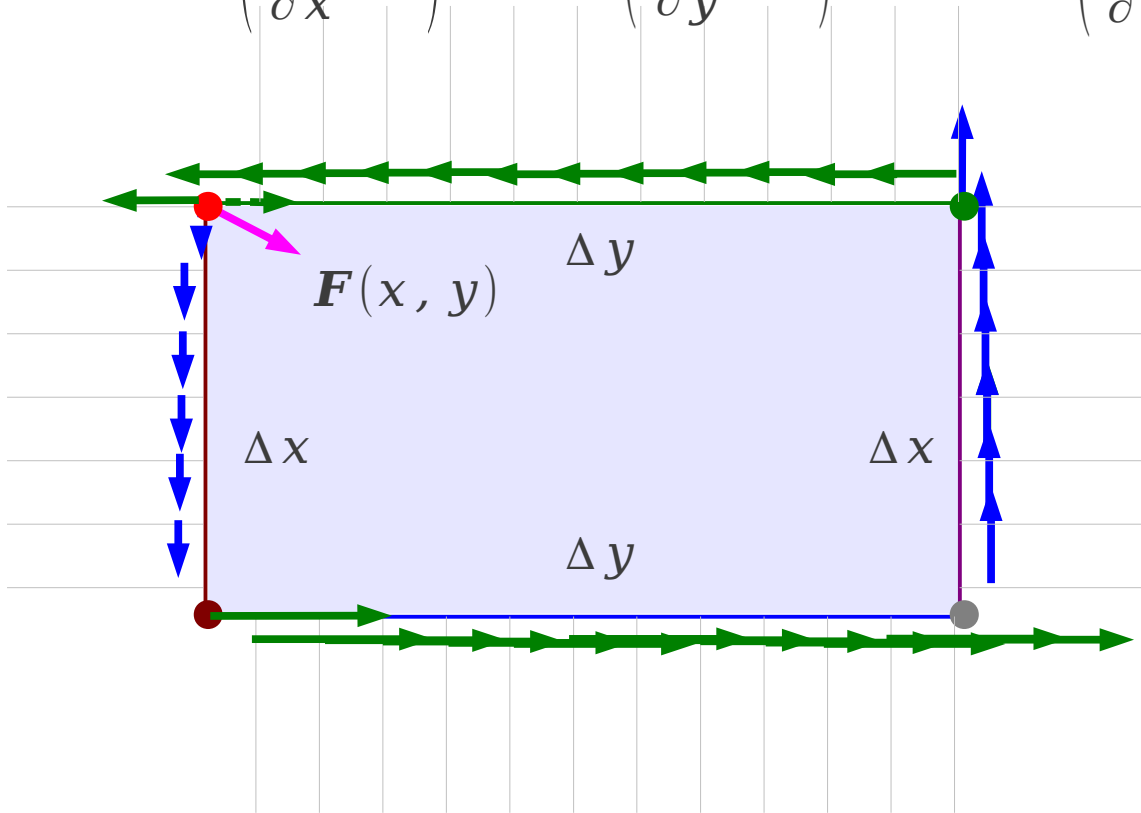
k-component
Curl of \mathbf{F}

Circulation Density

2-D Curl (d)

Circulation around rectangle boundary

$$\approx \left(\frac{\partial N}{\partial x} \Delta x \right) \Delta y - \left(\frac{\partial M}{\partial y} \Delta y \right) \Delta x = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \Delta x \Delta y$$



