## DFT (4A)

- CTFS and DFT
- CTFT and DFT
- DTFT and DFT
- CTFT and DFT

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## CTFS with Complex Coefficients

$$
x(t)=a_{0}+\sum_{\boldsymbol{k}=1}^{\infty}\left(a_{\boldsymbol{k}} \cos \left(\boldsymbol{k} \omega_{0} t\right)+b_{\boldsymbol{k}} \sin \left(\boldsymbol{k} \omega_{0} t\right)\right)
$$

$$
a_{0}=\frac{1}{T} \int_{0}^{T} x(t) d t
$$

$$
a_{\boldsymbol{k}}=\frac{2}{T} \int_{0}^{T} x(t) \cos \left(\boldsymbol{k} \omega_{0} t\right) d t
$$

$$
b_{\boldsymbol{k}}=\frac{2}{T} \int_{0}^{T} x(t) \sin \left(\boldsymbol{k} \omega_{0} t\right) d t
$$

$$
\boldsymbol{k}=1,2, \ldots
$$

$$
x(t)=A_{0}+\sum_{\boldsymbol{k}=1}^{\infty}\left(A_{\boldsymbol{k}} e^{j \boldsymbol{k} \omega_{0} t}+B_{\boldsymbol{k}} e^{-j \boldsymbol{k} \omega_{0} t}\right)
$$

$$
\begin{aligned}
A_{0} & =a_{0} \\
A_{\boldsymbol{k}} & =\frac{1}{2}\left(a_{\boldsymbol{k}}-j b_{\boldsymbol{k}}\right) \\
B_{\boldsymbol{k}} & =\frac{1}{2}\left(a_{\boldsymbol{k}}+j b_{\boldsymbol{k}}\right) \\
& \boldsymbol{k}=1,2, \ldots
\end{aligned}
$$

$$
\begin{gathered}
x(t)=\sum_{k=0}^{\infty}\left(A_{\boldsymbol{k}} e^{+j \boldsymbol{k} \omega_{0} t}+B_{\boldsymbol{k}} e^{-j \boldsymbol{k} \omega_{0} t}\right) \\
A_{\boldsymbol{k}}=\frac{1}{T} \int_{0}^{T} x(t) e^{-j \boldsymbol{k} \omega_{0} t} d t \\
\boldsymbol{k}=0,1,2, \ldots \\
B_{\boldsymbol{k}}=\frac{1}{T} \int_{0}^{T} x(t) e^{+j \boldsymbol{k} \omega_{0} t} d t \\
\boldsymbol{k}=1,2, \ldots
\end{gathered}
$$

$$
x(t)=\sum_{k=-\infty}^{+\infty} C_{k} e^{+j \boldsymbol{k} \omega_{0} t}
$$

$$
\begin{aligned}
C_{\boldsymbol{k}} & =\frac{1}{T} \int_{0}^{T} x(t) e^{-j \boldsymbol{k} \omega_{0} t} d t \\
& \boldsymbol{k}
\end{aligned}=-2,-1,0,+1,+2, \ldots .
$$

$$
C_{k}= \begin{cases}A_{0}=a_{0} & (\boldsymbol{k}=0) \\ A_{\boldsymbol{k}}=\frac{1}{2}\left(a_{\boldsymbol{k}}-j b_{\boldsymbol{k}}\right) & (\boldsymbol{k}>0) \\ B_{k}=\frac{1}{2}\left(a_{\boldsymbol{k}}+j b_{k}\right) & (\boldsymbol{k}<0)\end{cases}
$$

