Up-Sampling (5B)

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11/8/12

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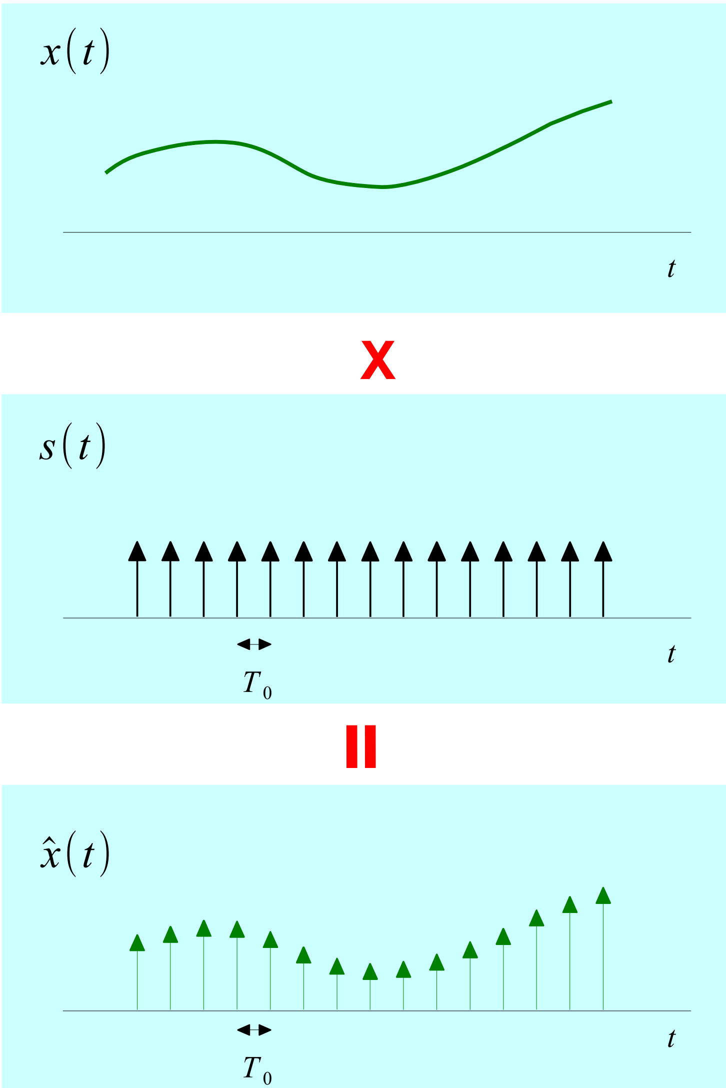
Spectrum Replication (1)

Ideal Sampling



$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0)\,\delta(t - nT_0)$$

$$s(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_0)$$

$$= \frac{1}{T_0} \sum_{m=-\infty}^{+\infty} e^{+j2\pi m f_s t}$$

$$\hat{x}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} x(t)\, e^{+j2\pi m f_s t}$$

Shift Property

$$\hat{X}(f) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} X(f - m f_s)$$

Spectrum Replication (2)

