

DFT Frequency (4B)

- Negative Frequency
- Angular Frequency
- Fundamental Frequency
- Harmonic Frequency
- Sampling Frequency
- Normalized Frequency
- Examples of $N=8$ DFT Matrix

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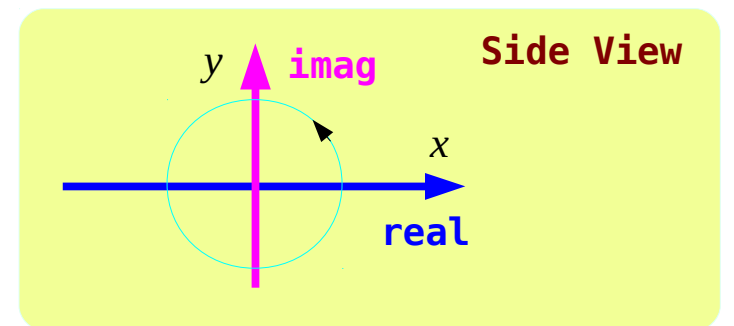
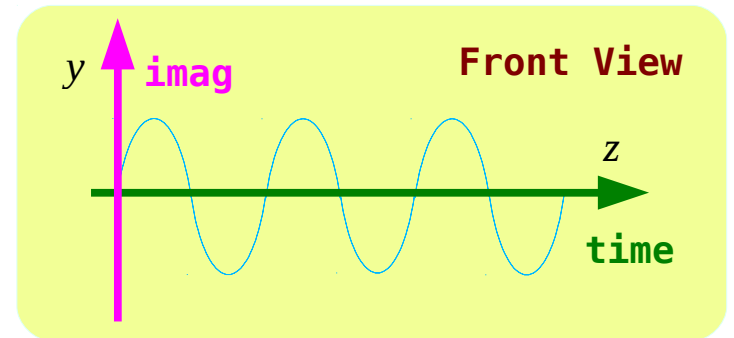
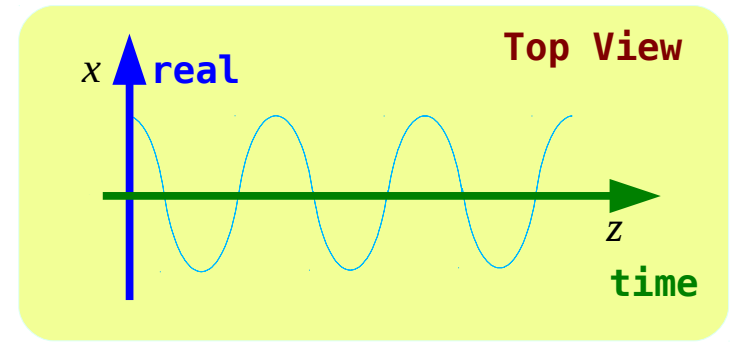
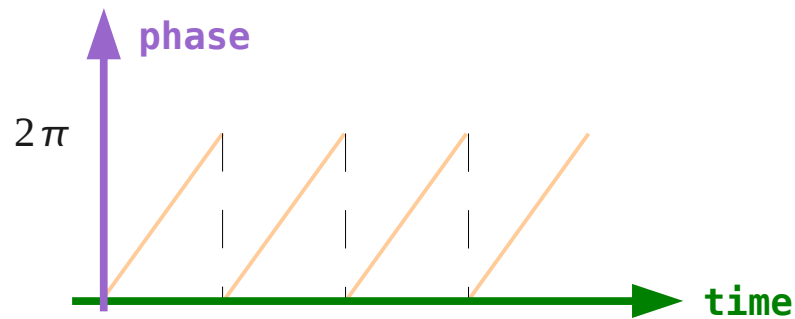
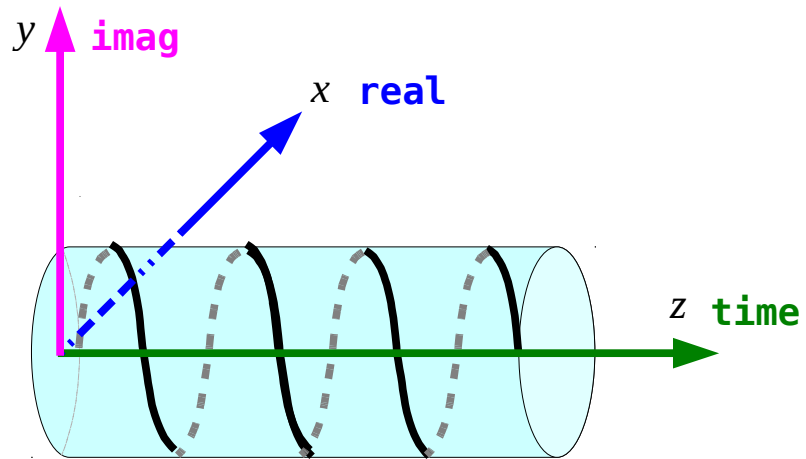
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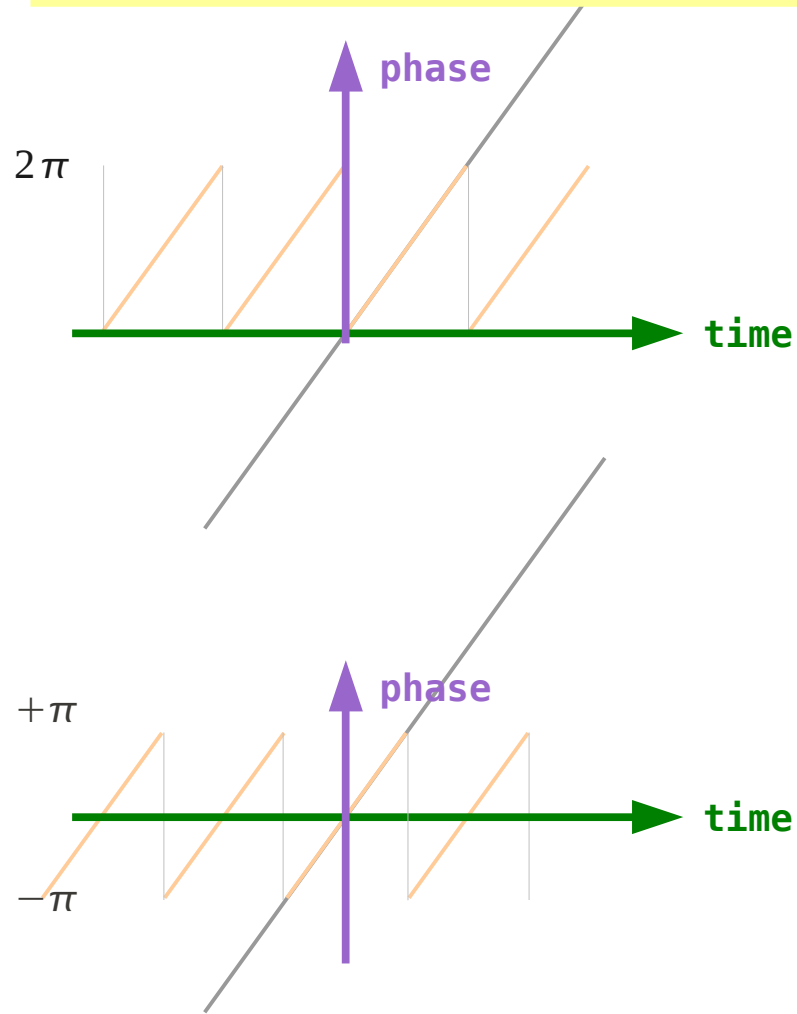
Euler Equation

$$e^{+j\omega t} = \cos \omega t + j \sin \omega t$$

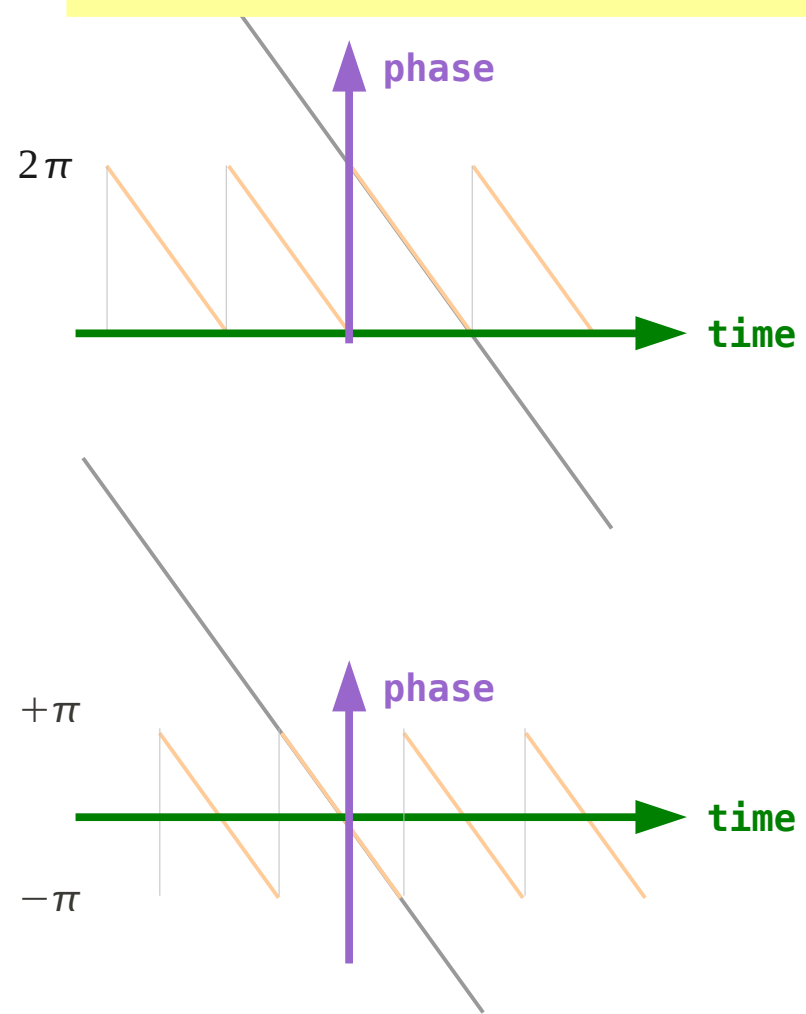


Linear Phase (1)

$$\Phi = \omega t \quad (\omega > 0)$$

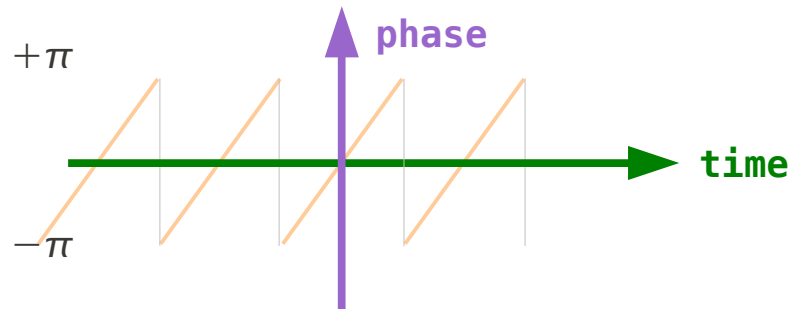


$$\Phi = \omega' t \quad (\omega' < 0)$$

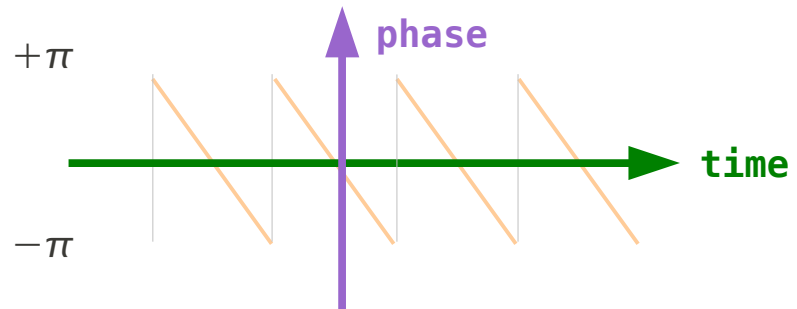


Linear Phase (2)

$$\Phi = \omega t \quad (\omega > 0)$$



$$\Phi = \omega' t \quad (\omega' < 0)$$



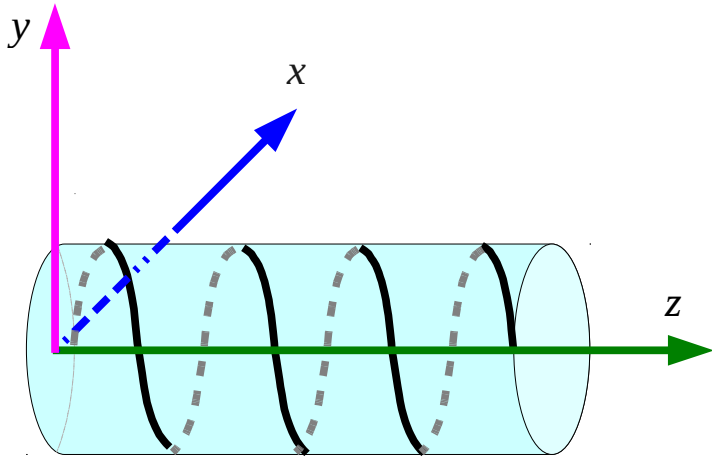
$$\omega' = -\omega$$

$$\cos(\omega' t) = \cos(-\omega t) \quad \Rightarrow \quad \cos(\omega' t) = \cos(\omega t)$$

$$\sin(\omega' t) = \sin(-\omega t) \quad \Rightarrow \quad \sin(\omega' t) = -\sin(\omega t)$$

