General Vector Space (3A)

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Vector Space

V: non-empty set of objects

defined operations: addition $\mathbf{u} + \mathbf{v}$

scalar multiplication $k\,\mathbf{u}$

if the following axioms are satisfied
for all object $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and all scalar k, m

⇨ V: vector space
objects in V: vectors

1. if $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ (zero vector)
5. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + (\mathbf{u}) = \mathbf{0}$
6. if k is any scalar and $\mathbf{u}$ is objects in V, then $k\mathbf{u}$ is in V
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

Test for a Vector Space

1. Identify the set V of objects
2. Identify the addition and scalar multiplication on V
3. Verify $\mathbf{u} + \mathbf{v}$ is in V and $k\mathbf{u}$ is in V

   closure under **addition** and **scalar multiplication**
4. Confirm other axioms.

1. if $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ (zero vector)
5. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + (\mathbf{u}) = \mathbf{0}$
6. if k is any scalar and $\mathbf{u}$ is objects in V, then $k\mathbf{u}$ is in V
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

Subspace

a subset W of a vector space V

If the subset W is itself a vector space ➡ the subset W is a **subspace** of V

1. if **u** and **v** are objects in W, then **u** + **v** is in W
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. **0** + **u** = **u** + **0** = **u** (zero vector)
5. **u** + (–**u**) = (–**u**) + (**u**) = **0**
6. if *k* is any scalar and **u** is objects in W, then *k***u** is in W
7. *k*(**u** + **v**) = *k***u** + *k***v**
8. (*k* + *m*)**u** = *k***u** + *m***u**
9. *k*(*m***u**) = (*km*)**u**
10. *1*(**u**) = **u**

Subspace Example (1)

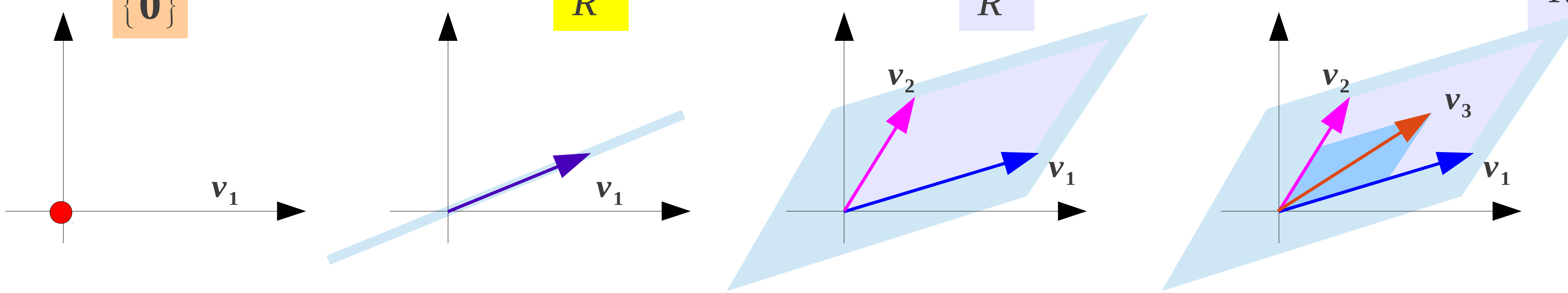
In vector space R^2

any one vector	(linearly indep.)	**spans**	R^1	line <u>through **0**</u>
any two non-collinear vectors	(linearly indep.)	**spans**	R^2	plane
any three or more vectors	(linearly dep.)	**spans**	R^2	plane