$$S(f) = \frac{1}{T}\sum_{m=-\infty}^{+\infty}\delta(f - m f_s)$$

Convolution in Frequency

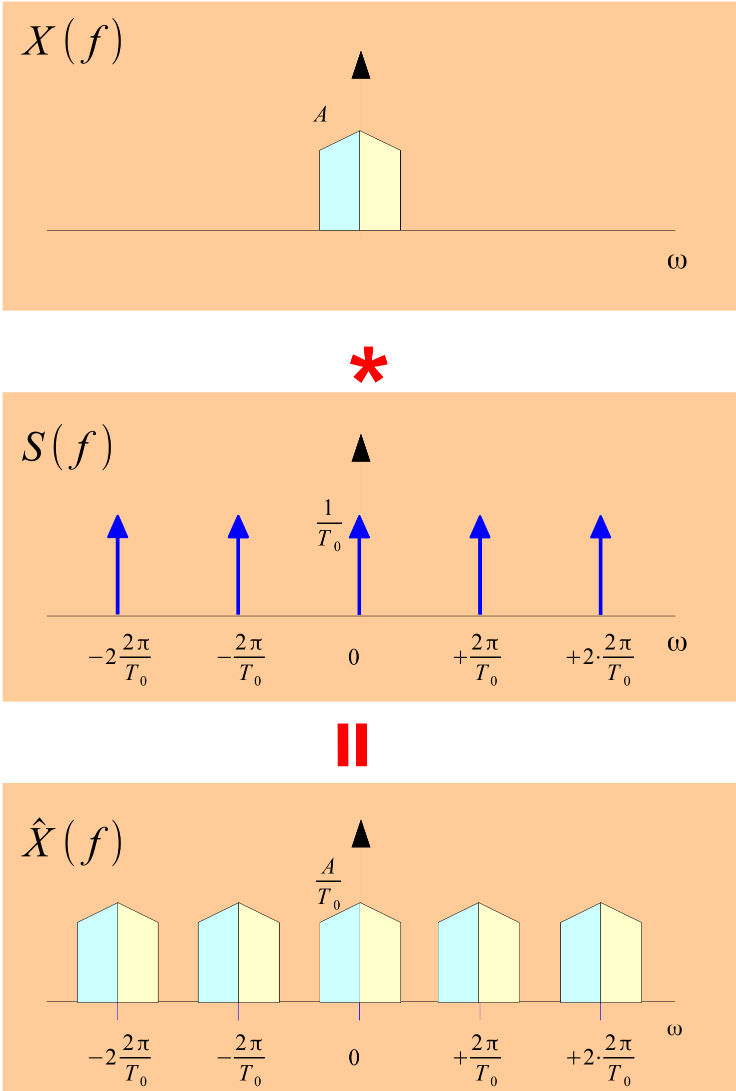
$$\hat{X}(f) = X(f) * S(f)$$

$$= \int_{-\infty}^{+\infty} X(f - f')S(f')\,d f'$$

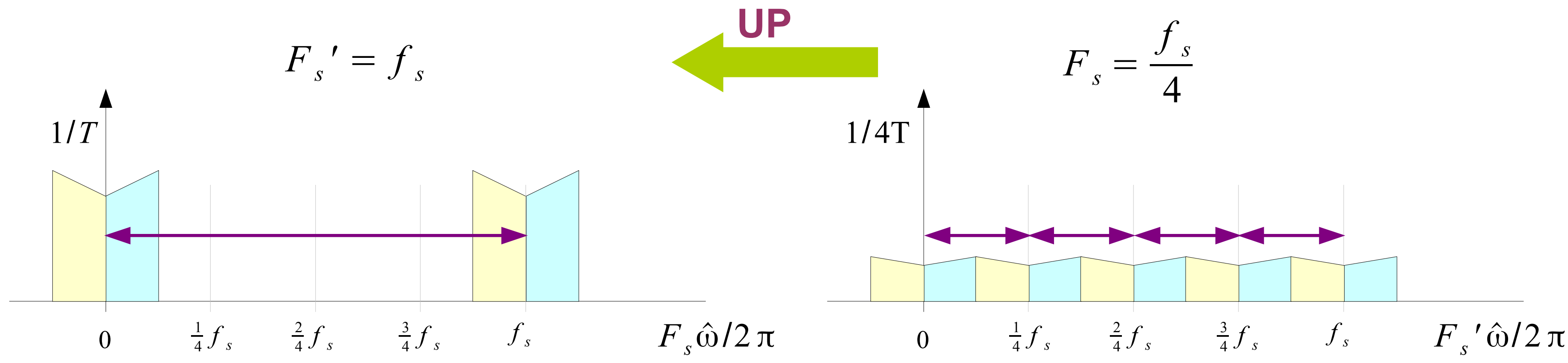
$$= \frac{1}{T_0}\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{+\infty} X(f - f')\delta(f' - m f_s)\,d f'$$

