

# Down-Sampling (4B)

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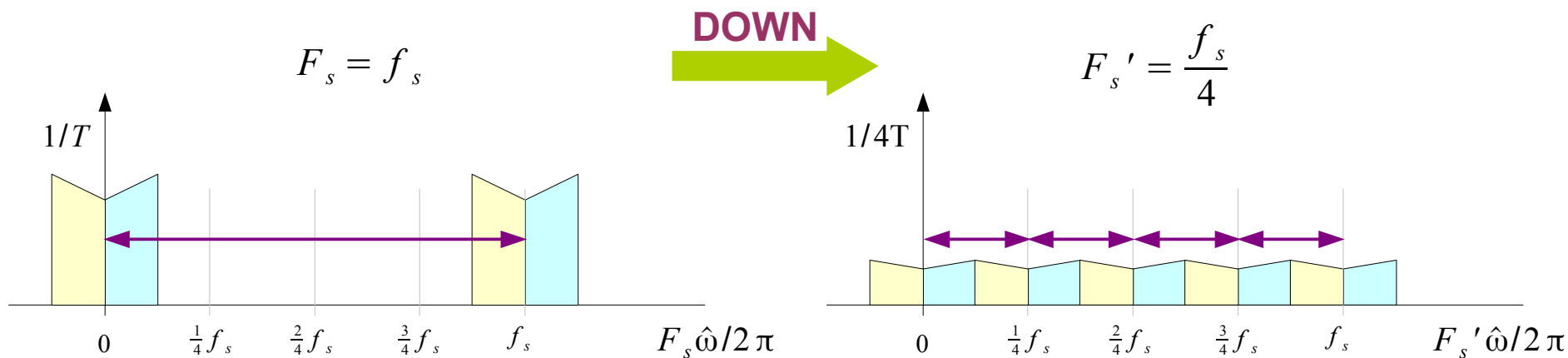
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Please send corrections (or suggestions) to [youngwlim@hotmail.com](mailto:youngwlim@hotmail.com).

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# Decreasing Sampling Frequency



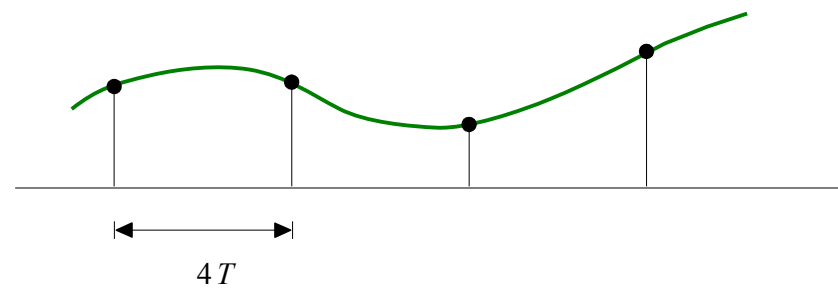
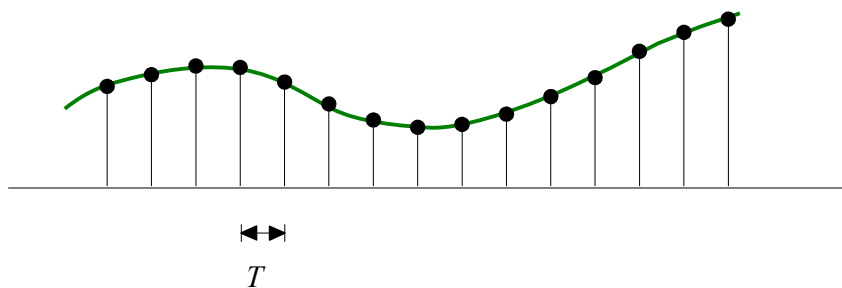
Sampling Frequency  $F_s = f_s$

Sampling Time  $T = \frac{1}{f_s}$

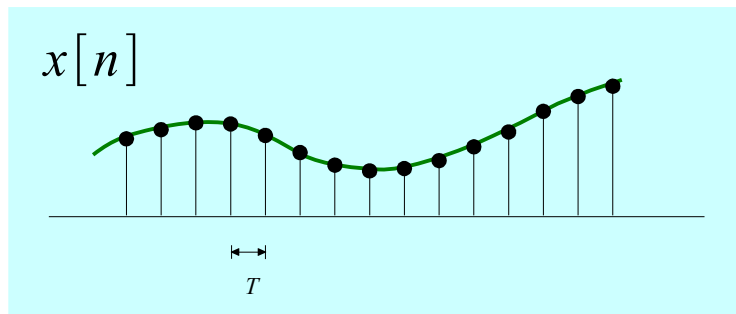


Sampling Frequency  $F_s' = \frac{f_s}{4}$

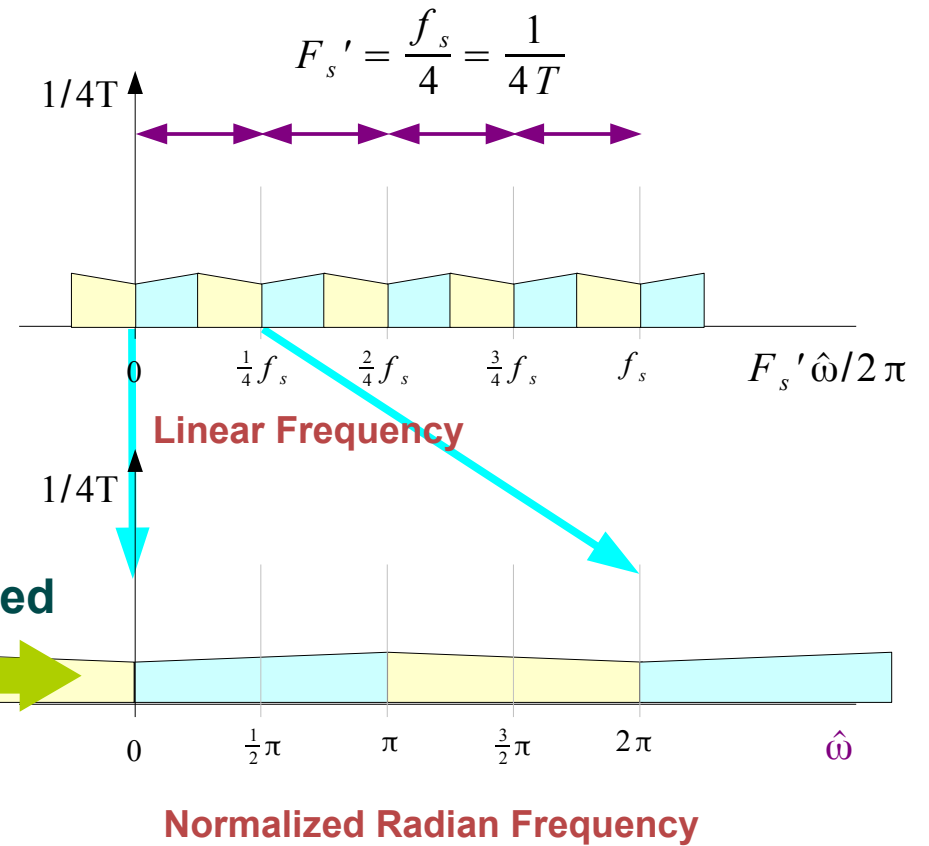
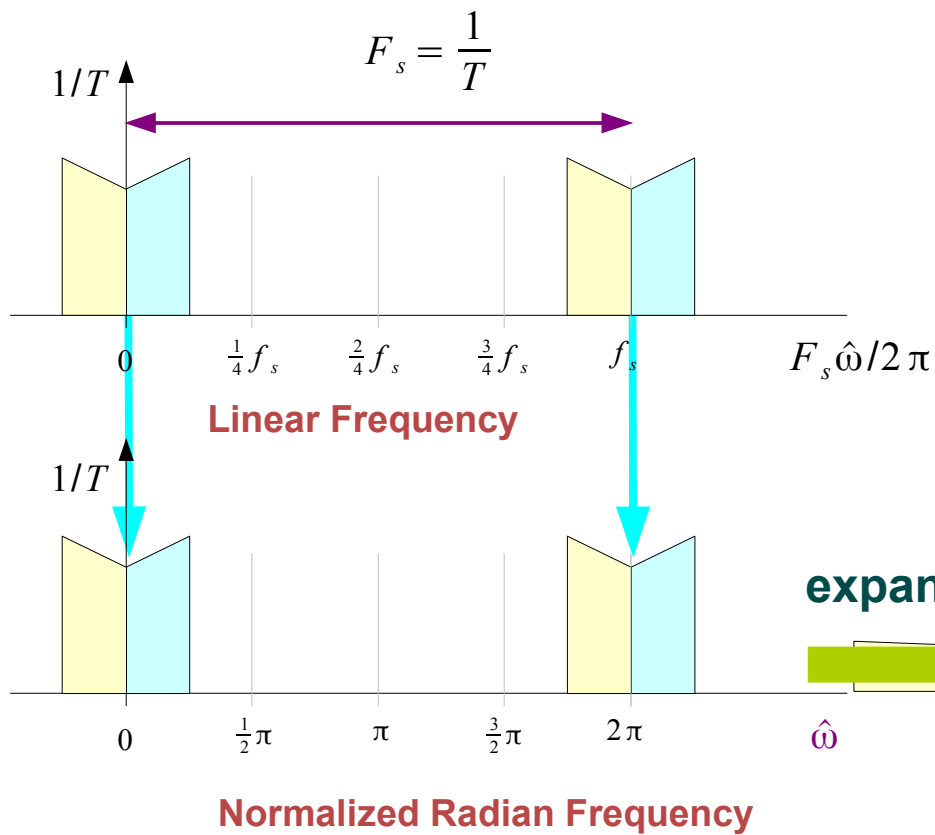
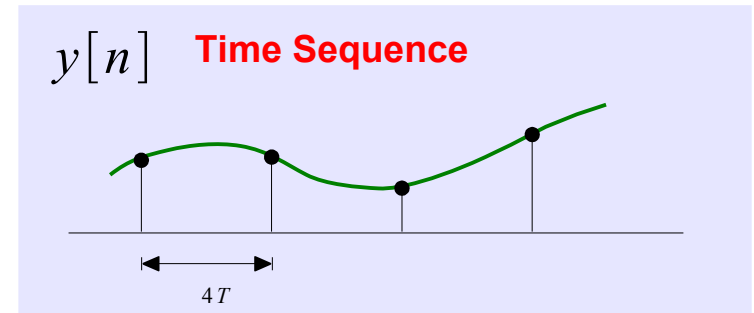
Sampling Time  $T' = 4T$



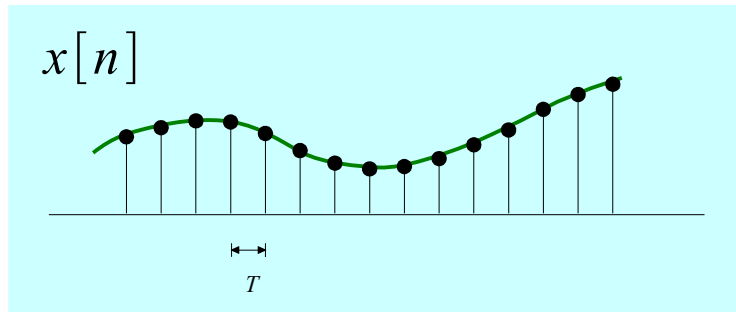
# Coarse Sequence & Spectrum



**DOWN**



# Normalized Radian Frequency

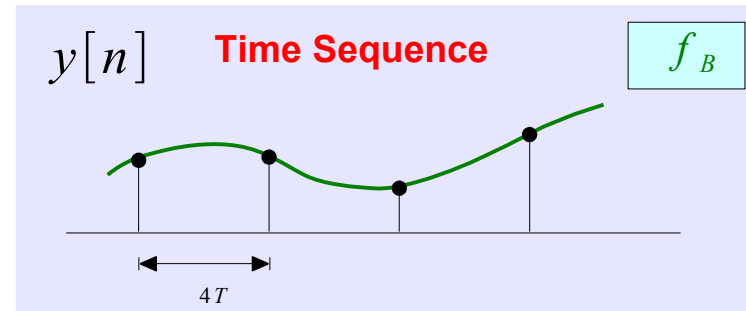


$$\hat{\omega} = \omega \cdot T_s = \frac{\omega}{1/T_s}$$

$$\hat{\omega} = \frac{\omega}{f_s} = 2\pi \frac{f}{f_s}$$

↑                      ↑  
Normalized to  $f_s$

Normalized Radian Frequency

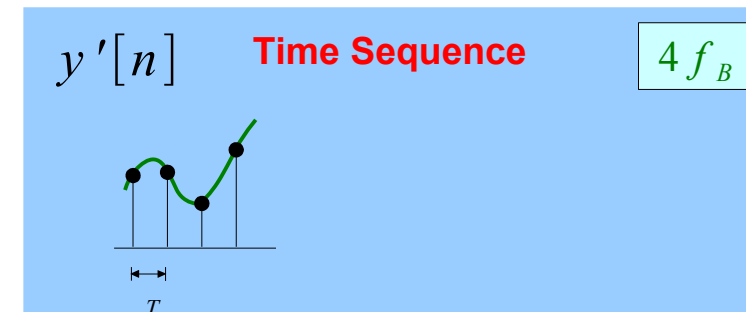


$$\frac{f}{f_s} = \frac{f_B}{1/4T} = f_B \cdot 4T$$

The Same

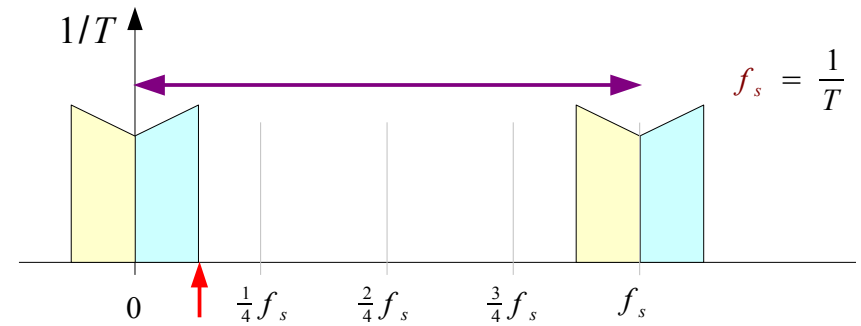
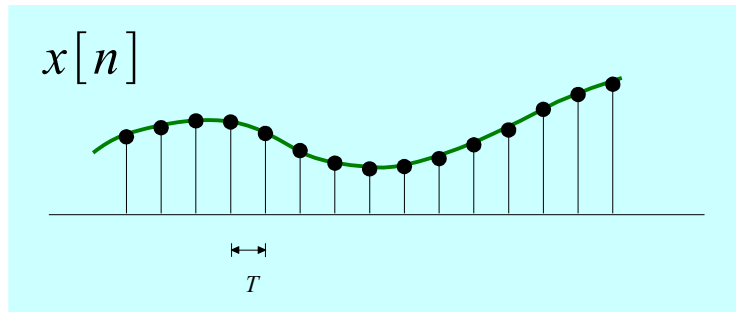
- Time Sequence
- Normalized Radian Frequency

$$\frac{f}{f_s} = \frac{4f_B}{1/T} = f_B \cdot 4T$$

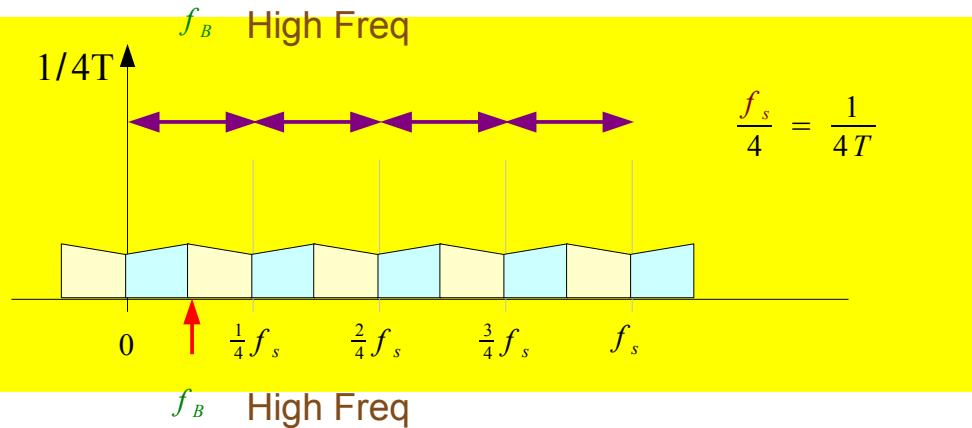
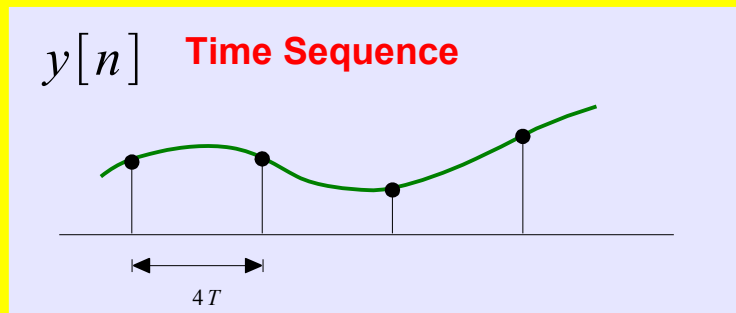


