Group Delay and Phase Delay (1A)

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Phase Shift and Time Shift

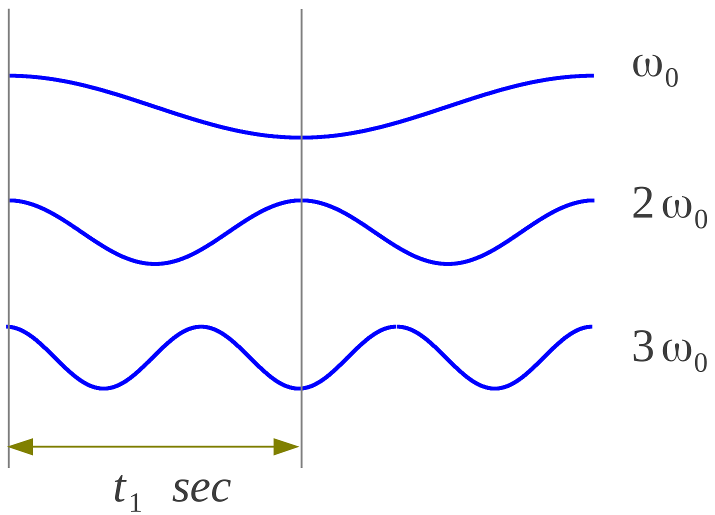
measure phase shift not <u>in second</u>
*but **<u>in portions</u>** of a cosine wave cycle*

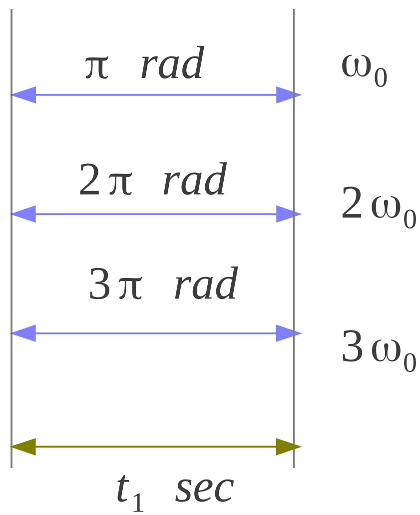
within phase change in one cycle

Phase Shift → in radians, degrees
Delay → in seconds (time)

Given time shift (delay) $t_1 \ sec$

The actual phase shift is different according to the frequency $\pi, 2\pi, 3\pi \ rad$





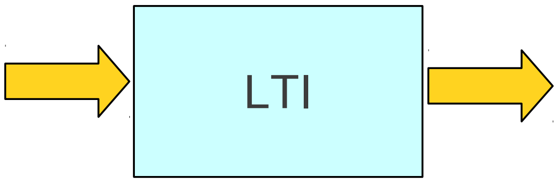
The same <u>delay</u>
applied to all frequencies

The different <u>phase shift</u>
to the different frequency

Frequency Response

Frequency Response $H(e^{j\omega})$

$e^{j\omega t}$

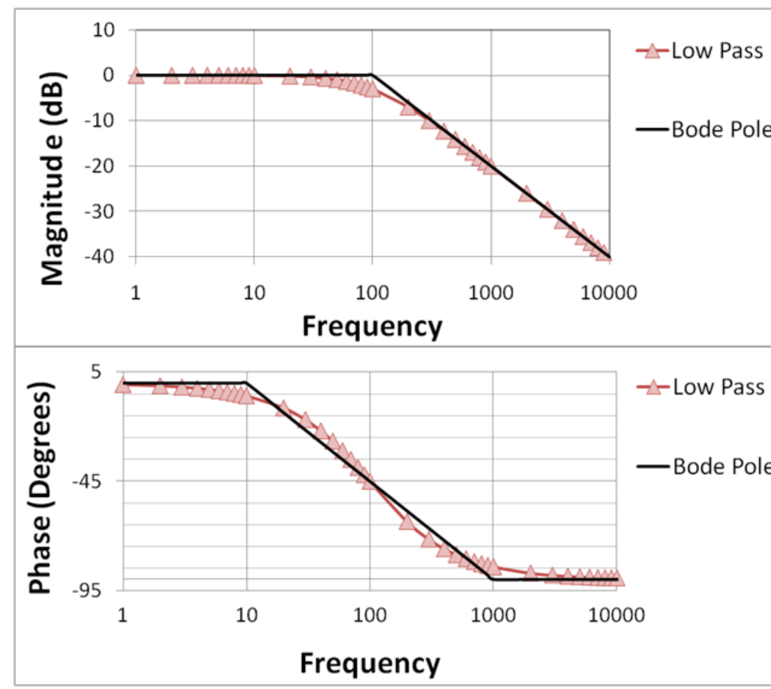


$H(e^{j\omega})\, e^{j\omega t}$

LPF example

$|H(e^{j\omega})|$ Magnitude Response

$\angle H(e^{j\omega})$ Phase Response



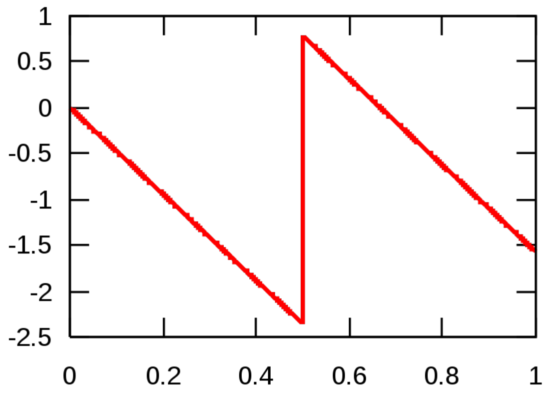
Linear Phase System

Linear Phase System

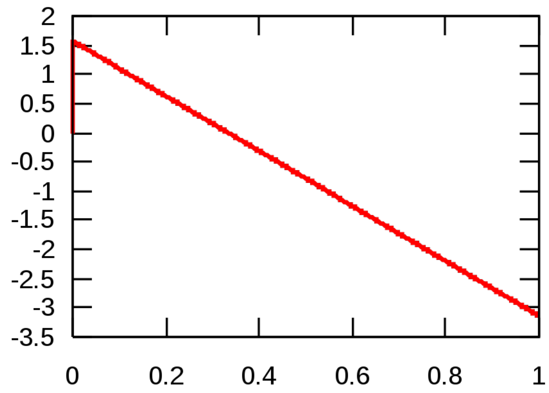
Phase Shift $\propto$ Frequency

$$\angle H\left(e^{j\omega}\right) \propto \omega$$

a) FIR Filter (Type II) having Linear Phase

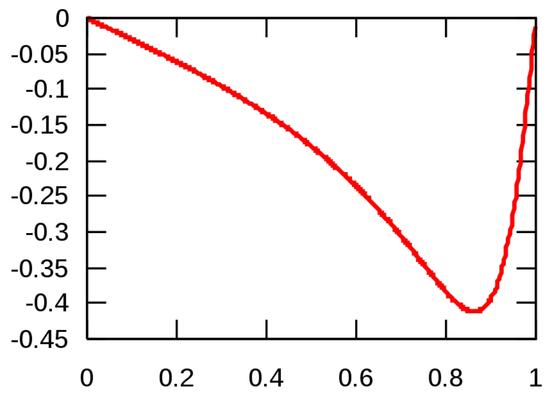


b) FIR Filter (Type IV) having Linear Phase

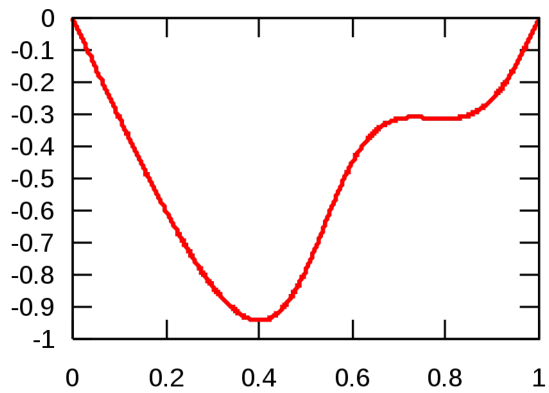


Non-Linear Phase System

c) IIR Filter having Non-Linear Phase

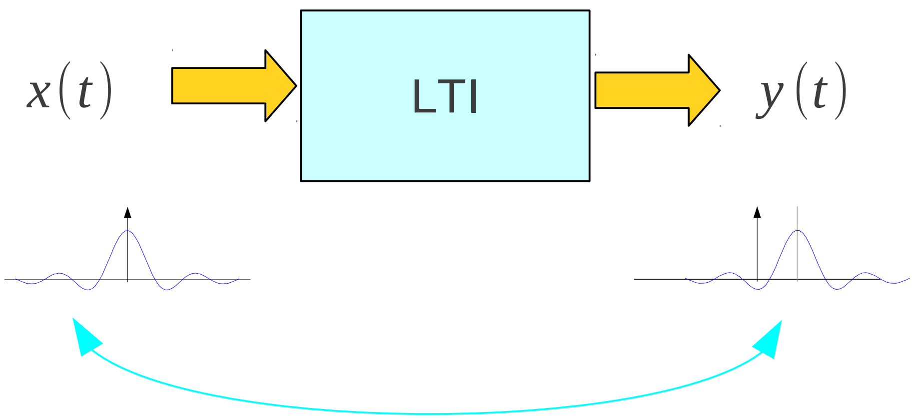


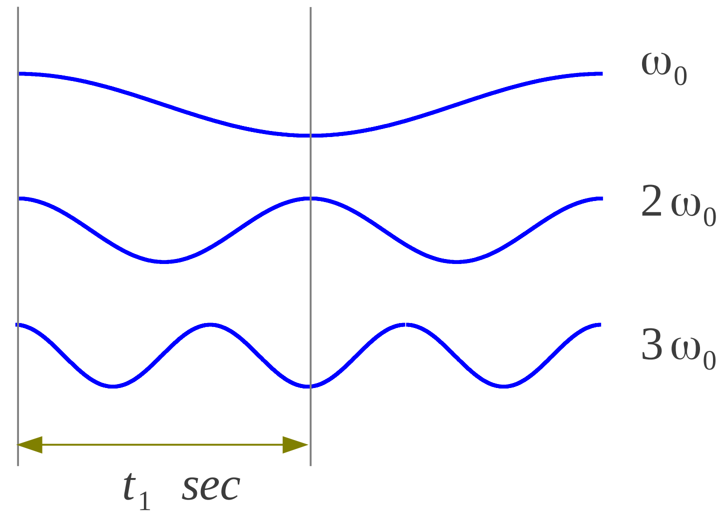
d) FIR Filter having Non-Linear Phase



Uniform Time Delay (1)

Frequency Response $H(e^{j\omega})$



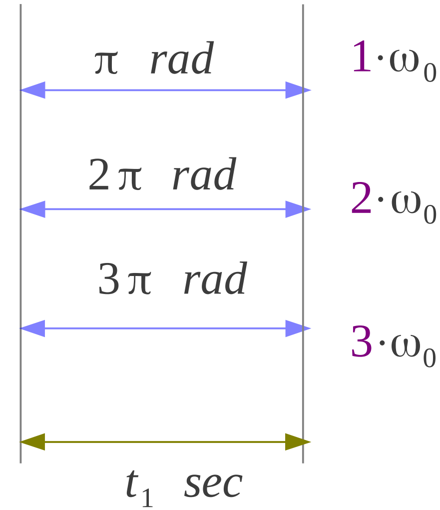


The waveform shape can be preserved.

