General Vector Space (3A)

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Vector Space

V: non-empty set of objects

defined operations: addition $\mathbf{u} + \mathbf{v}$

scalar multiplication $k\,\mathbf{u}$

if the following axioms are satisfied

for all object $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and all scalar k, m

⇒ V: vector space

objects in V: vectors

1. if $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ (zero vector)
5. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + (\mathbf{u}) = \mathbf{0}$
6. if k is any scalar and $\mathbf{u}$ is objects in V, then $k\mathbf{u}$ is in V
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

Test for a Vector Space

1. Identify the set V of objects
2. Identify the addition and scalar multiplication on V
3. Verify $\mathbf{u} + \mathbf{v}$ is in V and $k\mathbf{u}$ is in V

    closure under **addition** and **scalar multiplication**
4. Confirm other axioms.

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Subspace

a subset W of a vector space V

If the subset W is itself a vector space ⇒ the subset W is a **subspace** of V

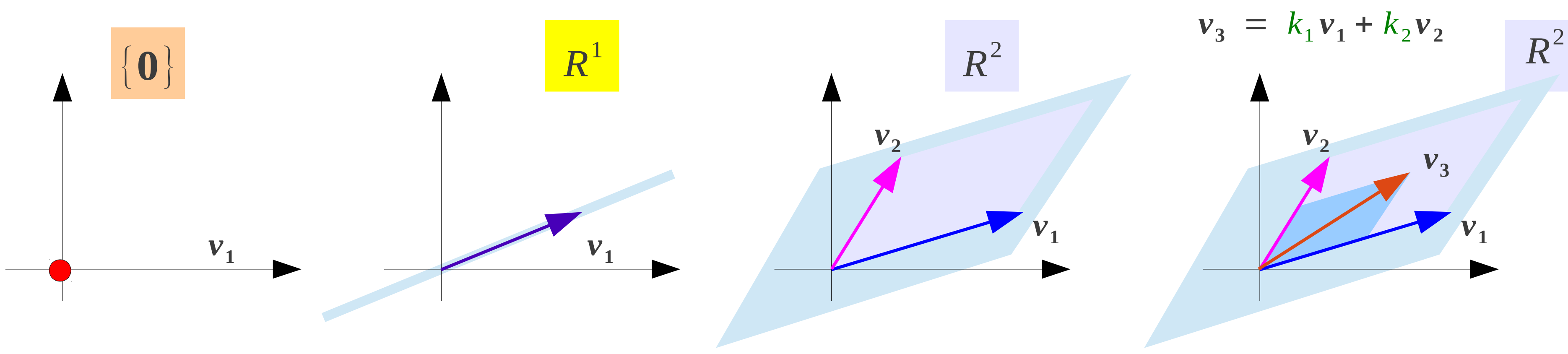
1. if **u** and **v** are objects in W, then **u** + **v** is in W
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. **0** + **u** = **u** + **0** = **u** (zero vector)
5. **u** + (–**u**) = (–**u**) + (**u**) = **0**
6. if *k* is any scalar and **u** is objects in W, then *k***u** is in W
7. *k*(**u** + **v**) = *k***u** + *k***v**
8. (*k* + *m*)**u** = *k***u** + *m***u**
9. *k*(*m***u**) = (*km*)**u**
10. *1*(**u**) = **u**

Subspace Example (1)

In vector space R^2

any one vector	(linearly indep.)	**spans**	R^1	line through **0**
any two non-collinear vectors	(linearly indep.)	**spans**	R^2	plane
any three or more vectors	(linearly dep.)	**spans**	R^2	plane