The Highest Frequency:

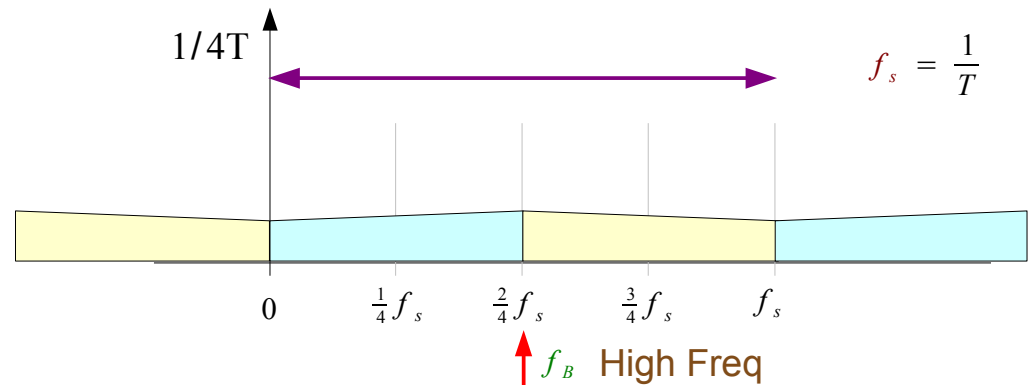
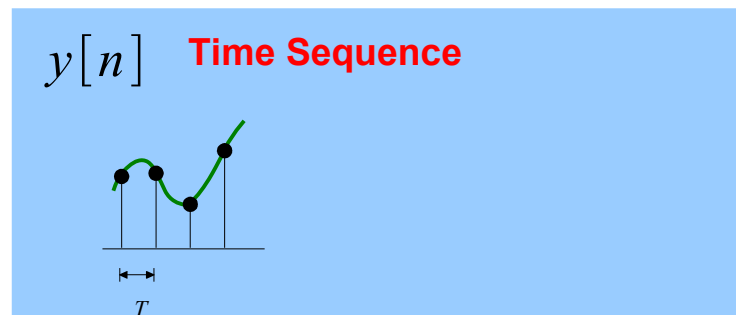
# Coarse Sequence Spectrum – Linear Frequency



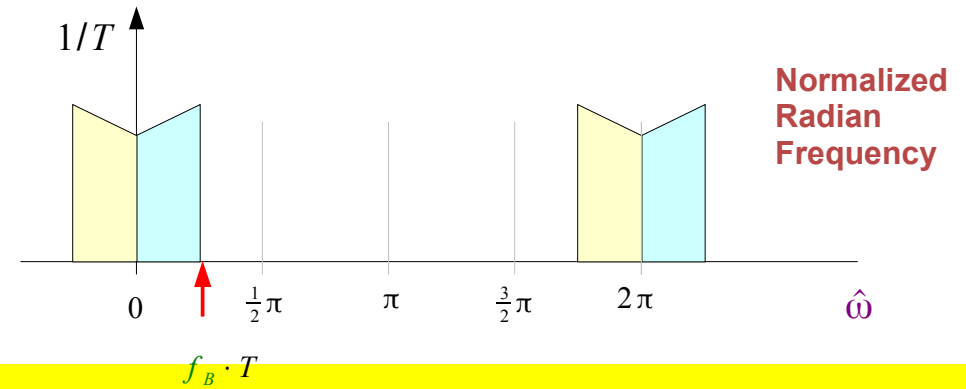
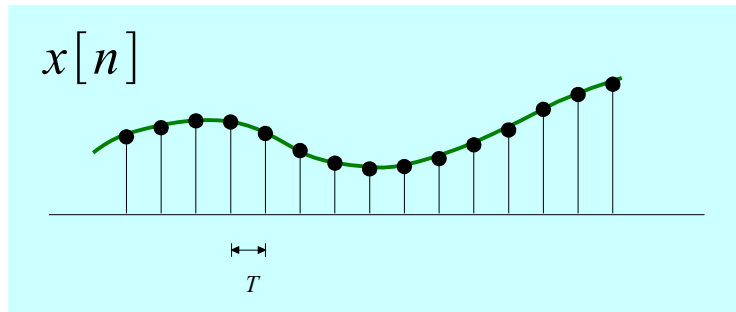
**DOWN**



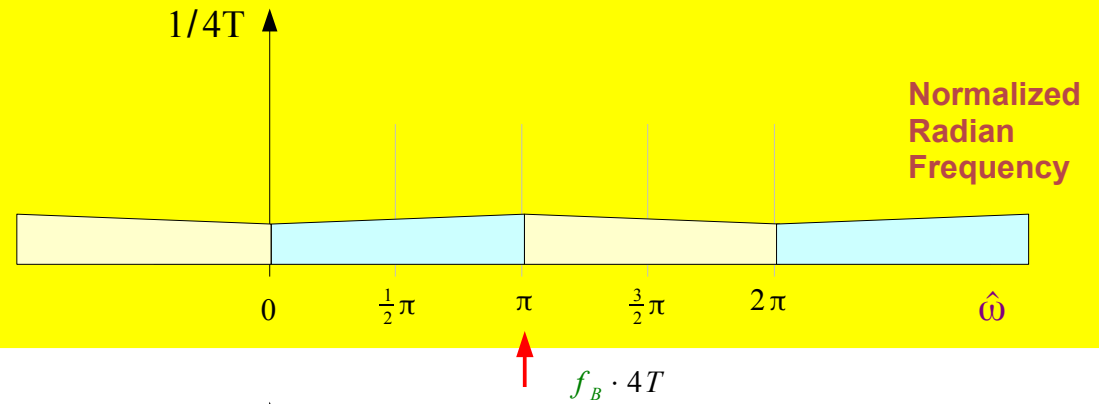
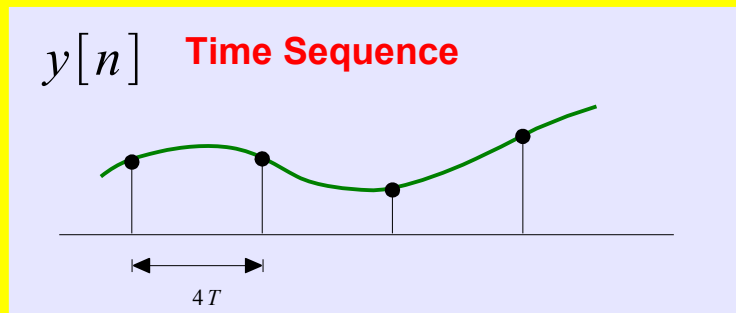
**||** The Same Time Sequence



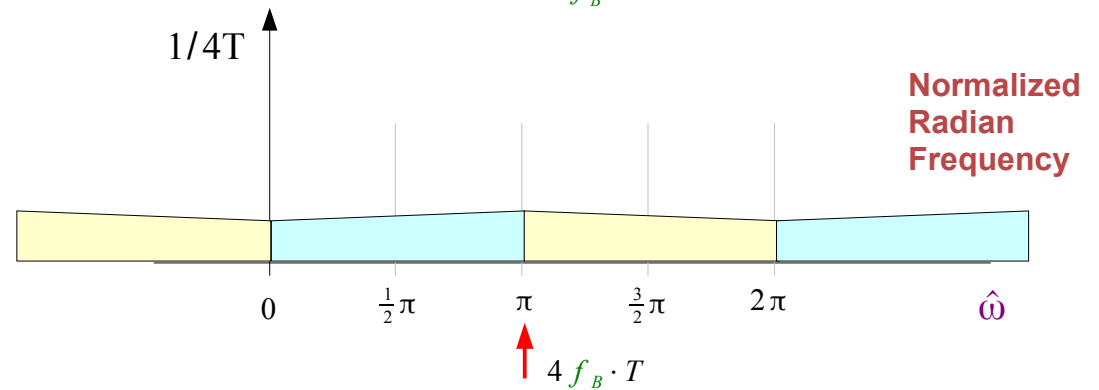
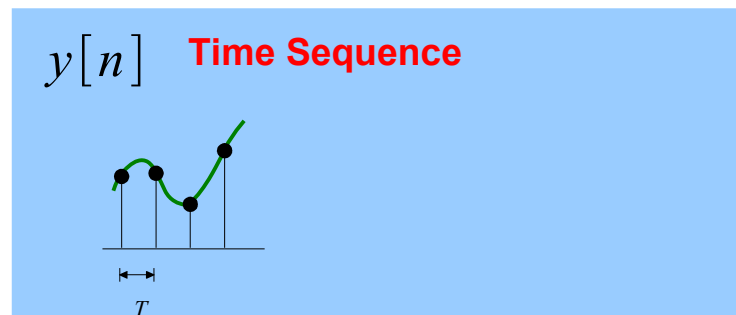
# Coarse Sequence Spectrum – Normalized Frequency



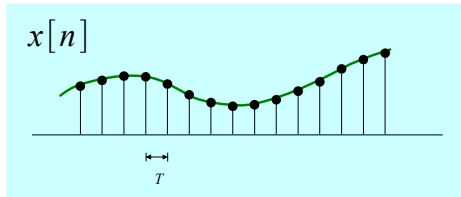
**DOWN**



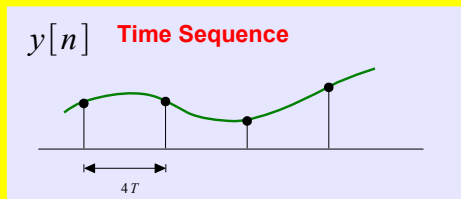
**||** The Same Time Sequence



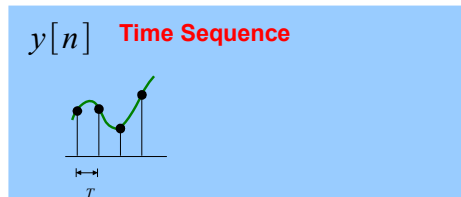
# Coarse Sequence Spectrum – Linear Frequency



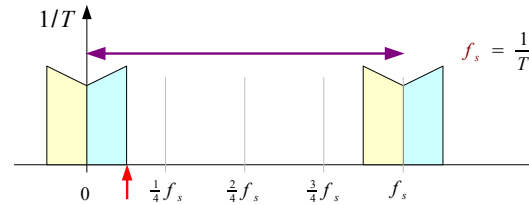
DOWN



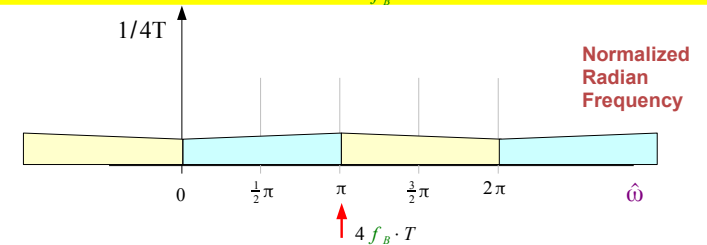
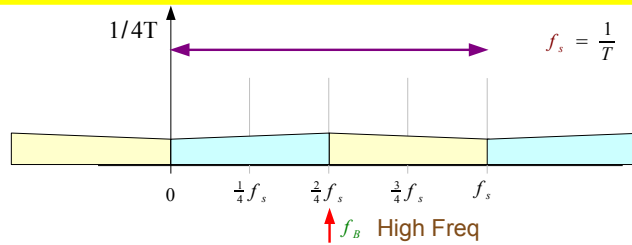
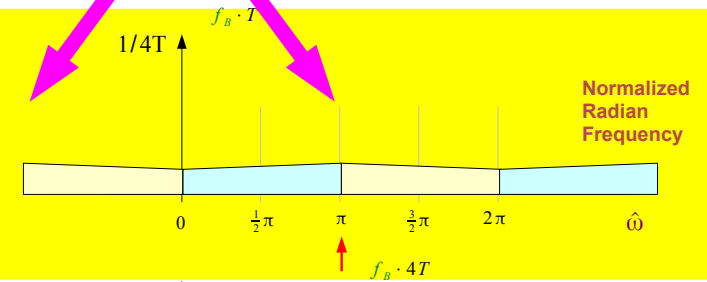
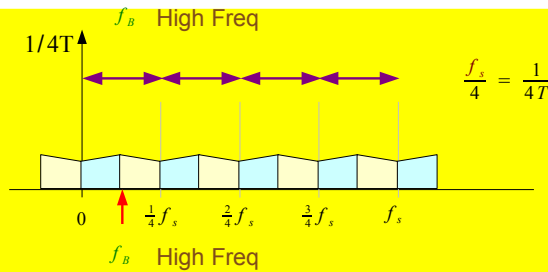
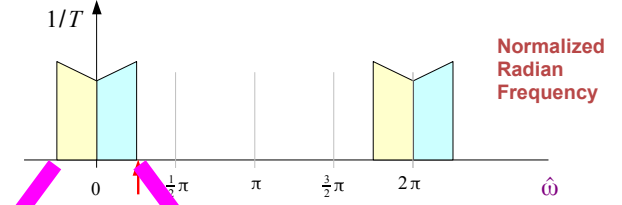
|| The Same Time Sequence