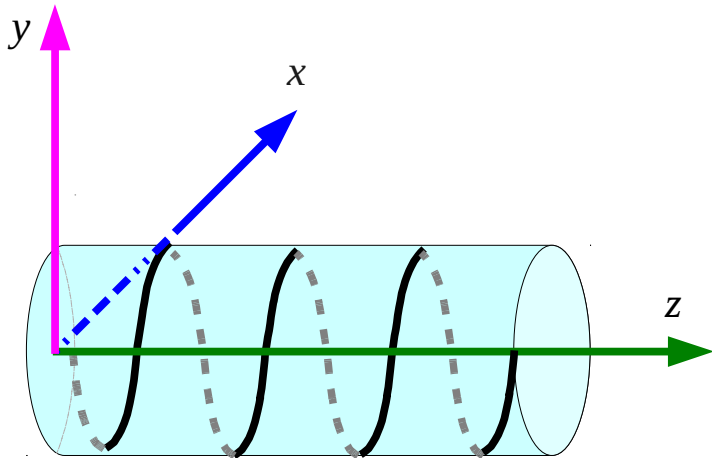
$$e^{j\omega' t} = e^{-j\omega t} \quad \Rightarrow \quad e^{j\omega' t} = \cos(\omega t) - j\sin(\omega t)$$

Linear Phase (3)



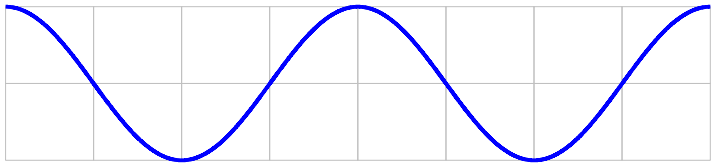
$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$\omega' = -\omega$$

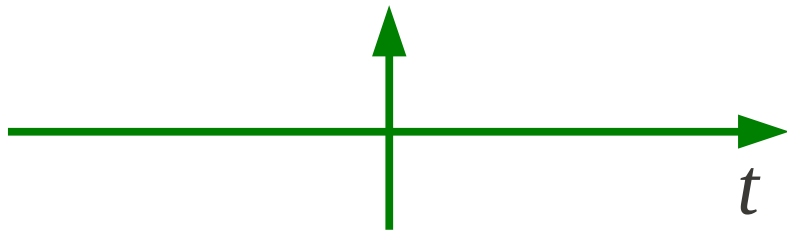


$$e^{j\omega' t} = \cos(\omega t) - j\sin(\omega t)$$

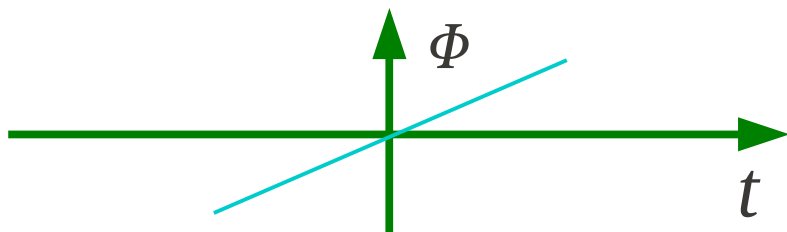
Negative Frequency (1)



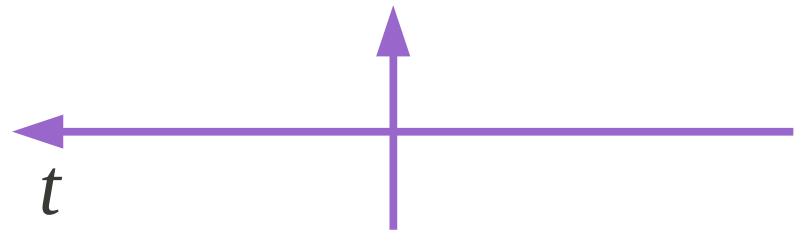
Coordinate (A)



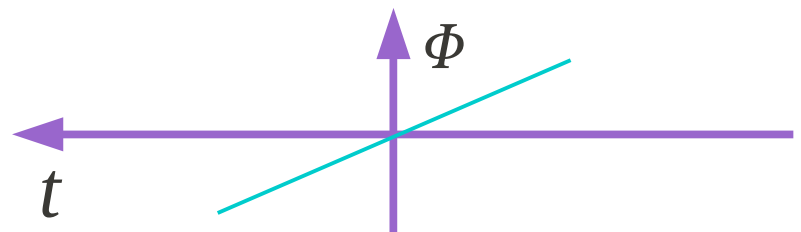
*As t increases,
the phase increases.*



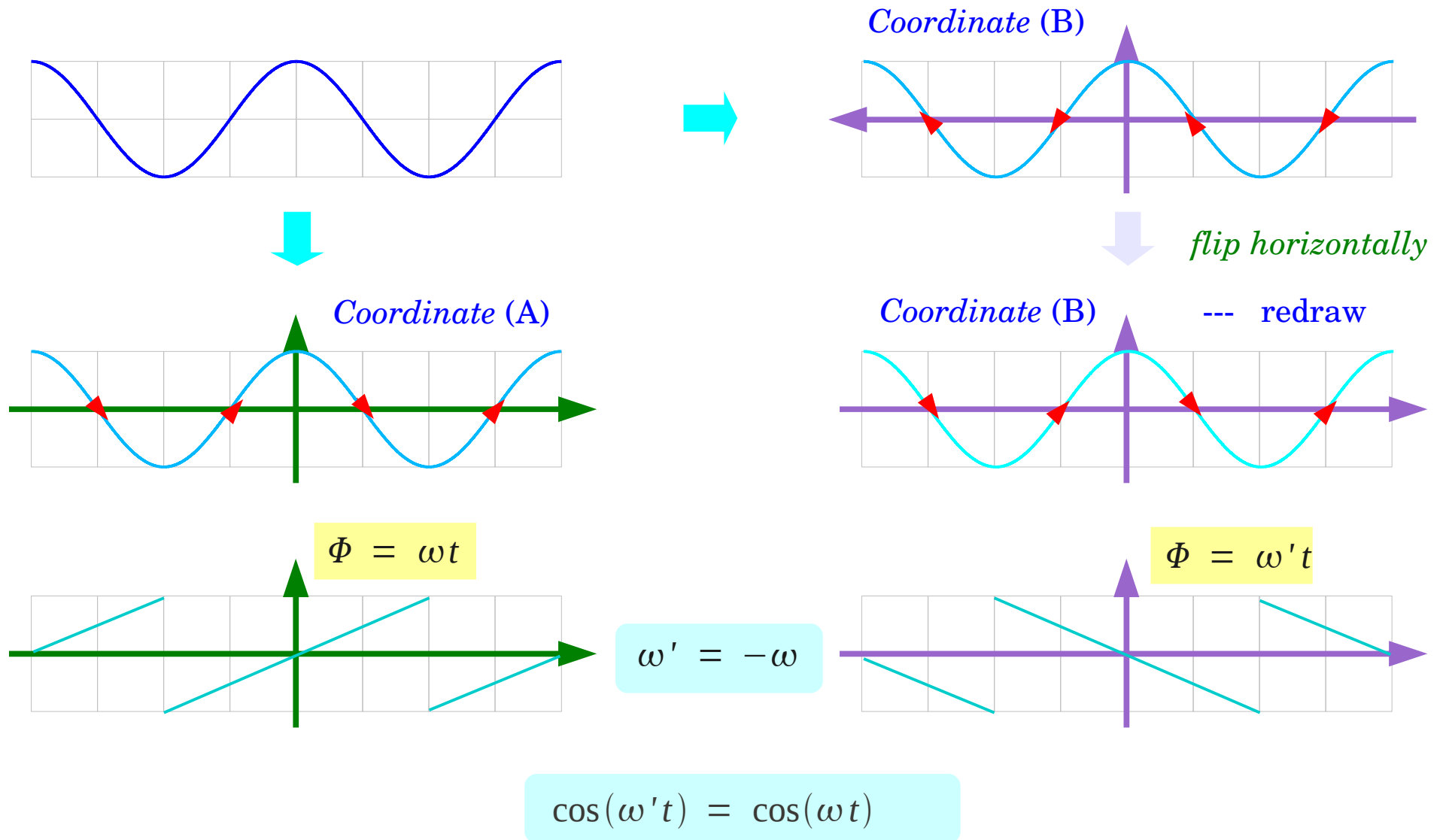
Coordinate (B)



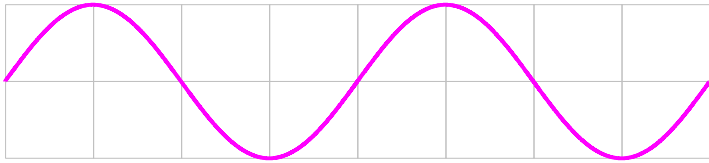
*As t increases,
the phase decreases.*



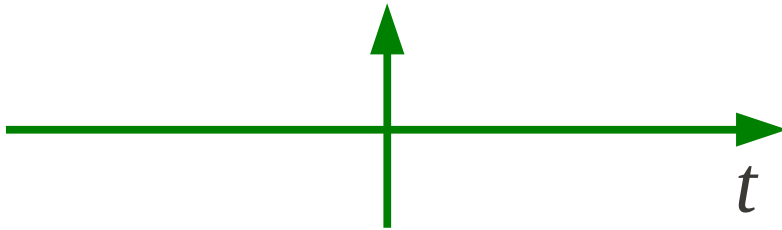
Negative Frequency (2)



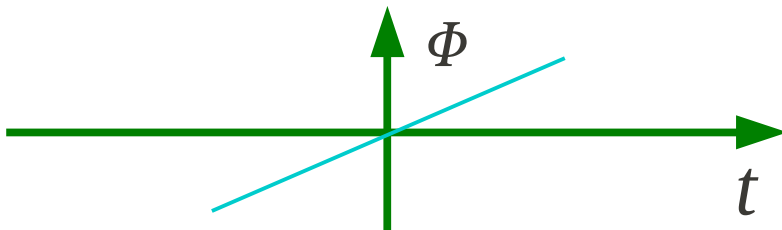
Negative Frequency (3)



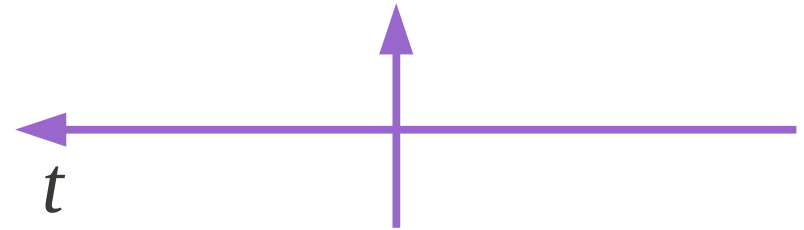
Coordinate (A)