Circulation density = $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$

k-component
Curl of \mathbf{F}

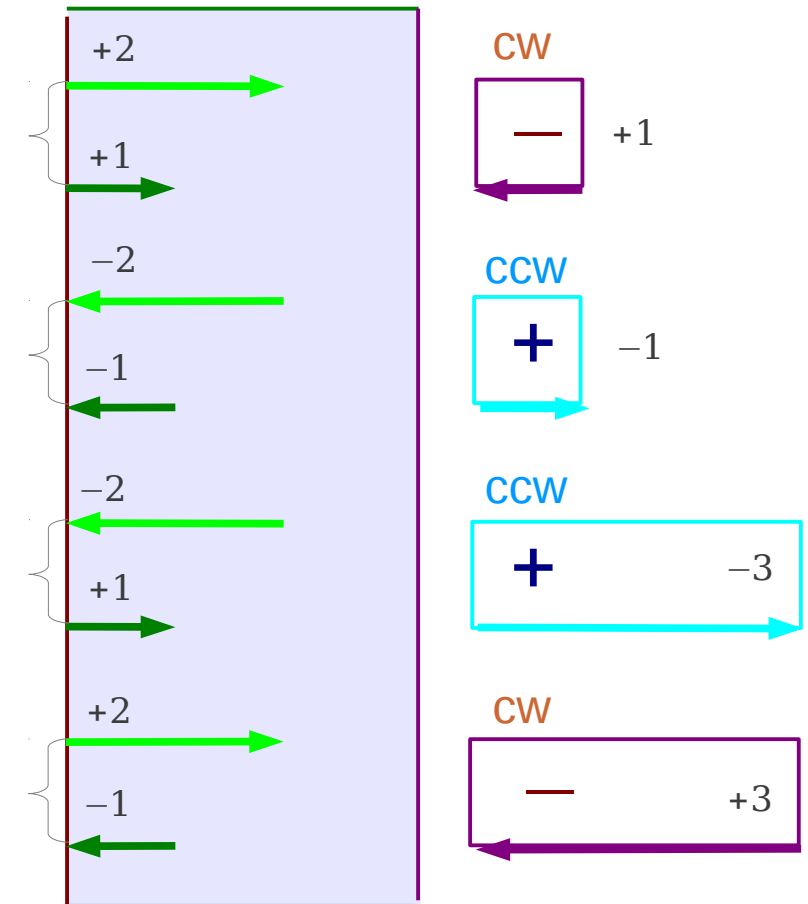
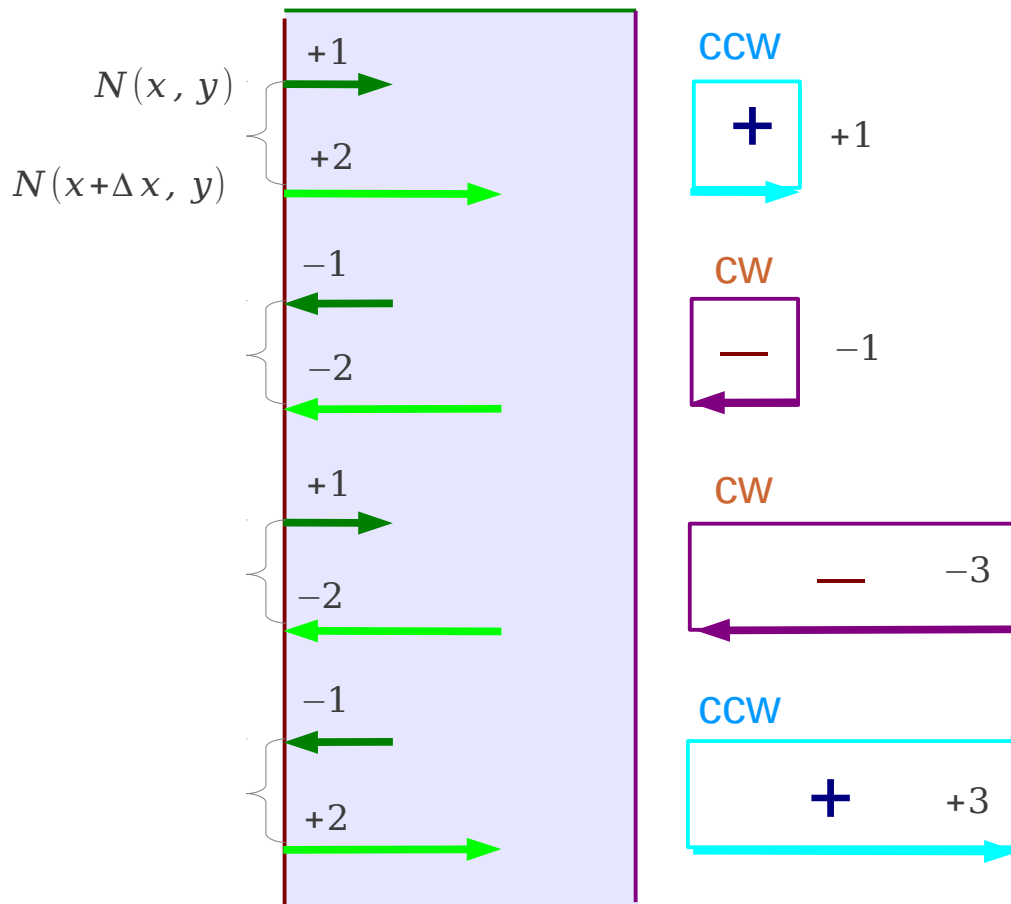
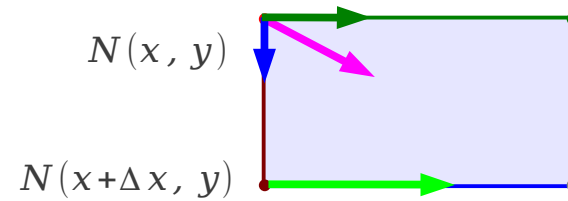
Circulation Density