- uniform magnitude $|H(e^{j\omega})| = c$
- uniform time delay

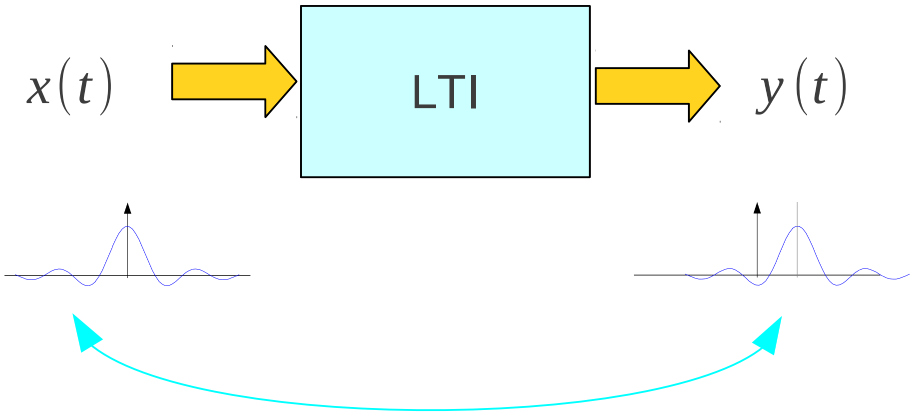
⇕

linear phase $\angle H(e^{j\omega}) = k\omega$



Uniform Time Delay (2)

Frequency Response $H(e^{j\omega})$



The waveform shape can be preserved.

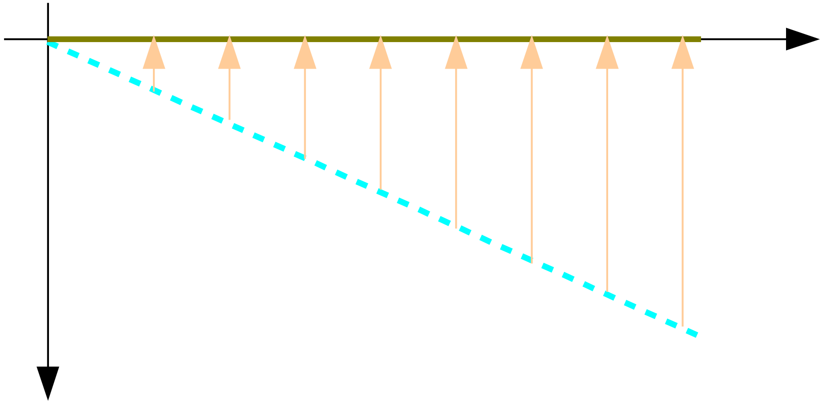
uniform magnitude $|H(e^{j\omega})| = c$

uniform time delay

linear phase $\angle H(e^{j\omega}) = k\,\omega$

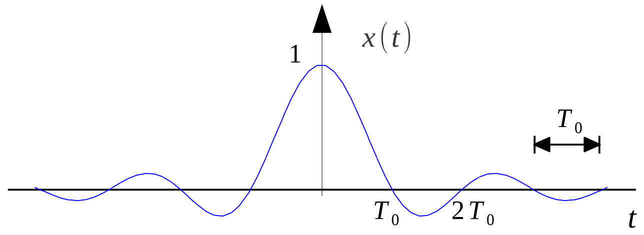
Uniform Time Delay

Could remove delay
from the <u>phase response</u>
to achieve a horizontal line at
***zero degree** (No delay)*



CTFT of Sinc Function

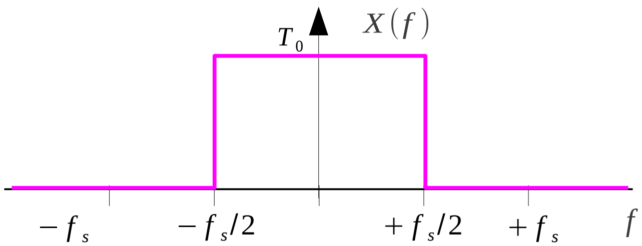
$t = \pm T_0,\ \pm 2T_0,\ \pm 3T_0,\ \cdots \Rightarrow x(t) = 0$



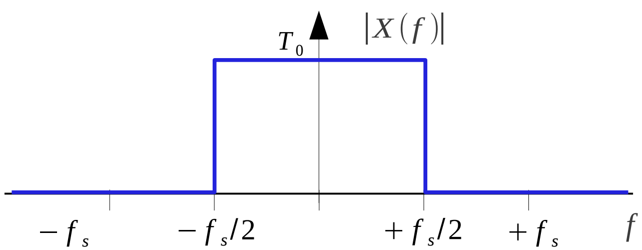
$$x(t) = \frac{\sin(\pi t / T_0)}{\pi t / T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

$$\frac{1}{T_0} \equiv f_s$$

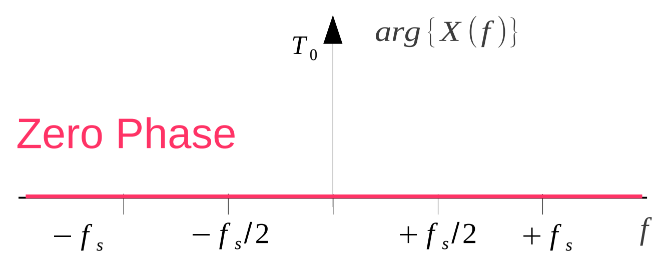
CTFT



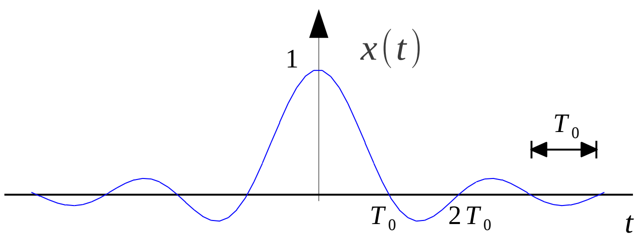
$$H(f) = \begin{cases} T_0, & |f| \le f_s/2 \\ 0, & otherwise \end{cases}$$



Real Symmetric Signal

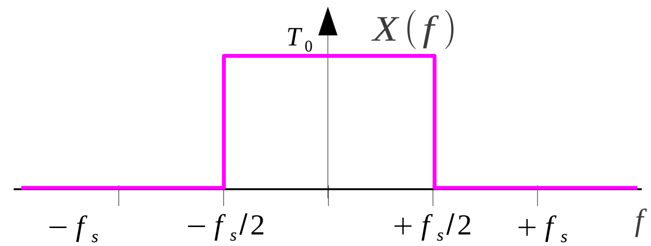


CTFT Time Shifting Property



$$\frac{1}{T_0} \equiv f_s$$

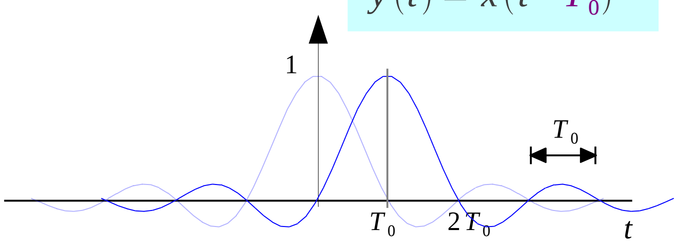
CTFT



$$x(t) = \frac{\sin(\pi t/T_0)}{\pi t/T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

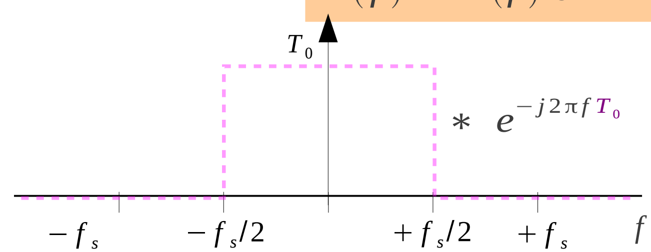
$$X(f) = \begin{cases} T_0, & |f| \le f_s/2 \\ 0, & otherwise \end{cases}$$

$$y(t) = x(t - T_0)$$

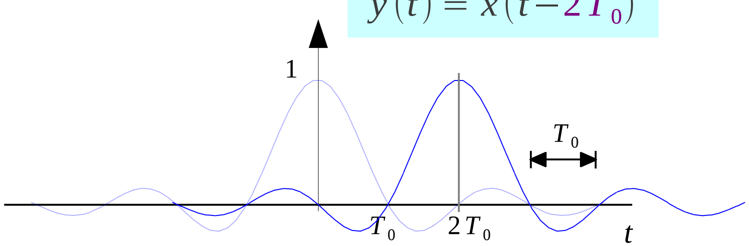


CTFT

$$Y(f) = X(f) \cdot e^{-j2\pi f T_0}$$

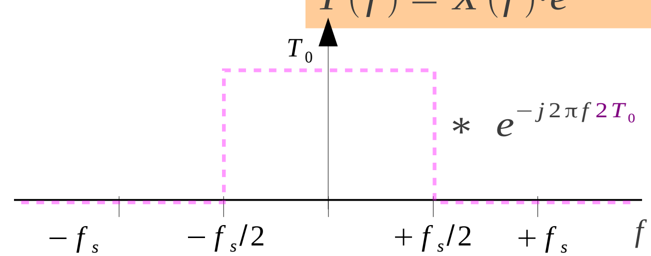


$$y(t) = x(t - 2T_0)$$

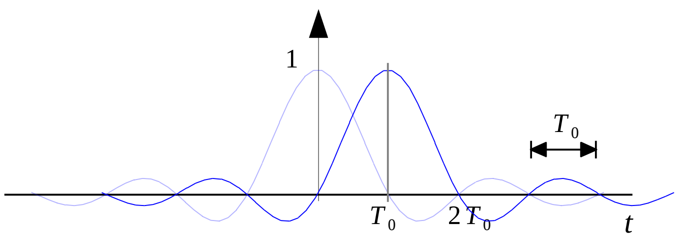


CTFT

$$Y(f) = X(f) \cdot e^{-j2\pi f 2T_0}$$

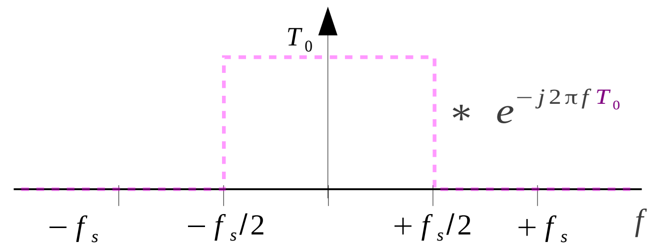


CTFT of Sinc Function Shifted by T_0



$$\frac{1}{T_0} \equiv f_s$$

CTFT



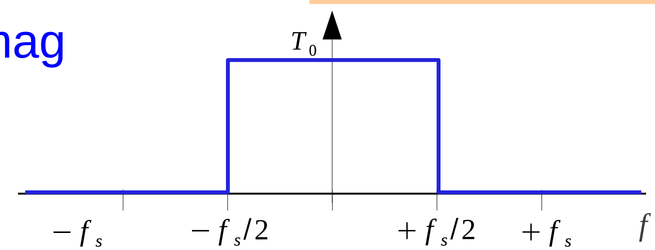
$$x(t) = \frac{\sin(\pi t/T_0)}{\pi t/T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