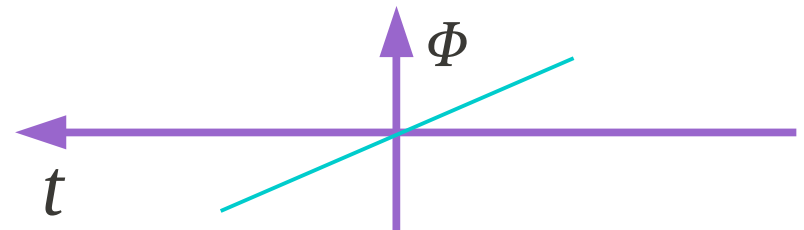
*As t increases,
the phase increases.*



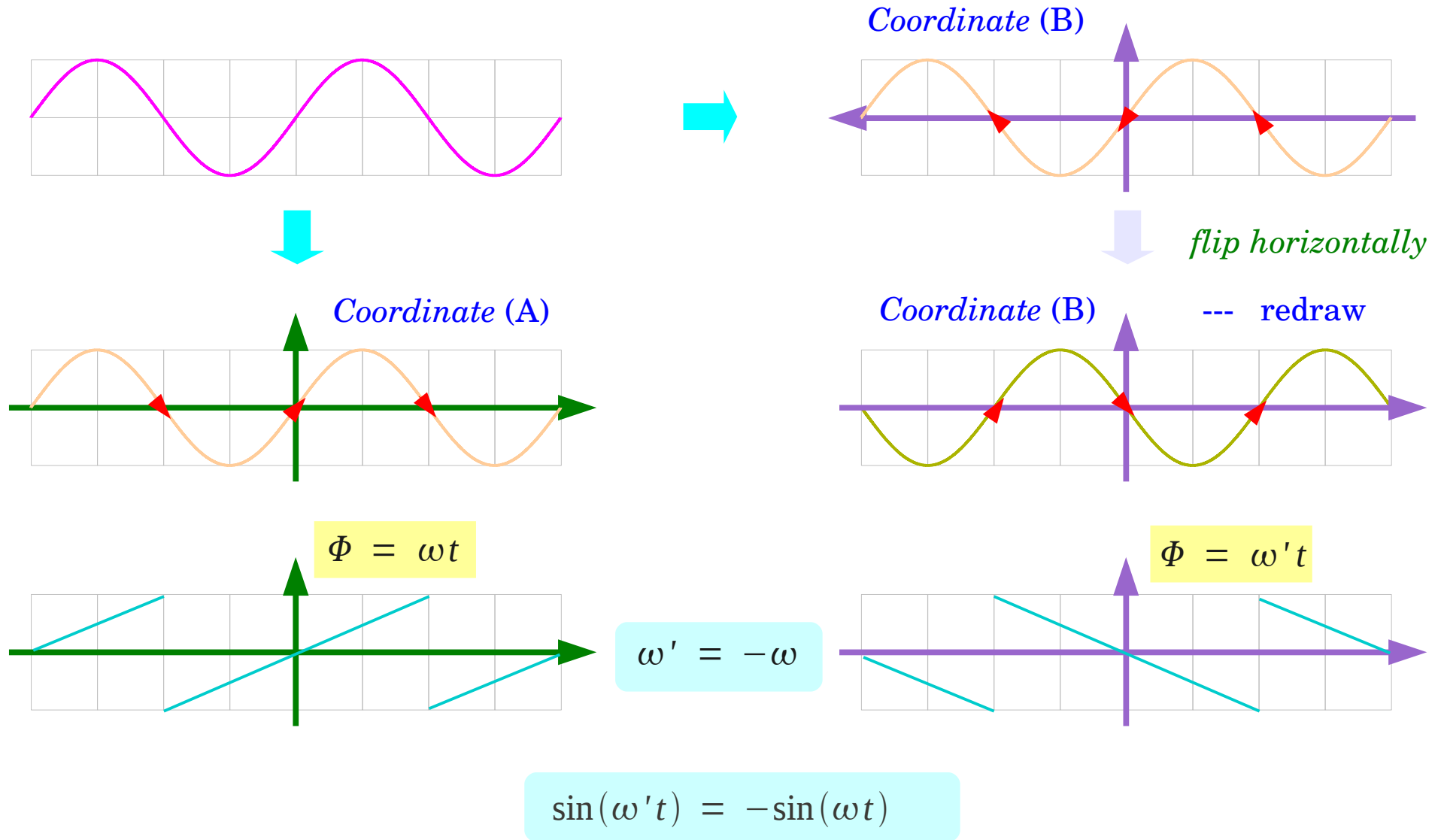
Coordinate (B)



*As t increases,
the phase decreases.*

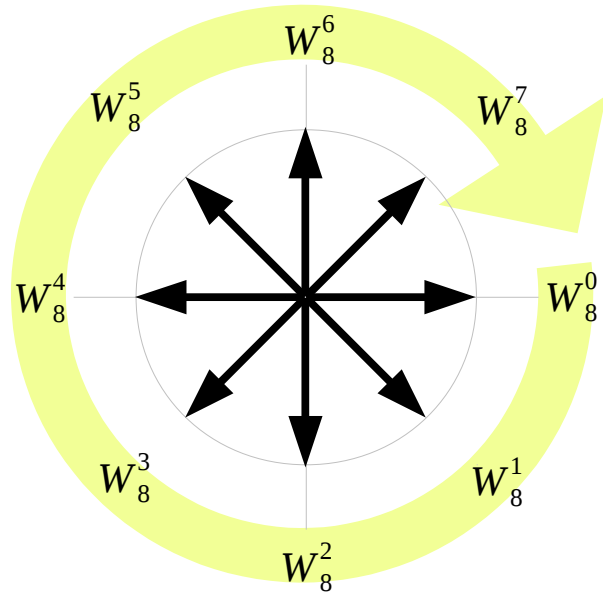


Negative Frequency (4)

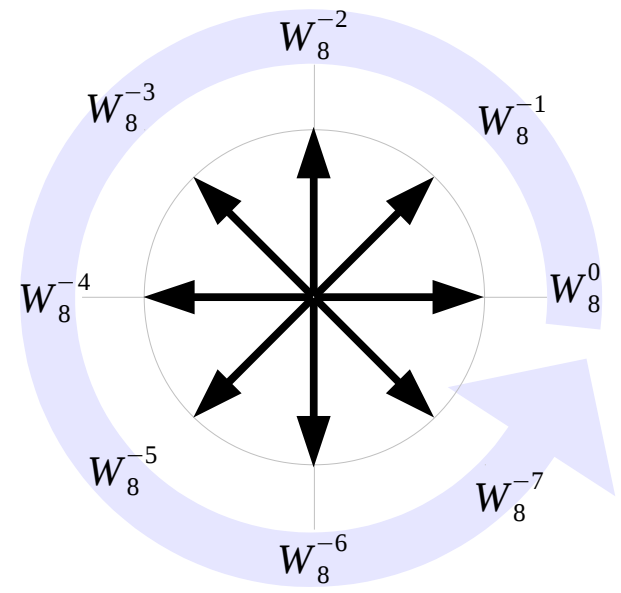


Complex Phase Factor

$$W_8^k = e^{-j\left(\frac{2\pi}{8}\right)k}$$



$$W_8^{-k} = e^{+j\left(\frac{2\pi}{8}\right)k}$$



$$W_8^1 = W_8^{-7}$$

$$= W_8^1$$

$$W_8^2 = W_8^{-6}$$

$$= W_8^2$$

$$W_8^3 = W_8^{-5}$$

$$= W_8^3$$

$$W_8^4 = W_8^{-4}$$

$$= W_8^4$$

$$W_8^5 = W_8^{-3}$$

$$= W_8^{-3}$$

$$W_8^6 = W_8^{-2}$$

$$= W_8^{-2}$$

$$W_8^7 = W_8^{-1}$$

$$= W_8^{-1}$$

$$W_N^{k \pm N} = W_N^k$$

DFT Matrix (1)

W_8^0	W_8^0	W_8^0	W_8^0	W_8^0	W_8^0	W_8^0	W_8^0
W_8^0	W_8^1	W_8^2	W_8^3	W_8^4	W_8^5	W_8^6	W_8^7
W_8^0	W_8^2	W_8^4	W_8^6	W_8^0	W_8^2	W_8^4	W_8^6
W_8^0	W_8^3	W_8^6	W_8^1	W_8^4	W_8^7	W_8^2	W_8^5
W_8^0	W_8^4	W_8^0	W_8^4	W_8^0	W_8^4	W_8^0	W_8^4
W_8^0	W_8^5	W_8^2	W_8^7	W_8^4	W_8^1	W_8^6	W_8^3
W_8^0	W_8^6	W_8^4	W_8^2	W_8^0	W_8^6	W_8^4	W_8^2
W_8^0	W_8^7	W_8^6	W_8^5	W_8^4	W_8^3	W_8^2	W_8^1

$$W_8^{kn} = e^{j\left(\frac{2\pi}{8}\right)kn}$$

	n=0	n=1	n=2	n=3	n=4	n=5	n=6	n=7
k=0	0	0	0	0	0	0	0	0
k=1	0	-1	-2	-3	-4	-5	-6	-7
k=2	0	-2	-4	-6	0	-2	-4	-6
k=3	0	-3	-6	-1	-4	-7	-2	-5
k=4	0	-4	0	-4	0	-4	0	-4
k=5	0	-5	-2	-7	-4	-1	-6	-3
k=6	0	-6	-4	-2	0	-6	-4	-2
k=7	0	-7	-6	-5	-4	-3	-2	-1

DFT Matrix (2)

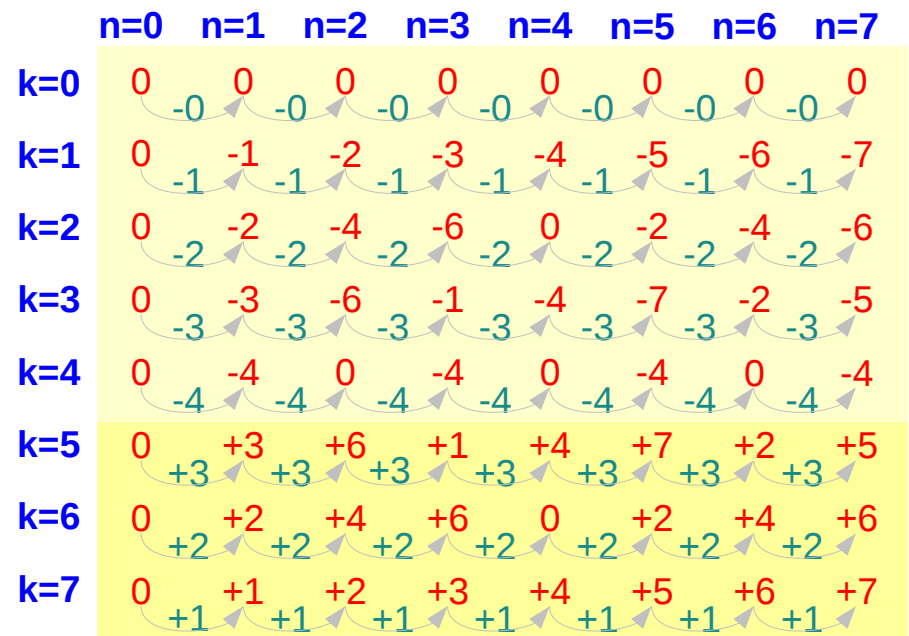
$$\begin{matrix}
 W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
 W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\
 W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
 W_8^0 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\
 W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4
 \end{matrix}$$

still symmetric matrix

$$\begin{matrix}
 W_8^0 & W_8^{-3} & W_8^{-6} & W_8^{-1} & W_8^{-4} & W_8^{-7} & W_8^{-2} & W_8^{-5} \\
 W_8^0 & W_8^{-2} & W_8^{-4} & W_8^{-6} & W_8^0 & W_8^{-2} & W_8^{-4} & W_8^{-6} \\
 W_8^0 & W_8^{-1} & W_8^{-2} & W_8^{-3} & W_8^{-4} & W_8^{-5} & W_8^{-6} & W_8^{-7}
 \end{matrix}$$