Clock-Wise & Counter-Clock-Wise

Clock-Wise & Counter-Clock-Wise - X (i)

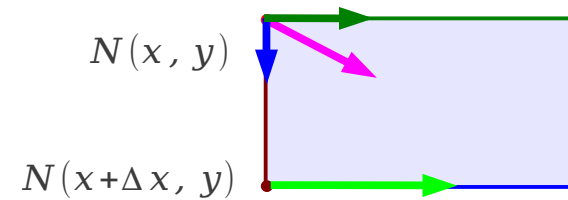
$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

$$\{N(x+\Delta x, y) - N(x, y)\}$$



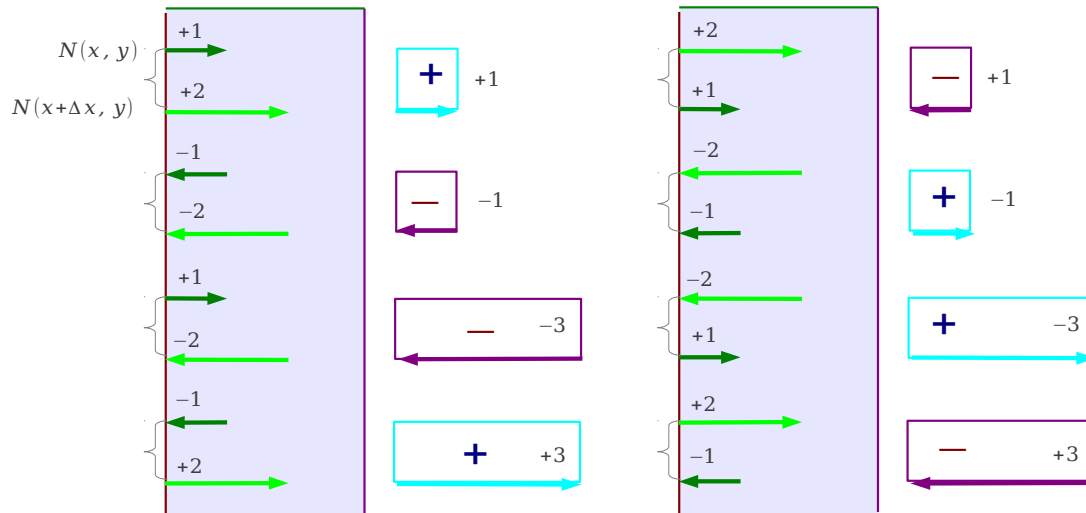
Clock-Wise & Counter-Clock-Wise - X (ii)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$



$$\{N(x + \Delta x, y) - N(x, y)\} > 0 \quad \Delta x > 0 \quad \text{CCW bound net flow}$$

$$\{N(x + \Delta x, y) - N(x, y)\} < 0 \quad \Delta x > 0 \quad \text{CW bound net flow}$$