$$X(f) = \begin{cases} T_0, & |f| \le f_s/2 \\ 0, & otherwise \end{cases}$$

$$y(t) = x(t - T_0)$$

$$Y(f) = X(f) \cdot e^{-j2\pi f T_0}$$

mag

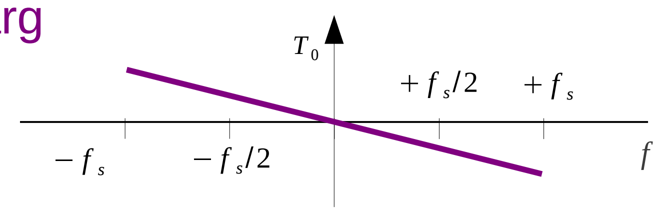


$$Arg \Rightarrow \Phi(f)$$

$$slope = \frac{d\Phi}{df} = -2\pi T_0 \Rightarrow \frac{d\Phi}{d\omega} = -T_0$$

Group Delay $\quad -\dfrac{d\Phi}{d\omega} = T_0$

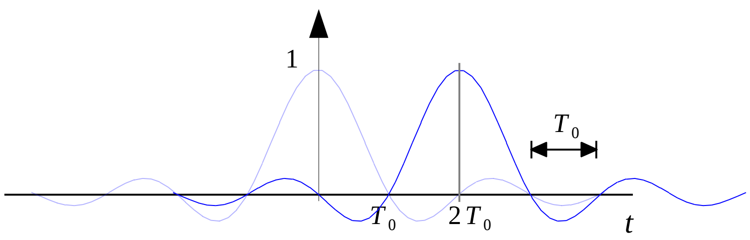
arg



Pure Delay (No Dispersion) → Linear Phase Change

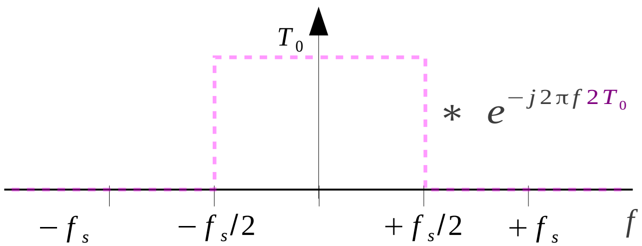
$slope = -2\pi T_0$

CTFT of Sinc Function Shifted by $2T_0$



$$\frac{1}{T_0} \equiv f_s$$

CTFT



$$* \; e^{-j2\pi f 2T_0}$$

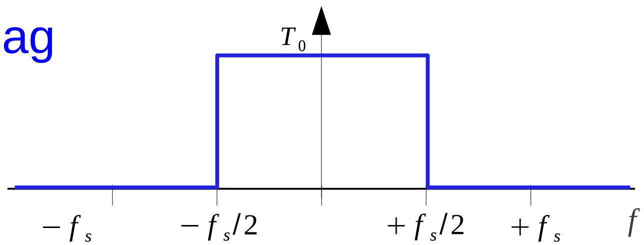
$$x(t) = \frac{\sin(\pi t/T_0)}{\pi t/T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

$$X(f) = \begin{cases} T_0, & |f| \le f_s/2 \\ 0, & otherwise \end{cases}$$

$$y(t) = x(t - 2T_0)$$

$$Y(f) = X(f) \cdot e^{-j2\pi f 2T_0}$$

mag

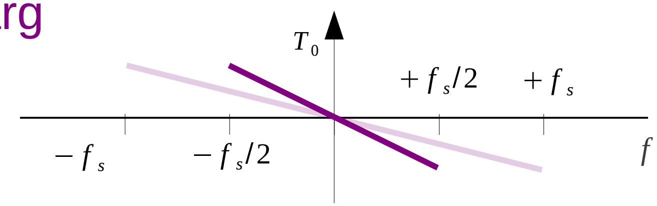


Arg ➡ $\Phi(f)$

$$slope = \frac{d\Phi}{df} = -2\pi 2T_0 \;➡\; \frac{d\Phi}{d\omega} = -2T_0$$

Group Delay $\quad -\frac{d\Phi}{d\omega} = 2T_0$

arg



Pure Delay (No Dispersion) ➡ Linear Phase Change

$$slope = -2\pi 2T_0$$

Group Delay (1)

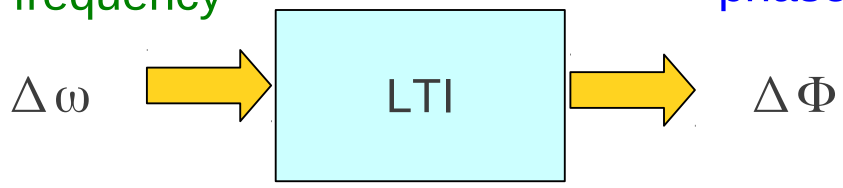
Consider the cosine components
at *closely spaced frequencies* and
their phase shifts in relation to each other

→

Group Delay:
The phase shift changes
for small changes in frequency

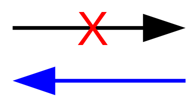
small changes in frequency

phase shift changes



A uniform, waveform preserving
phase response → linear

Constant Group Delay



Uniform Time Delay (linear phase)

Group Delay (2)

Constant slope → Constant Group Delay

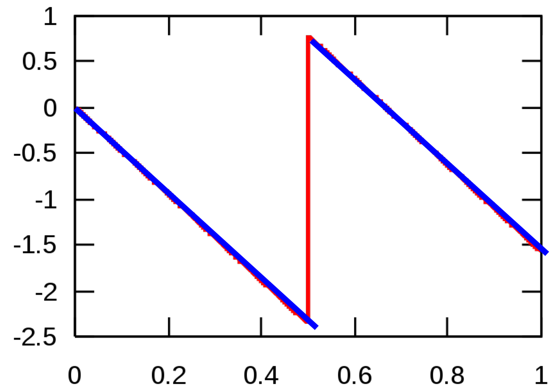
Linear Phase System

Phase Shift $\propto$ Frequency

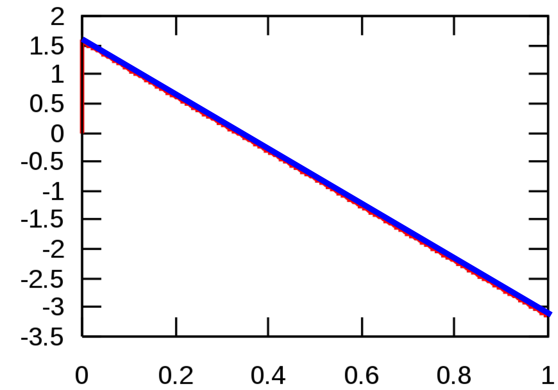
$$\angle H\left(e^{j\omega}\right) \propto \omega$$

No dispersion

a) FIR Filter (Type II) having Linear Phase



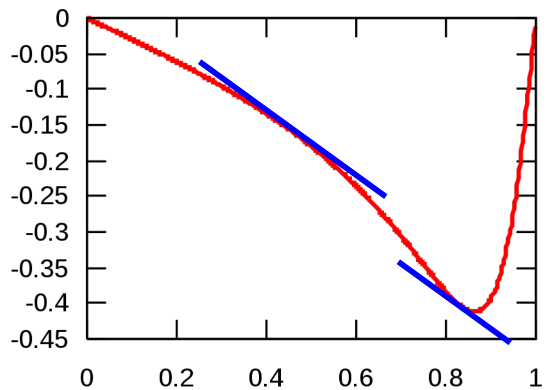
b) FIR Filter (Type IV) having Linear Phase



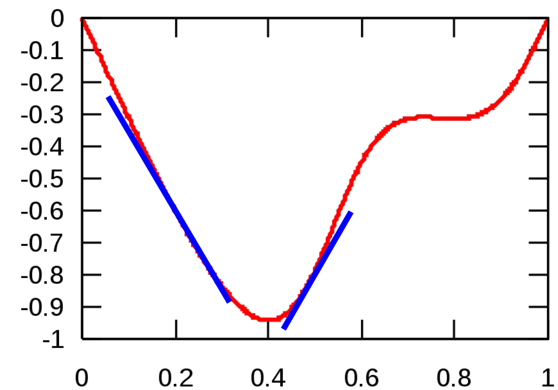
Non-Linear Phase System

Dispersion

c) IIR Filter having Non-Linear Phase



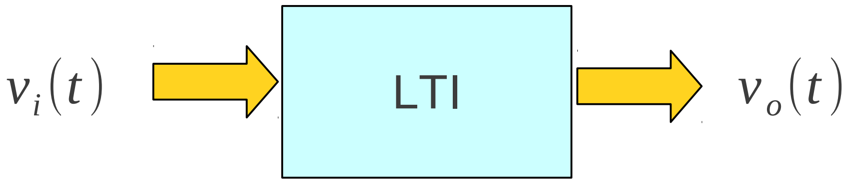
d) FIR Filter having Non-Linear Phase

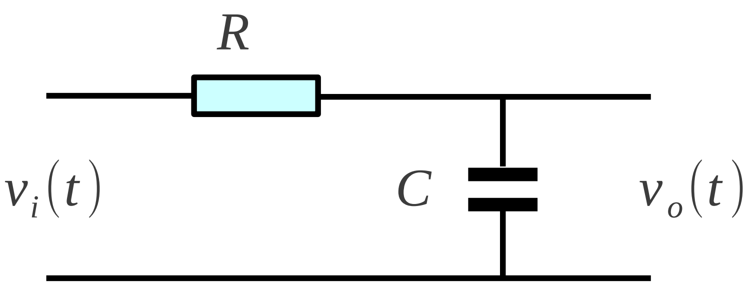


Varying slope → Varying Group Delay

Simple Low Pass Filter (1)