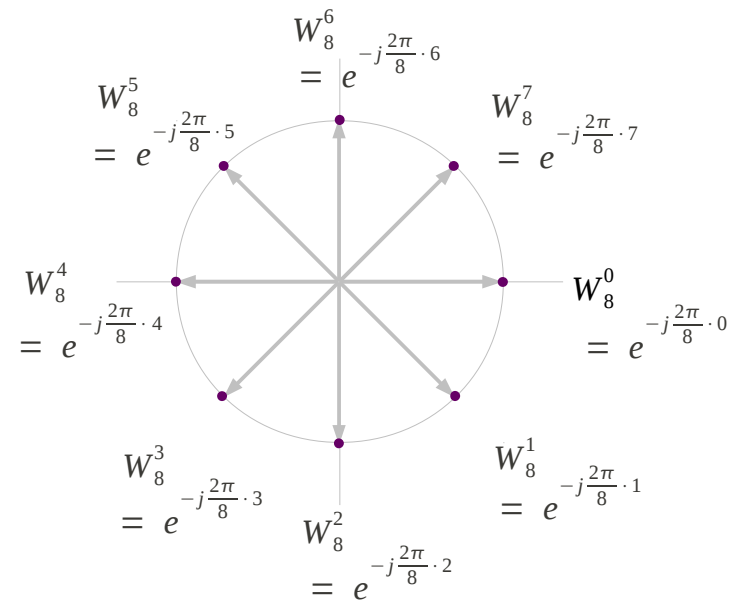
$$W_N^{nk+N} = W_N^{nk}$$

$$W_8^{nk} = e^{j\left(\frac{2\pi}{8}\right)nk}$$



DFT Matrix (3)

	n=0	n=1	n=2	n=3	n=4	n=5	n=6	n=7
k=0	0	0	0	0	0	0	0	0
k=1	0	-1	-2	-3	-4	-5	-6	-7
k=2	0	-2	-4	-6	0	-2	-4	-6
k=3	0	-3	-6	-3	-4	-7	-3	-5
k=4	0	-4	0	-4	0	-4	0	-4
k=5	0	-5	-2	-7	-4	-1	-6	-3
k=6	0	-6	-4	-6	0	-6	-4	-2
k=7	0	-7	-6	-7	-4	-3	-2	-1

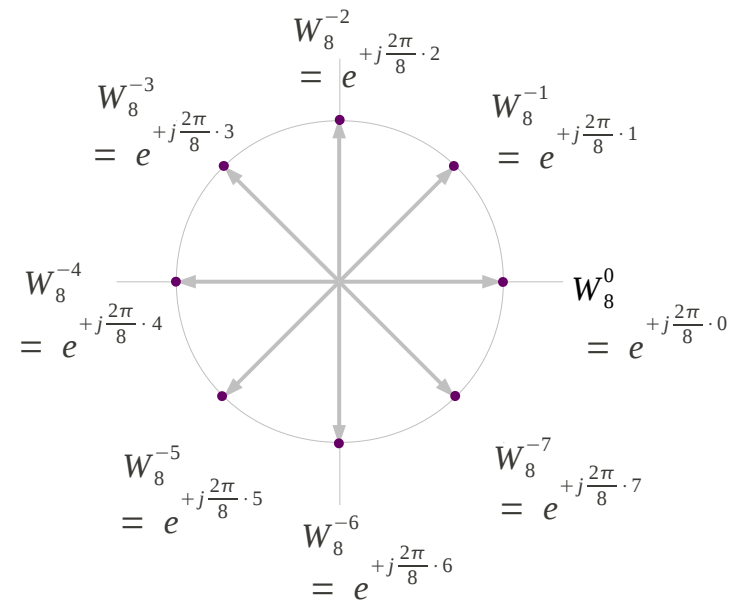


k=0	stride = 0	cw angular speed = 0
k=1	stride = -1	cw angular speed = -1ω
k=2	stride = -2	cw angular speed = -2ω
k=3	stride = -3	cw angular speed = -3ω
k=4	stride = -4	cw angular speed = -4ω
k=5	stride = -5	cw angular speed = -5ω
k=6	stride = -6	cw angular speed = -6ω
k=7	stride = -7	cw angular speed = -7ω

-7

DFT Matrix (4)

	n=0	n=1	n=2	n=3	n=4	n=5	n=6	n=7
k=0	0	0	0	0	0	0	0	0
k=1	0	-1	-2	-3	-4	-5	-6	-7
k=2	0	-2	-4	-6	0	-2	-4	-6
k=3	0	-3	-6	-1	-4	-7	-2	-5
k=4	0	-4	0	-4	0	-4	0	-4
k=5	0	+3	+6	+1	+4	+7	+2	+5
k=6	0	+2	+4	+6	0	+2	+4	+6
k=7	0	+1	+2	+3	+4	+5	+6	+7



k=0	stride = 0	cw angular speed = 0			
k=1	stride = -1	cw angular speed = -1ω			
k=2	stride = -2	cw angular speed = -2ω			
k=3	stride = -3	cw angular speed = -3ω			
k=4	stride = -4	cw angular speed = -4ω			
k=5	stride = -5	cw angular speed = -5ω	↔	stride = +3	ccw angular speed = $+3\omega$
k=6	stride = -6	cw angular speed = -6ω	↔	stride = +2	ccw angular speed = $+2\omega$
k=7	stride = -7	cw angular speed = -7ω	↔	stride = +1	ccw angular speed = $+1\omega$

Angular Frequency

Frequency $f = \frac{1}{T}$ (Hz: cycles per second)

1Hz → event repeats once per second

Angular Frequency $\omega = \frac{2\pi}{T}$ (radians per second)

One revolution = 2π radian

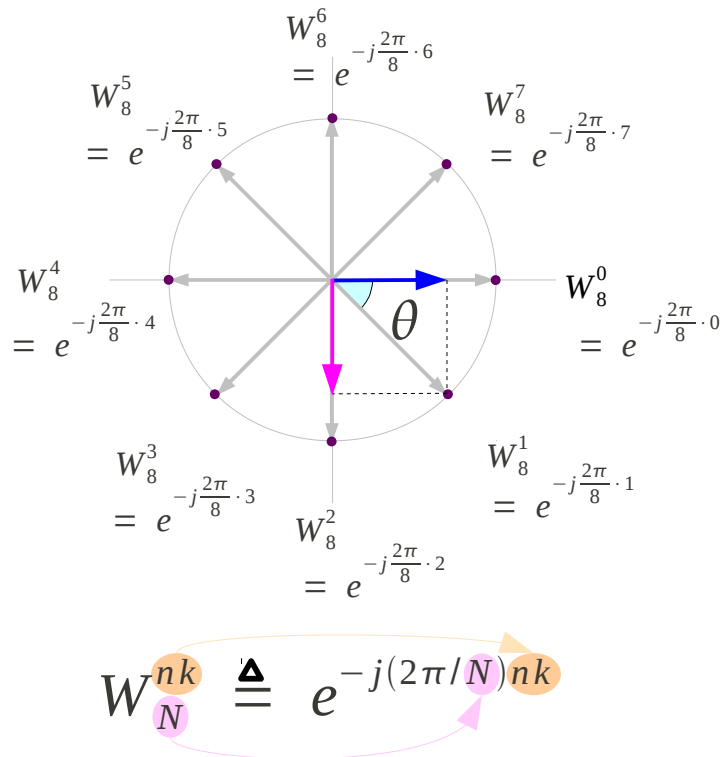
$\omega = 2\pi f = 2\pi \frac{1}{T}$ → **Angular Speed**

$\theta = \omega t = 2\pi f t$ → **Phase**

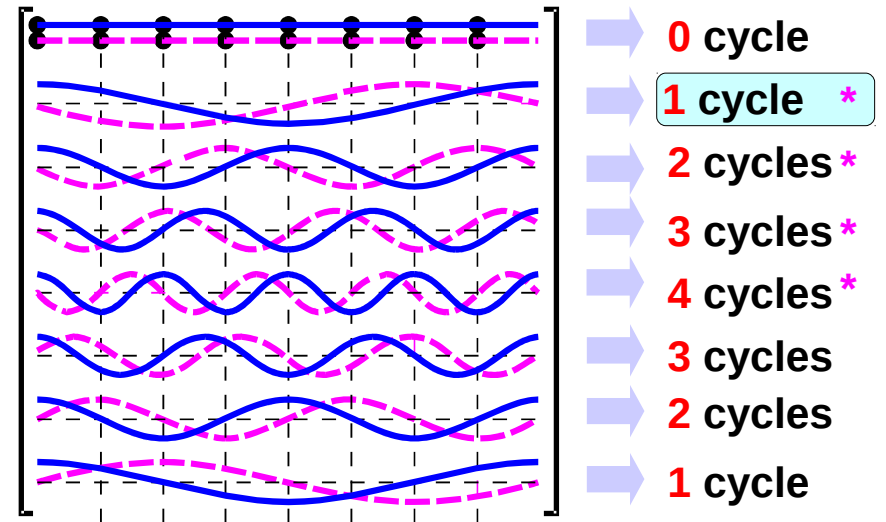
Fundamental Frequency

N=8 → 8 complex phases →
DFT

View as 8 samples in time domain



time →



Period: T

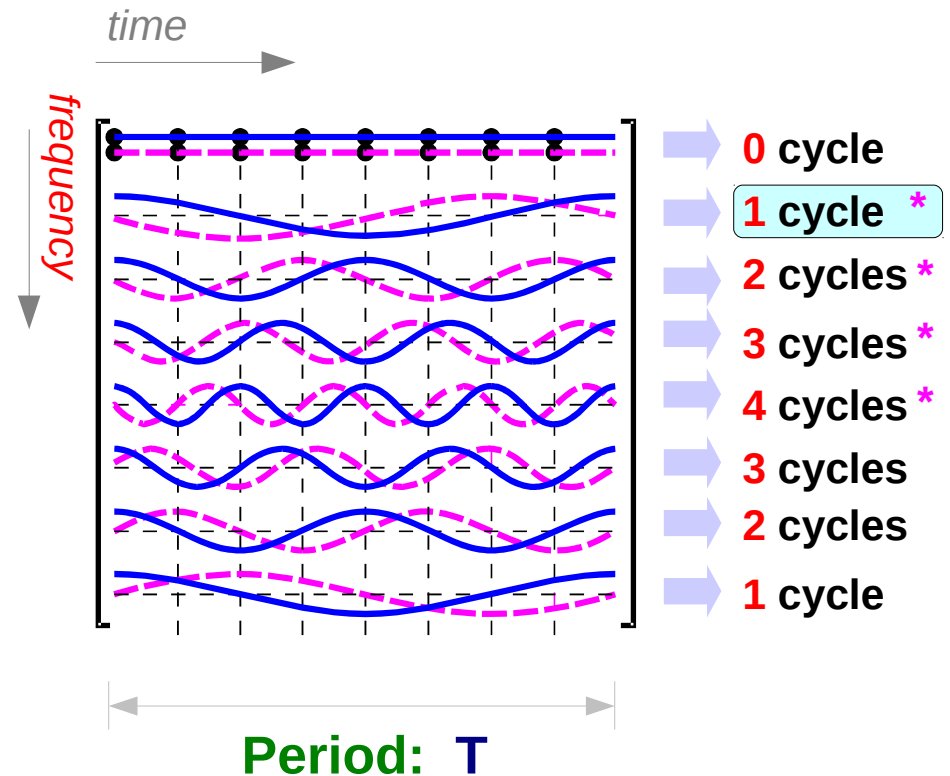
Fundamental Frequency : $f = 1 / T$

Harmonic Frequency

N=8 → 8 complex phases → View as 8 samples in time domain
DFT

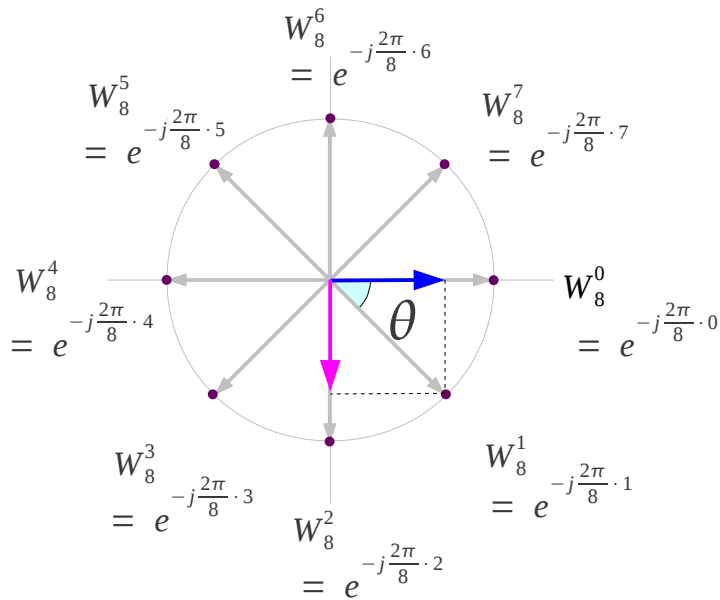
Measuring Frequency	
	0
1 st harmonic	+f
2 nd harmonic	+2f
3 rd harmonic	+3f
4 th harmonic	+4f
5 th harmonic	-3f
6 th harmonic	-2f
7 th harmonic	-f

stride	angular speed
0	CW 0
-1	CW -1ω
-2	CW -2ω
-3	CW -3ω
-4	CW -4ω
+3	CCW $+3\omega$
+2	CCW $+2\omega$
+1	CCW $+1\omega$