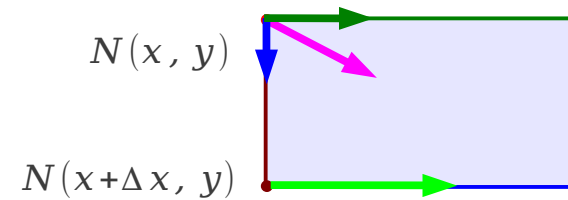


$$\frac{\{N(x + \Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} > 0 \quad \text{CCW}$$

$$\frac{\{N(x + \Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} < 0 \quad \text{CW}$$

Clock-Wise & Counter-Clock-Wise - X (iii)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$



$$\frac{\{N(x+\Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} > 0$$

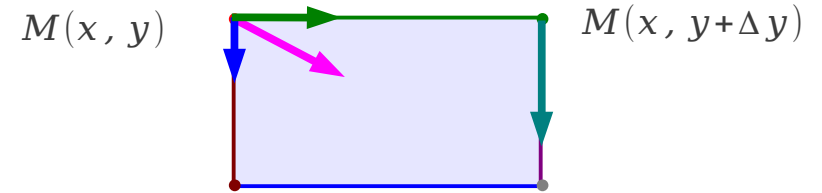
- **Positive Slope** of a tangent line parallel to the x axis
- **CCW** bound net flow along the z axis

$$\frac{\{N(x+\Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} < 0$$

- **Negative Slope** of a tangent line parallel to the x axis
- **CW** bound net flow along the z axis

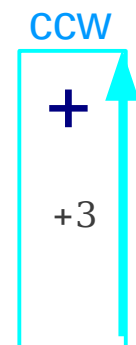
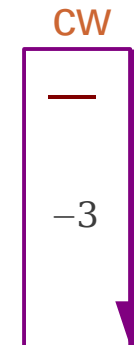
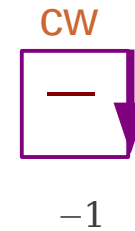
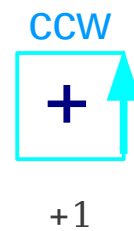
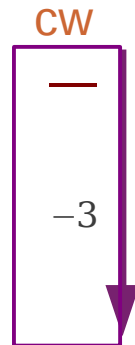
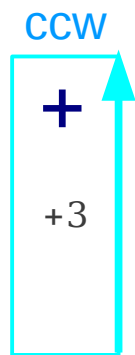
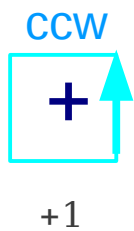
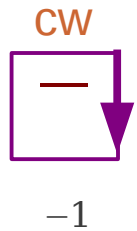
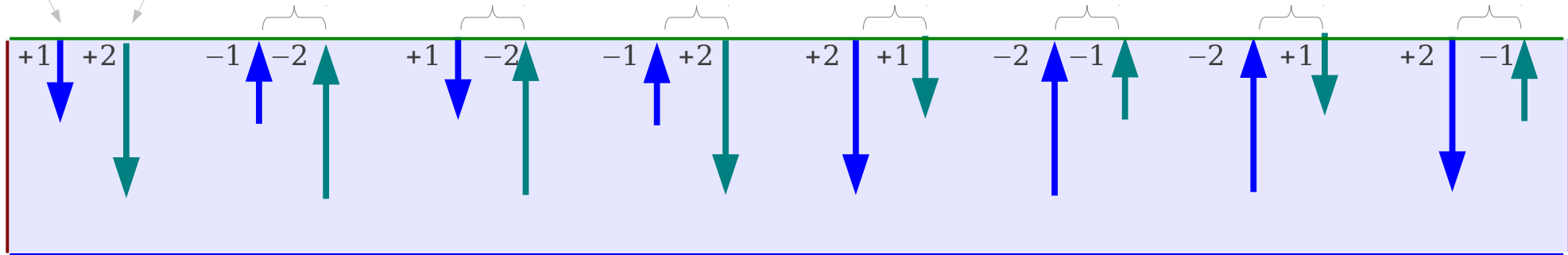
Clock-Wise & Counter-Clock-Wise - Y (i)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$