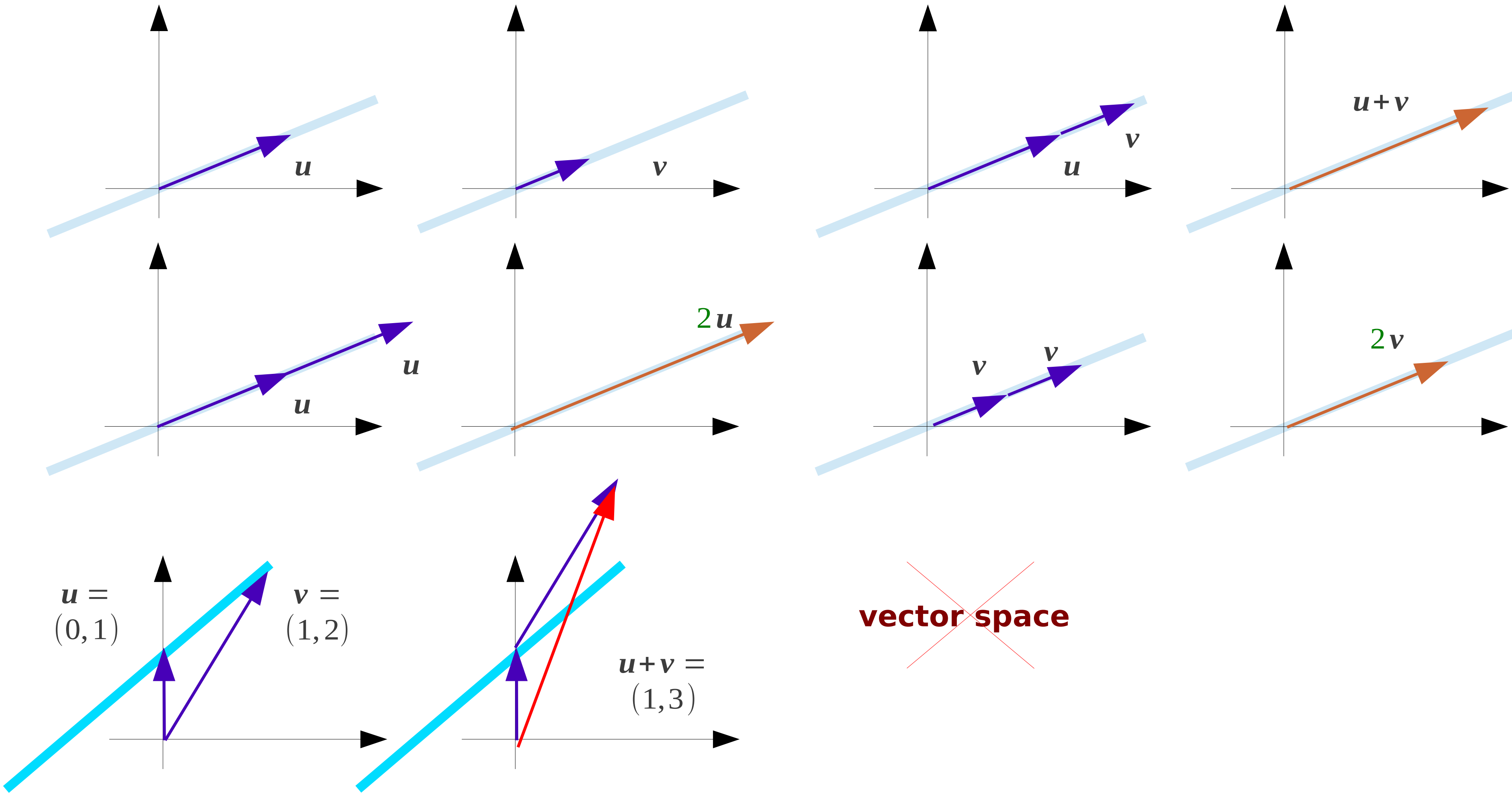
Subspaces of R^2



Subspace Example (2)

In vector space R^2

any one vector (linearly indep.) **spans** R^1 line through **0**



Subspace Example (3)

In vector space R^3

any one vector	(linearly indep.)	**spans**	R^1	line through $\mathbf{0}$
any two non-collinear vectors	(linearly indep.)	**spans**	R^2	plane through $\mathbf{0}$
any three vectors non-collinear, non-coplanar	(linearly indep.)	**spans**	R^3	3-dim space
any four or more vectors	(linearly dep.)	**spans**	R^3	3-dim space

Subspaces of R^2

$\{\mathbf{0}\}$	R^1	R^2	R^3
	line through $\mathbf{0}$	plane through $\mathbf{0}$	3-dim space

Row & Column Spaces

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

ROW Space **subspace** of R^n

$$= span\{\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_m\}$$

COLUMN Space **subspace** of R^m

$$= span\{\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n\}$$

$$\boldsymbol{r}_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix}$$

$$\boldsymbol{r}_2 = \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}$$

$$\vdots$$

$$\boldsymbol{r}_m = \begin{pmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$\boldsymbol{c}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \quad \boldsymbol{c}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \quad \cdots \quad \boldsymbol{c}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Null Space

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{0}$$

NULL Space R^n **subspace**

$$= span\{\boldsymbol{x_1}, \boldsymbol{x_2}, \cdots, \boldsymbol{x_n}\}$$

solution space

ROW Space **subspace** of R^n

$$= span\{\boldsymbol{r_1}, \boldsymbol{r_2}, \cdots, \boldsymbol{r_m}\}$$

COLUMN Space **subspace** of R^m

$$= span\{\boldsymbol{c_1}, \boldsymbol{c_2}, \cdots, \boldsymbol{c_n}\}$$

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] D.G. Zill, “Advanced Engineering Mathematics”