$$\boxed{\hat{X}(f) = \frac{1}{T_0}\sum_{n=-\infty}^{+\infty} X(f - m f_s)}$$

Frequency Domain



Increasing Sampling Frequency

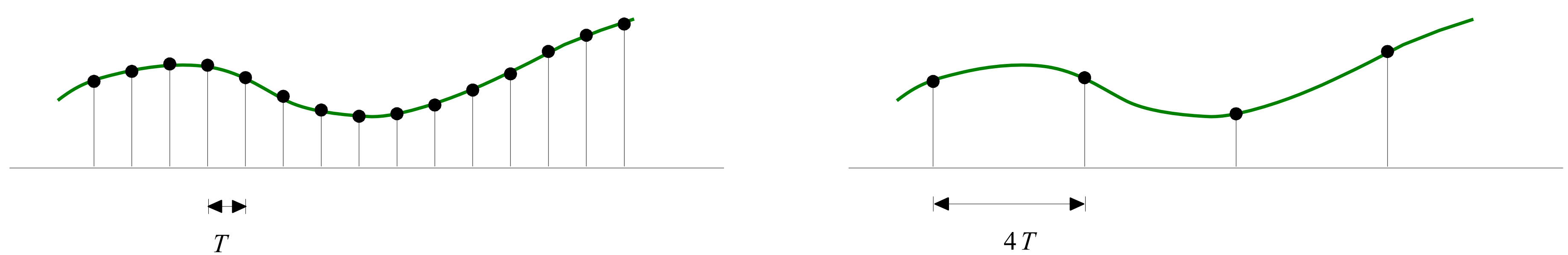


Sampling Frequency $F_s' = \boxed{f_s}$

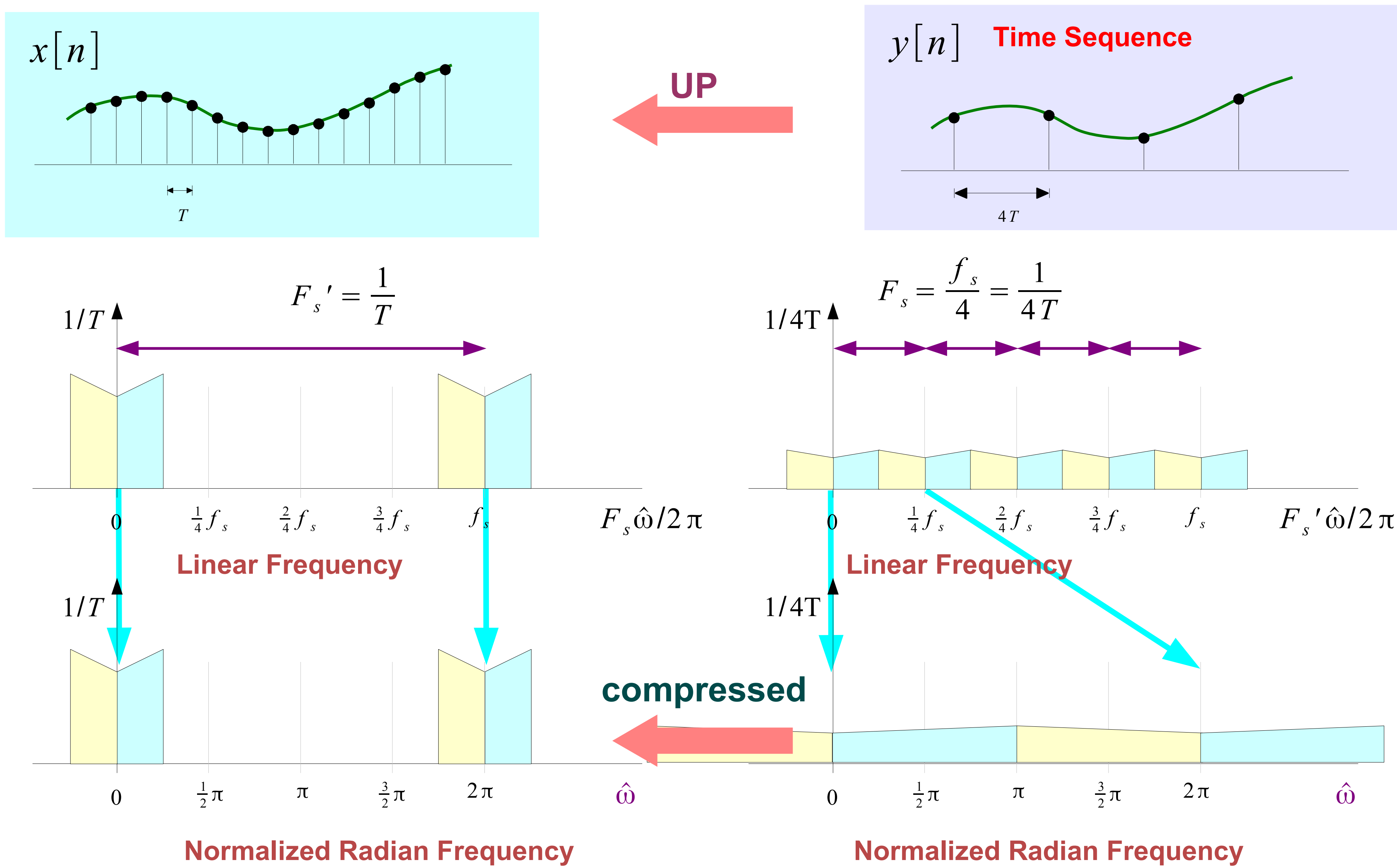
Sampling Time $T' = \boxed{\frac{T}{4}}$

Sampling Frequency $F_s = \boxed{\frac{f_s}{4}}$

Sampling Time $T = \boxed{\frac{4}{f_s}}$

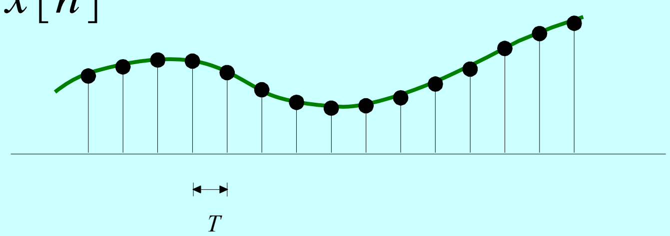


Fine Sequence & Spectrum



Normalized Radian Frequency

$x[n]$



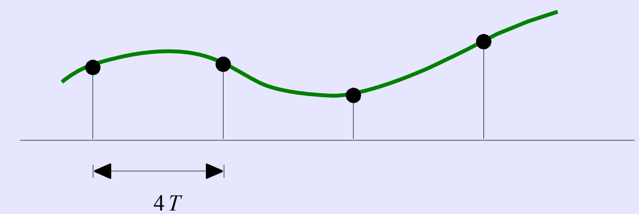
$$\hat{\omega} = \omega \cdot T_s = \frac{\omega}{1/T_s}$$