Linear Frequency



Normalized Radian Frequency



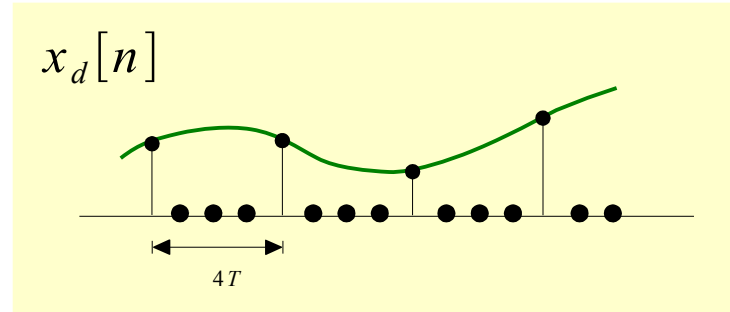
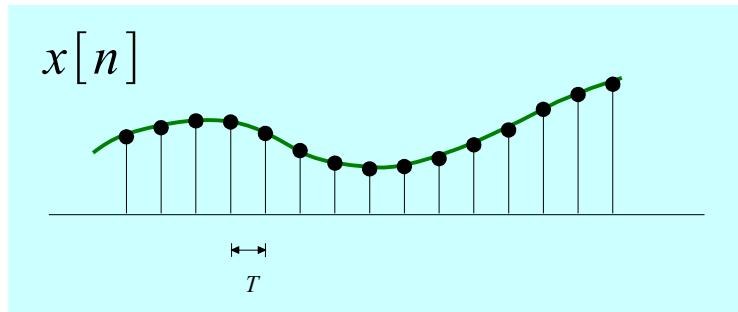
Coarse Sequence Spectrum

➔ Expanded

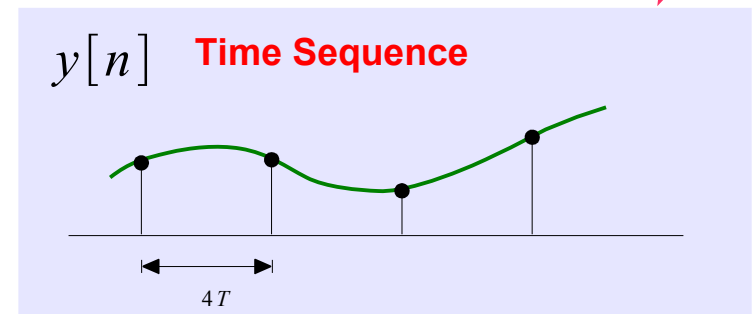
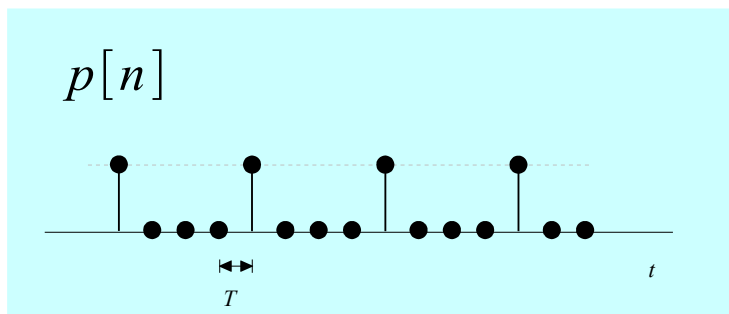
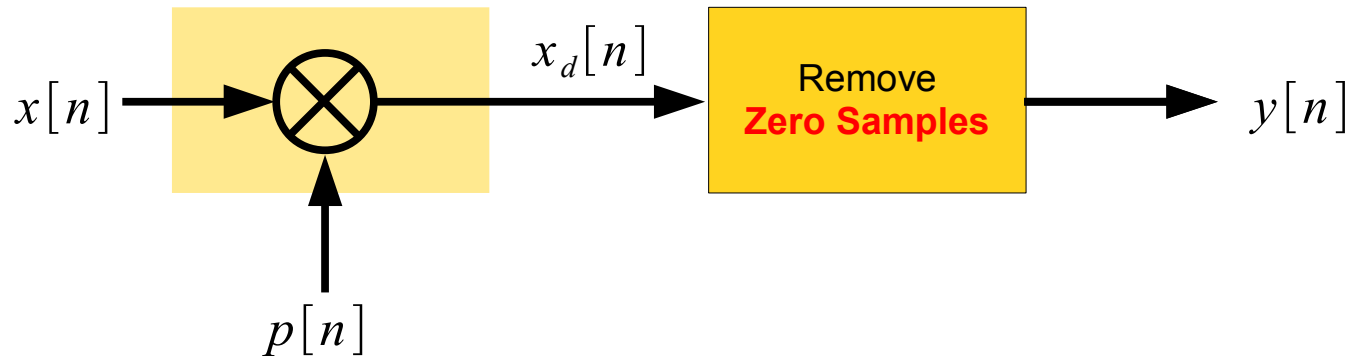
Normalized Radian Frequency



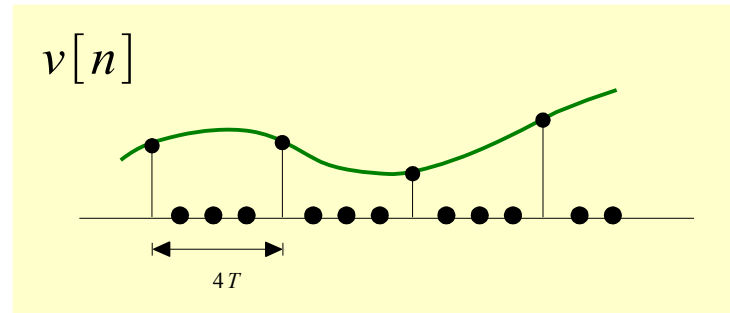
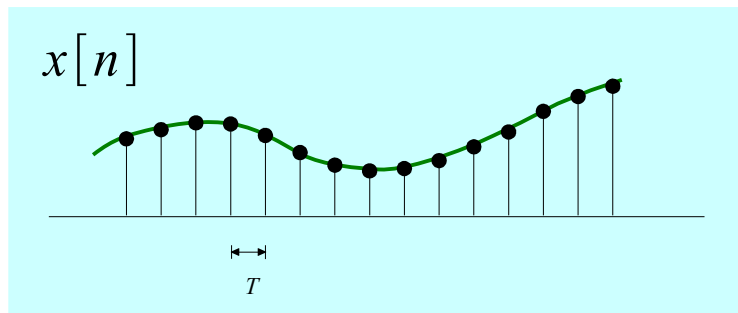
# Coarse Sequence Generation



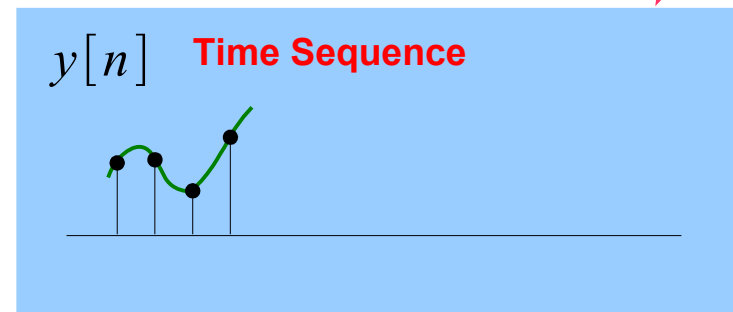
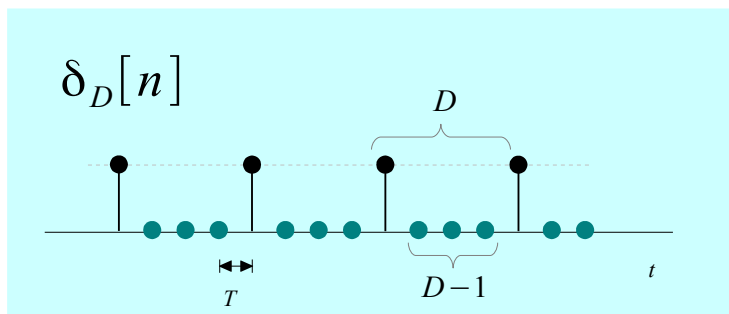
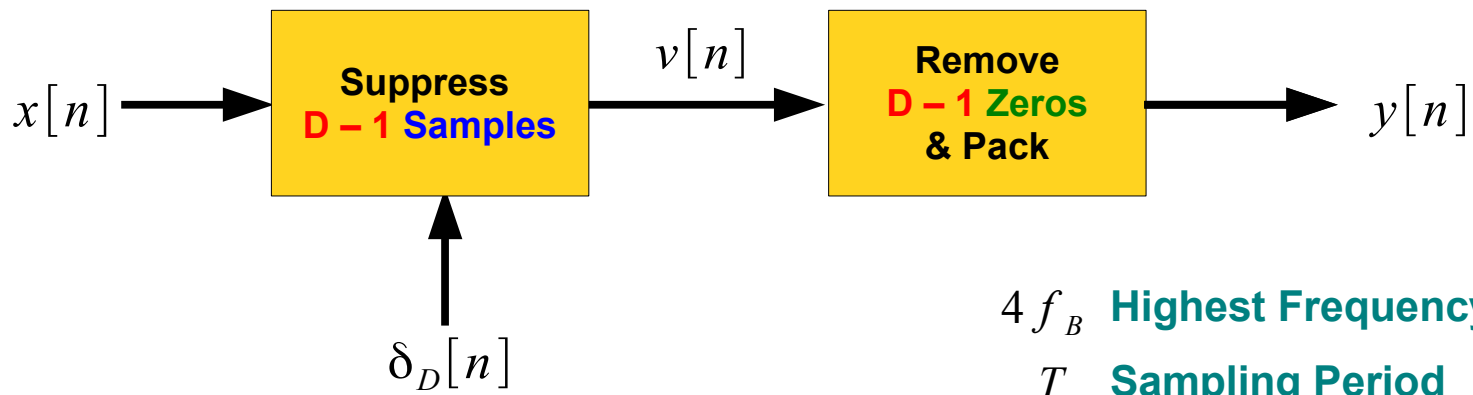
$T$  Sampling Period



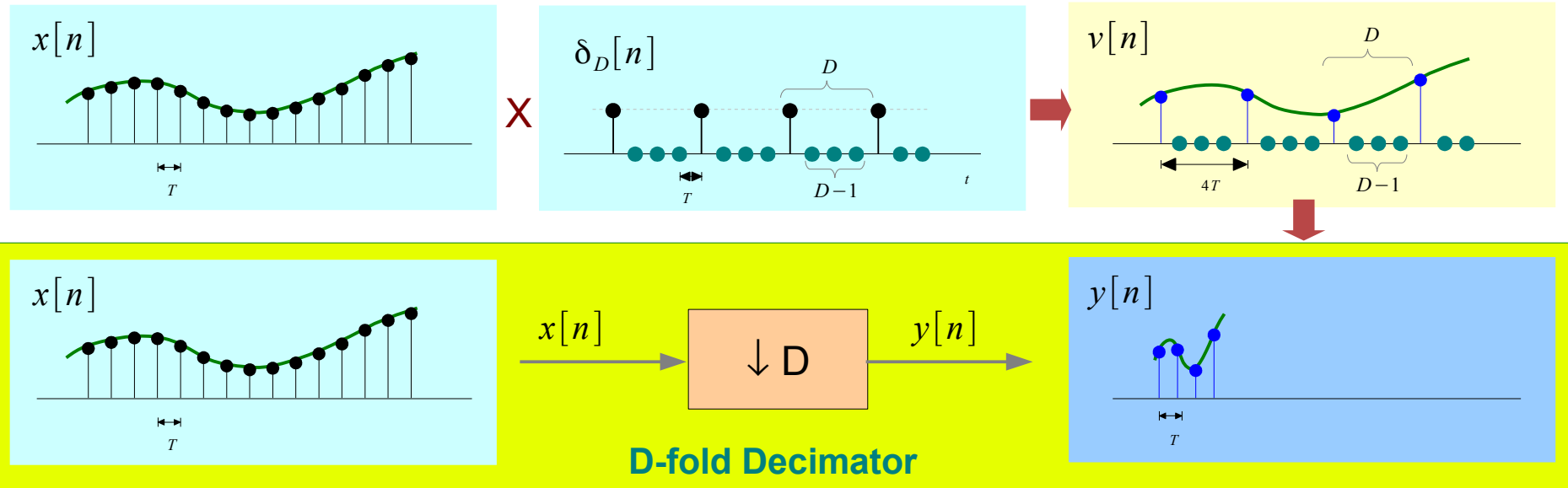
# Down Sampling in Two Steps



$T$  Sampling Period



# Down-Sampling Operator



$$y[n] = S_{1/D} x[n] = x[nD]$$

Decrease  
sampling  
frequency  
by a factor of  $1/D$

Increase  
sampling  
period  
by a factor of  $D$

$$D = 2$$

$$y[0] = x[0 \cdot 2]$$

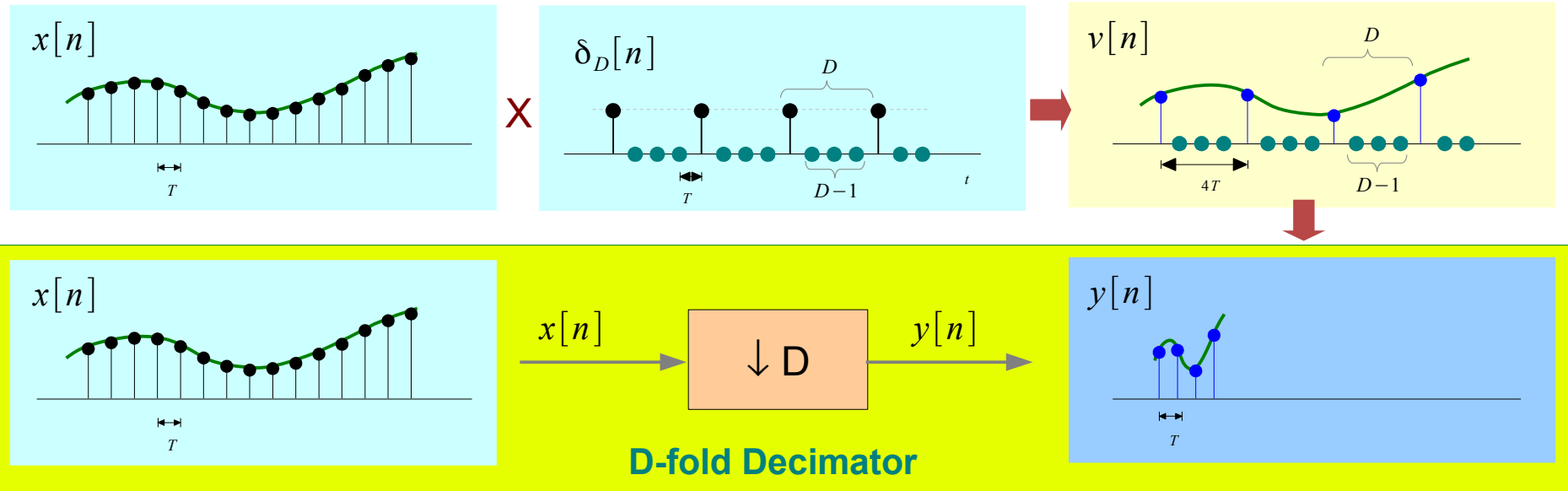
$$y[1] = x[1 \cdot 2]$$

$$y[2] = x[2 \cdot 2]$$

$$y[3] = x[3 \cdot 2]$$

...