## CTFS and DTFS

$$
\begin{aligned}
& x(t)=\sum_{k=-\infty}^{+\infty} C_{k} e^{+j k \omega_{0} t} \quad \text { CTFS } \\
& C_{k}=\frac{1}{T} \int_{0}^{T} x(t) e^{-j k \omega_{0} t} d t \\
& \boldsymbol{k}=-2,-1,0,+1,+2, \ldots \\
& x(t) \approx \sum_{k=-M}^{+M} C_{k} e^{+j k \omega_{0} t} \quad N=2 \mathrm{M}+1 \\
& j \boldsymbol{k} \omega_{0} t \rightarrow \boldsymbol{k}\left(\frac{2 \pi}{T}\right) \boldsymbol{n}\left(\frac{T}{N}\right)=\left(\frac{2 \pi}{T}\right) \boldsymbol{n} \boldsymbol{k} \\
& x[\boldsymbol{n}]=\sum_{k=-M}^{+M} \gamma_{k} e^{+j\left(\frac{2 \pi}{N}\right) n k} \\
& \boldsymbol{n}=0,1,2, \ldots, N-1 \text {, } \\
& \gamma_{k}=\frac{1}{N} \sum_{n=0}^{N-1} x[\boldsymbol{n}] e^{-j\left(\frac{2 \pi}{N}\right) n k \omega_{0} t} \\
& \boldsymbol{k}=-M, \ldots, 0, \ldots,+M \\
& x_{F S}(t)=\sum_{k=-\infty}^{+\infty} \gamma_{k} e^{+j k \omega_{0} t} \quad \text { DTFS }
\end{aligned}
$$

## CTFT and DFT

Continuous Time Fourier Transform

$$
X(j \omega)=\int_{-\infty}^{+\infty} x(t) e^{-j \omega t} d t \Leftrightarrow x(t)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} X(j \omega) e^{+j \omega t} d \omega
$$

Discrete Fourier Transform

$$
X[k]=\sum_{n=0}^{N-1} x[n] e^{-j(2 \pi / N) k n} \Leftrightarrow x[n]=\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{+j(2 \pi / N) k n}
$$

## From CTFT to DFT (1)

## Continuous Time Fourier Transform

$$
X(j \omega)=\int_{-\infty}^{+\infty} x(t) e^{-j \omega t} d t \Leftrightarrow x(t)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} X(j \omega) e^{+j \omega t} d \omega
$$

Time Samples


$$
t \rightarrow n T_{s} \quad d t \rightarrow T_{s} \quad \int \rightarrow \sum
$$

$$
0 \leq n<L \quad T_{s} \rightarrow 0
$$

Frequency Samples

$$
\begin{array}{c:c}
\frac{2 \pi}{T_{s}} \frac{1}{N} & \omega_{k}=\frac{2 \pi}{T_{s}} \frac{k}{N} \\
\hdashline \mathbf{~} & \omega_{1} \\
0 & \omega_{2}
\end{array}
$$

$$
\omega \rightarrow \omega_{k} \quad d \omega \rightarrow \frac{2 \pi}{T_{s}} \frac{1}{N} \quad \int \rightarrow \sum
$$

$$
0 \leq k<N
$$

$$
0 \leq \omega_{k}<\frac{2 \pi}{T_{s}}
$$

## From CTFT to DFT (2)

## Continuous Time Fourier Transform

$$
\begin{gathered}
X(j \omega)=\int_{-\infty}^{+\infty} x(t) e^{-j \omega t} d t \Leftrightarrow x(t)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} X(j \omega) e^{+j \omega t} d \omega \\
t \rightarrow n T_{s} d t \rightarrow T_{s} \quad \int \rightarrow \sum \quad 0 \leq n<L \\
\hat{X}(j \omega)=\sum_{n=-\infty}^{+\infty} x\left(n T_{s}\right) e^{-j \omega n T_{s}} \cdot T_{s} \Leftrightarrow x\left(n T_{s}\right)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} \hat{X}(j \omega) e^{+j \omega n T_{s}} d \omega \\
\omega \rightarrow \omega_{k} d \omega \rightarrow \frac{2 \pi}{T_{s}} \frac{1}{N} \quad \int \rightarrow \sum 0 \leq k<N \quad \omega_{k}=\frac{2 \pi}{T_{s}} \frac{k}{N} \\
\hat{X}\left(j \omega_{k}\right)=T_{s} \sum_{n=0}^{L-1} x[n] e^{-j \omega_{k} n T_{s}} \quad \Leftrightarrow x[n]=\frac{1}{2 \pi} \sum_{k=0}^{N-1} \hat{X}\left(j \omega_{k}\right) e^{+j \omega_{k} n T_{s}} \frac{2 \pi}{T_{s}} \frac{1}{N}
\end{gathered}
$$