$$\hat{\omega} = \frac{\omega}{f_s} = 2\pi \frac{f}{f_s}$$

Normalized to f_s

Normalized Radian Frequency

$y[n]$ **Time Sequence** f_B

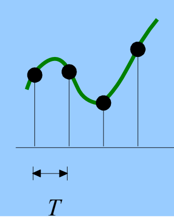


$$\frac{f}{f_s} = \frac{f_B}{1/\mathbf{4T}} = f_B \cdot 4T$$

The Same
- **Time Sequence**
- **Normalized Radian Frequency**

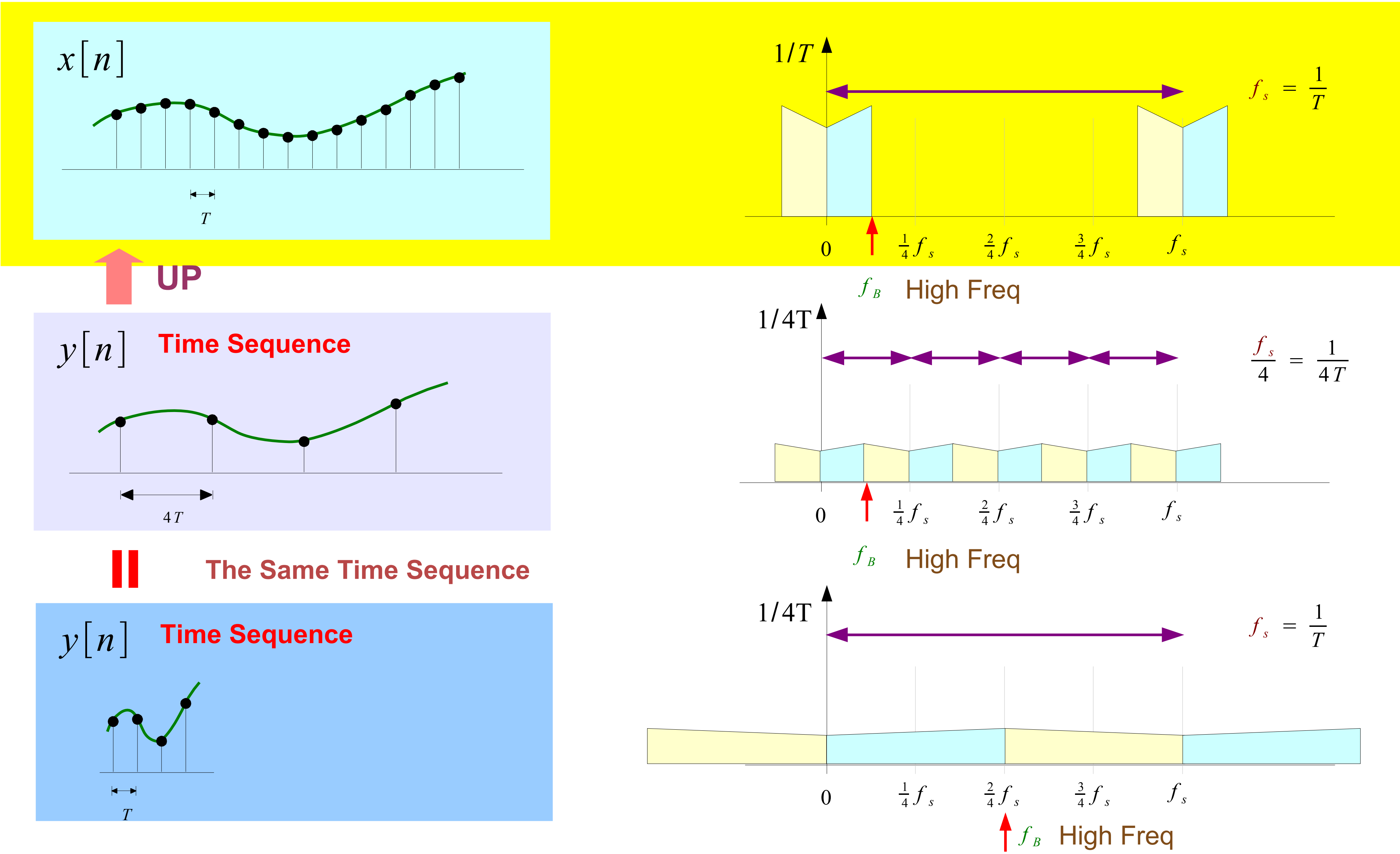
$$\frac{f}{f_s} = \frac{\mathbf{4 f_B}}{1/T} = f_B \cdot 4T$$