# Down-Sampling Operator



$$y[n] = S_{1/D} x[n] = x[nD]$$

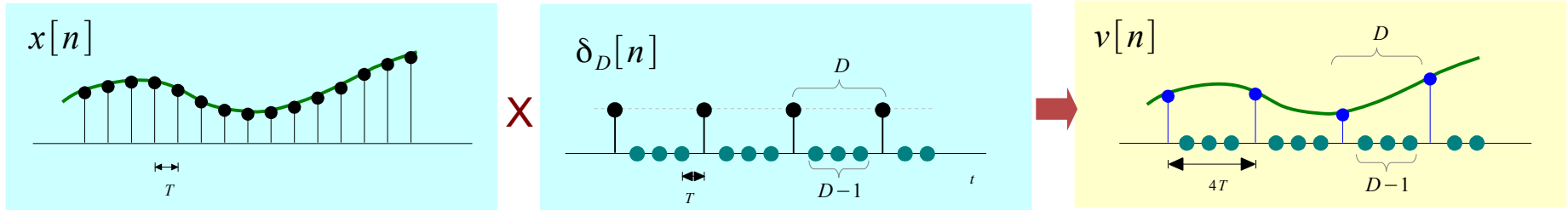
$$x[n] = e^{j\hat{\omega}n} \quad \rightarrow \quad y[n] = e^{j\hat{\omega}Dn}$$

$$-\pi \leq \hat{\omega} \leq +\pi \quad \quad -\pi \leq \hat{\omega}D \leq +\pi \quad \text{stretched}$$

$$-\frac{\pi}{D} \leq \hat{\omega}_1 \leq +\frac{\pi}{D} \quad \quad -\pi \leq \hat{\omega}_1 D \leq +\pi \quad \text{w/o aliasing}$$

$$\hat{\omega}_2 > +\frac{\pi}{D} \quad \quad \hat{\omega}_2 D > +\pi \quad \text{with aliasing}$$

# Suppressing D-1 Samples (1)



$$\delta_D[n] = \frac{1}{D} \sum_{k=0}^{D-1} e^{-j2\pi kn/D} = \begin{cases} \frac{1}{D} D = 1 & \text{if } \text{mod}(n, D) = 0 \\ \frac{1}{D} \frac{1 - e^{-j2\pi n}}{1 - e^{-j2\pi n/D}} = 0 & \text{otherwise} \end{cases} \quad (e^{-j2\pi k(n/D)} = e^{-j2\pi km} = 1)$$

$$v[n] = \delta_D[n] x[n] = \frac{1}{D} \sum_{k=0}^{D-1} x[n] e^{-j2\pi kn/D}$$

$$\hat{\omega}_k = 2\pi k/D$$

$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

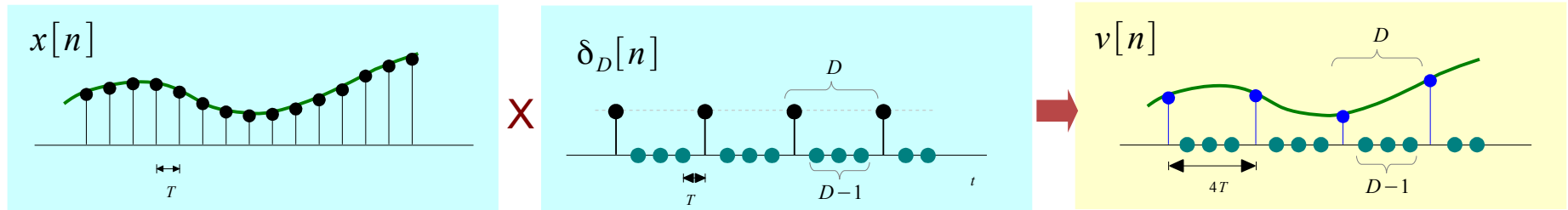
$$Z\{x[n] e^{-j\hat{\omega}_k n}\} = \sum_{n=-\infty}^{+\infty} x[n] (e^{j\hat{\omega}_k} \cdot z)^{-n} = X(e^{j\hat{\omega}_k} \cdot z)$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

$$X(e^{-j2\pi k/D} \cdot z) = X(e^{-j\omega_k} e^{j\hat{\omega}}) = X(e^{j(\hat{\omega} - \omega_k)})$$

$$z = e^{j\hat{\omega}}$$

# Removing D-1 Samples and Packing (2)



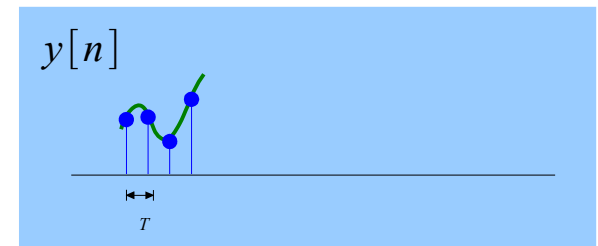
$$V(z) = \dots + v[0]z^0 + 0 + \dots + 0 + v[D]z^{-D} + 0 + \dots + 0 + v[2D]z^{-2D} + \dots$$

$$V(z) = \sum_{n=-\infty}^{+\infty} v[n]z^{-n} = \sum_{m=-\infty}^{+\infty} v[mD]z^{-mD} = F(z^D) \quad \Leftrightarrow \quad V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

$$Y(z) = \dots + v[0]z^0 + v[D]z^{-1} + v[2D]z^{-2} + v[3D]z^{-3} + \dots$$

$$= \dots + y[0]z^0 + y[1]z^{-1} + y[2]z^{-2} + y[3]z^{-3} + \dots$$

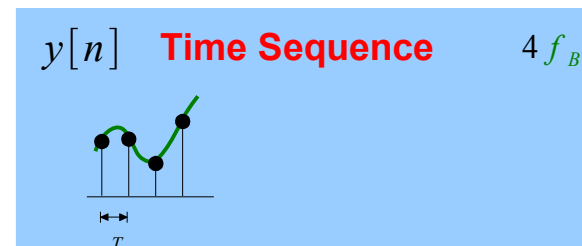
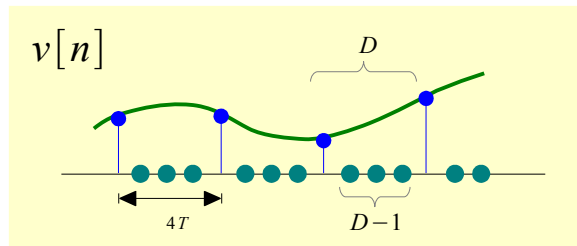
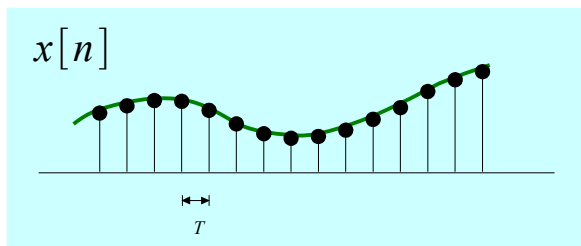
$$V(z^{1/D}) = \sum_{n=-\infty}^{+\infty} v[n]z^{-n/D} = \sum_{m=-\infty}^{+\infty} v[mD]z^{-m}$$



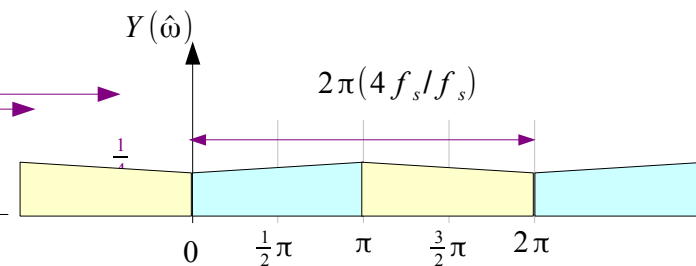
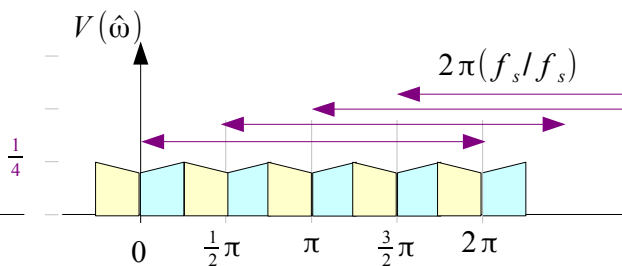
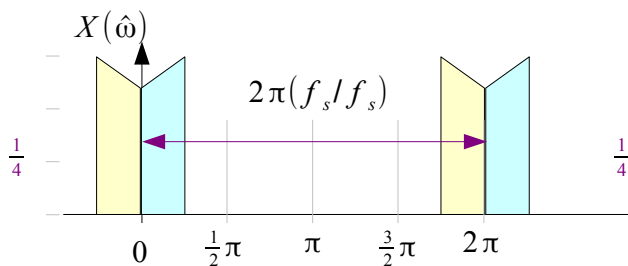
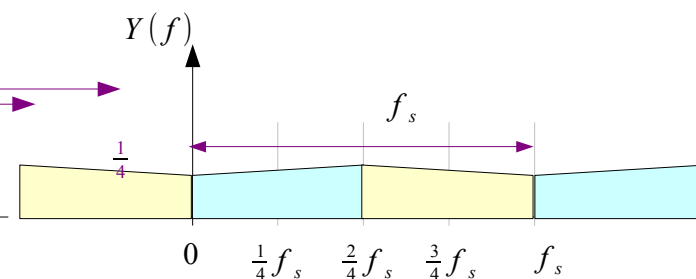
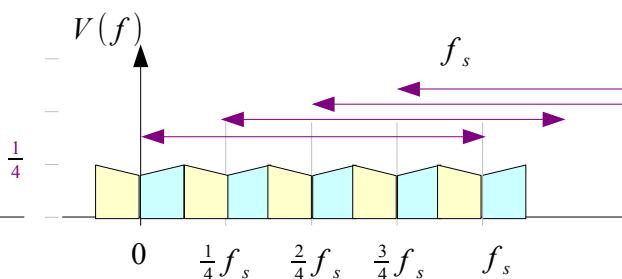
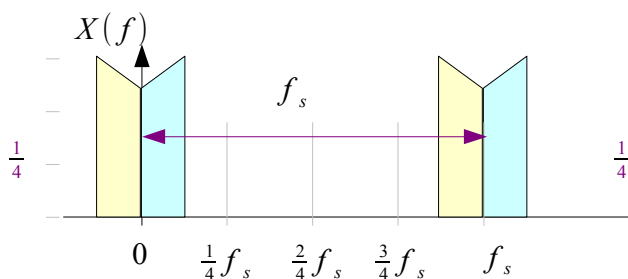
$$Y(z) = V(z^{1/D}) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D})$$

$$Y(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(\frac{\hat{\omega}}{D} - k \frac{2\pi}{D}\right)$$

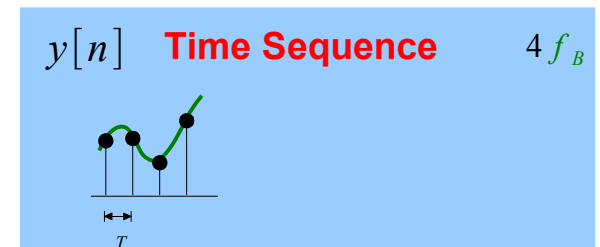
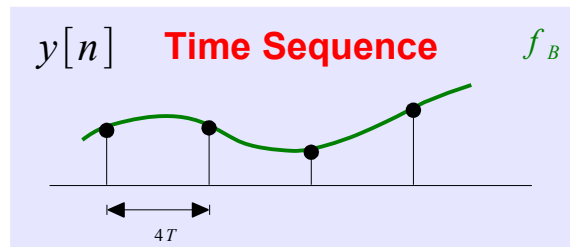
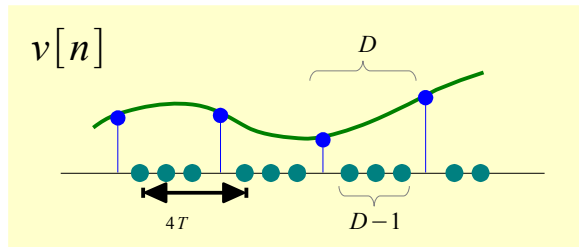
# Down-Sampling Spectra (1)



$$\begin{aligned}
 X(z) & & V(z) &= \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z) & Y(z) &= \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D}) \\
 X(\hat{\omega}) & & V(\hat{\omega}) &= \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D) & Y(\hat{\omega}) &= \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)
 \end{aligned}$$



# Z-Transform & Removing D-1 Zeros (2)

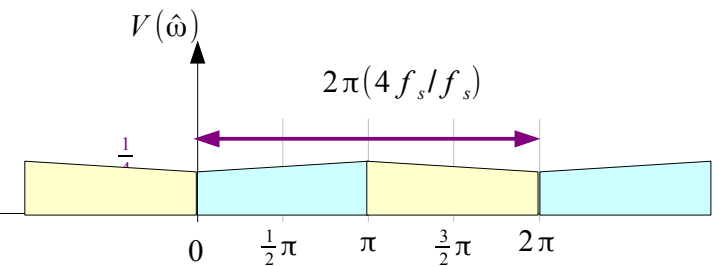
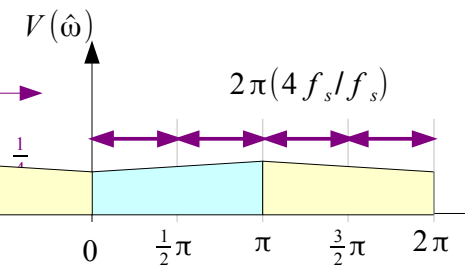
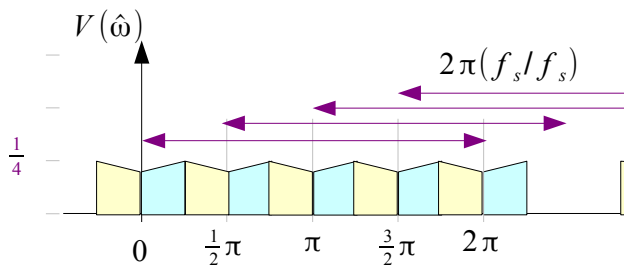
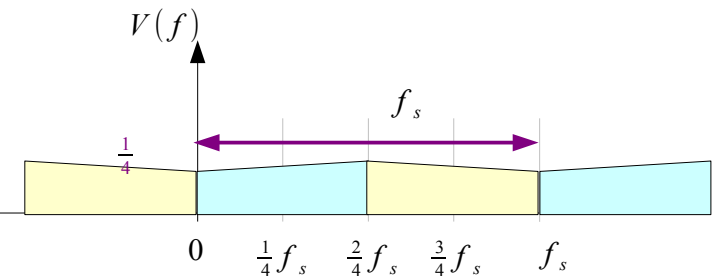
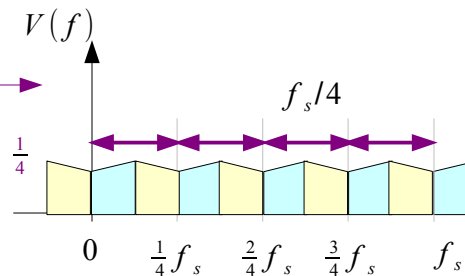
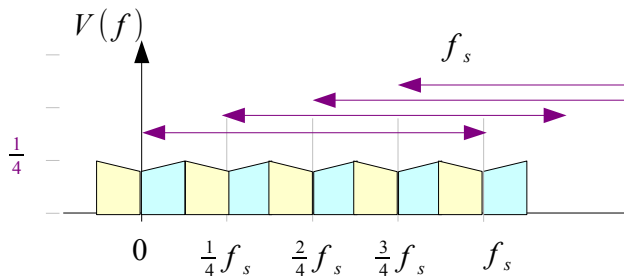