Frequency Response





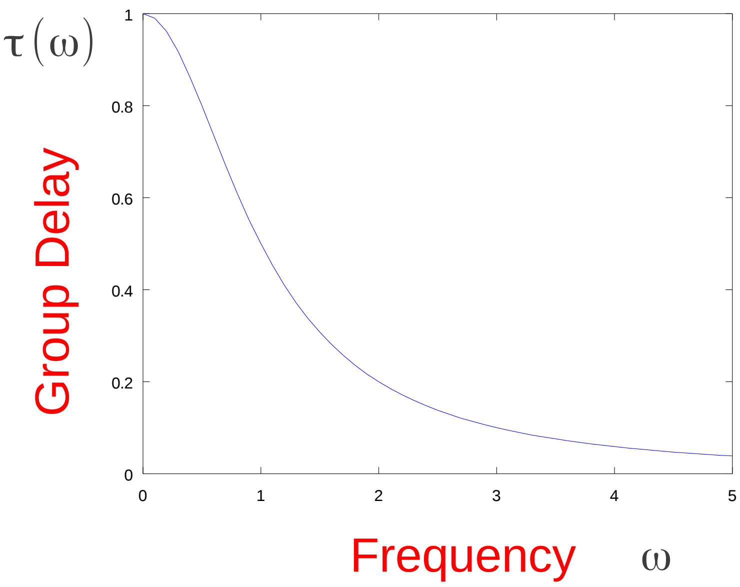
$$H(j\omega) = \frac{1}{1 + j\omega/\omega_0} \qquad \omega_0 = \frac{1}{RC}$$

$$H(j\omega) = A(j\omega)e^{j\phi(j\omega)}$$

$$A(j\omega) = \frac{1}{\sqrt{1+\omega^2/\omega_0^2}}$$

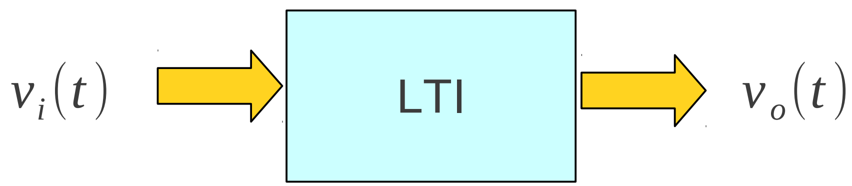
$$\phi(j\omega) = \tan^{-1}(-\omega/\omega_0)$$

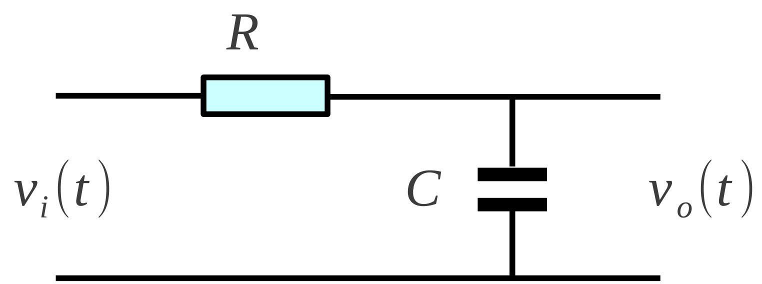
$$\tau(\omega) = -\frac{d\phi}{d\omega} = \frac{1}{1+\omega^2/\omega_0^2}$$



Simple Low Pass Filter (2)

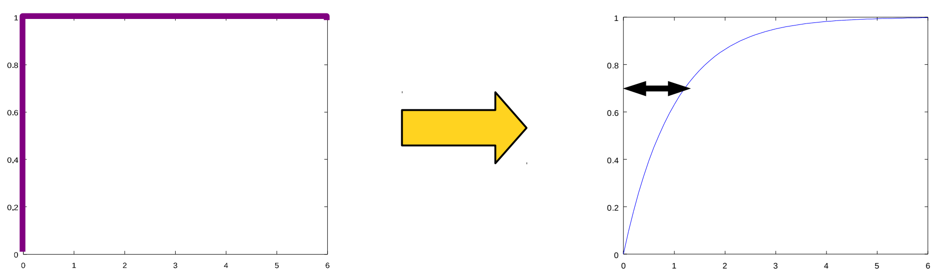
Frequency Response





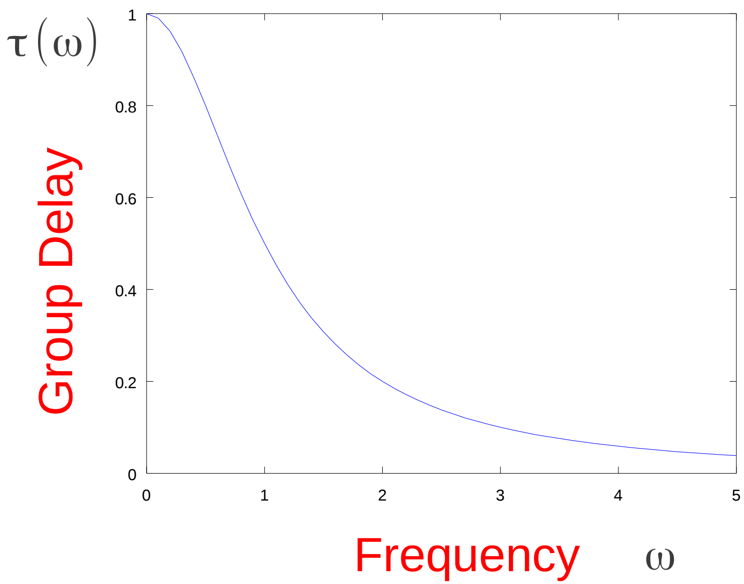
$$v_o(t) = 1 - e^{-\frac{t}{\tau}} \qquad \omega_0 = \frac{1}{RC} = \frac{1}{\tau}$$

Where is the group delay ?



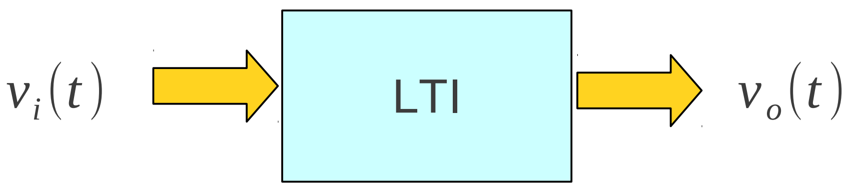
Group delay is not constant

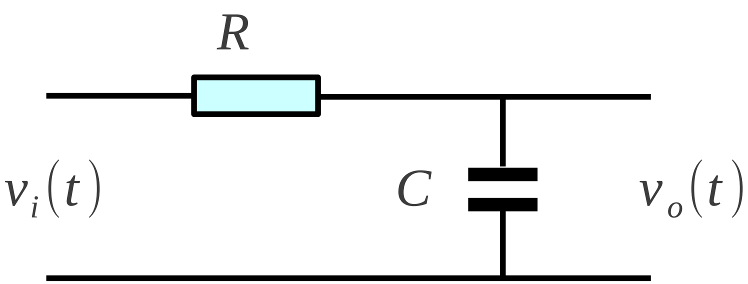
Dispersion



Simple Low Pass Filter (3)

Frequency Response





$$v_o(t) = 1 - e^{-\frac{t}{\tau}} \qquad \omega_0 = \frac{1}{RC} = \frac{1}{\tau}$$

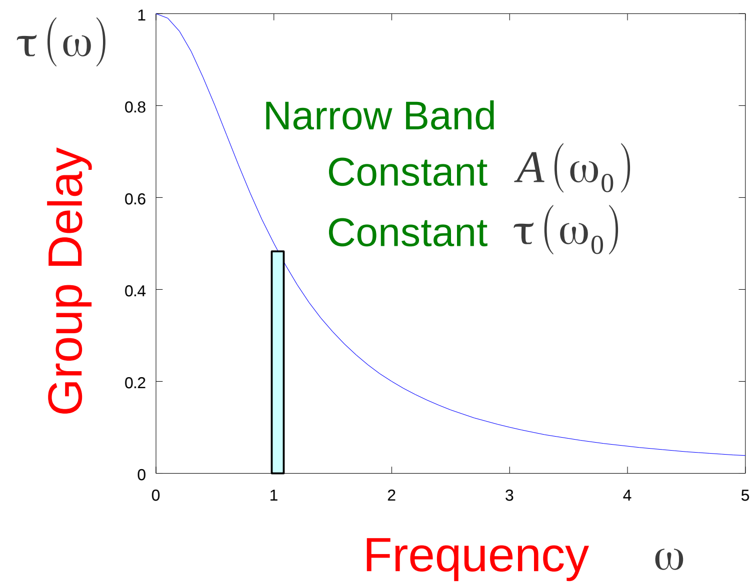
When focusing Narrow Band

Output

Time delayed by $\tau(\omega_0)$

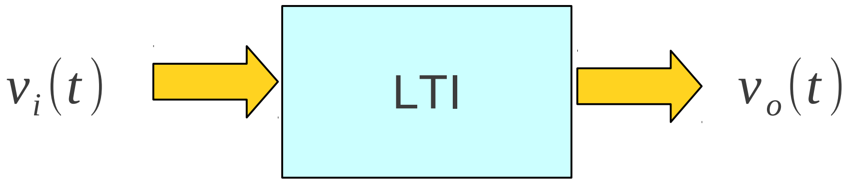
Amplitude scaled by $A(\omega_0)$

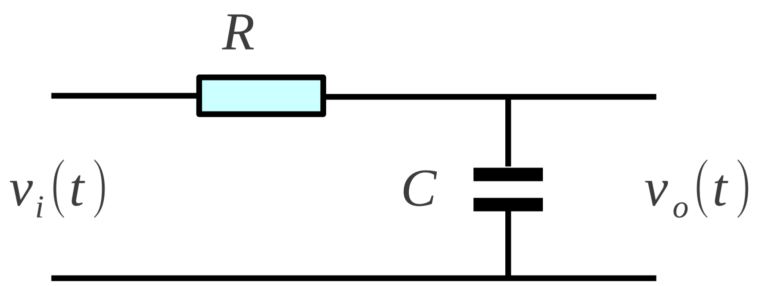
Phase shifted by $\phi(\omega_0)$



Simple Low Pass Filter (4)