Subspaces of R^2

$\{\mathbf{0}\}$ R^1 R^2

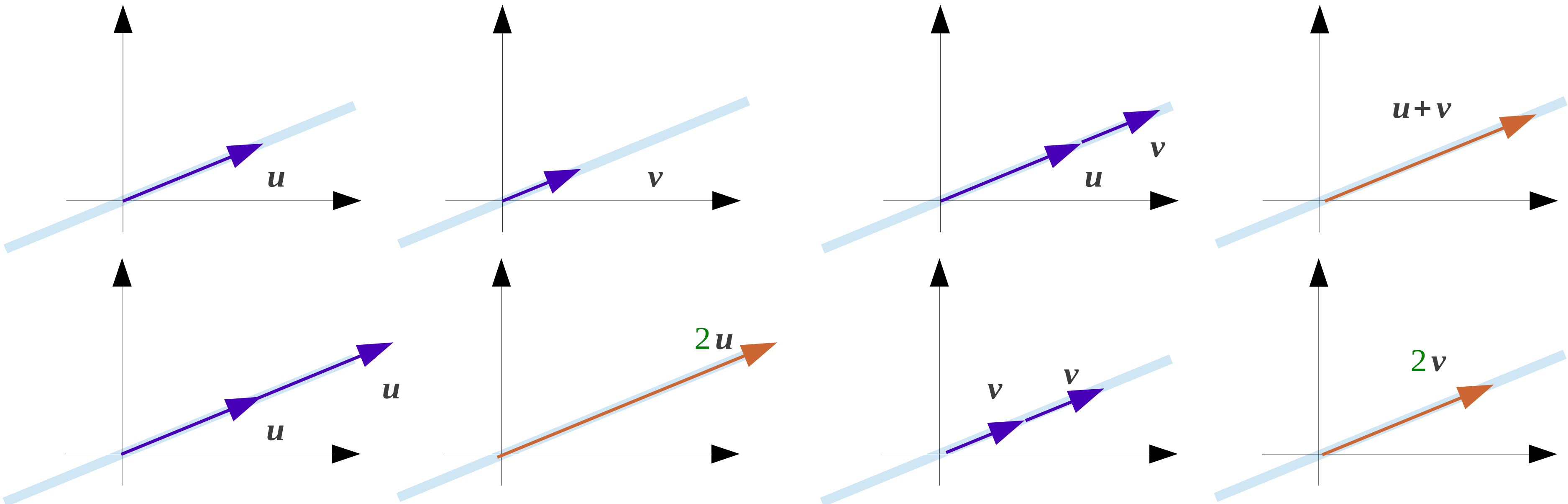
$\mathbf{v}_3 = k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2$ R^2

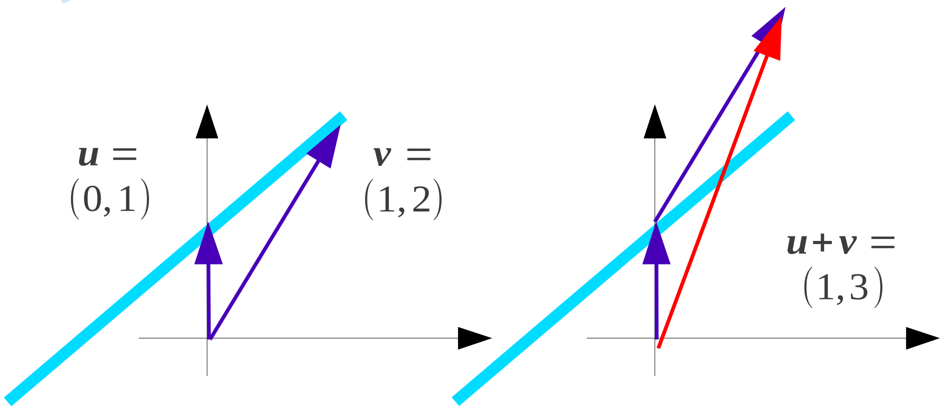


Subspace Example (2)

In vector space R^2

any one vector (linearly indep.) **spans** R^1 line through **0**





~~vector space~~

Subspace Example (3)

In vector space R^3

any one vector	(linearly indep.)	**spans**	R^1	line through **0**
any two non-collinear vectors	(linearly indep.)	**spans**	R^2	plane through **0**
any three vectors non-collinear, non-coplanar	(linearly indep.)	**spans**	R^3	3-dim space
any four or more vectors	(linearly dep.)	**spans**	R^3	3-dim space

Subspaces of R^2

$\{\mathbf{0}\}$	R^1	R^2	R^3
	line through **0**	plane through **0**	3-dim space

Row & Column Spaces

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

ROW Space **subspace** of R^n

$$= span\{\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_m\}$$

COLUMN Space **subspace** of R^m

$$= span\{\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n\}$$

$$\begin{aligned} \boldsymbol{r}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \boldsymbol{r}_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ & \quad\ \ \vdots \qquad \vdots \qquad\qquad\quad \vdots \\ \boldsymbol{r}_m &= \underbrace{\begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{\boldsymbol{n}} \end{aligned}$$

$\boldsymbol{r}_i \in R^n$

$$\begin{array}{cccc} \boldsymbol{c}_1 & \boldsymbol{c}_2 & & \boldsymbol{c}_n \\ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} & \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array} \ \boldsymbol{m}$$