$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

$$Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D})$$

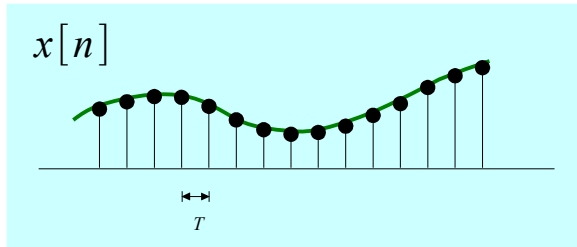
$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)$$

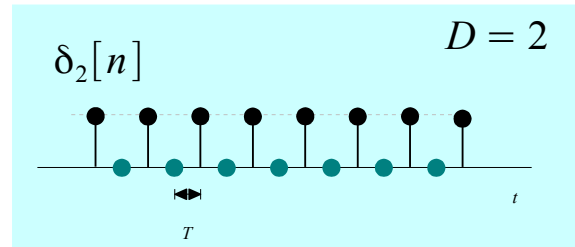
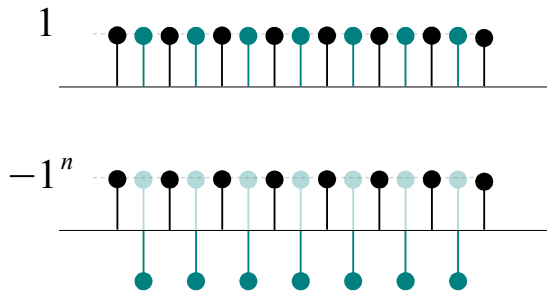




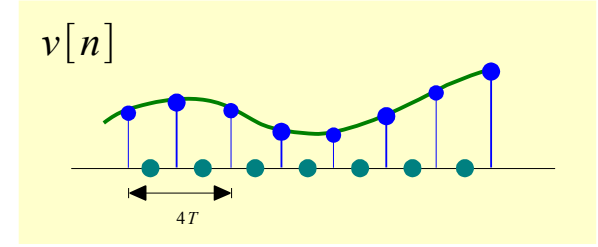
# Example When D=2 (1)



$$x[n] = e^{j\omega n}$$



$$\begin{aligned} \delta_2[n] &= \frac{1}{2}(1 + (-1)^n) \\ &= \frac{1}{2}(1 + e^{-j\pi n}) \\ &\quad (e^{-j\pi} = -1) \end{aligned}$$



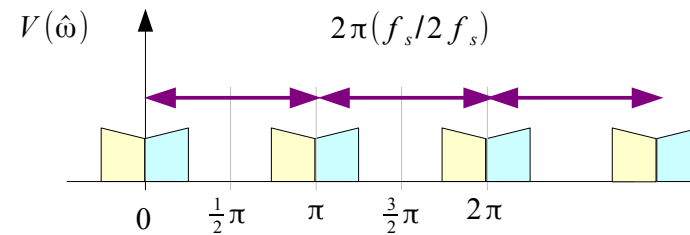
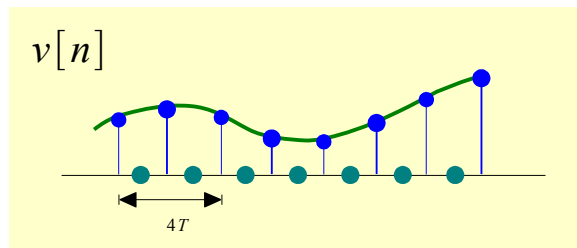
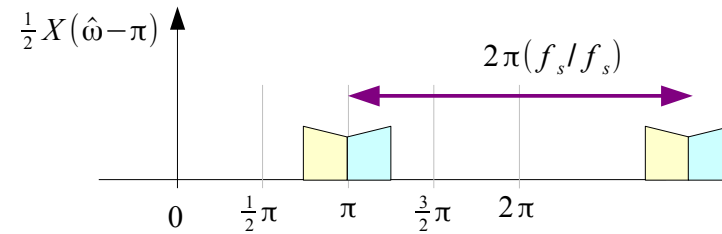
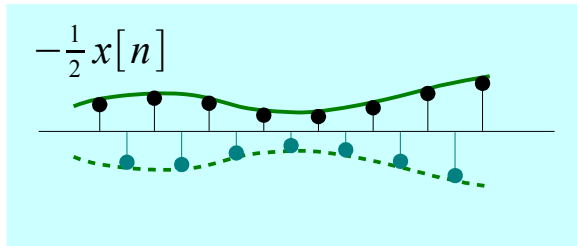
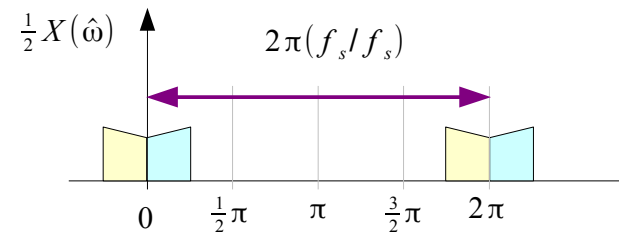
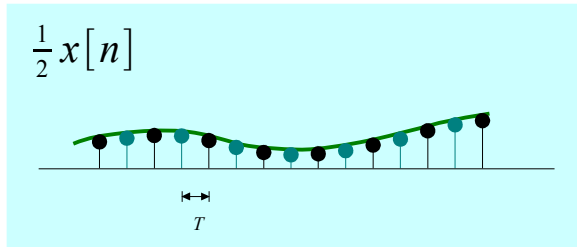
$$\begin{aligned} v[n] &= \frac{1}{2}x[n] + \frac{1}{2}e^{-j\pi n}x[n] \\ &= \frac{1}{2}e^{j\omega n} + \frac{1}{2}e^{-j\pi n}e^{j\omega n} \\ &= \frac{1}{2}e^{j\omega n} + \frac{1}{2}e^{+j(\omega-\pi)n} \end{aligned}$$

$$\begin{aligned} V(z) &= \frac{1}{2} \sum_{n=-\infty}^{+\infty} (x[n]z^{-n} + x[n](-z)^{-n}) \\ &= \frac{1}{2}X(z) + \frac{1}{2}X(-z) \end{aligned}$$

$$V(e^{j\hat{\omega}}) = \frac{1}{2}X(e^{j\hat{\omega}}) + \frac{1}{2}X(e^{-j\pi}e^{j\hat{\omega}})$$

$$V(\hat{\omega}) = \frac{1}{2}X(\hat{\omega}) + \frac{1}{2}X(\hat{\omega} - \pi)$$

# Example When D=2 (2)



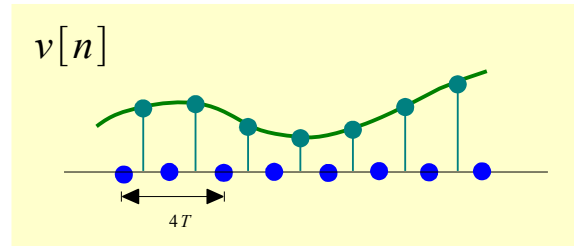
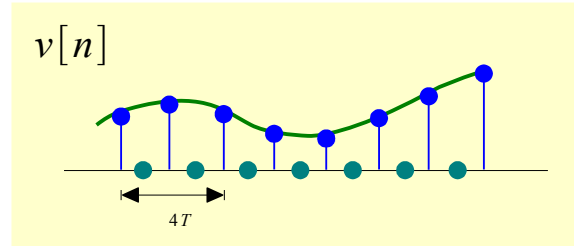
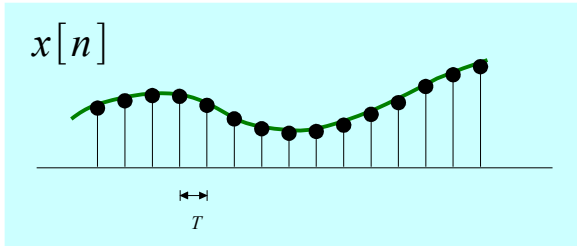
$$v[n] = \frac{1}{2}x[n] + \frac{1}{2}e^{-j\pi n}x[n]$$

$$V(z) = \frac{1}{2}X(z) + \frac{1}{2}X(-z)$$

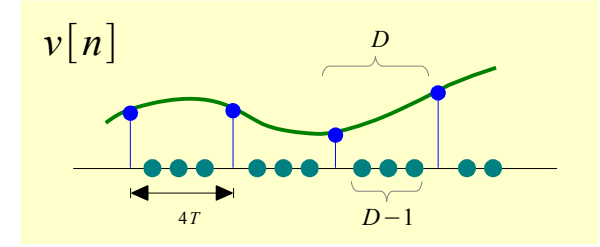
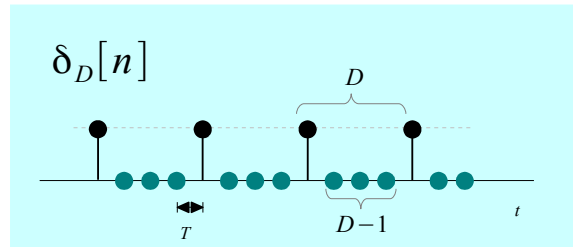
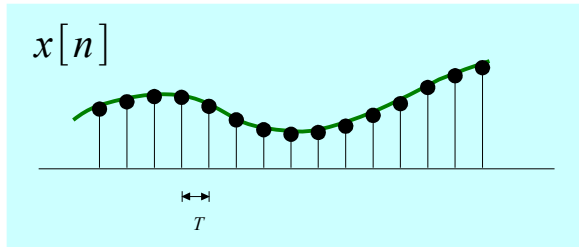
$$V(e^{j\hat{\omega}}) = \frac{1}{2}X(e^{j\hat{\omega}}) + \frac{1}{2}X(e^{-j\pi}e^{j\hat{\omega}})$$

$$V(\hat{\omega}) = \frac{1}{2}X(\hat{\omega}) + \frac{1}{2}X(\hat{\omega} - \pi)$$

# Example When $D=2$ (3)



# Z-Transform & Suppressing D-1 Samples (1)



$$\delta_D[n] = \frac{1}{D} \sum_{k=0}^{D-1} e^{-j2\pi kn/D} = \begin{cases} \frac{1}{D} D = 1 & \text{if } \text{mod}(n, D) = 0 \\ \frac{1}{D} \frac{1 - e^{-j2\pi n}}{1 - e^{-j2\pi n/D}} = 0 & \text{otherwise} \end{cases} \quad v[n] = \delta_D[n]x[n]$$

$$v[n] = \delta_D[n]x[n] = \frac{1}{D} \sum_{k=0}^{D-1} x[n] e^{-j2\pi kn/D} \quad Z\{x[n] e^{-j\hat{\omega}_k n}\} = \sum_{n=-\infty}^{+\infty} x[n] (e^{j\hat{\omega}_k \cdot} z)^{-n} = X(e^{j\hat{\omega}_k \cdot} z)$$

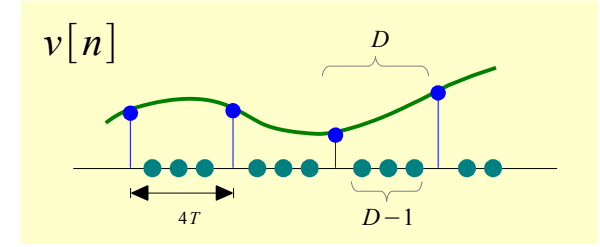
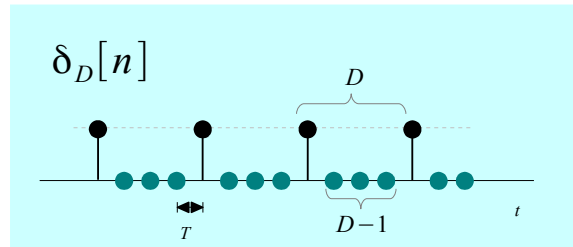
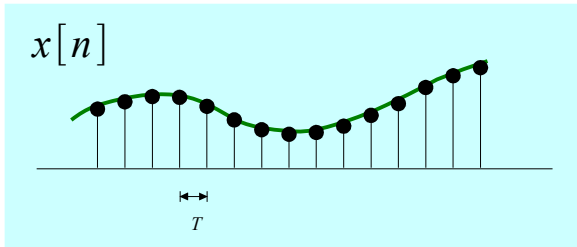
$$\hat{\omega}_k = 2\pi k/D$$

$$e^{-j2\pi kn/D} \cdot z = e^{-j2\pi k/D} e^{j\hat{\omega}_k n} = e^{j(\hat{\omega}_k - 2\pi k/D)n}$$

$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

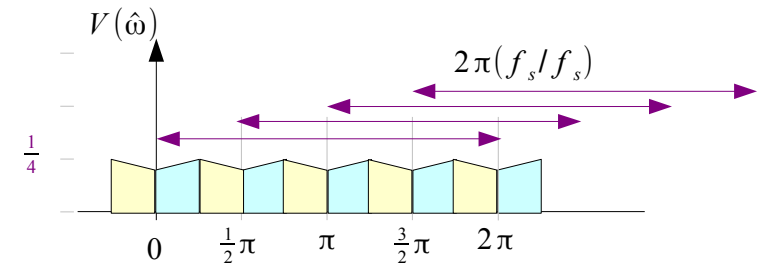
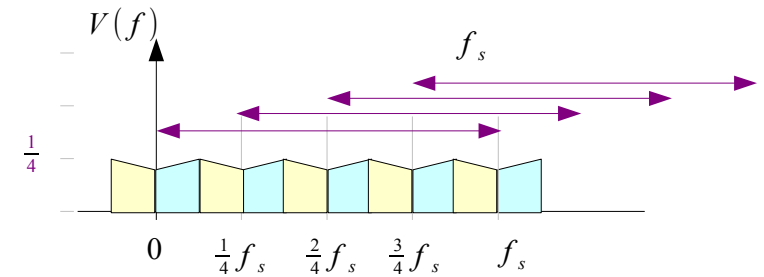
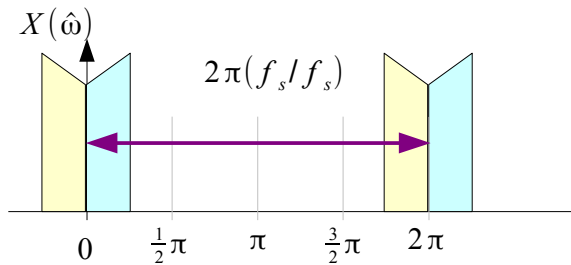
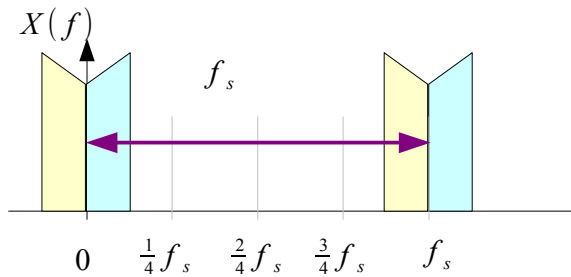
$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