Frequency Response





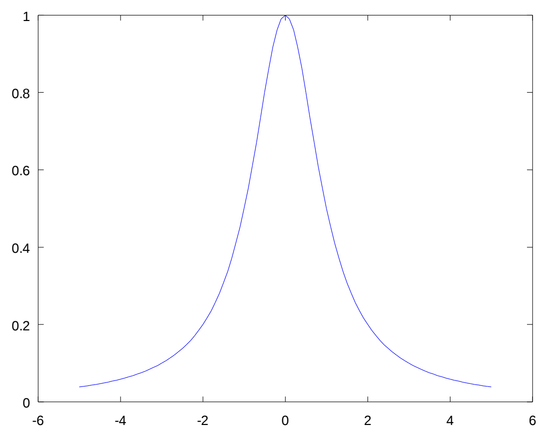
$$H(j\omega) = \frac{1}{1 + j\omega/\omega_0} \qquad \omega_0 = \frac{1}{RC}$$

$$A(j\omega) = |H(j\omega)| = \frac{1}{\sqrt{1+\omega^2/\omega_0^2}}$$

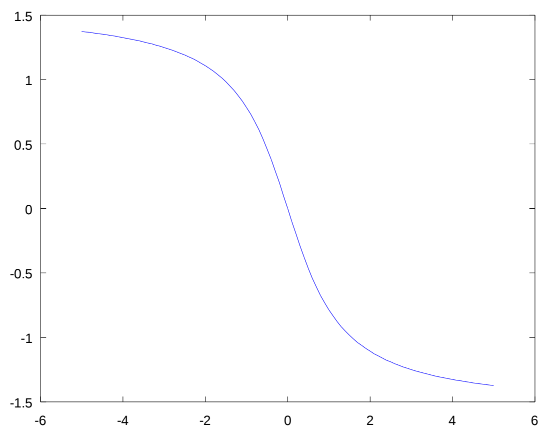
$$H(j\omega) = A(j\omega)e^{j\phi(j\omega)}$$

$$\phi(j\omega) = arg\{H(j\omega)\} = \tan^{-1}(-\omega/\omega_0)$$

Magnitude Response

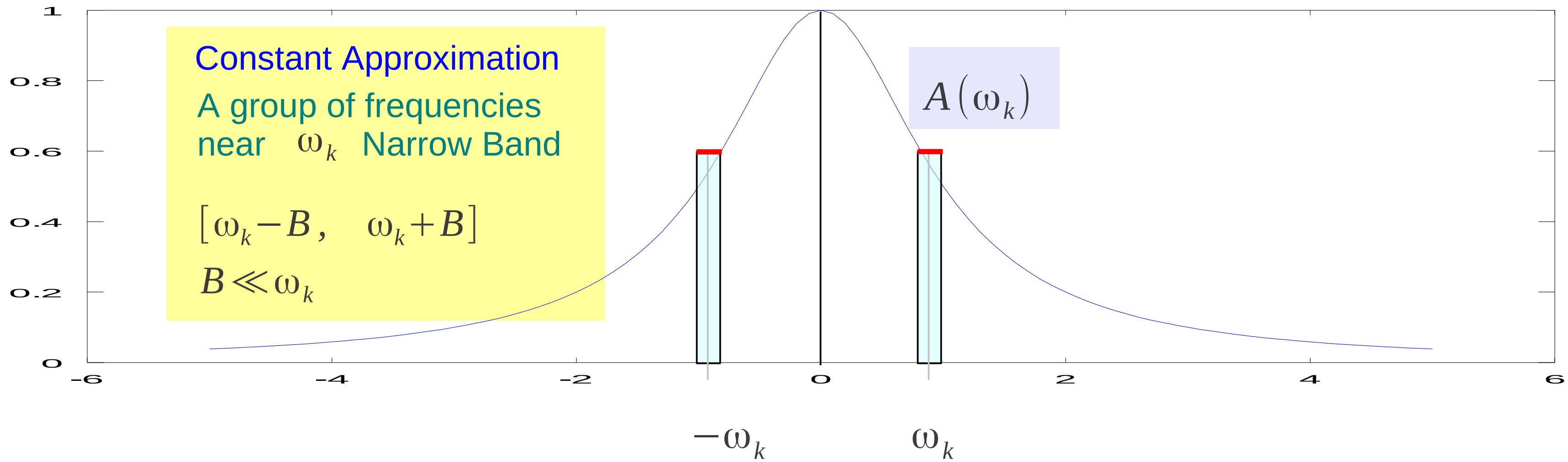


Phase Response

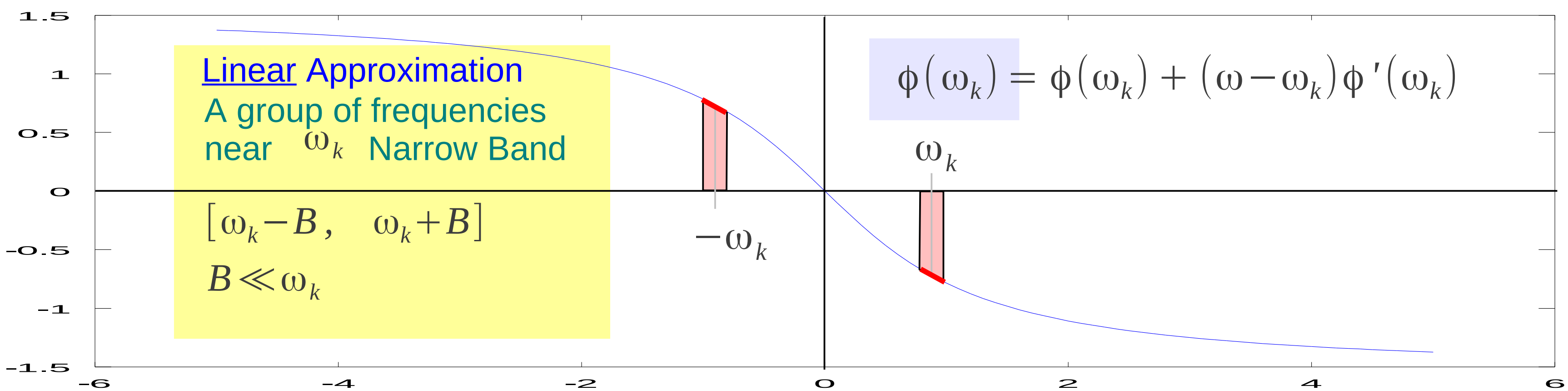


Simple Low Pass Filter (5)

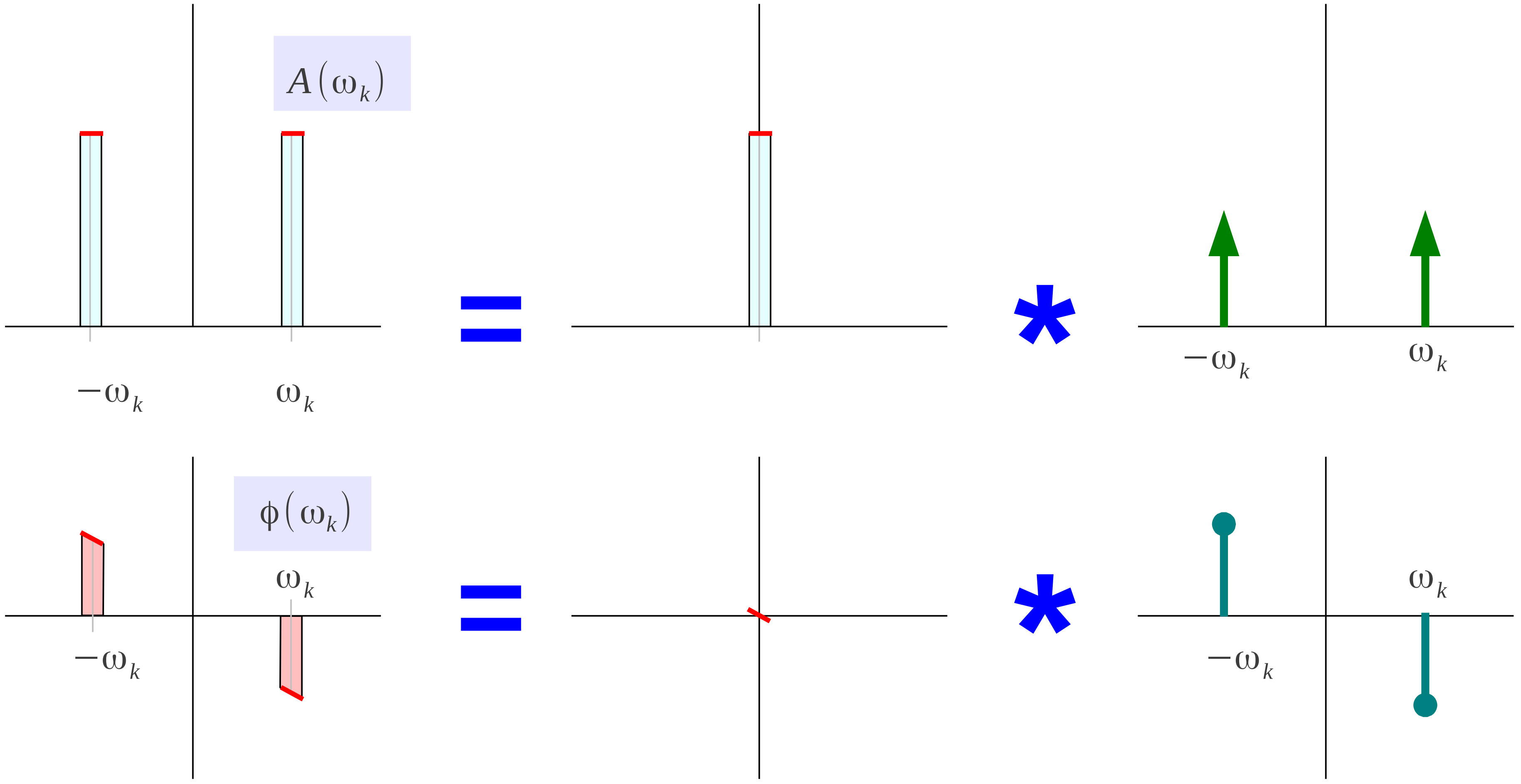
Magnitude Response $A(j\omega) = |H(j\omega)| = 1 \;/\; \sqrt{1+\omega^2/\omega_0^2}$ ➡ $\approx \sum_k A(\omega_k)$



Phase Response $\phi(j\omega) = arg\{H(j\omega)\} = \tan^{-1}(-\omega/\omega_0)$ ➡ $\approx \sum_k \phi(\omega_k)$



Simple Low Pass Filter (6)