$y'[n]$ **Time Sequence** $4f_B$

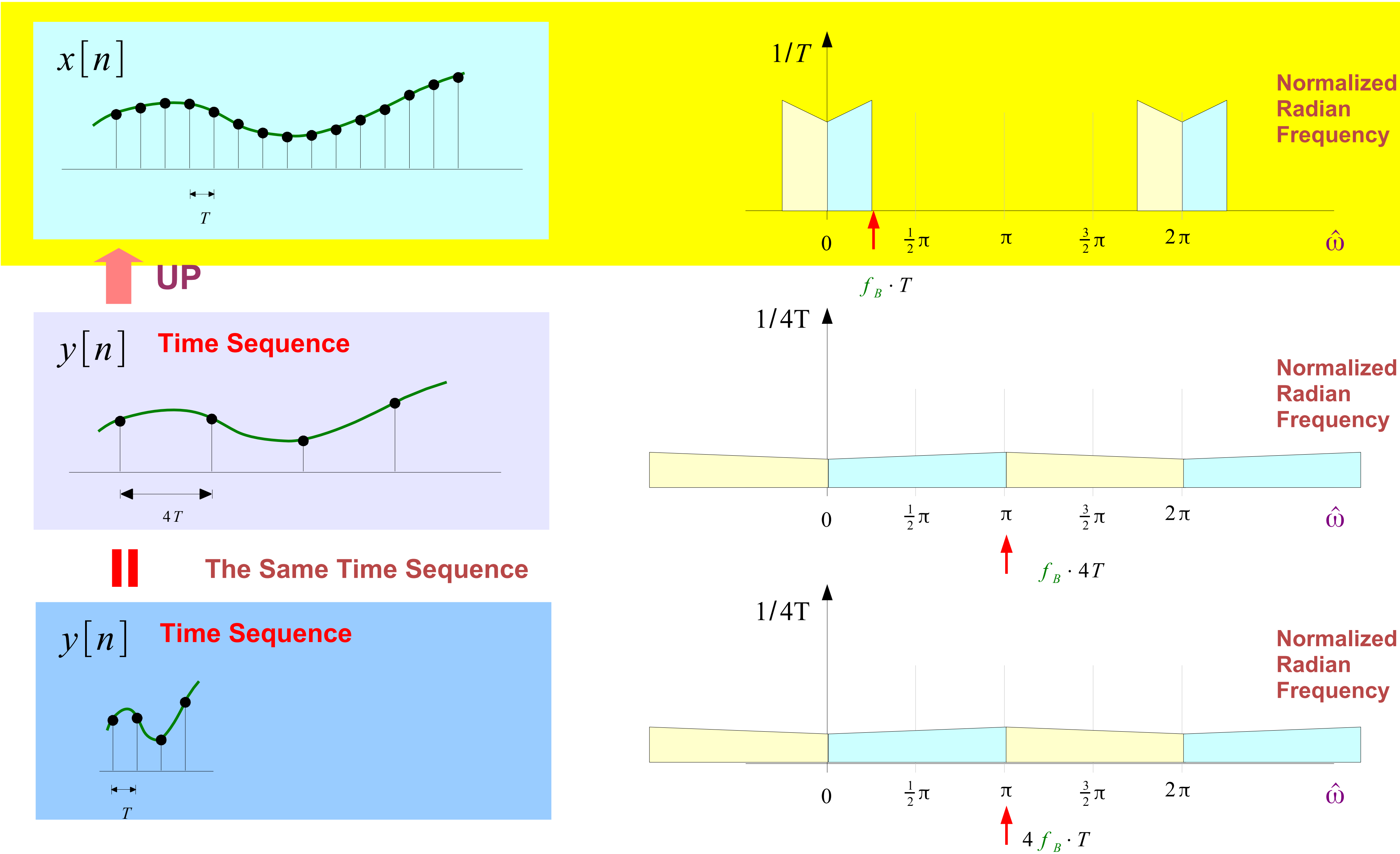


The Highest Frequency:

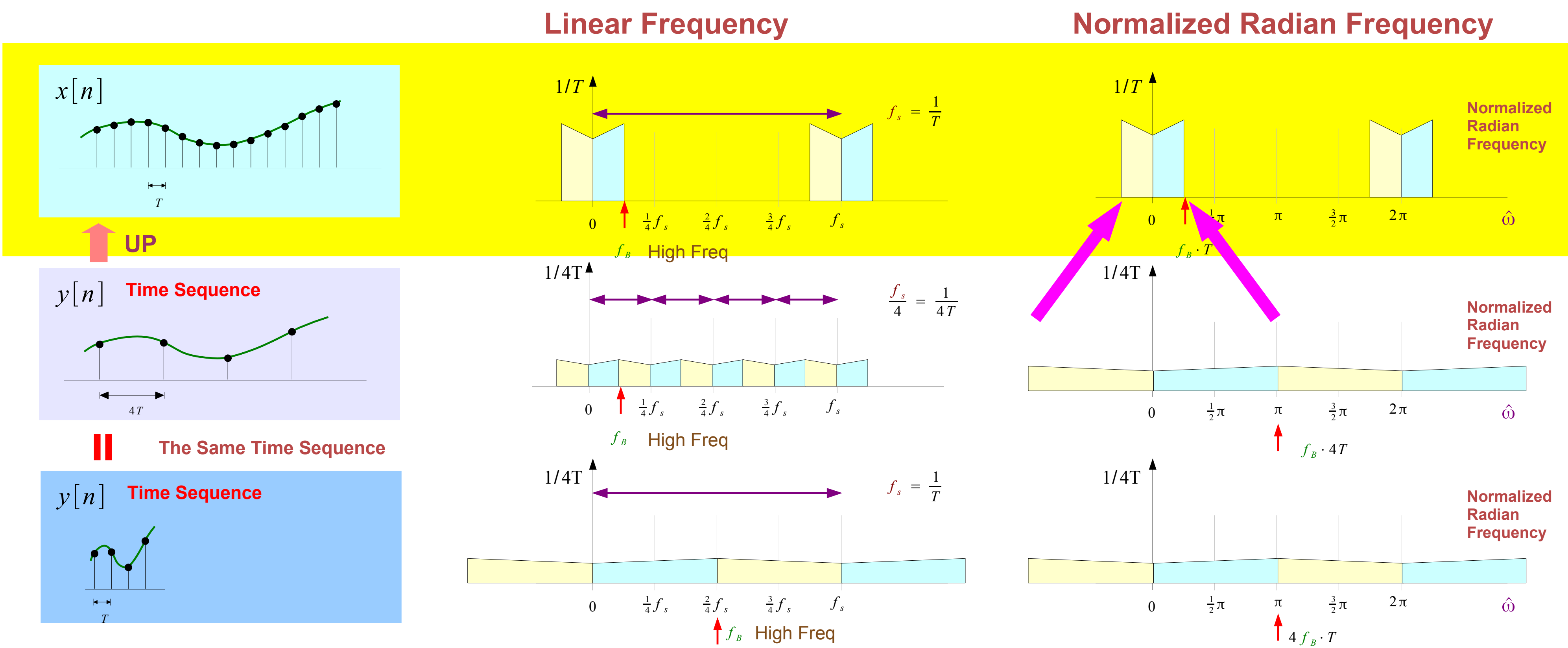
Fine Sequence Spectrum – Linear Frequency