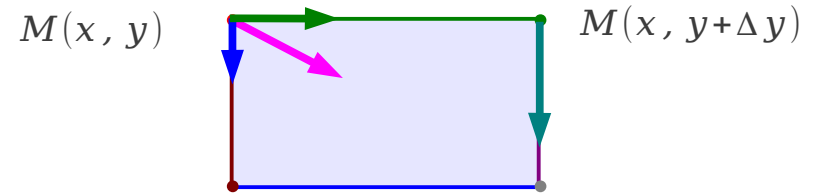
$$\{M(x, y + \Delta y) - M(x, y)\}$$

$M(x, y)$ $M(x, y + \Delta y)$



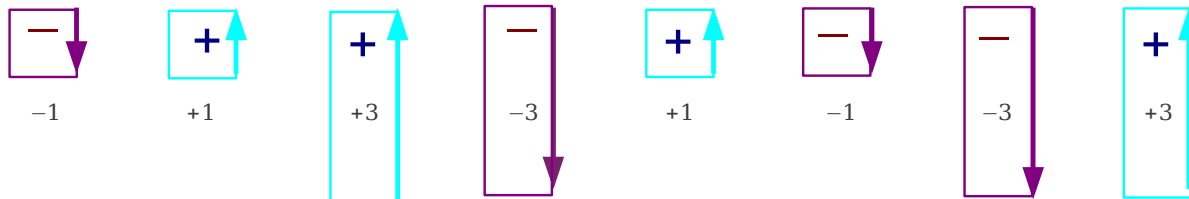
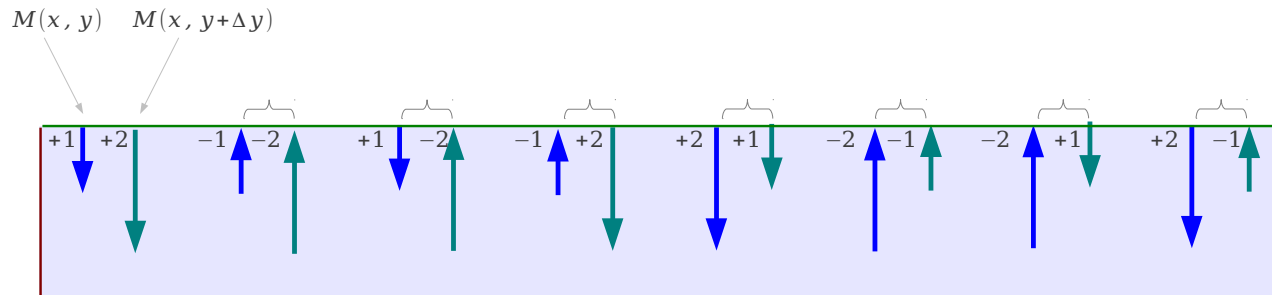
Clock-Wise & Counter-Clock-Wise - Y (ii)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$



$$\{M(x, y + \Delta y) - M(x, y)\} > 0 \quad \Delta x > 0 \quad \text{CCW bound net flow}$$

$$\{M(x, y + \Delta y) - M(x, y)\} < 0 \quad \Delta x > 0 \quad \text{CW bound net flow}$$