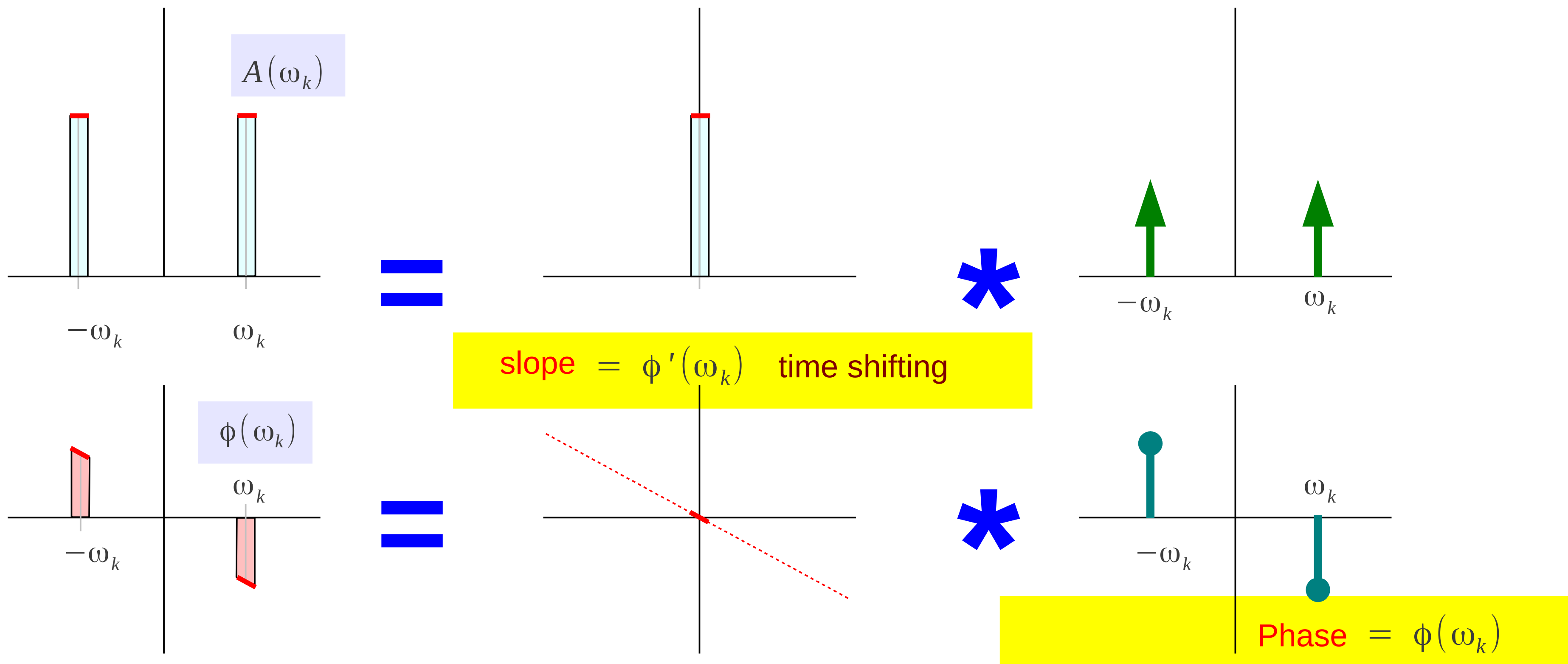


$$2A(\omega_k)B\frac{\sin[B(t+\phi'(\omega_k))]}{\pi B(t+\phi'(\omega_k))}$$

$$\begin{cases} A(\omega_k) & (-B \le \omega \le +B) \\ 0 & else \end{cases}$$

$$\pi[e^{+j\phi(\omega_k)}\delta(\omega-\omega_k) + e^{-j\phi(\omega_k)}\delta(\omega+\omega_k)]$$

Simple Low Pass Filter (6)



Frequency Domain

$$\begin{cases} A(\omega_k) & (-B \le \omega \le +B) \\ 0 & else \end{cases} \quad * \quad \pi[e^{+j\phi(\omega_k)}\delta(\omega-\omega_k) + e^{-j\phi(\omega_k)}\delta(\omega+\omega_k)]$$

Group Delay

Phase Delay

Time Domain

$$2A(\omega_k)B\frac{\sin[B(t+\phi'(\omega_k))]}{\pi B(t+\phi'(\omega_k))} \quad \times \quad \cos(\omega_k t+\phi(\omega_k))$$

Beat Signal

Very similar frequency signals

1.1 Hz $\cos(2\pi * 1.1 * t)$

0.9 Hz $\cos(2\pi * 0.9 * t)$

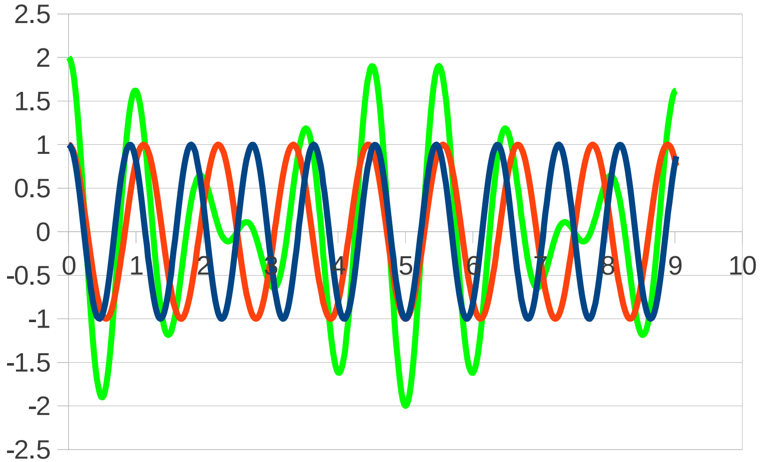
$$\cos(2\pi * 1.1 * t) + \cos(2\pi * 0.9 * t)$$

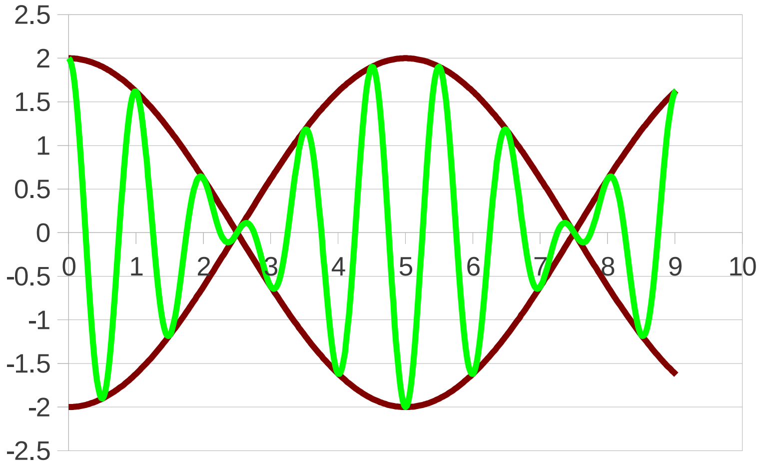
$$= \cos\left(2\pi * \frac{(1.1-0.9)}{2} * t\right) \cdot \cos\left(2\pi * \frac{(1.1+0.9)}{2} * t\right)$$

$$= \cos(2\pi * 0.1 * t) \cdot \cos(2\pi * 1.0 * t)$$

Slow moving envelop

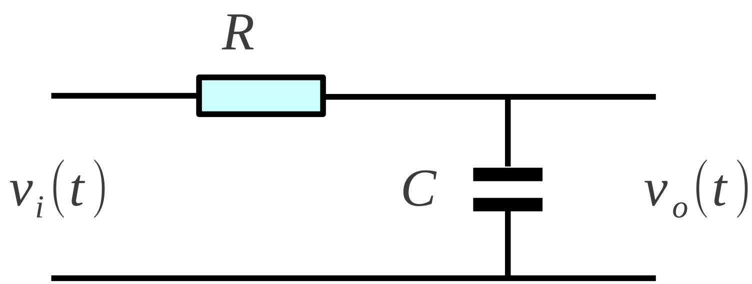
Fast moving carrier





Group Delay Example (1)

Frequency Response



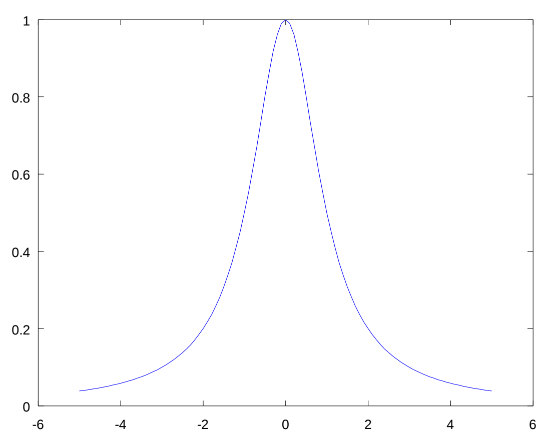
$$H(j\omega) = \frac{1}{1 + j\omega/\omega_0} \qquad \omega_0 = 1$$

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

$$arg\{H(j\omega)\} = \tan^{-1}(-\omega)$$

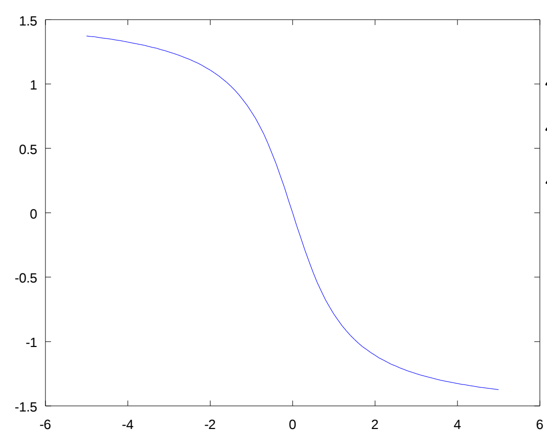
$$arg\{H(j\omega)\} = \tan^{-1}(-\omega)$$

Magnitude Response



$$|H(j\,0.9)| = 0.743$$
$$|H(j\,1.0)| = 0.707$$
$$|H(j\,1.1)| = 0.672$$

Phase Response

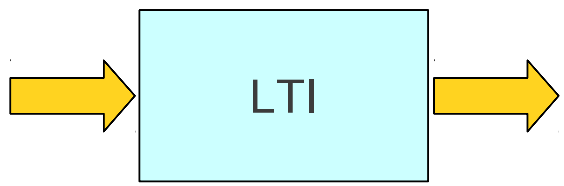


$$\angle H(j\,0.9) = -0.73$$
$$\angle H(j\,1.0) = -0.79$$
$$\angle H(j\,1.1) = -0.83$$

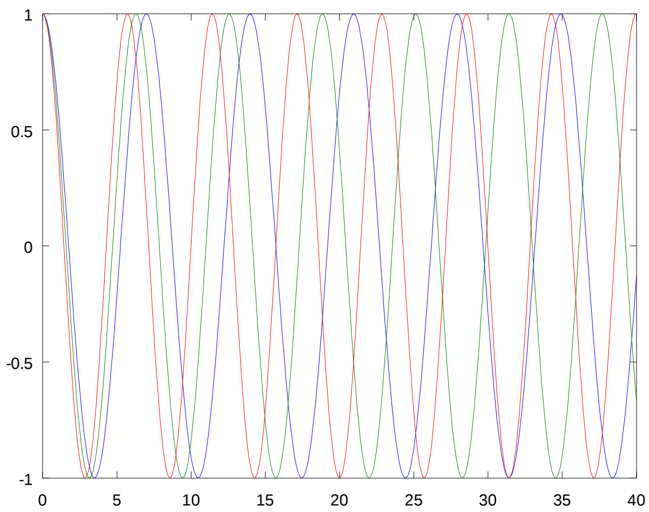
Group Delay Example (2)