Fine Sequence Spectrum – Normalized Frequency



Fine Sequence Spectrum – Linear Frequency

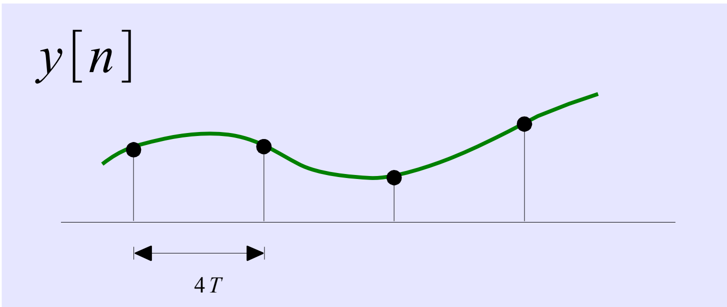


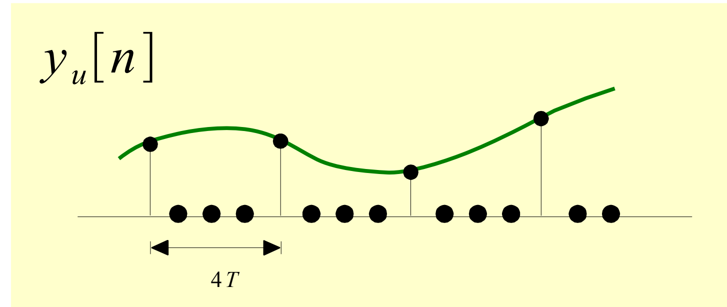
Fine Sequence Spectrum

➡ **Compressed**

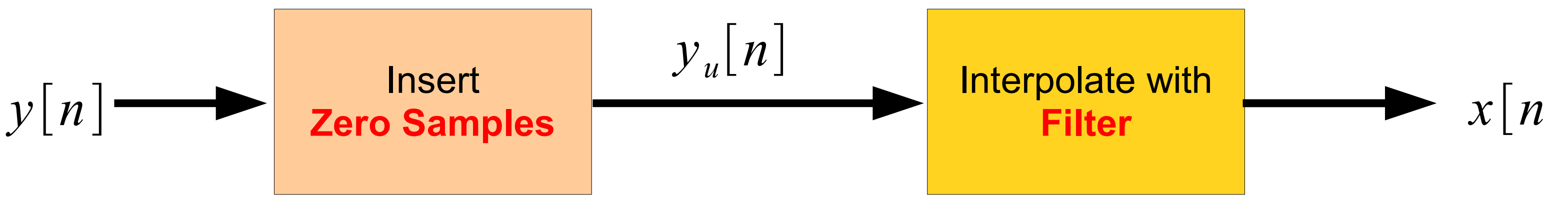
Normalized Radian Frequency

Fine Sequence Generation

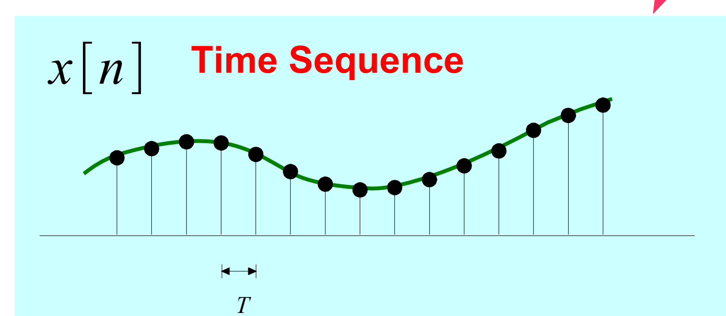




$4T$ **Sampling Period**

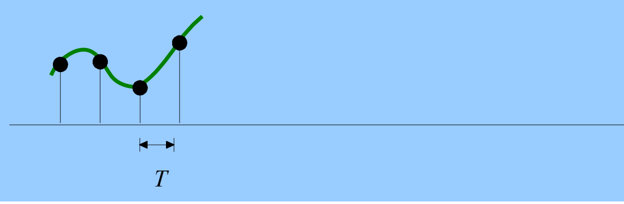


T **Sampling Period**

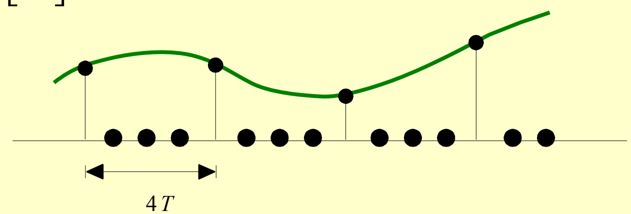


Up Sampling in Two Steps

$y[n]$ **Time Sequence**



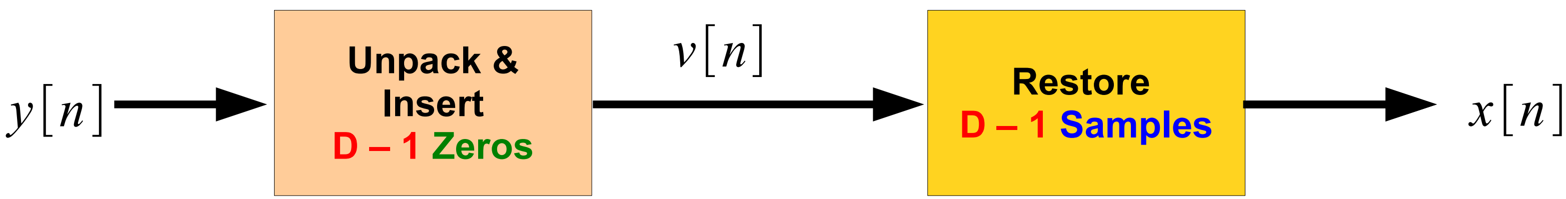
$v[n]$



$4f_B$ **Highest Frequency**

T **Sampling Period**

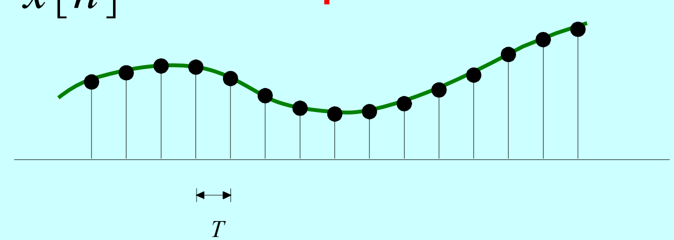
Interpolation



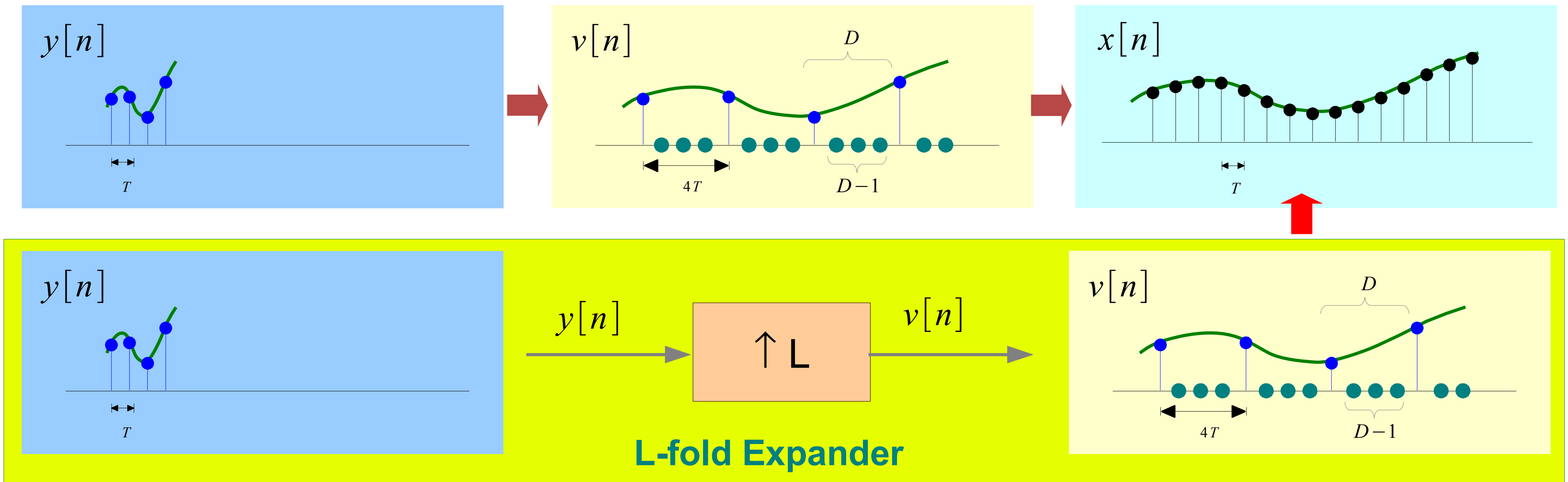
f_B **Highest Frequency**