# Z-Transform & Suppressing D-1 Samples (2)

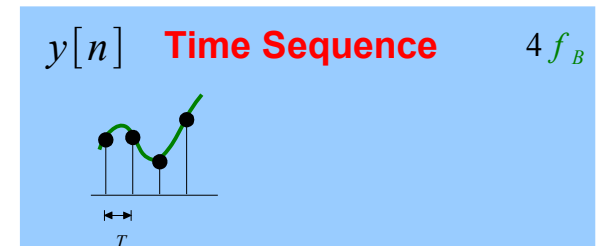
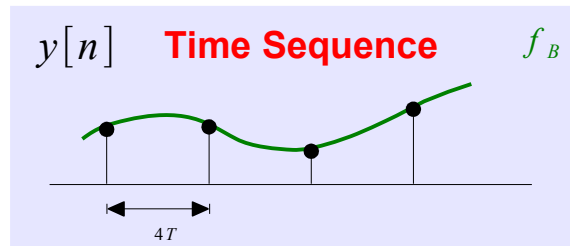
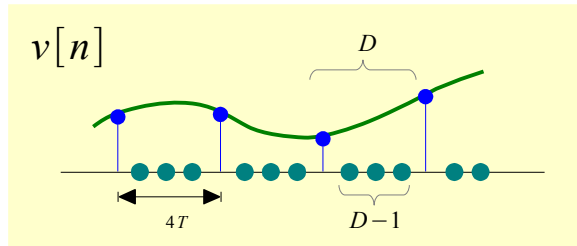


$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$



# Z-Transform & Removing D-1 Zeros (1)



$$\begin{aligned}
 Y(z) &= \cdots + \underline{y[0]} + \underline{y[1]}z^{-1} + \underline{y[2]}z^{-2} + \cdots + \underline{y[n]}z^{-n} + \cdots \\
 &= \cdots + \underline{v[0]} + \underline{v[D]}z^{-1} + \underline{v[2D]}z^{-2} + \cdots + \underline{v[nD]}z^{-n} + \cdots
 \end{aligned}$$

$$Y(z) = V(z^{1/D})$$

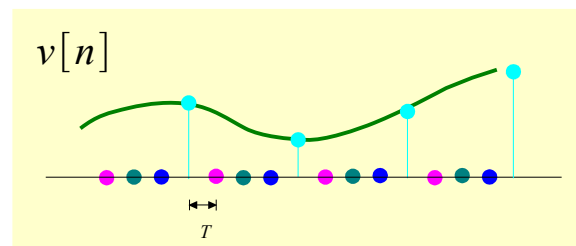
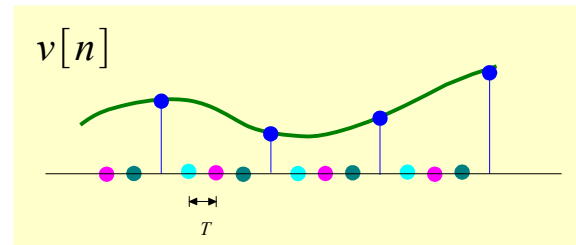
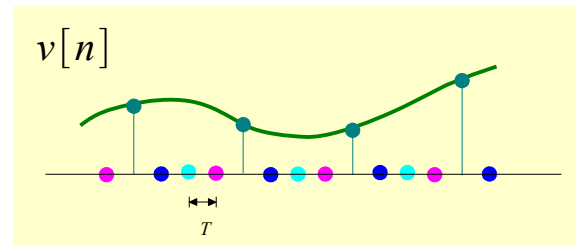
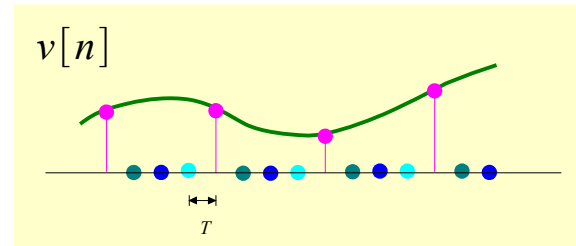
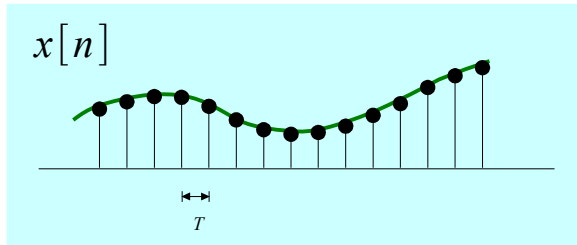
$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

$$Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D})$$

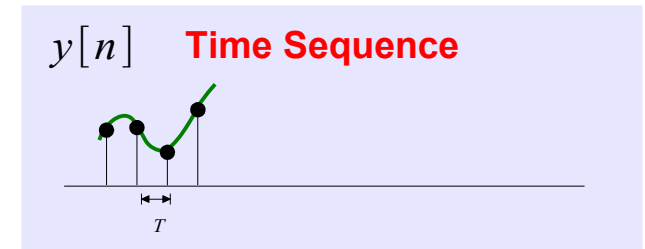
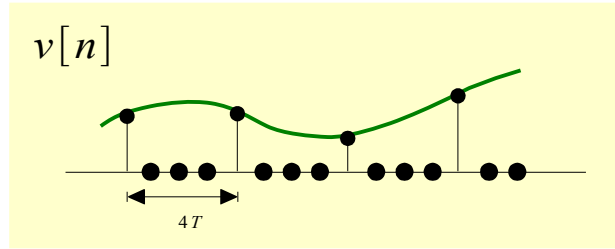
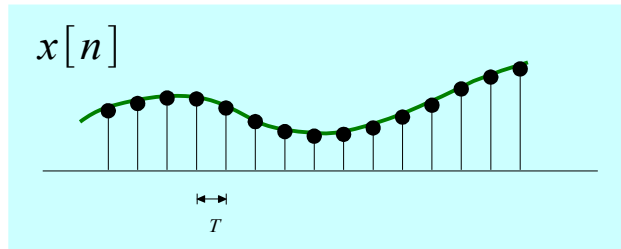
$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)$$

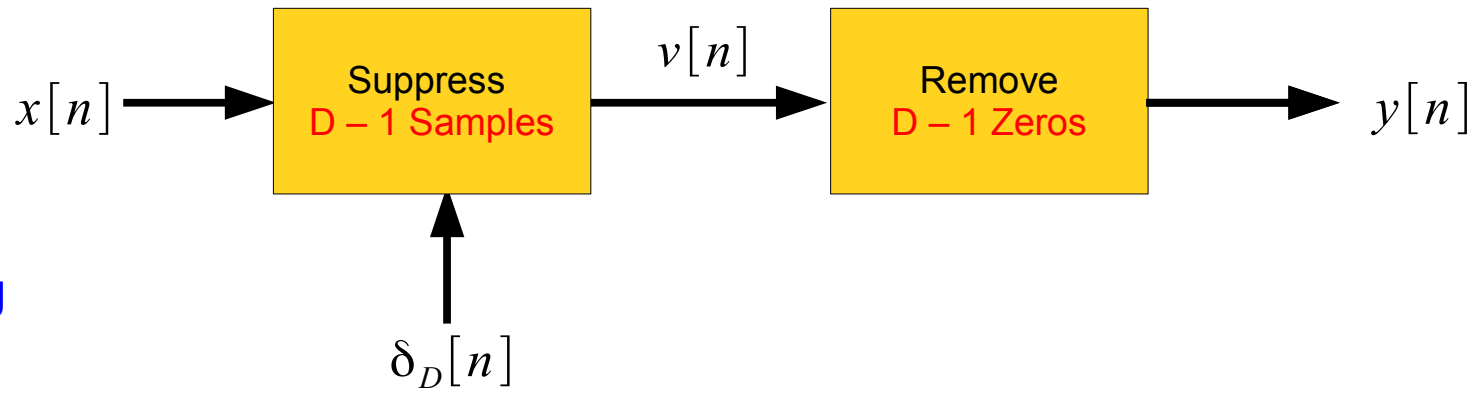
# Example When $D=2$ (3)



# Down Sampling



$T$  Sampling Period



**Ideal Sampling**

**Amplitude Axis Scaling**  
 $1 / D$

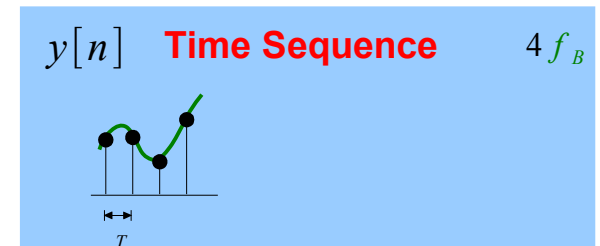
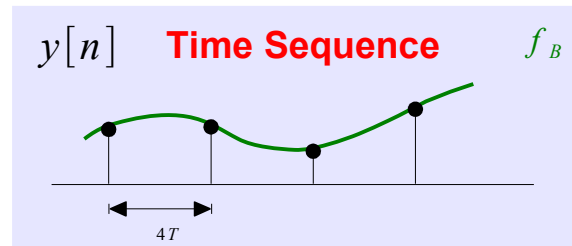
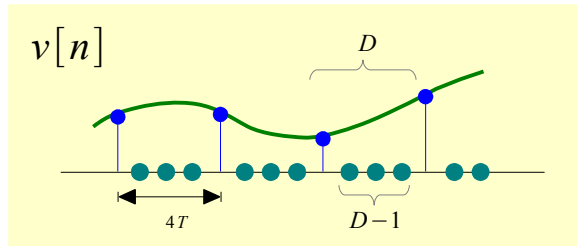
**Time Axis Scaling**  
 $1 / D$

**Spectrum Replication**  
 $f_s / D$

**Stretching Spectrum**  
 $\omega / D$



# Z-Transform & Removing D-1 Zeros (1)



$$\begin{aligned}
 Y(z) &= \cdots + \underline{y[0]} + \underline{y[1]}z^{-1} + \underline{y[2]}z^{-2} + \cdots + \underline{y[n]}z^{-n} + \cdots \\
 &= \cdots + \underline{v[0]} + \underline{v[D]}z^{-1} + \underline{v[2D]}z^{-2} + \cdots + \underline{v[nD]}z^{-n} + \cdots
 \end{aligned}$$

$$Y(z) = V(z^{1/D})$$

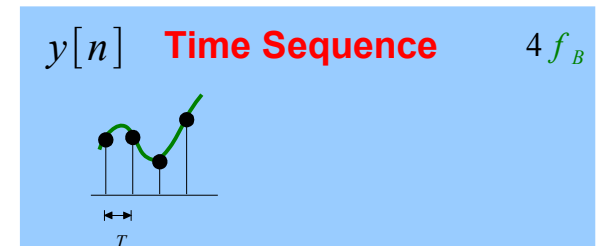
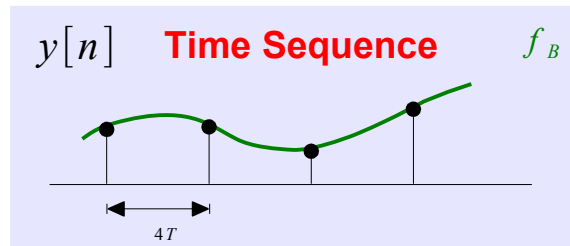
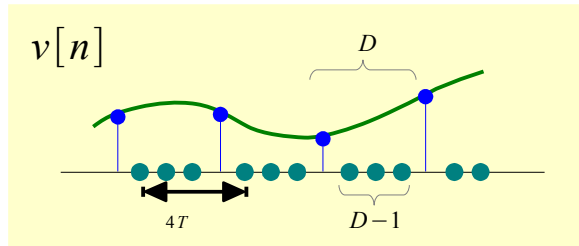
$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

$$Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D})$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)$$

# Z-Transform & Removing D-1 Zeros (2)

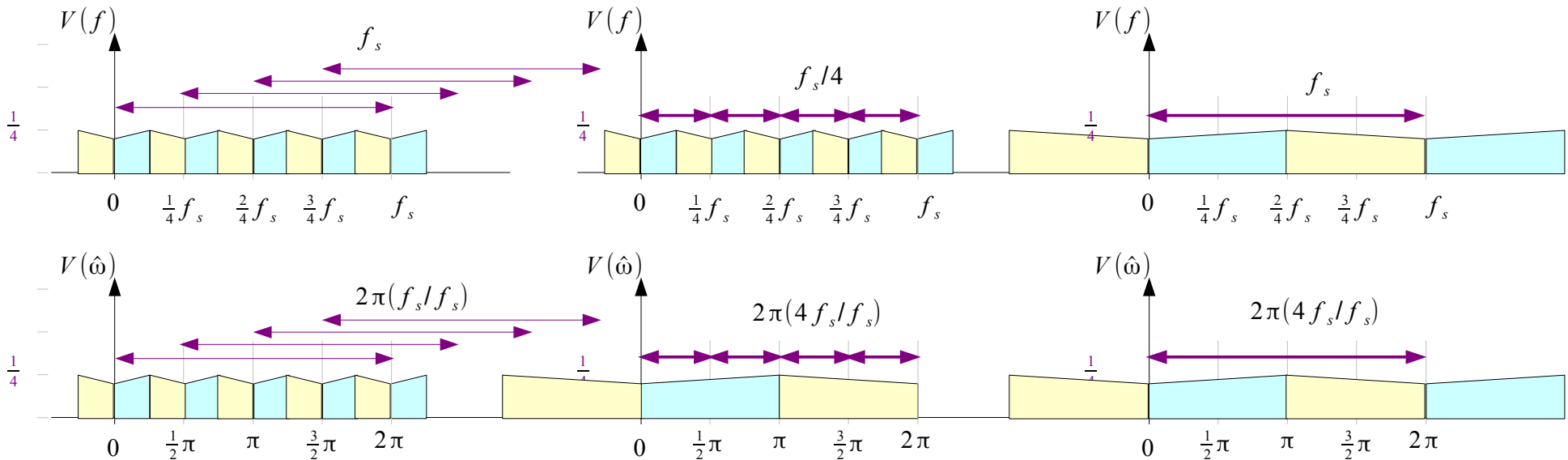


$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

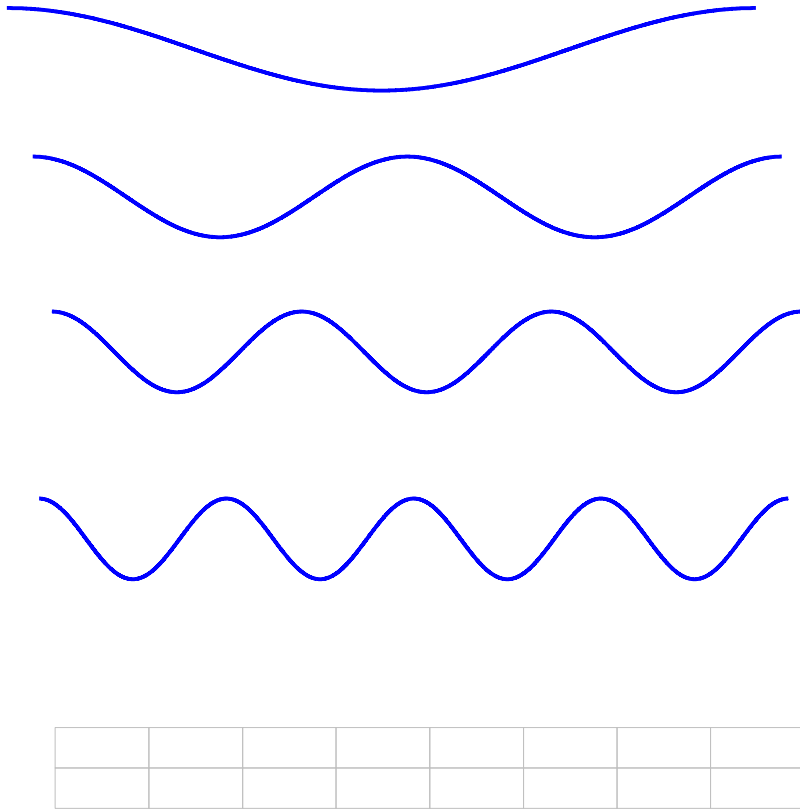
$$Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D})$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