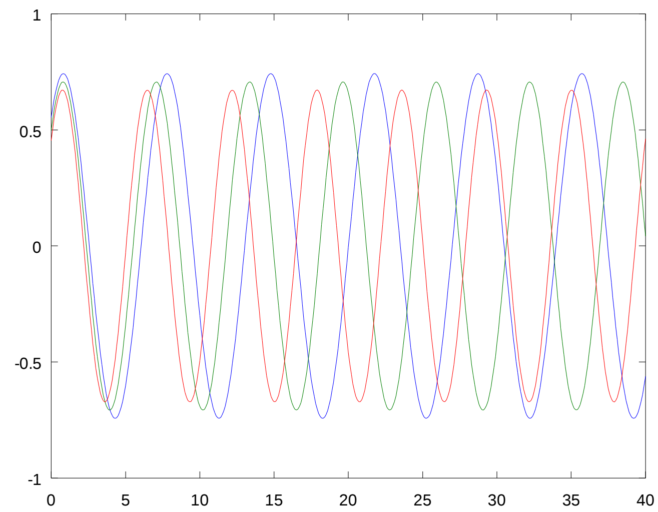
Frequency Response

$$\cos(0.9t)$$
$$\cos(1.0t)$$
$$\cos(1.1t)$$



$$0.743\cos(0.9t - 0.73)$$
$$0.707\cos(1.0t - 0.79)$$
$$0.672\cos(1.1t - 0.83)$$

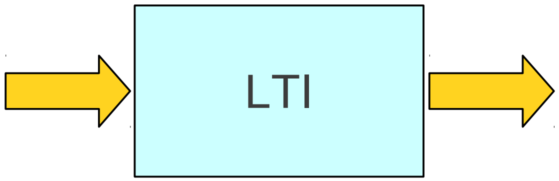




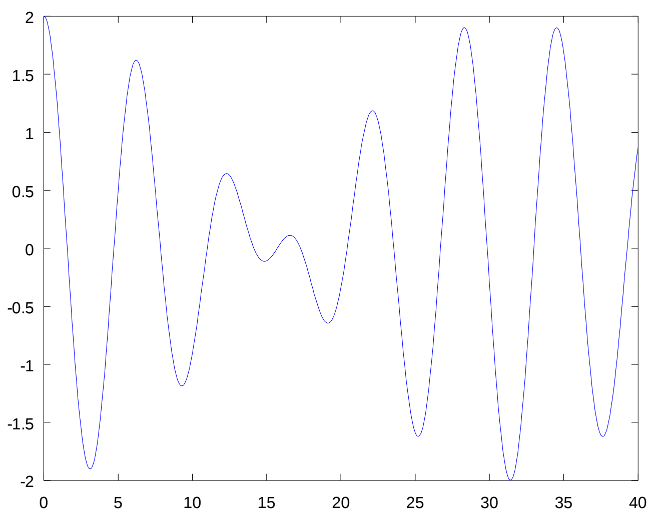
Group Delay Example (3)

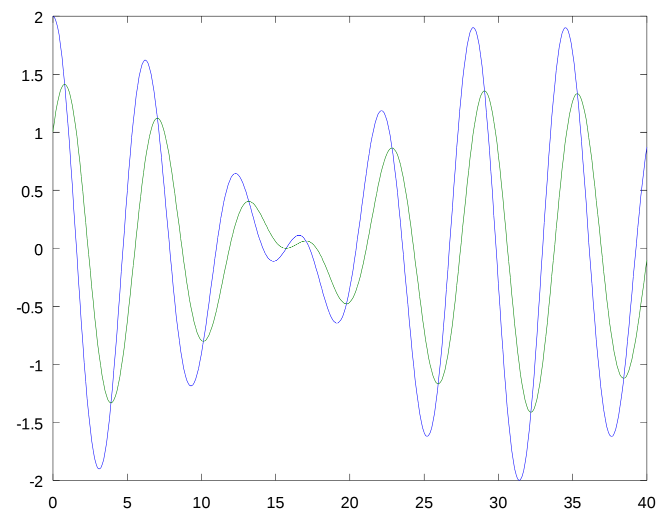
Frequency Response

$$\cos(0.9t) + \cos(1.1t)$$



$$0.743\cos(0.9t - 0.73) + 0.672\cos(1.1t - 0.83)$$



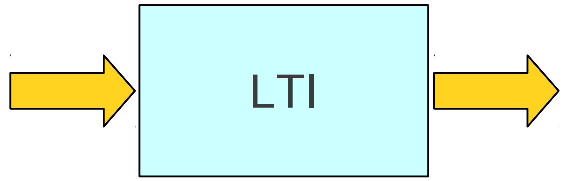


Group Delay Example (4)

Frequency Response

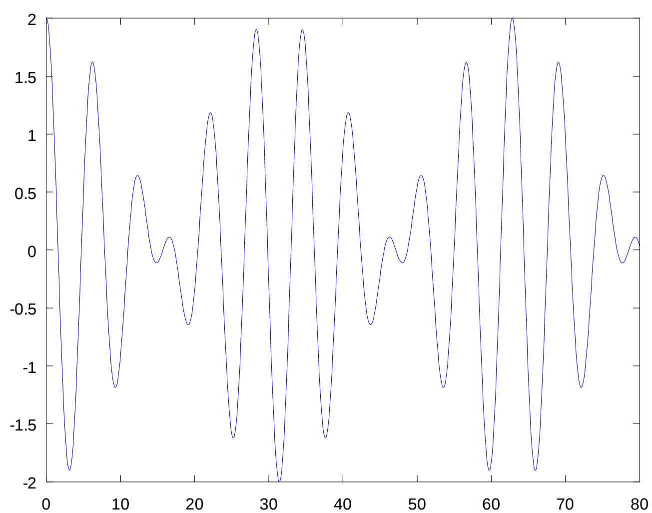
$$\tau(\omega) = -\frac{d\phi}{d\omega} = \frac{1}{1+\omega^2} \qquad \tau(1) = \frac{1}{2}$$

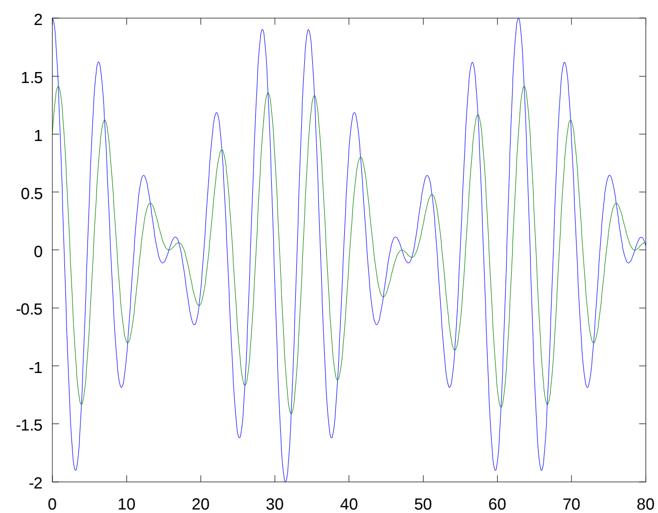
$$\cos(0.9t) + \cos(1.1t)$$



$$0.743\cos(0.9t - 0.73) + 0.672\cos(1.1t - 0.83)$$

$$= 2\cos(0.1t)\cos(1.0t)$$

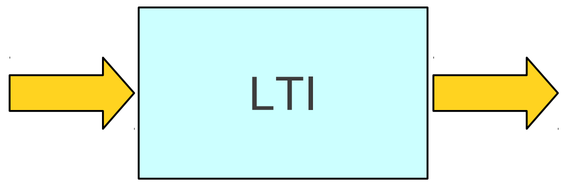




Group Delay Example (5)

Frequency Response $\tau(\omega) = -\dfrac{d\phi}{d\omega} = \dfrac{1}{1+\omega^2}$ $\tau(1) = \dfrac{1}{2}$