T **Sampling Period**

$x[n]$ **Time Sequence**



Up-Sampling Operator



$$v[n] = S_L\, y[n] = \begin{cases} y[n/L] & \text{if } \mathbf{mod}(n/L) = 0 \\ 0 & \text{otherwise} \end{cases}$$

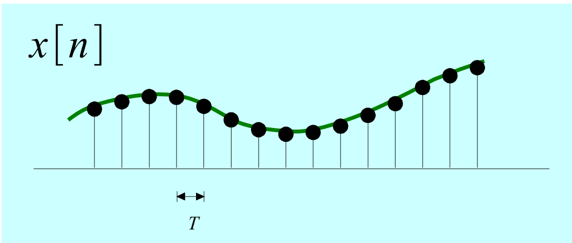
Increase sampling frequency by a factor of L

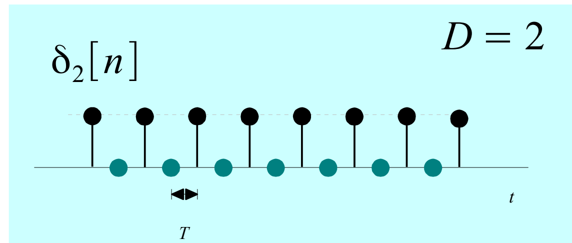
Decrease sampling period by a factor of 1/L

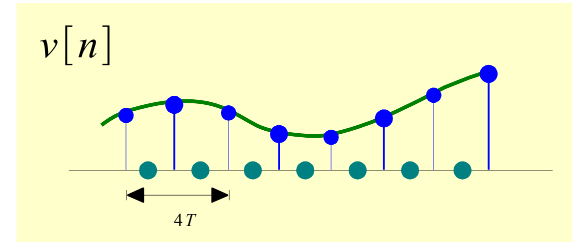
$D = 2$

$n=0\cdot 2=0$	$v[0] = y[0]$	$v[1] = 0$
$n=1\cdot 2=2$	$v[2] = y[1]$	$v[3] = 0$
$n=2\cdot 2=4$	$v[4] = y[2]$	$v[5] = 0$
$n=3\cdot 2=6$	$v[6] = y[3]$	$v[6] = 0$
	$\cdots$	$\cdots$

Example When D=2 (1)







$$x[n] = e^{j\omega n}$$

$$\delta_2[n] = \tfrac{1}{2}\left(1 + (-1)^n\right)$$

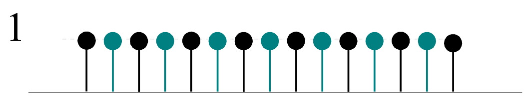
$$= \tfrac{1}{2}\left(1 + e^{-j\pi n}\right)$$