$$\frac{\{M(x, y + \Delta y) - M(x, y)\}}{\Delta y}$$

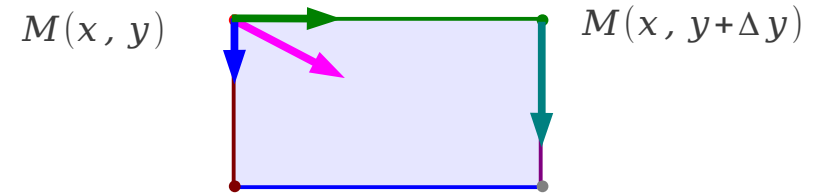
$$\approx \frac{\partial M}{\partial y} > 0 \quad \text{CCW}$$

$$\frac{\{M(x, y + \Delta y) - M(x, y)\}}{\Delta y}$$

$$\approx \frac{\partial M}{\partial y} < 0 \quad \text{CW}$$

Clock-Wise & Counter-Clock-Wise - Y (iii)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$



$$\frac{\{M(x, y + \Delta y) - M(x, y)\}}{\Delta y} \approx \frac{\partial M}{\partial y} > 0$$

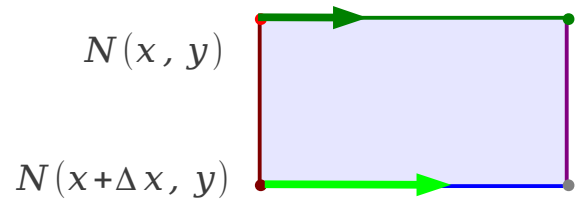
- **Positive Slope** of a tangent line parallel to the y axis
- **CCW** bound net flow along the z axis

$$\frac{\{M(x, y + \Delta y) - M(x, y)\}}{\Delta y} \approx \frac{\partial M}{\partial y} < 0$$

- **Negative Slope** of a tangent line parallel to the y axis
- **CW** bound net flow along the z axis

Clock-Wise & Counter-Clock-Wise

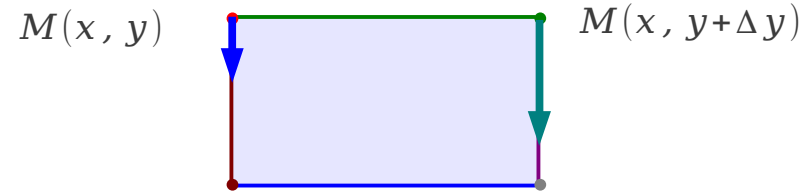
Clock-Wise & Counter-Clock-Wise



$$\frac{\{N(x+\Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} > 0 \quad \text{CCW}$$

$$\frac{\{N(x+\Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} < 0 \quad \text{CW}$$

$$\left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + 0 \mathbf{k} \right) \times (N \mathbf{j}) = \nabla \times (N \mathbf{j})$$



$$\frac{\{M(x, y+\Delta y) - M(x, y)\}}{\Delta y} \approx \frac{\partial M}{\partial y} > 0$$

$$\frac{\{M(x, y+\Delta y) - M(x, y)\}}{\Delta y} \approx \frac{\partial M}{\partial y} < 0$$

$$\left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + 0 \mathbf{k} \right) \times (M \mathbf{i}) = \nabla \times (M \mathbf{i})$$