## From CTFT to DFT (3)

## Continuous Time Fourier Transform

$$
X(j \omega)=\int_{-\infty}^{+\infty} x(t) e^{-j \omega t} d t \Leftrightarrow x(t)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} X(j \omega) e^{+j \omega t} d \omega
$$

Time Samples

$$
t \rightarrow n T_{s} \quad d t \rightarrow T_{s} \quad \int \rightarrow \sum
$$

$$
\hat{X}\left(j \omega_{k}\right)=T_{s} \sum_{n=0}^{L-1} x[n] e^{-j \omega_{k} n T_{s}} \Leftrightarrow x[n]=\frac{1}{2 \pi} \sum_{k=0}^{N-1} \hat{X}\left(j \omega_{k}\right) e^{+j \omega_{k} n T} \frac{2 \pi}{T_{s}} \frac{1}{N}
$$

$$
\omega_{k} T_{s} \rightarrow \frac{2 \pi}{N} k \quad \omega_{k} n T_{s} \rightarrow \frac{2 \pi}{N} k n
$$

$$
\omega_{k}=\frac{2 \pi}{T_{s}} \frac{k}{N}
$$

$$
\frac{1}{T_{s}} \hat{X}\left(j \omega_{k}\right)=\sum_{n=0}^{L-1} x[n] e^{-j\left(\frac{2 \pi}{N}\right) k n} \Leftrightarrow x[n]=\frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{T_{s}} \hat{X}\left(j \omega_{k}\right) e^{+j\left(\frac{2 \pi}{N}\right) k n}
$$

## From CTFT to DFT (4)

## Continuous Time Fourier Transform

$$
X(j \omega)=\int_{-\infty}^{+\infty} x(t) e^{-j \omega t} d t \Leftrightarrow x(t)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} X(j \omega) e^{+j \omega t} d \omega
$$

Time Samples

$$
t \rightarrow n T_{s} \quad d t \rightarrow T_{s} \quad \int \rightarrow \sum
$$

$$
\omega_{k} T_{s} \rightarrow \frac{2 \pi}{N} k
$$

$$
\omega_{k} n T_{s} \rightarrow \frac{2 \pi}{N} k n
$$

$$
\omega_{k}=\frac{2 \pi}{T_{s}} \frac{k}{N}
$$

$$
\frac{1}{T_{s}} \hat{X}\left(j \omega_{k}\right)=\sum_{n=0}^{L-1} x[n] e^{-j\left(\frac{2 \pi}{N}\right) k n} \Leftrightarrow x[n]=\frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{T_{s}} \hat{X}\left(j \omega_{k}\right) e^{+j\left(\frac{2 \pi}{N}\right) k n}
$$

## From CTFT to DFT (5)

## Continuous Time Fourier Transform

$$
X(j \omega)=\int_{-\infty}^{+\infty} x(t) e^{-j \omega t} d t \Leftrightarrow x(t)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} X(j \omega) e^{+j \omega t} d \omega
$$

$$
\frac{1}{T_{s}} \hat{X}\left(j \omega_{k}\right)=\sum_{n=0}^{L-1} x[n] e^{-j\left(\frac{2 \pi}{N}\right) k n} \Leftrightarrow x[n]=\frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{T_{s}} \hat{X}\left(j \omega_{k}\right) e^{+j\left(\frac{2 \pi}{N}\right) k n}
$$

