CLTI Laplace Transform

-

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Causal LTI Systems (1)

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^N} + \cdots + a_1 \frac{d\, y(t)}{dt} + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^M} + \cdots + b_1 \frac{d\, x(t)}{dt} + b_0 x(t)$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t)$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) y(t) = \left(D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N\right) x(t)$$

$$Q(D)\, y(t) = P(D)\, x(t)$$

$M = N$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t)$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) y(t) = \left(b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N\right) x(t)$$

$$Q(D)\, y(t) = P(D)\, x(t)$$

Causal LTI Systems (2)

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^N} + \cdots + a_1 \frac{d\,y(t)}{dt} + a_0\, y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^M} + \cdots + b_1 \frac{d\,x(t)}{dt} + b_0\, x(t)$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) y(t) = \left(D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N\right) x(t)$$

$$Q(D)\, y(t) = P(D)\, x(t)$$

- **Zero Input Response**
- **Zero State Response (Convolution with h(t))**

- **Natural Response (Homogeneous Solution)**
- **Forced Response (Particular Solution)**

Finding ZIR & ZSR Using Laplace Transform (1)

$$\frac{d^2 y(t)}{dt^2} + 5\frac{d\,y(t)}{dt} + 6\,y(t) = \frac{d\,x(t)}{dt} + x(t)$$

$$(D^2+5D+6)\,y(t) = (D+1)\,x(t)$$

initial condition $y(0^-) = 2, \quad \dot{y}(0^-) = 1$

input $x(t) = e^{-4t}\,u(t)$

$$y(t) \Leftrightarrow Y(s)$$

$$\frac{d\,y(t)}{dt} \Leftrightarrow sY(s) - y(0^-)$$

$$\frac{d^2 y(t)}{dt^2} \Leftrightarrow s^2 Y(s) - sy(0^-) - \dot{y}(0^-)$$

$$x(t) \Leftrightarrow X(s) = \frac{1}{s+4}$$

$$\frac{d\,x(t)}{dt} \Leftrightarrow sX(s) - x(0^-) = \frac{s}{s+4}$$

($x(0^-)$ → 0)

$$[s^2 Y(s) - 2s - 1] + 5[s\,Y(s) - 2] + 6\,Y(s) = \frac{s}{s+4} + \frac{1}{s+4}$$

initial condition terms

Finding ZIR & ZSR Using Laplace Transform (2)

$$\frac{d^2 y(t)}{dt^2} + 5\frac{d\,y(t)}{dt} + 6\,y(t) = \frac{d\,x(t)}{dt} + x(t)$$

$$(D^2 + 5D + 6)\,y(t) = (D+1)\,x(t)$$

initial condition $y(0^-) = 2, \quad \dot{y}(0^-) = 1$

input $x(t) = e^{-4t}\,u(t)$

$$[s^2 Y(s) - 2s - 1] + 5[sY(s) - 2] + 6Y(s) = \frac{s}{s+4} + \frac{1}{s+4}$$

$$(s^2 + 5s + 6)Y(s) - (2s + 11) = \frac{s+1}{s+4}$$

$$(s^2 + 5s + 6)Y(s) = (2s + 11) + \frac{s+1}{s+4} = \frac{2s^2 + 20s + 45}{s + 4}$$

$$Y(s) = \frac{2s^2 + 20s + 45}{(s + 4)(s^2 + 5s + 6)} = \frac{2s^2 + 20s + 45}{(s + 2)(s + 3)(s + 4)}$$

init cond terms

input terms

$$Y(s) = \frac{13/2}{s+2} - \frac{3}{s+3} - \frac{3/2}{s+4} \quad \Leftrightarrow \quad y(t) = \left(\frac{13}{2}e^{-2t} - 3e^{-3t} - \frac{3}{2}e^{-4t}\right)u(t)$$