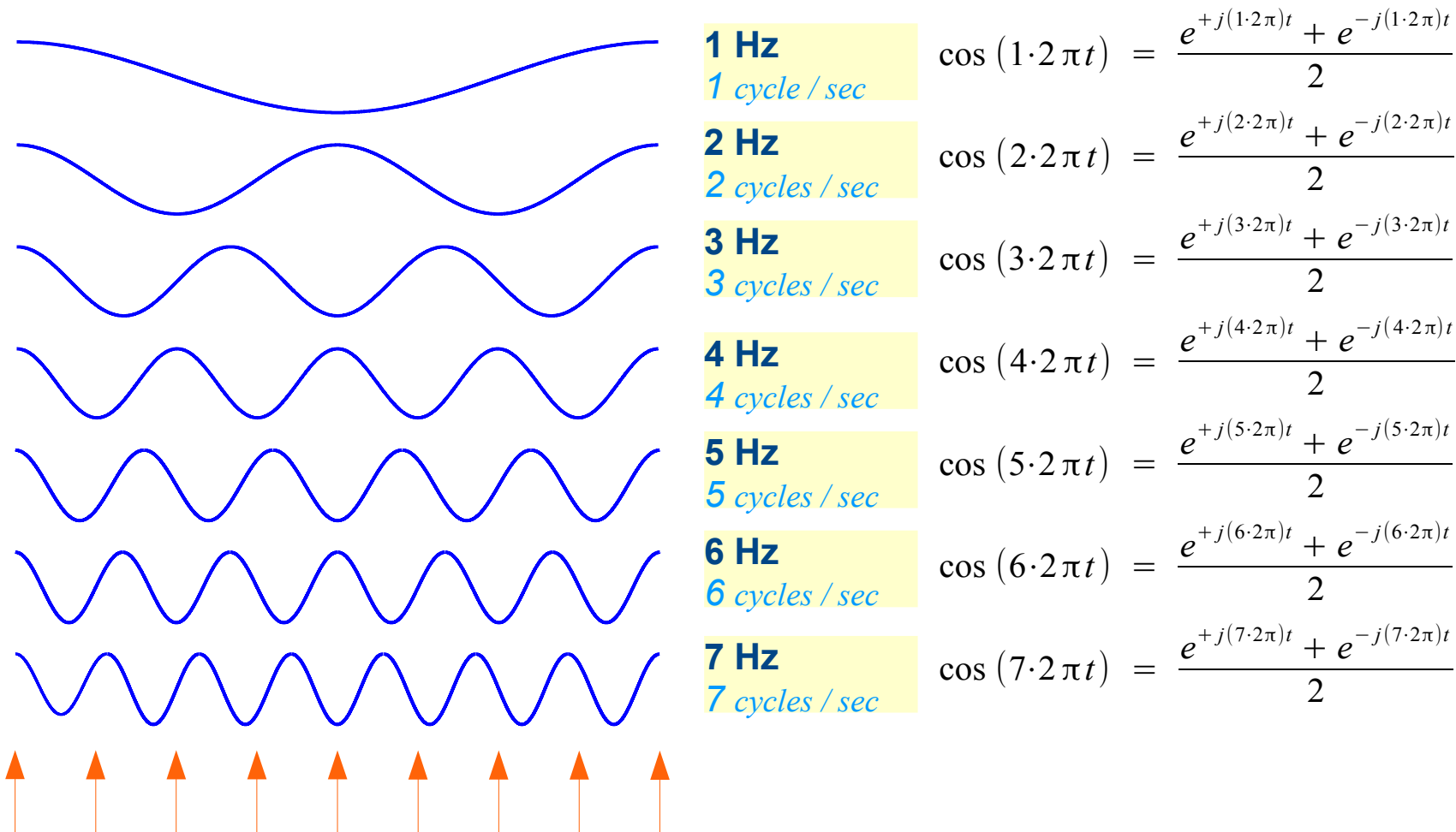
$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)$$



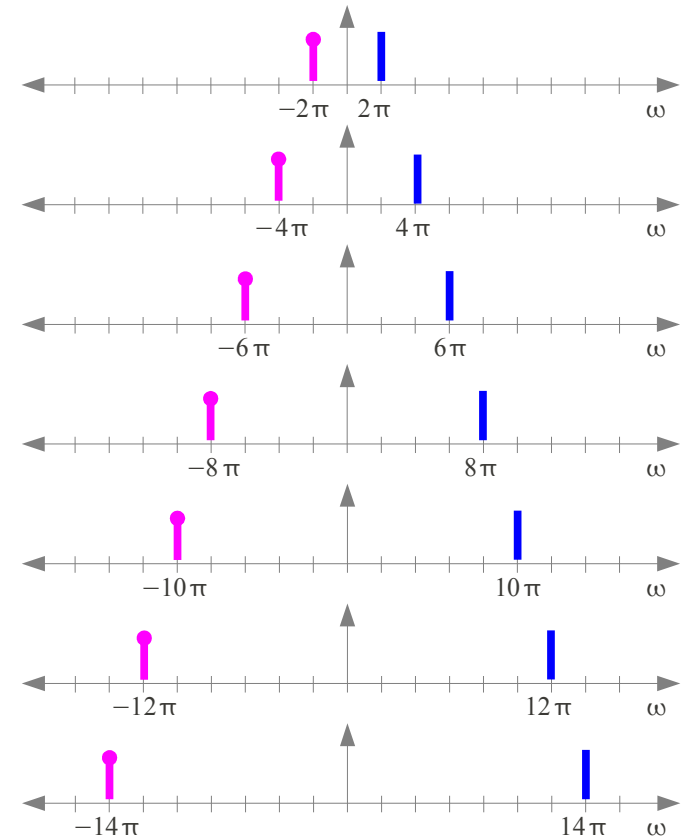
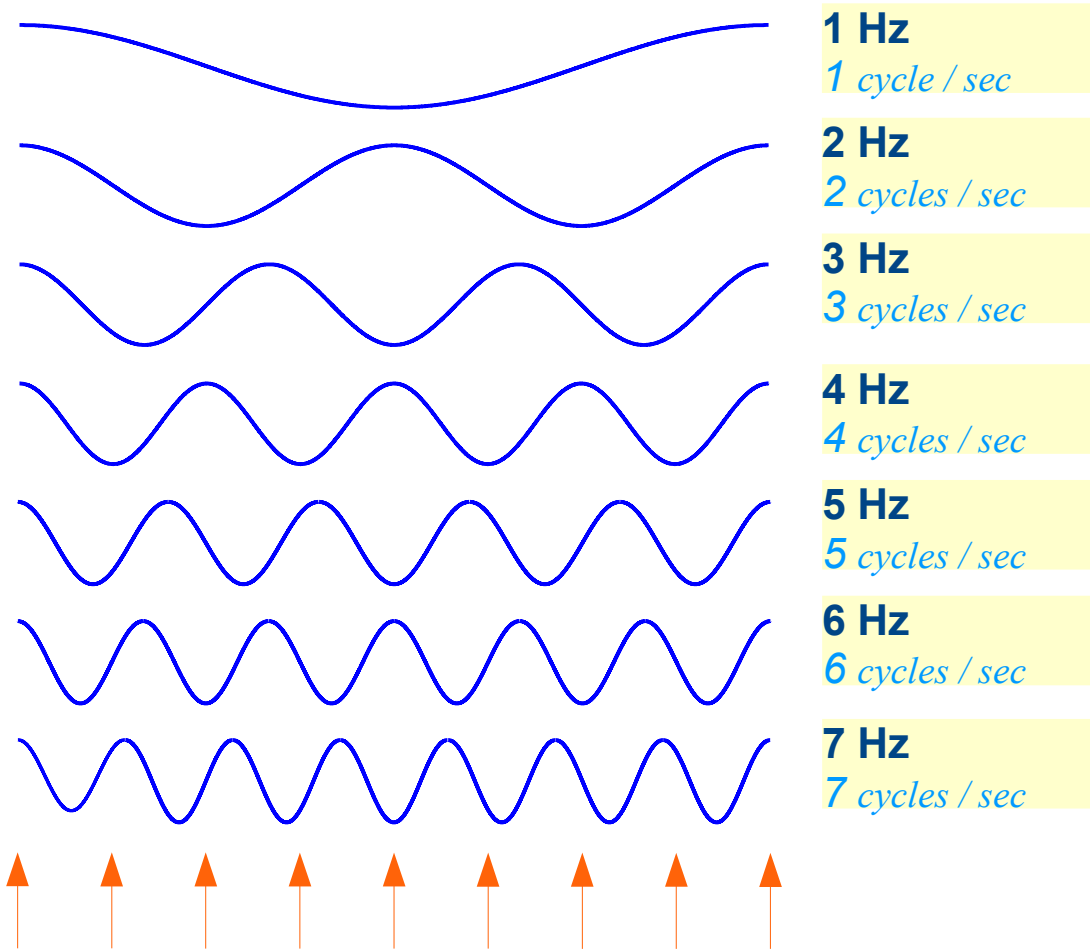
# Measuring Rotation Rate



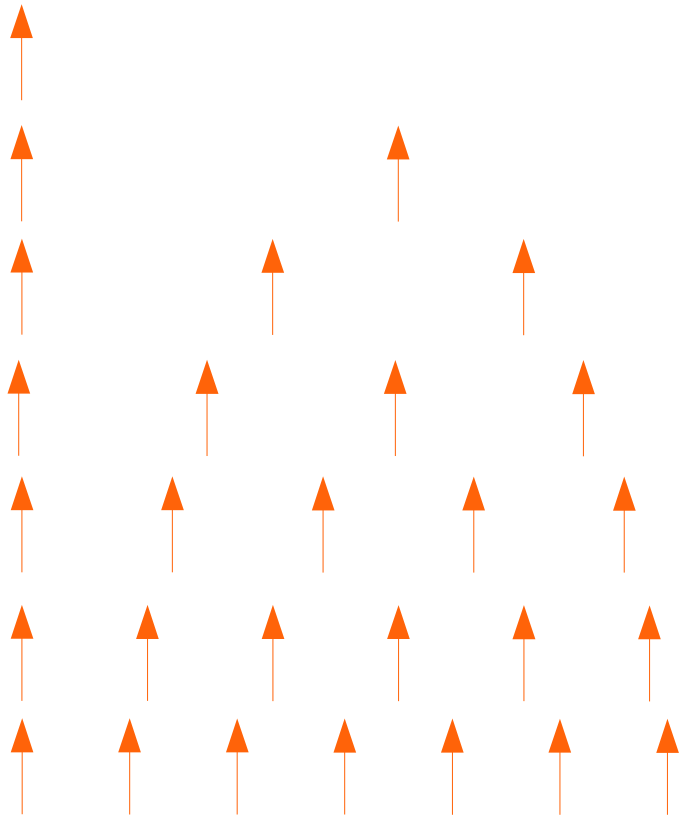
# Signals with Harmonic Frequencies (1)



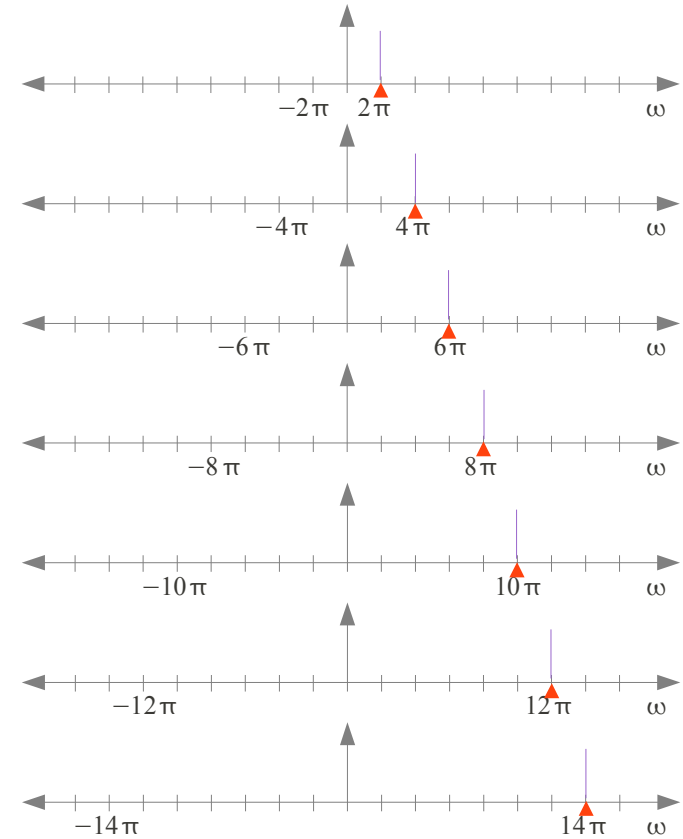
# Signals with Harmonic Frequencies (2)