$\boldsymbol{c}_i \in R^m$

Row Space

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

ROW Space **subspace** of R^n

$$= span\{\boldsymbol{r_1}, \boldsymbol{r_2}, \cdots, \boldsymbol{r_m}\}$$

$\boldsymbol{r_i} \in R^n$

$$\boldsymbol{r_1} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$
$$\boldsymbol{r_2} = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$
$$\vdots$$
$$\boldsymbol{r_m} = \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\xleftrightarrow{\quad \boldsymbol{n} \quad}$$

$$k_1\boldsymbol{r_1} + k_2\boldsymbol{r_2} + \cdots + k_m\boldsymbol{r_m}$$

$$= k_1\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} + k_2\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} + \cdots + k_m\begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Column Spaces

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

COLUMN Space **subspace** of R^m

$$= span\{\mathbf{c_1}, \mathbf{c_2}, \cdots, \mathbf{c_n}\}$$

$\mathbf{c}_i \in R^m$

$$\begin{array}{cccccc} & \mathbf{c_1} & \mathbf{c_2} & & \mathbf{c_n} \\ \mathbf{m} & \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} & \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array}$$

$$k_1\mathbf{c_1} + k_2\mathbf{c_2} + \cdots + k_n\mathbf{c_n}$$

$$= k_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + k_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots + k_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Null Space

$$\begin{array}{cc} & n \\ m & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \begin{array}{cc} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} & n \end{array} = \begin{array}{cc} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & m \end{array}$$

NULL Space

subspace of R^n

solution space

$$= \begin{pmatrix} a_{11}x_1 + & a_{12}x_2 + & \cdots & a_{1n}x_n \\ a_{21}x_1 + & a_{22}x_2 + & \cdots & a_{2n}x_n \\ \vdots & \vdots & & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \cdots & a_{mn}x_n \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$\boldsymbol{A}\boldsymbol{x} = x_1 \boldsymbol{c}_1 + x_2 \boldsymbol{c}_2 + \cdots + x_n \boldsymbol{c}_n = \boldsymbol{0} \qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$$

$$\boldsymbol{A}\boldsymbol{x} = x_1 \boldsymbol{c}_1 + x_2 \boldsymbol{c}_2 + \cdots + x_n \boldsymbol{c}_n = \boldsymbol{b} \qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

Null Space

$$\begin{array}{c} \quad n \\ m \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} m$$

NULL Space — **subspace** of R^n

solution space — $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$

Invertible A — $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{0} = \boldsymbol{0}$ — only trivial solution — $\{\boldsymbol{0}\}$

Non-invertible A — $\cancel{\boldsymbol{A}^{-1}}$

zero row(s) in a RREF	free variables	parameters *s, t, u, …*	
one	one	a line through the origin	R^1
two	two	a plane through the origin	R^2
three	three	a 3-dim space through the origin	R^3

Solution Space of **Ax=b** (1)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 1$$

$$\cancel{0 = 1}$$

$$1\cdot x_1 + 3\cdot x_3 = -1$$
$$1\cdot x_2 - 4\cdot x_3 = 2$$

$$1\cdot x_1 - 5\cdot x_2 + 1\cdot x_3 = 4$$

Solve for a leading variable

$$x_1 = -1 - 3\cdot x_3 \qquad x_2 = 2 + 4\cdot x_3$$

$$x_1 = 4 + 5\cdot x_2 - 1\cdot x_3$$

Treat a free variable
as a parameter

$$x_3 = t$$

$$x_2 = s \qquad x_3 = t$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

Solution Space of **Ax=b** (2)

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ \boxed{x_3 = t} \end{cases}$$

free variable

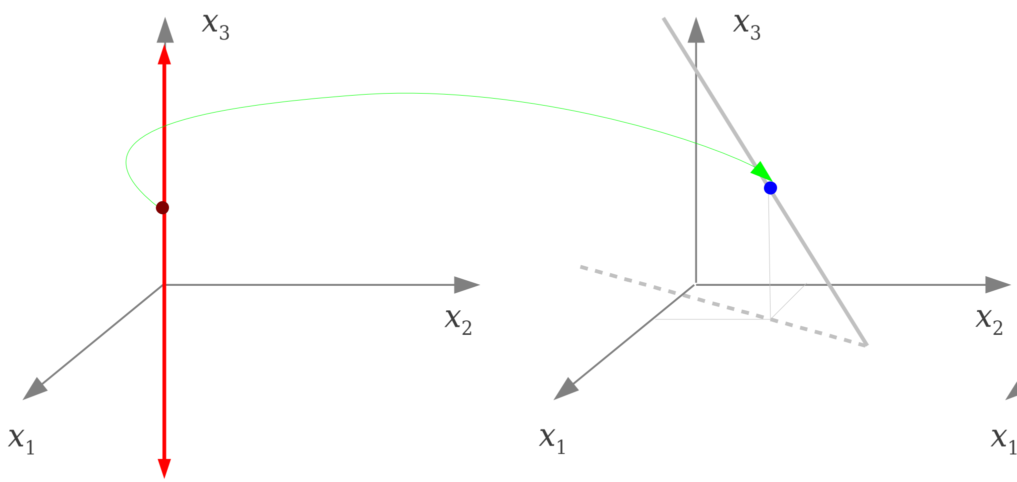
$$\begin{cases} x_1 = 4 + 5s - 1t \\ \boxed{\begin{aligned} x_2 &= s \\ x_3 &= t \end{aligned}} \end{cases}$$