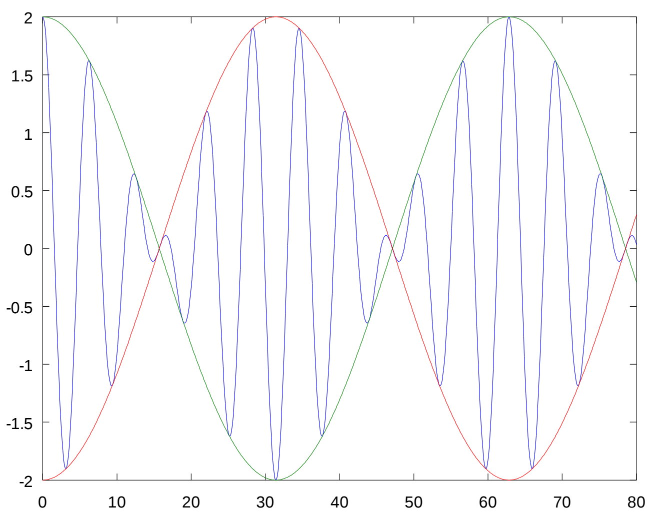
$$\cos(0.9t) + \cos(1.1t)$$

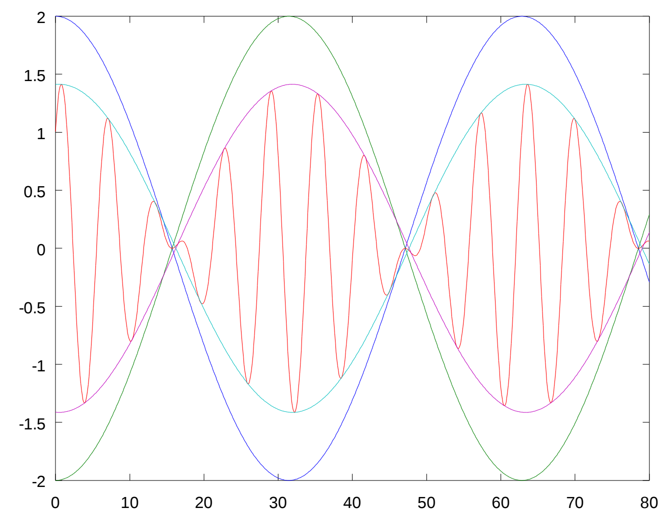


$$0.743\cos(0.9t-0.73) + 0.672\cos(1.1t-0.83)$$

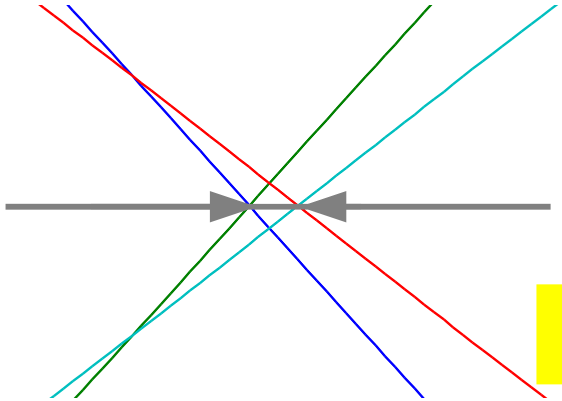
$$= 2\cos(0.1t)\cos(1.0t)$$

$$\approx 1.41\cos(0.1(t-0.5))\cos(1.0t-\pi/4)$$

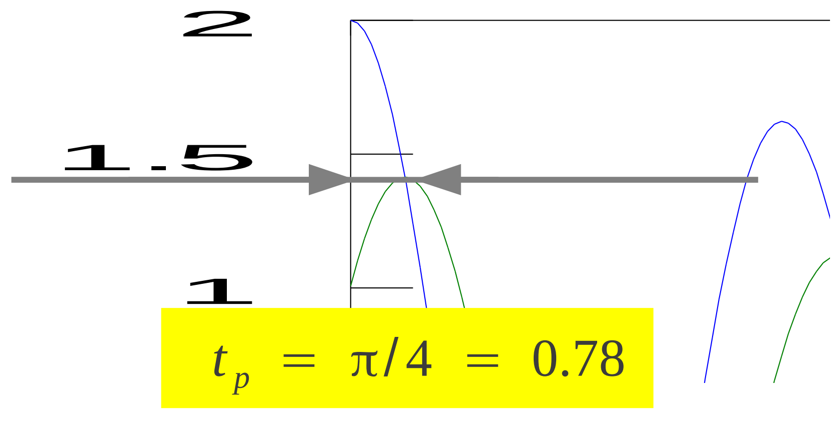




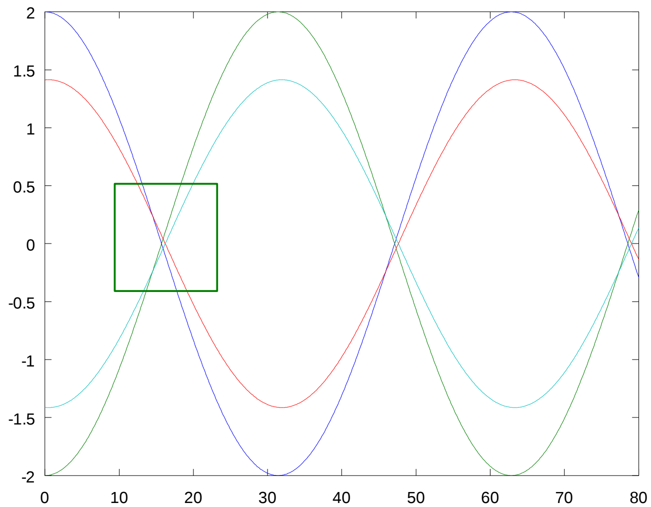
Group Delay Example (6)

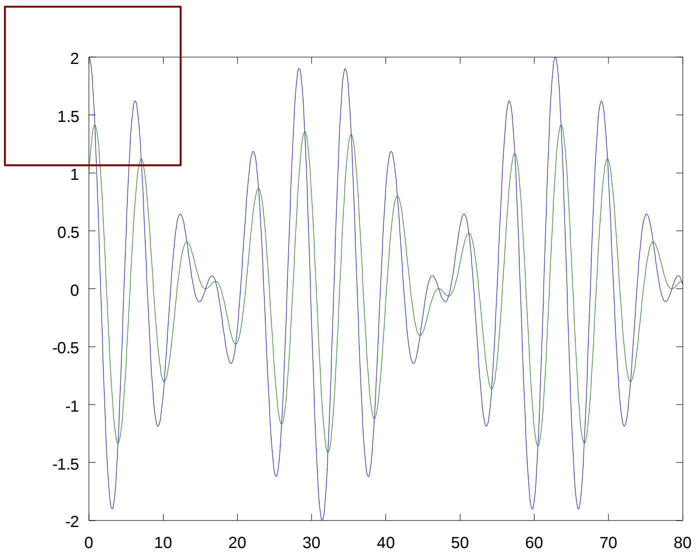


$t_g = 0.5$



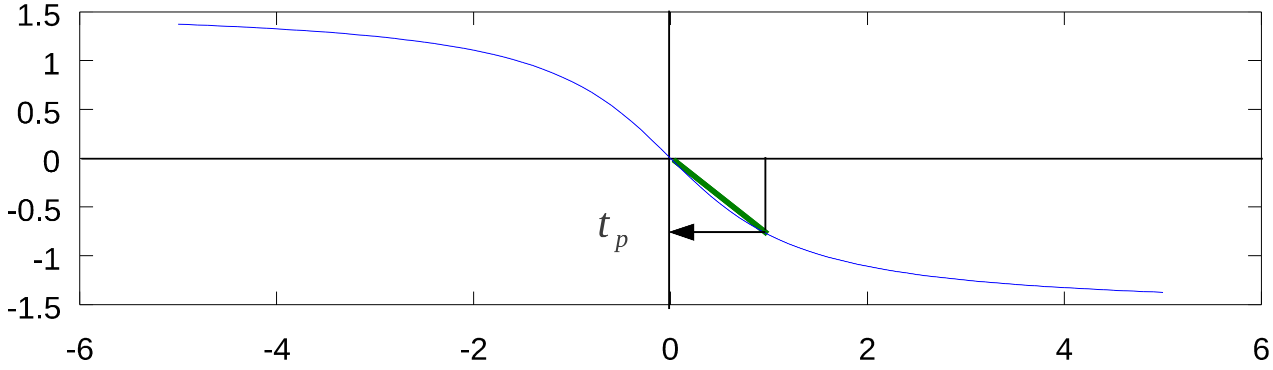
$t_p = \pi/4 = 0.78$





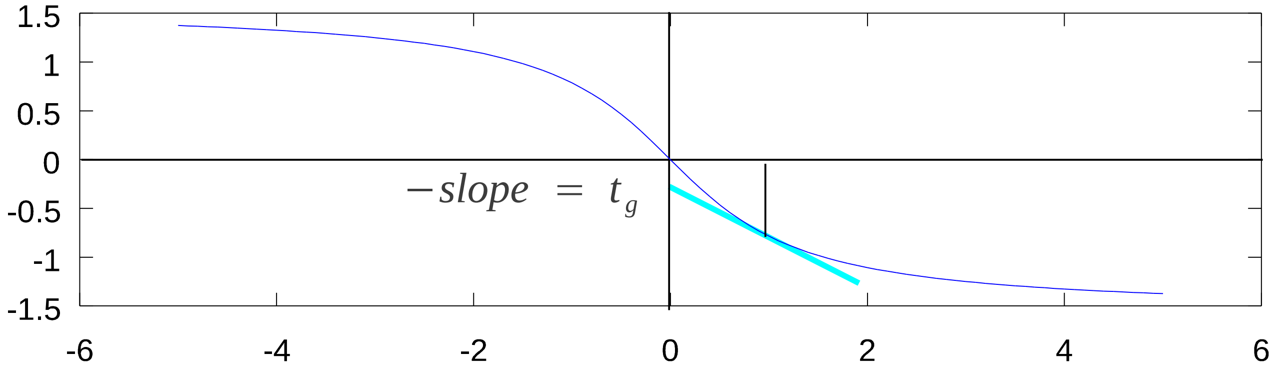
Phase & Group Delay from Phase Response

Phase Response



Phase Delay

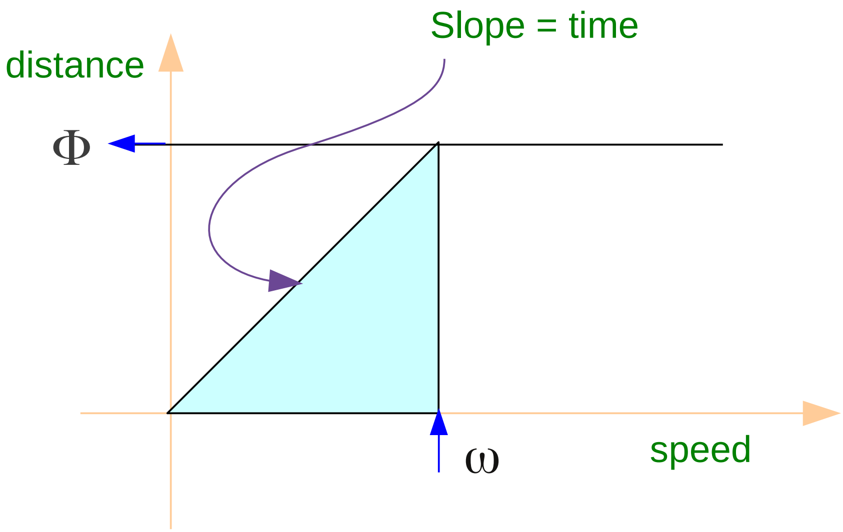
$$t_p = -\tan^{-1}(1)$$
$$= -\pi/4 = 0.785$$



Group Delay

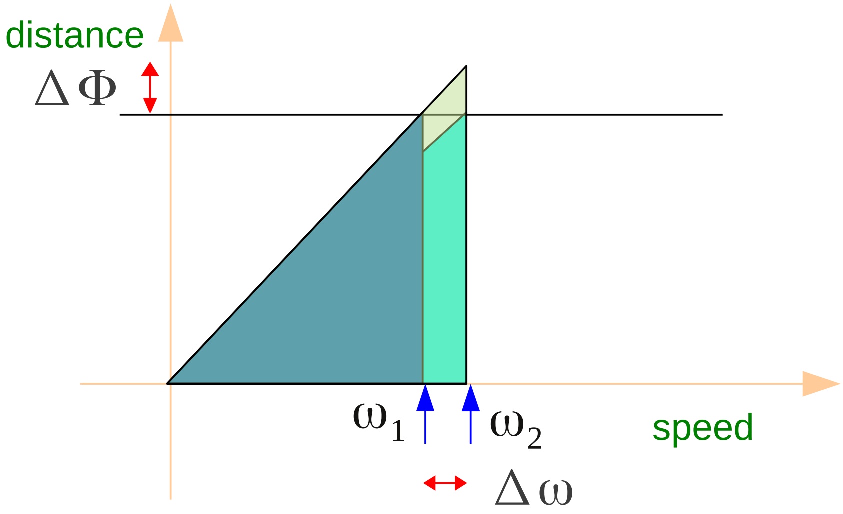
$$t_g = 0.5$$

Angle and Angular Speed



$$\Phi = \omega \cdot t$$

$$t = \frac{\Phi}{\omega}$$



$$\Delta\Phi = \Delta\omega \cdot \Delta t$$

$$\Delta t = \frac{\Delta\Phi}{\Delta\omega}$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] http://www.libinst.com/tpfd.htm