Fundamental Frequency : $f = 1 / T$

Sampling Time



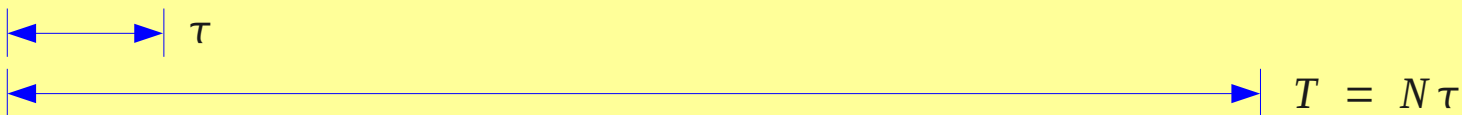
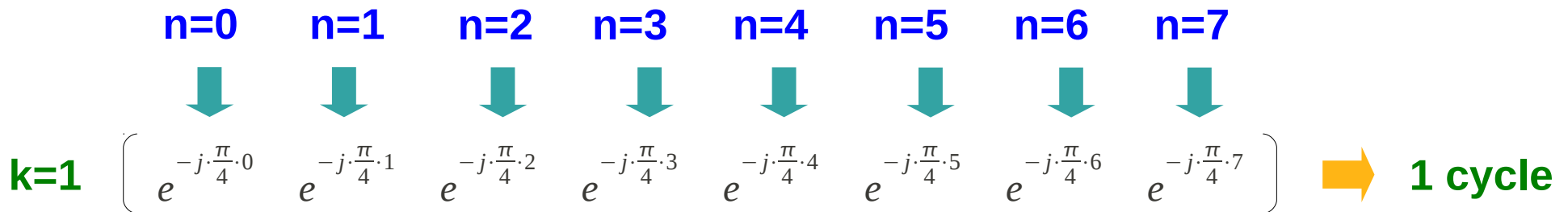
N=8
DFT

8 complex phases

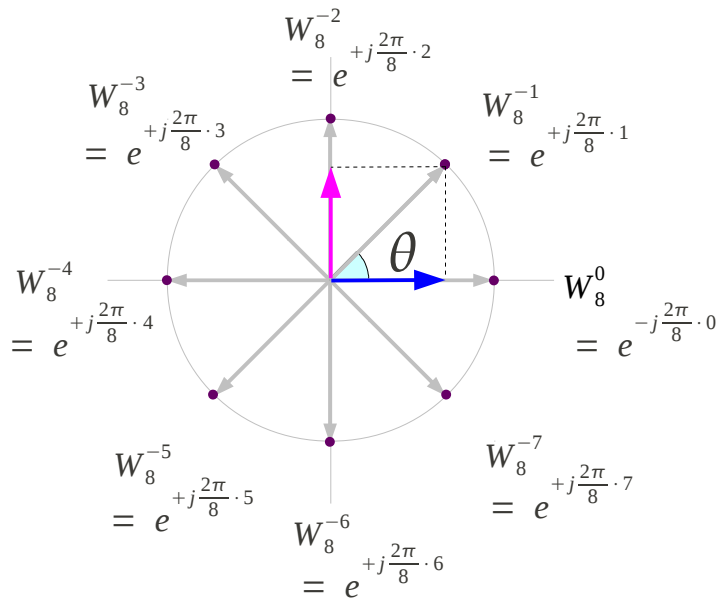
8 samples in time domain

Sampling Time : τ

Period: T $T = N\tau$

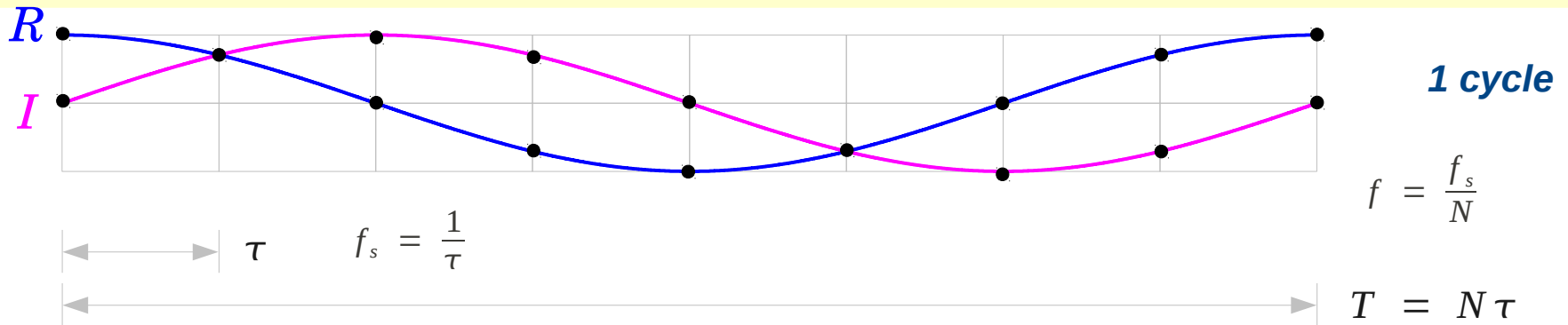


Sampling Frequency

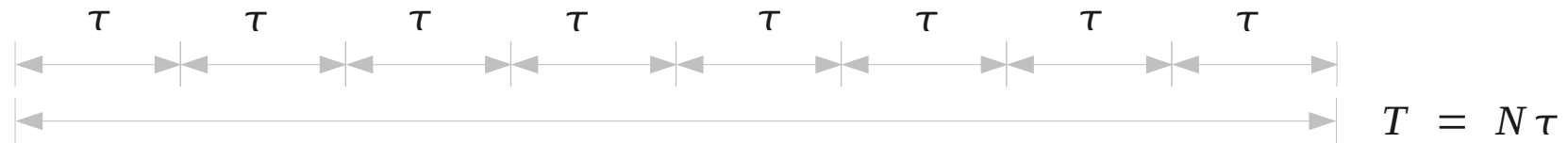


Sampling Time	τ
Sampling Frequency	$f_s = \frac{1}{\tau}$ (samples per second)
Period	$T = N\tau$
Fundamental Freq	$f = \frac{1}{T}$ (cycles per second)
	$f = \frac{f_s}{N} \left(= \frac{1}{N\tau} \right)$

$$\left(e^{+j\frac{\pi}{4}\cdot 0} \quad e^{+j\frac{\pi}{4}\cdot 1} \quad e^{+j\frac{\pi}{4}\cdot 2} \quad e^{+j\frac{\pi}{4}\cdot 3} \quad e^{+j\frac{\pi}{4}\cdot 4} \quad e^{+j\frac{\pi}{4}\cdot 5} \quad e^{+j\frac{\pi}{4}\cdot 6} \quad e^{+j\frac{\pi}{4}\cdot 7} \right)$$



Cycles / Sample



τ *second / sample*

$1/\tau$ *sample / second*

$\frac{0}{N\tau}$ (<i>cycles / second</i>)	➔	0 cycle	over N sample periods	= 0 / N (<i>cycles / sample</i>)
$\frac{1}{N\tau}$ (<i>cycles / second</i>)	➔	1 cycle	“ “	= 1 / N (<i>cycles / sample</i>)
$\frac{2}{N\tau}$ (<i>cycles / second</i>)	➔	2 cycles	“ “	= 2 / N (<i>cycles / sample</i>)
$\frac{3}{N\tau}$ (<i>cycles / second</i>)	➔	3 cycles	“ “	= 3 / N (<i>cycles / sample</i>)
$\frac{4}{N\tau}$ (<i>cycles / second</i>)	➔	4 cycles	“ “	= 4 / N (<i>cycles / sample</i>)

Normalized Frequency

$$\frac{f_n}{f_s} = \frac{n}{N}$$

(cycles per sample)

(cycles per second)

(samples per second)

Normalized Frequency (1)



Sampling Time

$$\tau$$

Sequence Time Length

$$T = N\tau$$

Sampling Frequency

$$f_s = \frac{1}{\tau}$$

(samples per second)

Normalized Frequency

Fundamental Frequency

$$f_0 = \frac{1}{T} = \frac{1}{N\tau}$$

Harmonic Frequencies

$$\begin{cases} f_1 = 1 \cdot f_0 \\ f_2 = 2 \cdot f_0 \\ f_3 = 3 \cdot f_0 \\ \dots \\ f_{N-1} = (N-1) \cdot f_0 \end{cases}$$

$$\frac{f_n}{f_s} = \frac{n}{N}$$

(cycles per sample)

$$1/N$$

$$2/N$$

$$3/N$$

...

$$(N-1)/N$$

Normalized Frequencies

Normalized Frequency (2)



Sampling Time

$$\tau$$

(seconds per sample)

Sequence Time Length

$$T = N\tau$$

Sampling Frequency

$$f_s = \frac{1}{\tau}$$

(samples per second)

1st Harmonic Freq

$$f_1 = \frac{1}{T} = \frac{1}{N\tau} = \frac{1}{N}f_s$$

n^{th} Harmonic Freq

$$f_n = \frac{n}{T} = \frac{n}{N\tau} = \frac{n}{N}f_s \quad n = 0, 1, 2, \dots, \frac{N}{2}$$

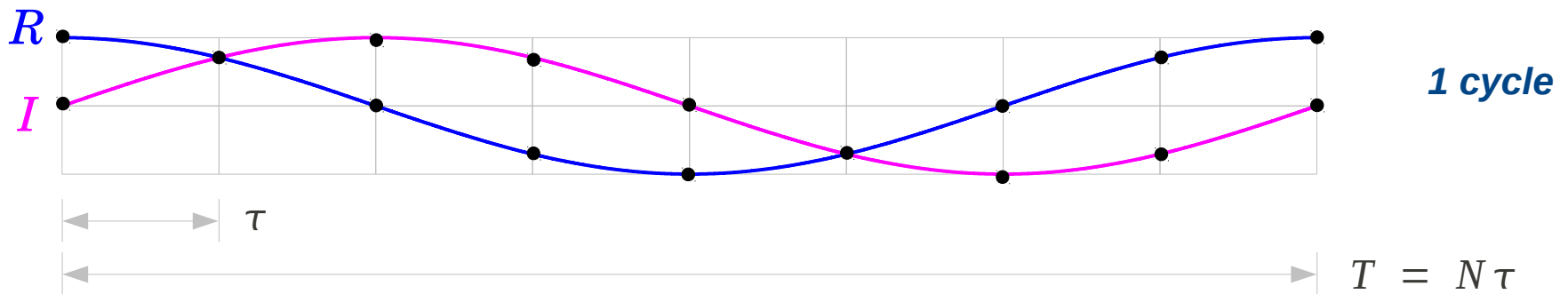
Normalized Frequency

$$\frac{f_n}{f_s} = \frac{n}{N}$$

(cycles per sample)