Discrete Fourier Transform

$$
L=N
$$

$$
X[k]=\sum_{n=0}^{N-1} x[n] e^{-j(2 \pi / N) k n} \Leftrightarrow x[n]=\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{+j(2 \pi / N) k n}
$$

## DTFT and CTFT

$$
\begin{aligned}
& \begin{array}{l}
X_{s}(j \omega)=\sum_{n=-\infty}^{+\infty} X_{c}\left(n T_{s}\right) e^{-j \omega n T_{s}} \\
=\sum_{n=-\infty}^{+\infty} x[n] e^{-j \omega n T_{s}}
\end{array}=\begin{array}{c}
X_{s}(j \omega)=\frac{1}{T_{s}} \sum_{k=-\infty}^{+\infty} X_{c}\left(j\left(\omega-k \omega_{s}\right)\right) \\
\omega_{s}=\frac{2 \pi}{T_{s}} \\
X\left(e^{j \hat{\omega}}\right) \\
=\sum_{n=-\infty}^{+\infty} x[n] e^{-j \hat{\omega} n} \\
\text { DTFT of a sampled signal } \\
X\left(e^{j \hat{\omega}}\right) \\
\hat{\omega}=\omega T_{s}=X\left(e^{j \omega T_{s}}\right)
\end{array} \\
&=\frac{1}{T_{s}} \sum_{k=-\infty}^{+\infty} X_{c}\left(j\left(\omega-k \omega_{s}\right)\right) \\
&=\frac{1}{T_{s}} \sum_{k=-\infty}^{+\infty} X_{c}\left(j\left(\omega-\frac{2 \pi k}{T_{s}}\right)\right)
\end{aligned}
$$

CTFT of a sampled signal

## DTFT and DFT

DTFT of a sampled signal

$$
\begin{aligned}
& X\left(e^{j \hat{\omega}}\right)=\sum_{n=-\infty}^{+\infty} x[n] e^{-j \hat{\omega} n} \\
& \hat{\omega}=\omega T_{s} \\
& \hat{\omega} \rightarrow \hat{\omega}_{k} \quad 0 \leq \hat{\omega}_{k}<2 \pi \quad 0 \leq k<N \quad 0 \leq n<L \\
& X\left(e^{j \hat{\omega}_{k}}\right)=\sum_{n=0}^{L-1} x[n] e^{-j \hat{\omega}_{k} n} \\
& \hat{\omega}_{k}=\left(\frac{2 \pi}{N}\right) k
\end{aligned}
$$

DFT of a sampled signal

$$
X[k]=\quad X\left(e^{j(2 \pi / N) k}\right)=\sum_{n=0}^{L-1} x[n] e^{-j(2 \pi / N) k n}
$$

DTFT sampled in frequency

## CTFT and DFT

DFT of a sampled signal

$$
X[k] \quad=X\left(e^{j(2 \pi / N) k}\right)=\sum_{n=0}^{L-1} x[n] e^{-j(2 \pi / N) k n}
$$

DTFT sampled in frequency

$$
\begin{aligned}
& X\left(\left.e^{j \omega T_{s}}\right|_{\omega=\frac{2 \pi k}{N T_{s}} \quad \text { CTFT evaluated at } \omega=\frac{2 \pi k}{N T_{s}}} ^{\left.=\frac{1}{T_{s}} \sum_{l=-\infty}^{+\infty} X_{c}\left(j\left(\omega-l \omega_{s}\right)\right) \right\rvert\, \omega=\frac{2 \pi k}{N T_{s}}}\right. \\
& \left.=\frac{1}{T_{s}} \sum_{l=-\infty}^{+\infty} X_{c}\left(j\left(\omega-l \frac{2 \pi}{T_{s}}\right)\right) \right\rvert\, \omega=\frac{2 \pi k}{N T_{s}}
\end{aligned}
$$

## References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003