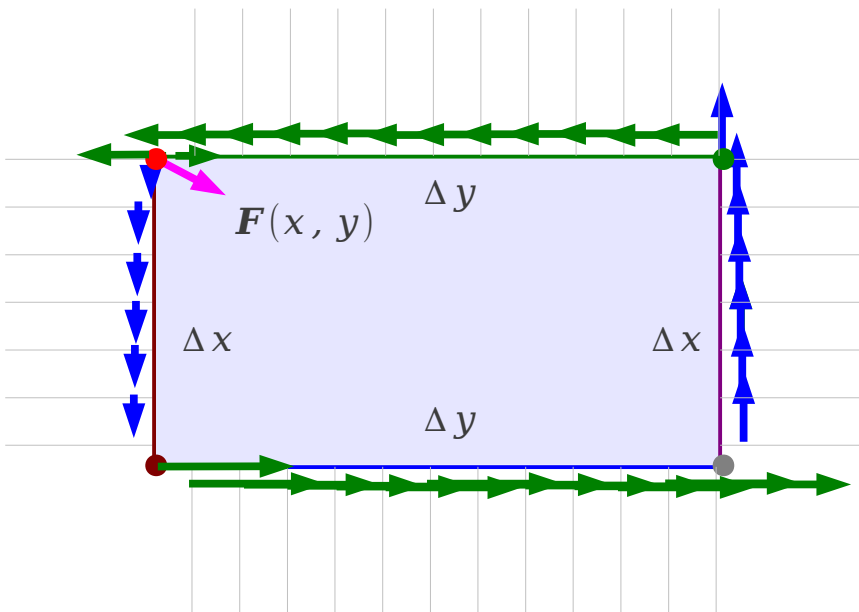
2-D Curl and Del Operator

2-D Vector Field

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

Circulation around rectangle boundary

$$\approx \left(\frac{\partial N}{\partial x} \Delta x \right) \Delta y - \left(\frac{\partial M}{\partial y} \Delta y \right) \Delta x = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \Delta x \Delta y$$



$$\begin{aligned} \text{Circulation density} &= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + 0\mathbf{k} \right) \times (M\mathbf{i} + N\mathbf{j} + 0\mathbf{k}) \end{aligned}$$

$$\text{Curl of } \mathbf{F} = \nabla \times \mathbf{F}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix}$$

3-D Curl and Del Operator

2-D Vector Field

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

Curl of \mathbf{F}

$$\begin{aligned} &= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} && \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + 0 \mathbf{k} \right) \times (M\mathbf{i} + N\mathbf{j} + 0\mathbf{k}) = \nabla \times \mathbf{F} \end{aligned}$$

3-D Vector Field

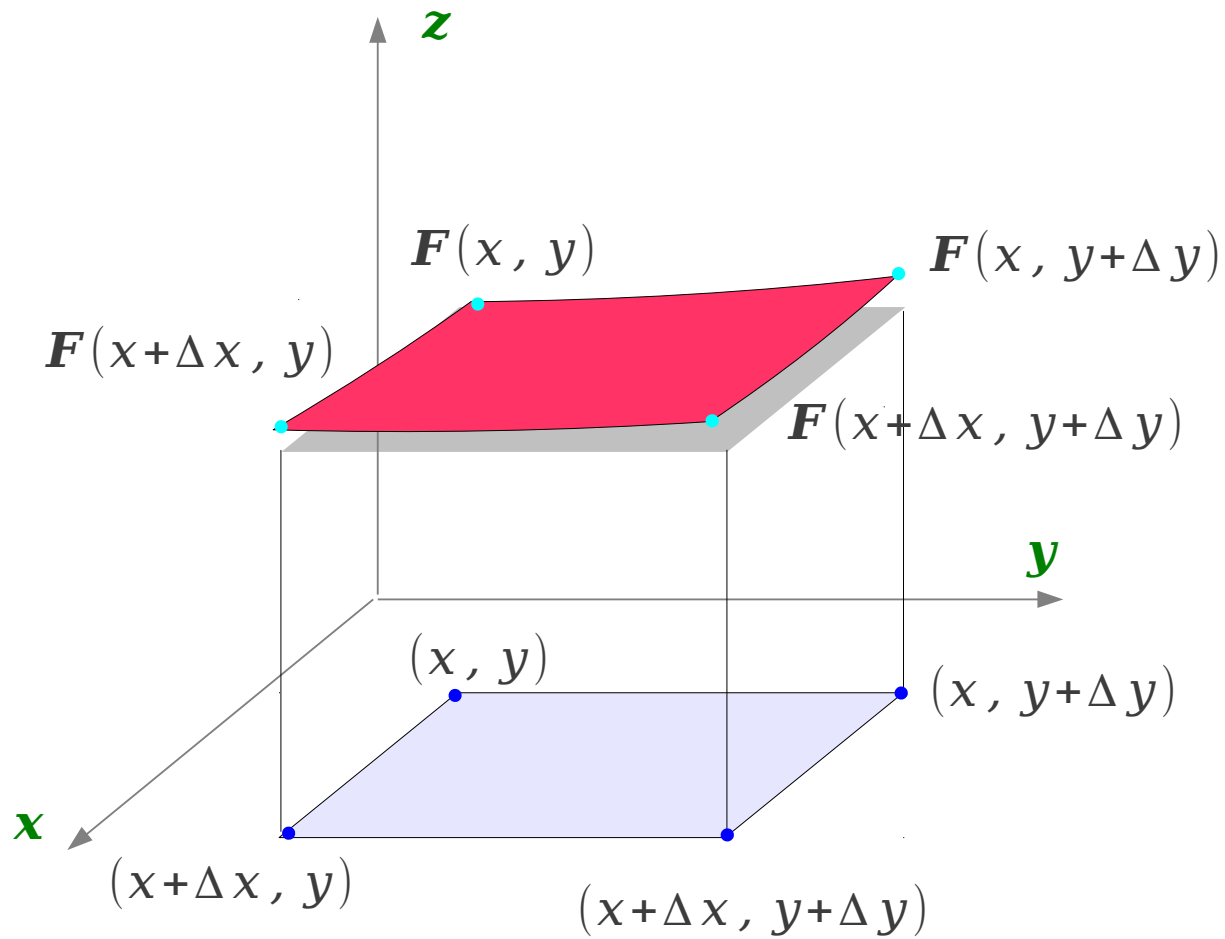
$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