free variable

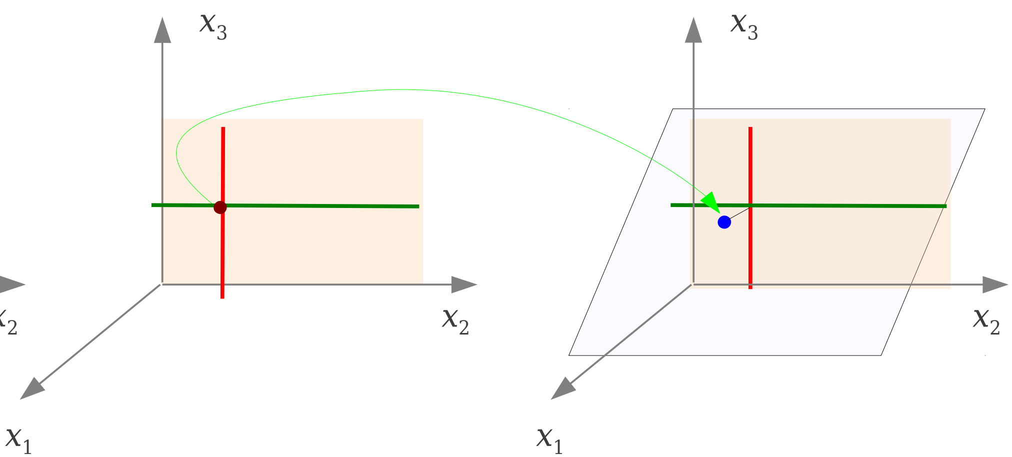
free variable

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$



infinitely many solutions



infinitely many solutions

Solution Space of **Ax=b** (3)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} + \boxed{t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}} + \boxed{s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

General Solution of $\boldsymbol{Ax = b}$ → Particular Solution of $\boldsymbol{Ax = b}$ + General Solution of $\boldsymbol{Ax = 0}$

Particular Solution of $\boldsymbol{Ax = b}$ + General Solution of $\boldsymbol{Ax = 0}$

Linear System & Inner Product (1)

Linear Equations

Corresponding Homogeneous Equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

$$\boldsymbol{a} = (a_1 , a_2 , \cdots , a_n)$$

$$\boldsymbol{x} = (x_1 , x_2 , \cdots , x_n)$$

normal vector

$$\boldsymbol{a} \cdot \boldsymbol{x} = b$$

$$\boldsymbol{a} \cdot \boldsymbol{x} = 0$$

each **solution** vector $\boldsymbol{x}$ of a **homogeneous** equation
orthogonal to the coefficient vector $\boldsymbol{a}$

Homogeneous Linear System

$$\begin{array}{l} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = 0 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = 0 \end{array}$$

$$\begin{array}{l} \boldsymbol{r_1} \cdot \boldsymbol{x} = 0 \\ \boldsymbol{r_2} \cdot \boldsymbol{x} = 0 \\ \cdots \\ \boldsymbol{r_m} \cdot \boldsymbol{x} = 0 \end{array}$$

Linear System & Inner Product (2)

Homogeneous Linear System

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{r_1} \cdot \boldsymbol{x} &= 0 \\
\boldsymbol{r_2} \cdot \boldsymbol{x} &= 0 \\
\cdots & \\
\boldsymbol{r_m} \cdot \boldsymbol{x} &= 0
\end{aligned}
$$

each **solution** vector $\boldsymbol{x}$ of a **homogeneous** equation
orthogonal to the row vector $\boldsymbol{r_i}$ of the coefficient matrix

Homogeneous Linear System $\boldsymbol{A} \cdot \boldsymbol{x} = 0$ $\boldsymbol{A} : m \times n$

solution set consists of all vectors in R^n
that are **orthogonal** to every row vector of $\boldsymbol{A}$

Linear System & Inner Product (3)