Finding ZIR & ZSR Using Laplace Transform (3)

$$\frac{d^2 y(t)}{dt^2} + 5\frac{d\,y(t)}{dt} + 6\,y(t) = \frac{d\,x(t)}{dt} + x(t)$$

$$(D^2+5D+6)\,y(t) = (D+1)\,x(t)$$

initial condition $y(0^-) = 2, \quad \dot{y}(0^-) = 1$

input $x(t) = e^{-4t}\,u(t)$

$$(s^2 + 5s + 6)Y(s) = (2s + 11) + \frac{s+1}{s+4}$$

$$Y(s) = \frac{(2s + 11)}{(s^2 + 5s + 6)} + \frac{s+1}{(s+4)(s^2 + 5s + 6)}$$

Zero Input comp *Zero State comp*

$$Y(s) = \left[\frac{7}{s+2} - \frac{5}{s+3}\right] + \left[\frac{-1/2}{s+2} + \frac{2}{s+3} - \frac{3/2}{s+4}\right]$$

$$\Leftrightarrow \quad y(t) = \left(7e^{-2t} - 5e^{-3t}\right)u(t) + \left(-\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-4t}\right)u(t)$$

Zero State Response (2)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$M = N$

$$Q(D) y(t) = P(D) x(t)$$

All initial conditions are zero $\quad y(0^-) = \dot{y}(0^-) = \ddot{y}(0^-) = \cdots = y^{(N-1)}(0^-) = 0$

$y(t) \Leftrightarrow Y(s)$ $\qquad \dfrac{d^r y(t)}{dt^r} \Leftrightarrow s^r Y(s)$

causal input: $\quad x(0^-) = \dot{x}(0^-) = \ddot{x}(0^-) = \cdots = x^{(N-1)}(0^-) = 0$

$x(t) \Leftrightarrow X(s)$ $\qquad \dfrac{d^k x(t)}{dt^k} \Leftrightarrow s^k X(s)$

Zero State Response (2)

$$\frac{d^N y(t)}{d t^N} + a_1 \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{d t} + a_N y(t) = b_0 \frac{d^M x(t)}{d t^M} + b_1 \frac{d^M x(t)}{d t^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{d t} + b_N x(t)$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) y(t) = \left(b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N\right) x(t)$$

$M = N$

$$Q(D)\, y(t) = P(D)\, x(t)$$

$$\left(s^N + a_1 s^{N-1} + \cdots + a_{N-1} s + a_N\right) Y(s) = \left(b_0 s^M + b_1 s^{M-1} + \cdots + b_{N-1} s + b_N\right) X(s)$$

$$Y(s) = \frac{\left(b_0 s^M + b_1 s^{M-1} + \cdots + b_{N-1} s + b_N\right)}{\left(s^N + a_1 s^{N-1} + \cdots + a_{N-1} s + a_N\right)} X(s)$$

Laplace

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1}\frac{dy(t)}{dt} + a_N y(t) = b_{N-M}\frac{d^M x(t)}{dt^M} + b_{N-M+1}\frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1}\frac{dx(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$Q(D)\, y_0(t) = 0 \quad \Rightarrow \quad (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_0(t) = 0$$

$$Q(\lambda) = 0 \quad \Leftrightarrow \quad \underbrace{(\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N)}_{=\,0}\ \underbrace{c e^{\lambda t}}_{\neq\,0} = 0$$

$$Q(\lambda) = (\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N) = 0$$

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N)$$

λ_i characteristic roots

$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \cdots + c_N e^{\lambda_N t} = y_0(t)$$