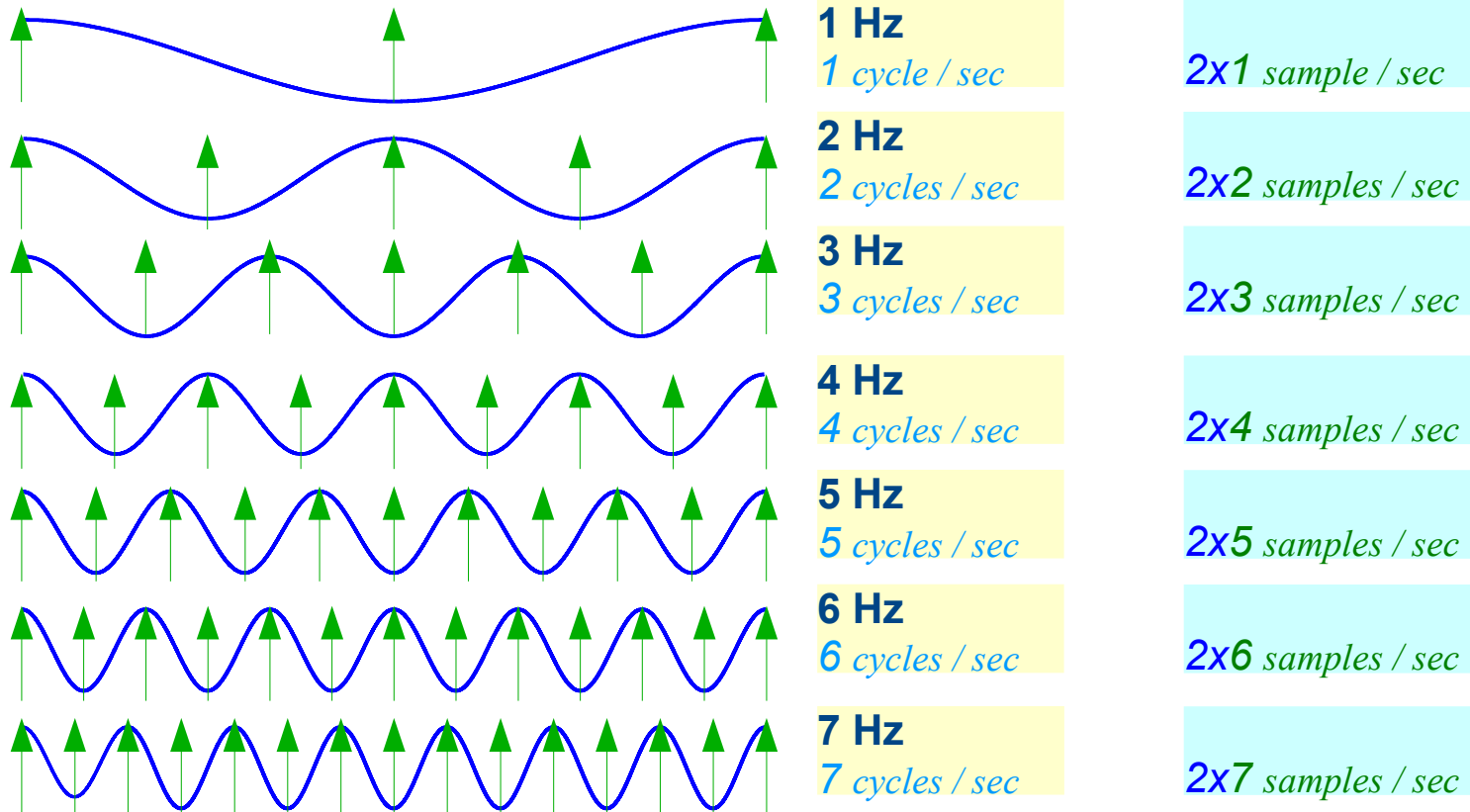
# Sampling Frequency



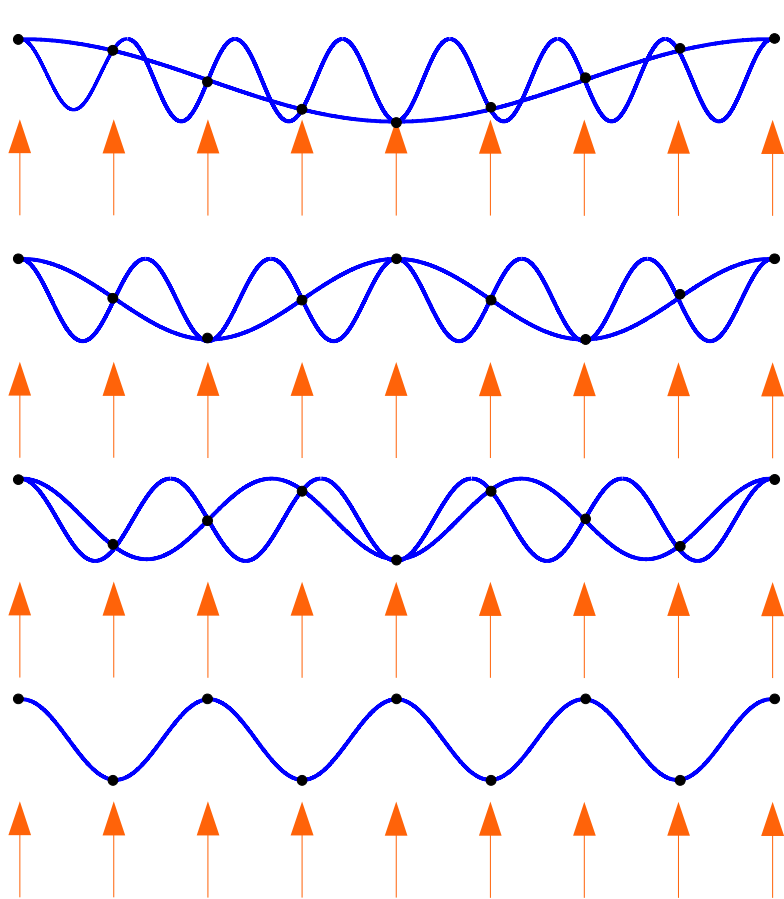
- 1 Hz  
*1 sample / sec*
- 2 Hz  
*2 samples / sec*
- 3 Hz  
*3 samples / sec*
- 4 Hz  
*4 samples / sec*
- 5 Hz  
*5 samples / sec*
- 6 Hz  
*6 samples / sec*
- 7 Hz  
*7 samples / sec*



# Nyquist Frequency



# Aliasing



1 Hz  
7 Hz

2x4 samples / sec

2 Hz  
6 Hz

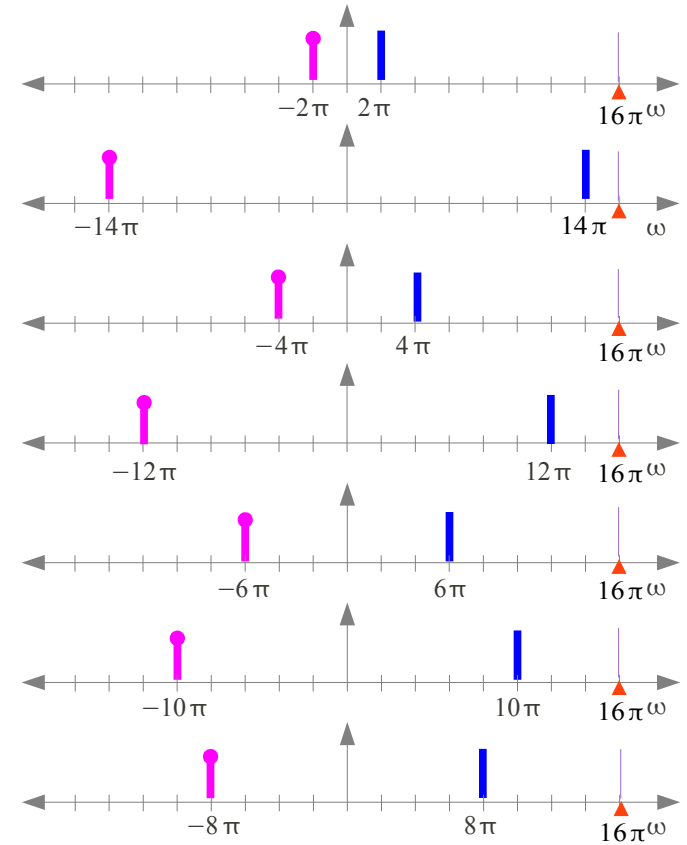
2x4 samples / sec

3 Hz  
5 Hz

2x4 samples / sec

4 Hz

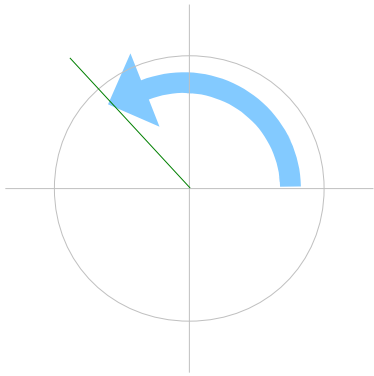
2x4 samples / sec





# Sampling

$$\omega_s = 2\pi f_s \text{ (rad/sec)}$$



$$\omega_1 = 2\pi f_1$$

$$\omega_1 = \frac{\omega_s}{2} \text{ (rad/sec)}$$

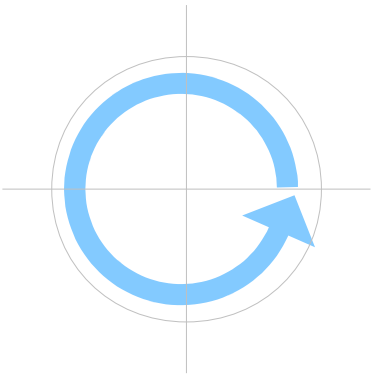
$$f_1 = \frac{f_s}{2} \text{ (rad/sec)}$$

$$\omega_2 = 2\pi f_2$$

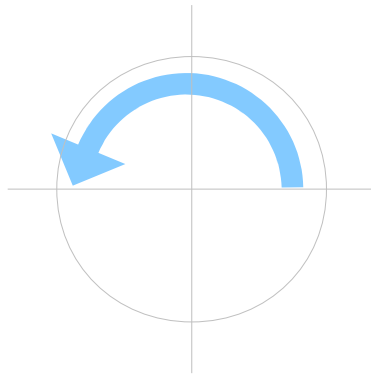
$$\omega_2 = -\frac{\omega_s}{2} \text{ (rad/sec)}$$

$$f_2 = -\frac{f_s}{2} \text{ (rad/sec)}$$

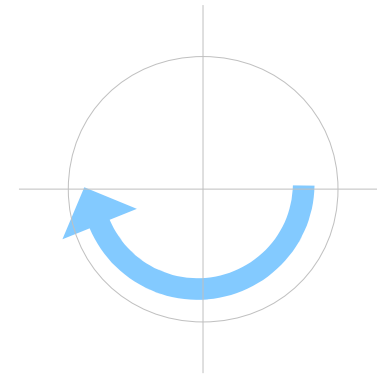
$$2\pi \text{ (rad)} / T_s \text{ (sec)}$$



$$\pi \text{ (rad)} / T_s \text{ (sec)}$$

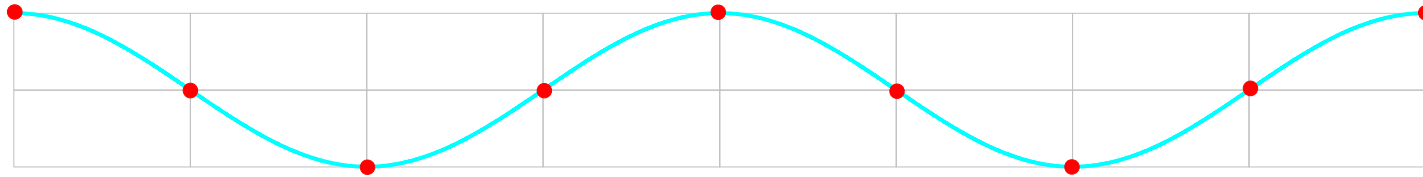


$$-\pi \text{ (rad)} / T_s \text{ (sec)}$$

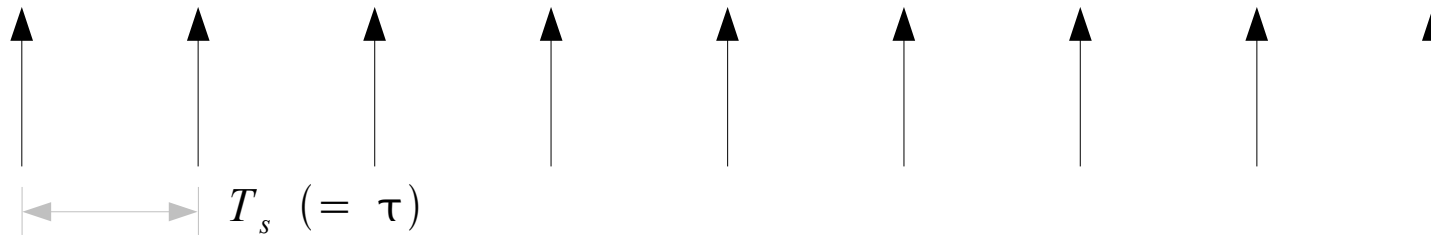


# Sampling

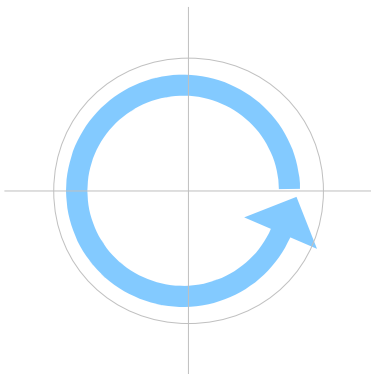
$$\omega_1 = 2\pi f_1 \text{ (rad/sec)}$$



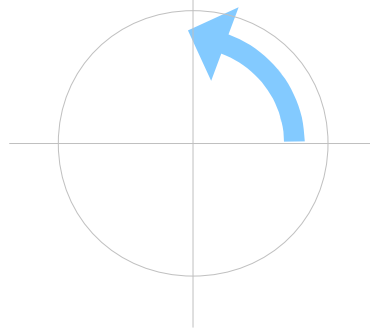
$$\omega_s = 2\pi f_s \text{ (rad/sec)}$$



$$2\pi \text{ (rad)} / T_s \text{ (sec)}$$



$$\frac{\pi}{2} \text{ (rad)} / T_s \text{ (sec)}$$



For the period of  $T_s$   
Angular displacement  $\frac{\pi}{2}$  (rad)

$$\begin{aligned} \hat{\omega} &= \omega \cdot T_s \text{ (rad)} \\ &= 2\pi f_1 \cdot T_s \text{ (rad)} \\ &= 2\pi \frac{f_s}{4} \cdot T_s \text{ (rad)} \\ &= \frac{\pi}{2} \text{ (rad)} \end{aligned}$$

# Angular Frequencies in Sampling

## continuous-time signals

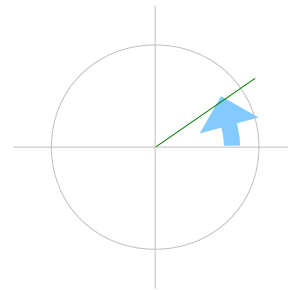
Signal Frequency

$$f_0 = \frac{1}{T_0}$$

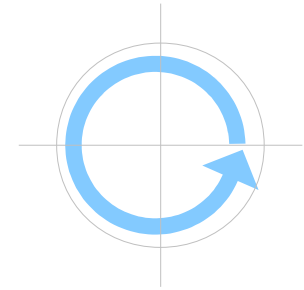
Signal Angular Frequency

$$\omega_0 = 2\pi f_0 \text{ (rad/sec)}$$

For 1 second  
 $2\pi f_0 \text{ (rad/sec)}$



For 1 revolution  
 $2\pi \text{ (rad)}$   
 $T_0 \text{ (sec)}$



## sampling sequence

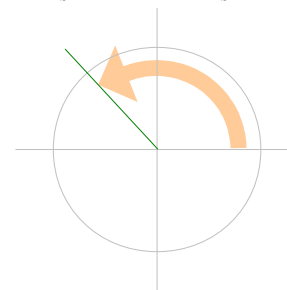
Sampling Frequency

$$f_s = \frac{1}{T_s}$$

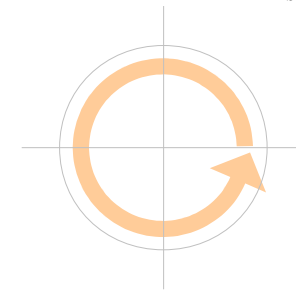
Sampling Angular Frequency

$$\omega_s = 2\pi f_s \text{ (rad/sec)}$$

For 1 second  
 $2\pi f_s \text{ (rad/sec)}$



For 1 revolution  
 $2\pi \text{ (rad)}$   
 $T_s \text{ (sec)}$









## References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] A "graphical interpretation" of the DFT and FFT, by Steve Mann
- [4] R. Cristi, "Modern Digital Signal Processing"