Curl of \mathbf{F}

$$\begin{aligned} &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} && \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \times (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) = \nabla \times \mathbf{F} \end{aligned}$$

i

2-D Divergence



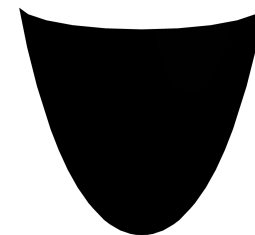
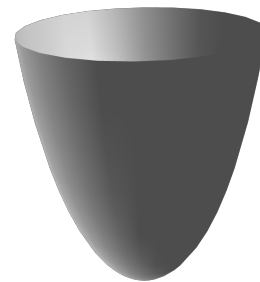
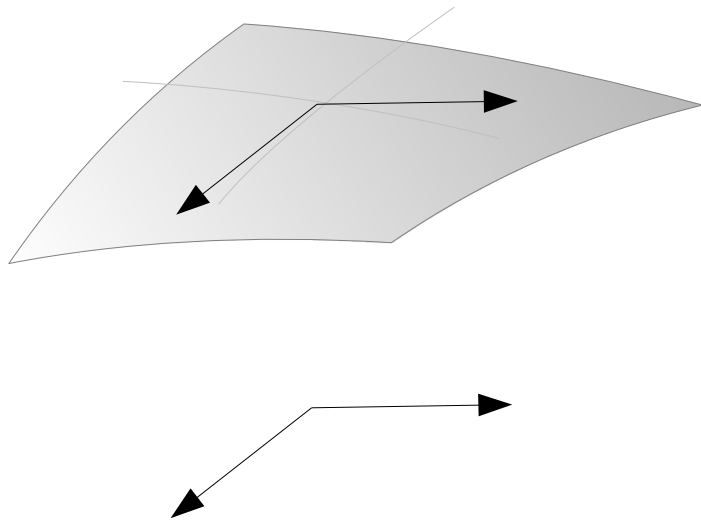
Chain Rule

Function of two variable

$$y = f(u, v)$$

$$u = g(x, y)$$

$$v = h(x, y)$$



References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [4] D.G. Zill, “Advanced Engineering Mathematics”