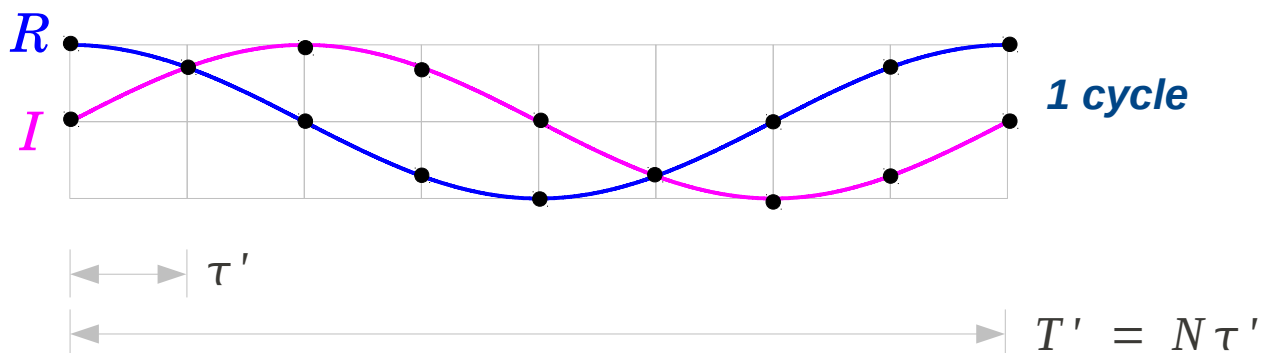
$\frac{\text{(cycles per second)}}{\text{(samples per second)}}$

Normalized Frequency (Ex 1)



1st Harmonic Freq $f_1 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$

Normalized Freq $\frac{f_1}{f_s} = \frac{1}{N}$

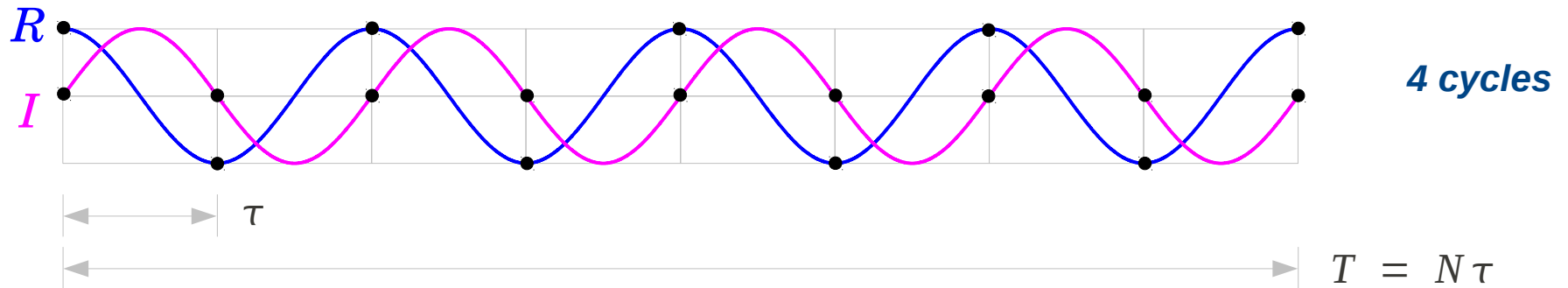


1st Harmonic Freq $f_1' = \frac{1}{T'} = \frac{1}{N\tau'} = \frac{f_s'}{N}$

Normalized Freq $\frac{f_1'}{f_s'} = \frac{1}{N}$

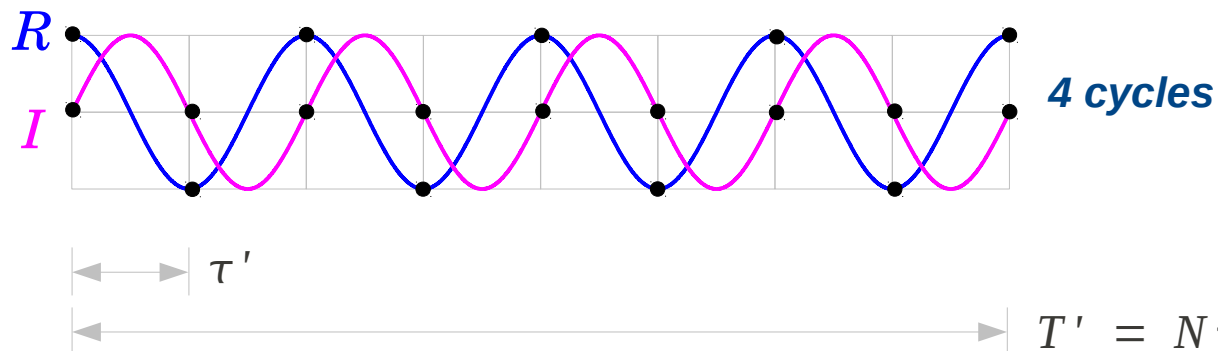


Normalized Frequency (Ex 2)



4th Harmonic Freq $f_1 = \frac{4}{T} = \frac{4}{N\tau} = \frac{4f_s}{N}$

Normalized Freq $\frac{f_1}{f_s} = \frac{4}{N}$



4th Harmonic Freq $f_1' = \frac{4}{T'} = \frac{4}{N\tau'} = \frac{4f_s'}{N}$

Normalized Freq $\frac{f_1'}{f_s'} = \frac{4}{N}$



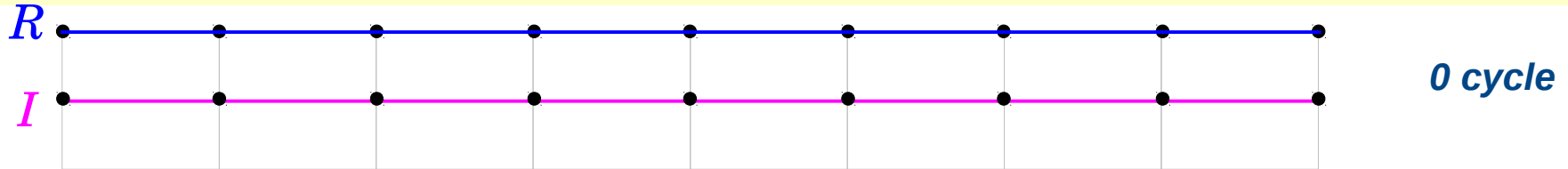
N=8 DFT

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$

$X[0]$	W_8^0	W_8^0	W_8^0	W_8^0	W_8^0	W_8^0	W_8^0	W_8^0	$x[0]$
$X[1]$	W_8^0	W_8^1	W_8^2	W_8^3	W_8^4	W_8^5	W_8^6	W_8^7	$x[1]$
$X[2]$	W_8^0	W_8^2	W_8^4	W_8^6	W_8^0	W_8^2	W_8^4	W_8^6	$x[2]$
$X[3]$	W_8^0	W_8^3	W_8^6	W_8^1	W_8^4	W_8^7	W_8^2	W_8^5	$x[3]$
$X[4]$	W_8^0	W_8^4	W_8^0	W_8^4	W_8^0	W_8^4	W_8^0	W_8^4	$x[4]$
$X[5]$	W_8^0	W_8^5	W_8^2	W_8^7	W_8^4	W_8^1	W_8^6	W_8^3	$x[5]$
$X[6]$	W_8^0	W_8^6	W_8^4	W_8^2	W_8^0	W_8^6	W_8^4	W_8^2	$x[6]$
$X[7]$	W_8^0	W_8^7	W_8^6	W_8^5	W_8^4	W_8^3	W_8^2	W_8^1	$x[7]$

N=8 DFT : The 1st Row of the DFT Matrix

$$\left(e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 0} \right)$$

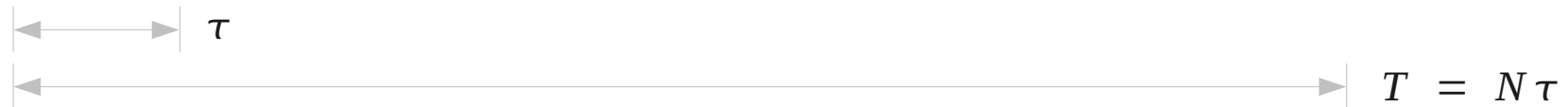


$$W_8^{kn} = e^{-j\frac{2\pi}{8}kn} \quad k = 0, \quad n = 0, 1, \dots, 7$$

$R \rightarrow$ samples of $\cos(-\omega t) = \cos(\omega t)$
 $I \rightarrow$ samples of $\sin(-\omega t) = -\sin(\omega t)$

$\left. \begin{array}{l} R \rightarrow \dots \\ I \rightarrow \dots \end{array} \right\} \xrightarrow{\text{measure}} \begin{array}{l} \omega t = 2\pi f t \\ 2\pi \cdot \left(\frac{0}{8}\right) \cdot f_s \cdot t \end{array}$

$X[0]$ measures how much of the $+0 \cdot \omega$ component is present in x .

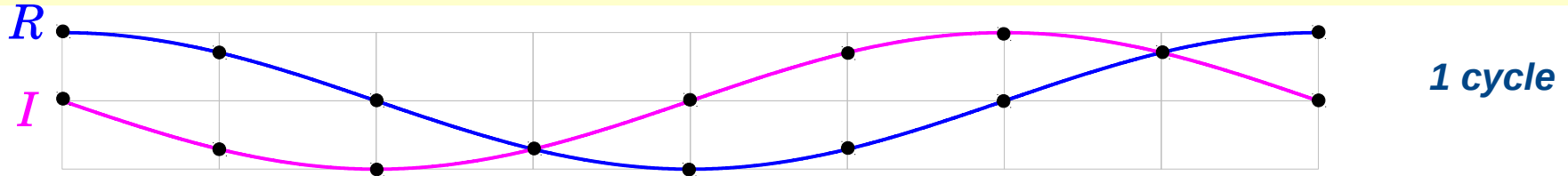


Sampling Time τ Sampling Frequency $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$ zero Frequency

N=8 DFT : The 2nd Row of the DFT Matrix

$$\left(e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 1} \quad e^{-j\frac{\pi}{4}\cdot 2} \quad e^{-j\frac{\pi}{4}\cdot 3} \quad e^{-j\frac{\pi}{4}\cdot 4} \quad e^{-j\frac{\pi}{4}\cdot 5} \quad e^{-j\frac{\pi}{4}\cdot 6} \quad e^{-j\frac{\pi}{4}\cdot 7} \right)$$



$$W_8^{kn} = e^{-j\frac{2\pi}{8}kn} \quad k = 1, \quad n = 0, 1, \dots, 7$$

R \rightarrow samples of $\cos(-\omega t) = \cos(\omega t)$

I \rightarrow samples of $\sin(-\omega t) = -\sin(\omega t)$

measure \rightarrow

$$\omega t = 2\pi f t$$

$$2\pi \cdot \left(\frac{1}{8}\right) \cdot f_s \cdot t$$

X[1] measures how much of the **+1· ω** component is present in **x**.



$$T = N\tau$$

Sampling Time τ

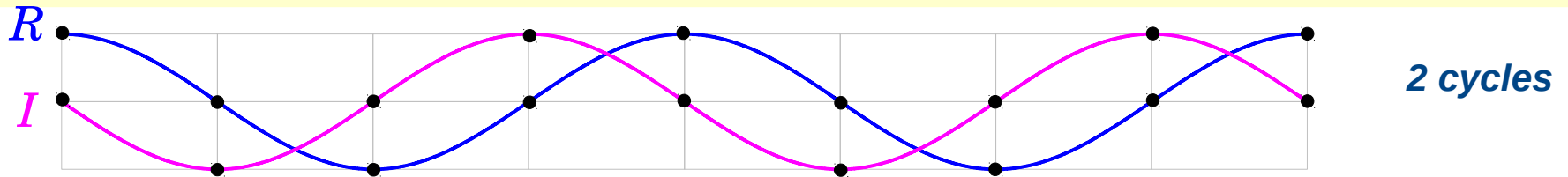
Sampling Frequency $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$

1st Harmonic Freq $f_1 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$

N=8 DFT : The 3rd Row of the DFT Matrix

$$\left(e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 2} \quad e^{-j\frac{\pi}{4}\cdot 4} \quad e^{-j\frac{\pi}{4}\cdot 6} \quad e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 2} \quad e^{-j\frac{\pi}{4}\cdot 4} \quad e^{-j\frac{\pi}{4}\cdot 6} \right)$$



$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \quad k = 2, \quad n = 0, 1, \dots, 7$$

$$\left. \begin{array}{l} R \rightarrow \text{samples of } \cos(-2\omega t) = \cos(2\omega t) \\ I \rightarrow \text{samples of } \sin(-2\omega t) = -\sin(2\omega t) \end{array} \right\} \xrightarrow{\text{measure}} \begin{array}{l} \omega t = 2\pi f t \\ 2\pi \cdot \left(\frac{2}{8}\right) \cdot f_s \cdot t \end{array}$$

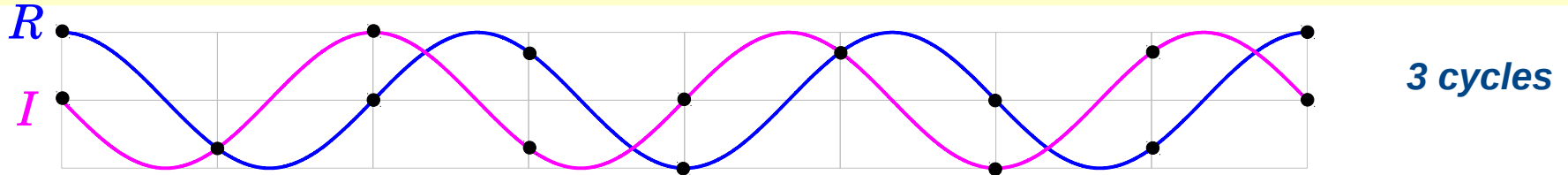
X[2] measures how much of the **+2·ω** component is present in **x**.