$e^{\lambda_i t}$ characteristic modes

ZIR: a linear combination of the characteristic modes of the system

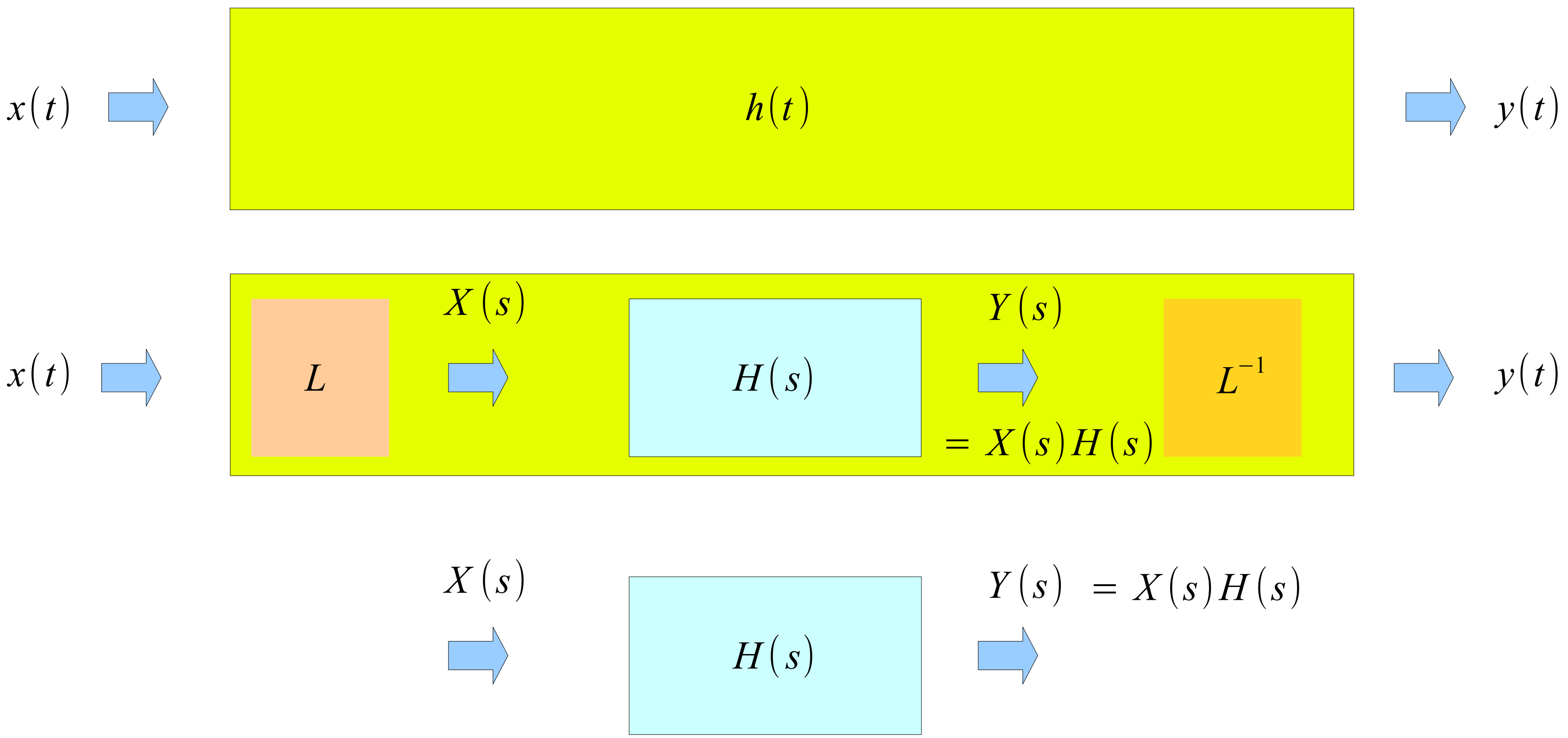
Laplace Transform (1)

e^{st}	→	$h(t)$	→	$H(s)\, e^{st}$
$x(t) = \sum_{k=1}^{K} X(s_i) e^{s_i t}$	→	$h(t)$	→	$y(t) = \sum_{k=1}^{K} X(s_i) H(S_i) e^{s_i t}$
$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st}\, ds$	→	$h(t)$	→	$y(t) = \frac{1}{2\pi j} \int_{c'-j\infty}^{c'+j\infty} X(s) H(s) e^{st}\, ds$

$$= L^{-1} X(s) H(s)$$

$$Y(s) = X(s) H(s)$$

Laplace Transform (2)



s is the complex frequency of e^{st}

Frequency domain method

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] B.P. Lathi, Linear Systems and Signals (2nd Ed)