Non-Homogeneous Linear System $\quad A \cdot x = b \quad A : m \times n$

Homogeneous Linear System $\quad A \cdot x = 0$

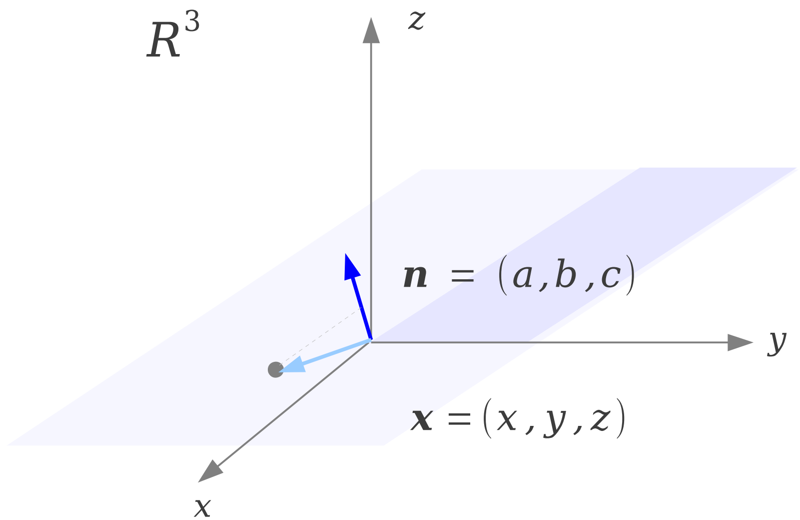
a particular solution

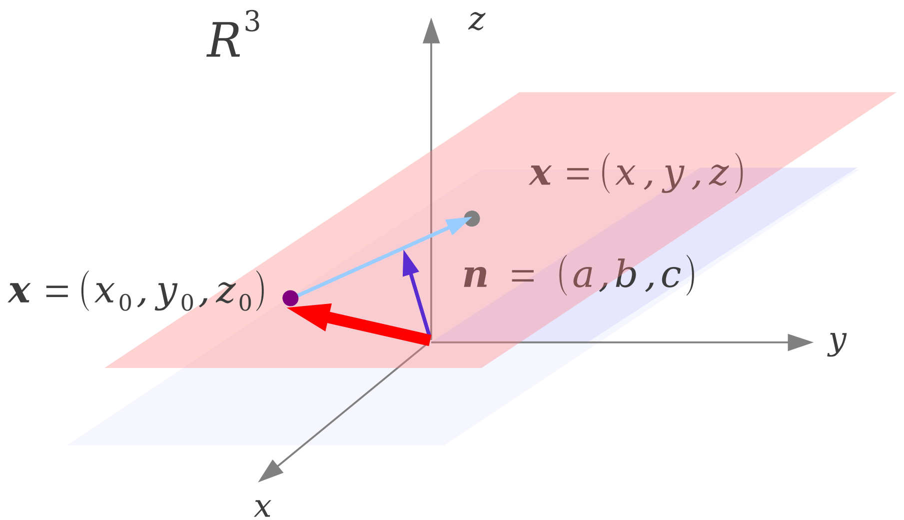
$A \cdot x = b$

solution set consists of all vectors in R^n that are **orthogonal** to every row vector of A

+

a <u>particular</u> solution $x_0 \quad A \cdot x_0 = b$





Linear System & Inner Product (4)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$3 \begin{cases} 2 \begin{cases} \boldsymbol{r_1} \cdot \boldsymbol{x} = 0 \\ \boldsymbol{r_2} \cdot \boldsymbol{x} = 0 \end{cases} \\ 1 \begin{cases} \text{a line through the origin} \quad R^1 \end{cases} \end{cases} \qquad 3 \begin{cases} 1 \begin{cases} \boldsymbol{r_1} \cdot \boldsymbol{x} = 0 \end{cases} \\ 2 \begin{cases} \text{a plane through the origin} \quad R^2 \end{cases} \end{cases}$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

Consistent Linear System **Ax=b**

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + & a_{12}x_2 + & \cdots & a_{1n}x_n \\ a_{21}x_1 + & a_{22}x_2 + & \cdots & a_{2n}x_n \\ \vdots & \vdots & & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \cdots & a_{mn}x_n \end{pmatrix}$$

$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ consistent $\Leftrightarrow$

$x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots + x_n\boldsymbol{c}_n = \boldsymbol{b}$

expressed in linear combination of column vectors

$\Leftrightarrow$ $\boldsymbol{b}$ is in the column space of $\boldsymbol{A}$

$$\boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots + x_n\boldsymbol{c}_n = \boldsymbol{b}$$

References

[1] http://en.wikipedia.org/
[2] Anton, et al., Elementary Linear Algebra, 10th ed, Wiley, 2011
[3] Anton, et al., Contemporary Linear Algebra,