Sampling Time	τ	Sampling Frequency	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	2 nd Harmonic Freq	$f_2 = \frac{2}{T} = \frac{2}{N\tau} = \frac{2f_s}{N}$

N=8 DFT : The 4th Row of the DFT Matrix

$$\left(e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 3} \quad e^{-j\frac{\pi}{4}\cdot 6} \quad e^{-j\frac{\pi}{4}\cdot 1} \quad e^{-j\frac{\pi}{4}\cdot 4} \quad e^{-j\frac{\pi}{4}\cdot 7} \quad e^{-j\frac{\pi}{4}\cdot 2} \quad e^{-j\frac{\pi}{4}\cdot 5} \right)$$



$$W_8^{kn} = e^{-j\frac{2\pi}{8}kn} \quad k = 3, \quad n = 0, 1, \dots, 7$$

$$\left. \begin{array}{l} R \rightarrow \text{samples of } \cos(-3\omega t) = \cos(3\omega t) \\ I \rightarrow \text{samples of } \sin(-3\omega t) = -\sin(3\omega t) \end{array} \right\} \xrightarrow{\text{measure}} \begin{array}{l} \omega t = 2\pi f t \\ 2\pi \cdot \left(\frac{3}{8}\right) \cdot f_s \cdot t \end{array}$$

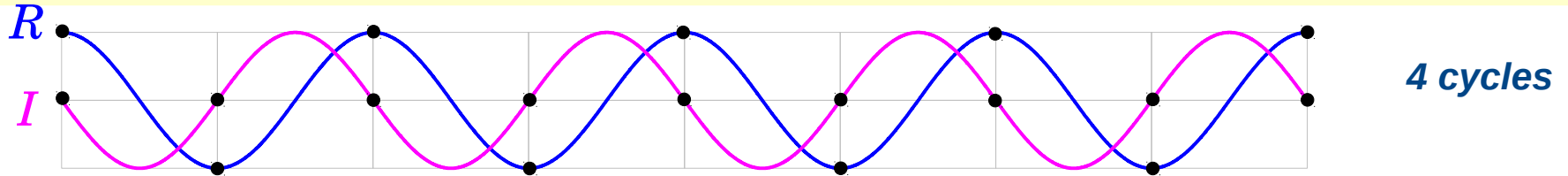
X[3] measures how much of the **+3·ω** component is present in **x**.



Sampling Time	τ	Sampling Frequency	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	3 rd Harmonic Freq	$f_3 = \frac{3}{T} = \frac{3}{N\tau} = \frac{3f_s}{N}$

N=8 DFT : The 5th Row of the DFT Matrix

$$\left(e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 4} \quad e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 4} \quad e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 4} \quad e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 4} \right)$$



$$W_8^{kn} = e^{-j\frac{2\pi}{8}kn} \quad k = 4, \quad n = 0, 1, \dots, 7$$

$$\left. \begin{array}{l} R \rightarrow \text{samples of } \cos(-4\omega t) = \cos(4\omega t) \\ I \rightarrow \text{samples of } \sin(-4\omega t) = -\sin(4\omega t) \end{array} \right\} \xrightarrow{\text{measure}} \begin{array}{l} \omega t = 2\pi f t \\ 2\pi \cdot \left(\frac{4}{8}\right) \cdot f_s \cdot t \end{array}$$

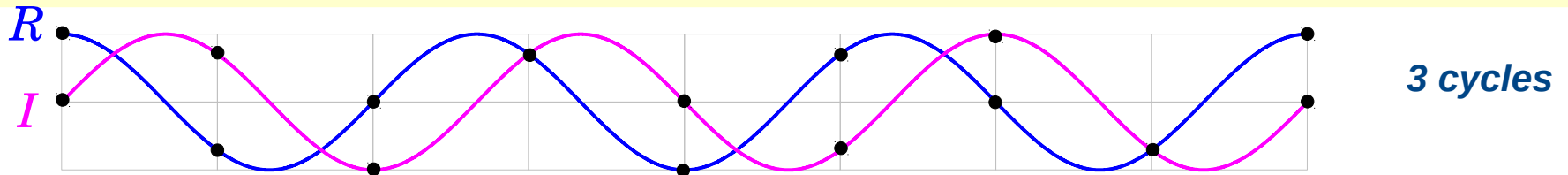
X[4] measures how much of the **+4·ω** component is present in **x**.



Sampling Time	τ	Sampling Frequency	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	4 th Harmonic Freq	$f_4 = \frac{4}{T} = \frac{4}{N\tau} = \frac{4f_s}{N}$

N=8 DFT : The 6th Row of the DFT Matrix

$$\left(e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 5} \quad e^{-j\frac{\pi}{4}\cdot 2} \quad e^{-j\frac{\pi}{4}\cdot 7} \quad e^{-j\frac{\pi}{4}\cdot 4} \quad e^{-j\frac{\pi}{4}\cdot 1} \quad e^{-j\frac{\pi}{4}\cdot 6} \quad e^{-j\frac{\pi}{4}\cdot 3} \right)$$



$$W_8^{kn} = e^{-j\frac{2\pi}{8}kn} \quad k = 5, \quad n = 0, 1, \dots, 7$$

$$\left. \begin{array}{l} R \rightarrow \text{samples of } \cos(-(-3\omega)t) = \cos(3\omega t) \\ I \rightarrow \text{samples of } \sin(-(-3\omega)t) = \sin(3\omega t) \end{array} \right\} \text{measure} \rightarrow \begin{array}{l} -\omega t = -2\pi f t \\ 2\pi \cdot \left(\frac{-3}{8}\right) \cdot f_s \cdot t \end{array}$$

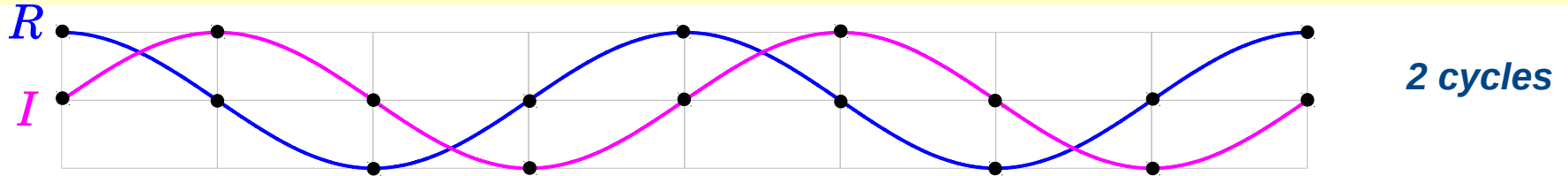
X[5] measures how much of the $-3 \cdot \omega$ component is present in x .



Sampling Time	τ	Sampling Frequency	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	-3^{rd} Harmonic Freq	$f_{-3} = \frac{-3}{T} = \frac{-3}{N\tau} = \frac{-3f_s}{N}$

N=8 DFT : The 7th Row of the DFT Matrix

$$\left(e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 6} \quad e^{-j\frac{\pi}{4}\cdot 4} \quad e^{-j\frac{\pi}{4}\cdot 2} \quad e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 6} \quad e^{-j\frac{\pi}{4}\cdot 4} \quad e^{-j\frac{\pi}{4}\cdot 2} \right)$$



$$W_8^{kn} = e^{-j\frac{(2\pi)}{8}kn} \quad k = 2, \quad n = 0, 1, \dots, 7$$

$$\left. \begin{array}{l} R \rightarrow \text{samples of } \cos(-(-2\omega)t) = \cos(2\omega t) \\ I \rightarrow \text{samples of } \sin(-(-2\omega)t) = \sin(2\omega t) \end{array} \right\} \text{measure} \rightarrow \begin{array}{l} -\omega t = -2\pi f t \\ 2\pi \cdot \left(\frac{-2}{8}\right) \cdot f_s \cdot t \end{array}$$

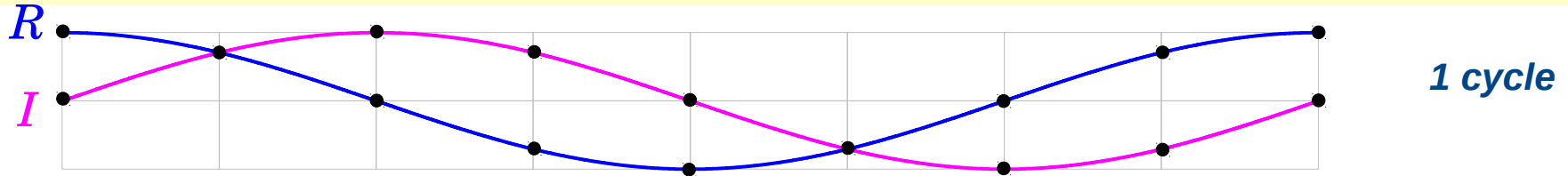
X[6] measures how much of the $-2\cdot\omega$ component is present in \mathbf{x} .



Sampling Time	τ	Sampling Frequency	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	-2^{nd} Harmonic Freq	$f_{-2} = \frac{-2}{T} = \frac{-2}{N\tau} = \frac{-2f_s}{N}$

N=8 DFT : The 8th Row of the DFT Matrix

$$\left(e^{-j\frac{\pi}{4}\cdot 0} \quad e^{-j\frac{\pi}{4}\cdot 7} \quad e^{-j\frac{\pi}{4}\cdot 6} \quad e^{-j\frac{\pi}{4}\cdot 5} \quad e^{-j\frac{\pi}{4}\cdot 4} \quad e^{-j\frac{\pi}{4}\cdot 3} \quad e^{-j\frac{\pi}{4}\cdot 2} \quad e^{-j\frac{\pi}{4}\cdot 1} \right)$$



$$W_8^{kn} = e^{-j\frac{2\pi}{8}kn} \quad k = 7, \quad n = 0, 1, \dots, 7$$

R \rightarrow samples of $\cos(-(-\omega)t) = \cos(\omega t)$

I \rightarrow samples of $\sin(-(-\omega)t) = \sin(\omega t)$

} *measure* \rightarrow
$$\begin{aligned} -\omega t &= -2\pi f t \\ 2\pi \cdot \left(\frac{-1}{8}\right) \cdot f_s \cdot t \end{aligned}$$

X[7] measures how much of the $-1 \cdot \omega$ component is present in **x**.



Sampling Time τ

Sampling Frequency $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$

-1^{st} Harmonic Freq $f_{-1} = \frac{-1}{T} = \frac{-1}{N\tau} = \frac{f_s}{N}$

N=8 DFT : DFT Matrix in + or - Frequencies

$$\omega_0 = 2\pi \cdot \frac{f_s}{N}$$



0th row:	<i>samples of</i>	$\cos(0\omega_0)t + j \cdot \sin(0\omega_0)t$	(0 cycle)
1th row:	<i>samples of</i>	$\cos(-1\omega_0)t + j \cdot \sin(-1\omega_0)t$	(1 cycle)
2th row:	<i>samples of</i>	$\cos(-2\omega_0)t + j \cdot \sin(-2\omega_0)t$	(2 cycles)
3th row:	<i>samples of</i>	$\cos(-3\omega_0)t + j \cdot \sin(-3\omega_0)t$	(3 cycles)
4th row:	<i>samples of</i>	$\cos(-4\omega_0)t + j \cdot \sin(-4\omega_0)t$	(4 cycles)
5th row:	<i>samples of</i>	$\cos(-5\omega_0)t + j \cdot \sin(-5\omega_0)t$	(5 cycles)
6th row:	<i>samples of</i>	$\cos(-6\omega_0)t + j \cdot \sin(-6\omega_0)t$	(6 cycles)
7th row:	<i>samples of</i>	$\cos(-7\omega_0)t + j \cdot \sin(-7\omega_0)t$	(7 cycles)

=



0th row:	<i>samples of</i>	$\cos(0\omega_0)t + j \cdot \sin(0\omega_0)t$	(0 cycle)
1th row:	<i>samples of</i>	$\cos(+7\omega_0)t + j \cdot \sin(+7\omega_0)t$	(7 cycles)
2th row:	<i>samples of</i>	$\cos(+6\omega_0)t + j \cdot \sin(+6\omega_0)t$	(6 cycles)
3th row:	<i>samples of</i>	$\cos(+5\omega_0)t + j \cdot \sin(+5\omega_0)t$	(5 cycles)
4th row:	<i>samples of</i>	$\cos(+4\omega_0)t + j \cdot \sin(+4\omega_0)t$	(4 cycles)
5th row:	<i>samples of</i>	$\cos(+3\omega_0)t + j \cdot \sin(+3\omega_0)t$	(3 cycles)
6th row:	<i>samples of</i>	$\cos(+2\omega_0)t + j \cdot \sin(+2\omega_0)t$	(2 cycles)
7th row:	<i>samples of</i>	$\cos(+1\omega_0)t + j \cdot \sin(+1\omega_0)t$	(1 cycles)

N=8 DFT : DFT Matrix in Both Frequencies

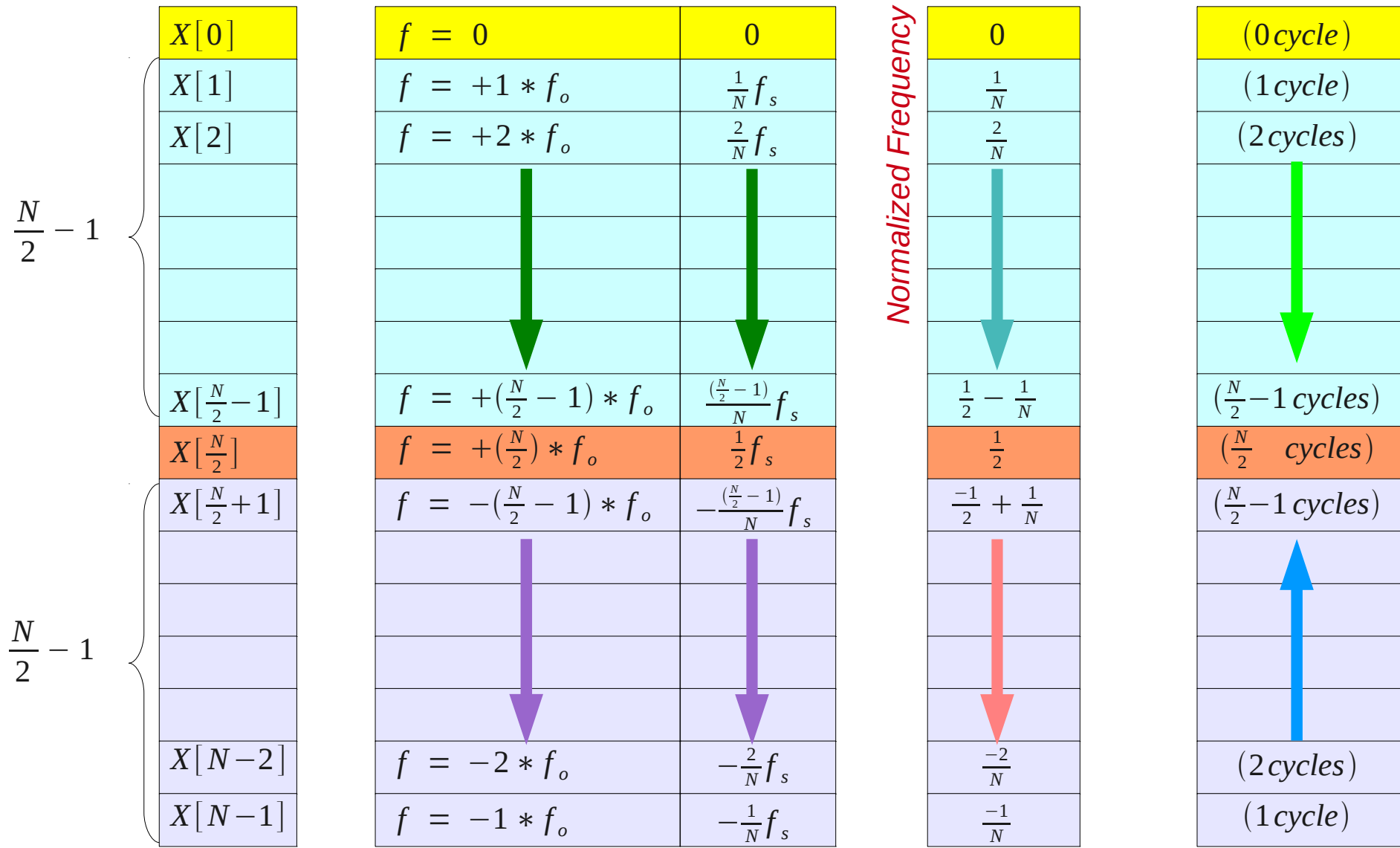
$$\omega_0 = 2\pi \cdot \frac{f_s}{N}$$

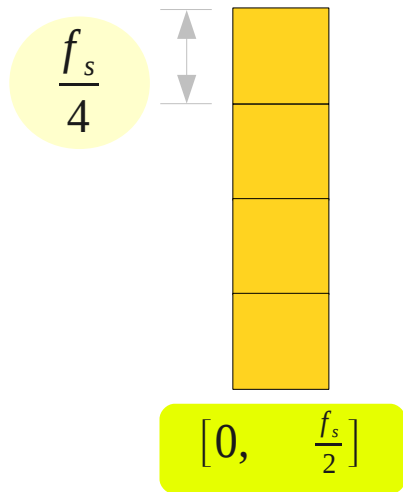
0th row:	<i>samples of</i>	$\cos(0\omega_0)t + j \cdot \sin(0\omega_0)t$	(0 cycle)
1th row:	<i>samples of</i>	$\cos(-1\omega_0)t + j \cdot \sin(-1\omega_0)t$	(1 cycle)
2th row:	<i>samples of</i>	$\cos(-2\omega_0)t + j \cdot \sin(-2\omega_0)t$	(2 cycles)
3th row:	<i>samples of</i>	$\cos(-3\omega_0)t + j \cdot \sin(-3\omega_0)t$	(3 cycles)
4th row:	<i>samples of</i>	$\cos(-4\omega_0)t + j \cdot \sin(-4\omega_0)t$	(4 cycles)
5th row:	<i>samples of</i>	$\cos(-5\omega_0)t + j \cdot \sin(-5\omega_0)t$	(5 cycles)
6th row:	<i>samples of</i>	$\cos(-6\omega_0)t + j \cdot \sin(-6\omega_0)t$	(6 cycles)
7th row:	<i>samples of</i>	$\cos(-7\omega_0)t + j \cdot \sin(-7\omega_0)t$	(7 cycles)

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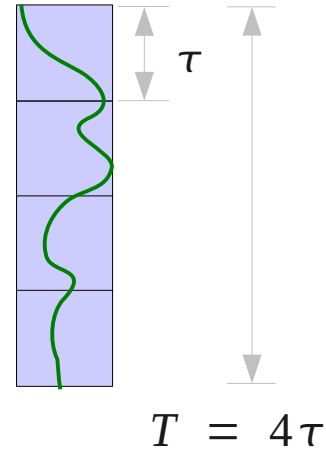
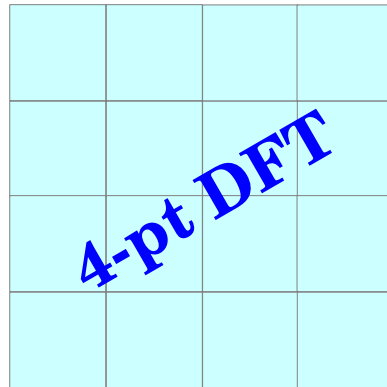
0th row:	<i>samples of</i>	$\cos(0\omega_0)t + j \cdot \sin(0\omega_0)t$	(0 cycle)
1th row:	<i>samples of</i>	$\cos(-1\omega_0)t + j \cdot \sin(-1\omega_0)t$	(1 cycle)
2th row:	<i>samples of</i>	$\cos(-2\omega_0)t + j \cdot \sin(-2\omega_0)t$	(2 cycles)
3th row:	<i>samples of</i>	$\cos(-3\omega_0)t + j \cdot \sin(-3\omega_0)t$	(3 cycles)
4th row:	<i>samples of</i>	$\cos(-4\omega_0)t + j \cdot \sin(-4\omega_0)t$	(4 cycles)
5th row:	<i>samples of</i>	$\cos(+3\omega_0)t + j \cdot \sin(+3\omega_0)t$	(3 cycles)
6th row:	<i>samples of</i>	$\cos(+2\omega_0)t + j \cdot \sin(+2\omega_0)t$	(2 cycles)
7th row:	<i>samples of</i>	$\cos(+1\omega_0)t + j \cdot \sin(+1\omega_0)t$	(1 cycles)

Frequency View of a X[i] Vector

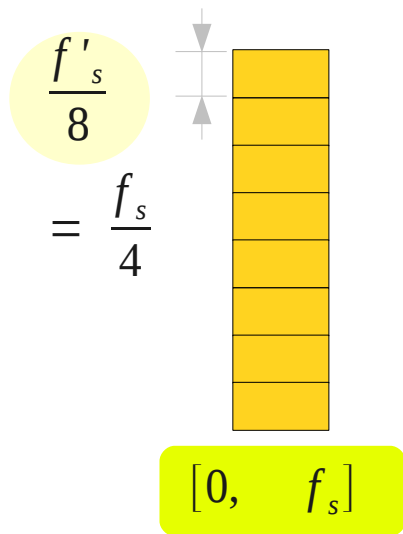




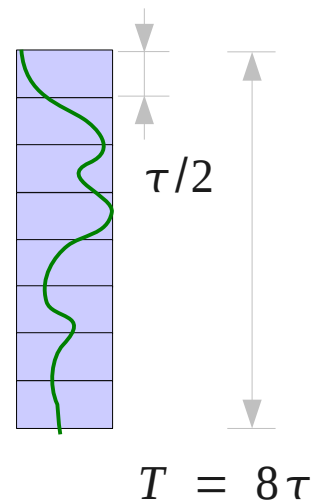
=



$f_s = \frac{1}{\tau}$



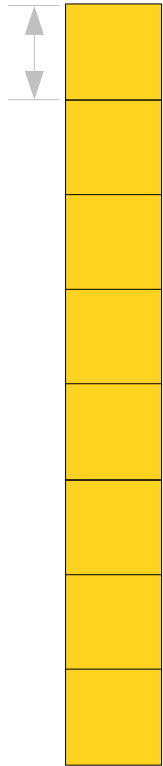
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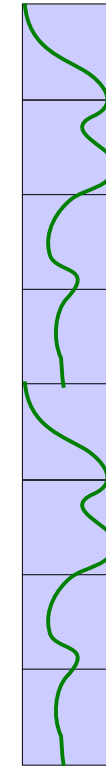
$f'_s = \frac{2}{\tau}$

$= 2f_s$

$$\frac{f_s}{8}$$



=



$$f_s = \frac{1}{\tau}$$

$$\left[0, \frac{f_s}{2}\right]$$

$$2T = 8\tau$$

References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] A “graphical interpretation” of the DFT and FFT, by Steve Mann