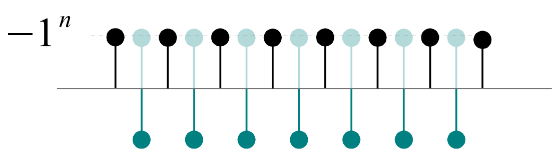
$$\left(e^{-j\pi} = -1\right)$$

$$v[n] = \tfrac{1}{2}x[n] + \tfrac{1}{2}e^{-j\pi n}x[n]$$

$$= \tfrac{1}{2}e^{j\omega n} + \tfrac{1}{2}e^{-j\pi n}e^{j\omega n}$$

$$= \tfrac{1}{2}e^{j\omega n} + \tfrac{1}{2}e^{+j(\omega-\pi)n}$$





$$V(z) = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\left(x[n]z^{-n} + x[n](-z)^{-n}\right) = \frac{1}{2}X(z) + \frac{1}{2}X(-z)$$

$$V(e^{j\hat{\omega}}) = \frac{1}{2}X(e^{j\hat{\omega}}) + \frac{1}{2}X(e^{-j\pi}e^{j\hat{\omega}})$$

$$V(\hat{\omega}) = \frac{1}{2}X(\hat{\omega}) + \frac{1}{2}X(\hat{\omega} - \pi)$$

Z-Transform Analysis

$$\delta_D[n] = \begin{cases} 1 & \text{if } n/D \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

$$v[n] = \delta_D[n]x[n]$$

$$V[z] = \cdots + v[0]z^0 + v[D]z^{-D} + v[2D]z^{-2D} + \cdots$$

$y[n]$

$$V[z] = \sum_{n=-\infty}^{+\infty} v[n]z^{-n} = \sum_{m=-\infty}^{+\infty} v[mD]z^{-mD} = F(z^D)$$

T **Sampling Period**

Z-Transform Analysis

$$\delta_2[n] = \frac{1}{2}\left(1 + (-1)^n\right) = \frac{1}{2}\left(1 + e^{-j\pi n}\right) = \begin{cases} 1 & \text{if } n/2 \text{ is an integer (even)} \\ 0 & \text{otherwise} \end{cases}$$

$$e^{-j\pi} = -1$$

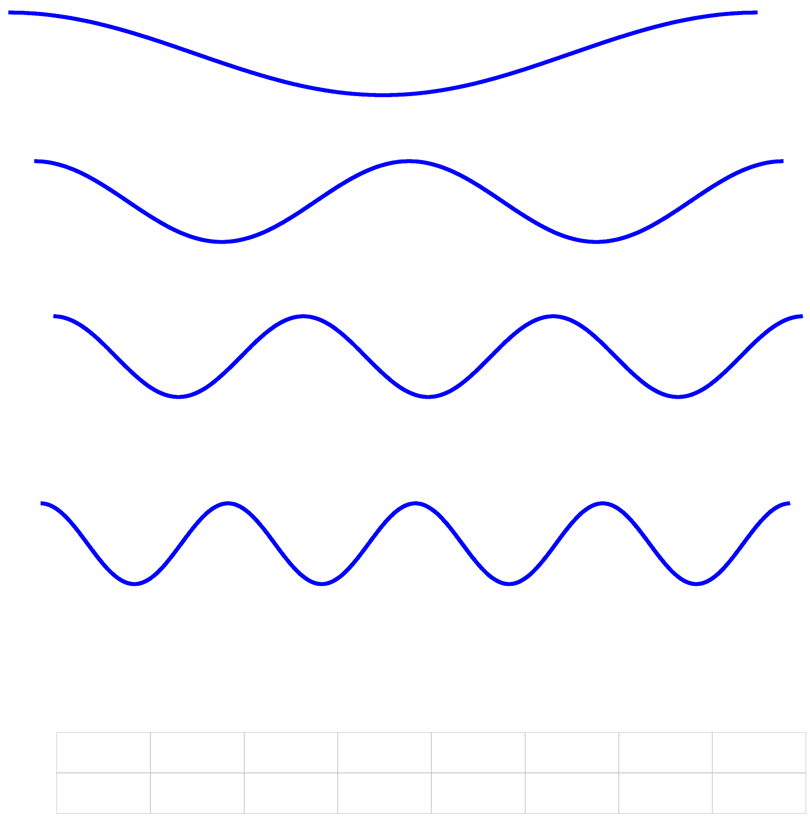
$$v[n] = \frac{1}{2}x[n] + \frac{1}{2}e^{-j\pi n}x[n] \qquad x[n] = e^{j\omega n}$$

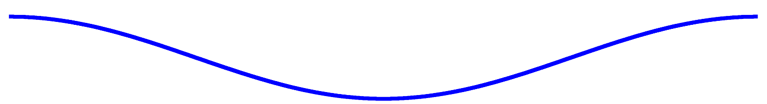
$$v[n] = \frac{1}{2}e^{j\omega n} + \frac{1}{2}e^{-j\pi n}e^{j\omega n} = \frac{1}{2}e^{j\omega n} + \frac{1}{2}e^{+j(\omega-\pi)n}$$

$$V(z) = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\left(x[n]z^{-n} + x[n](-z)^{-n}\right) = \frac{1}{2}X(z) + \frac{1}{2}X(-z)$$

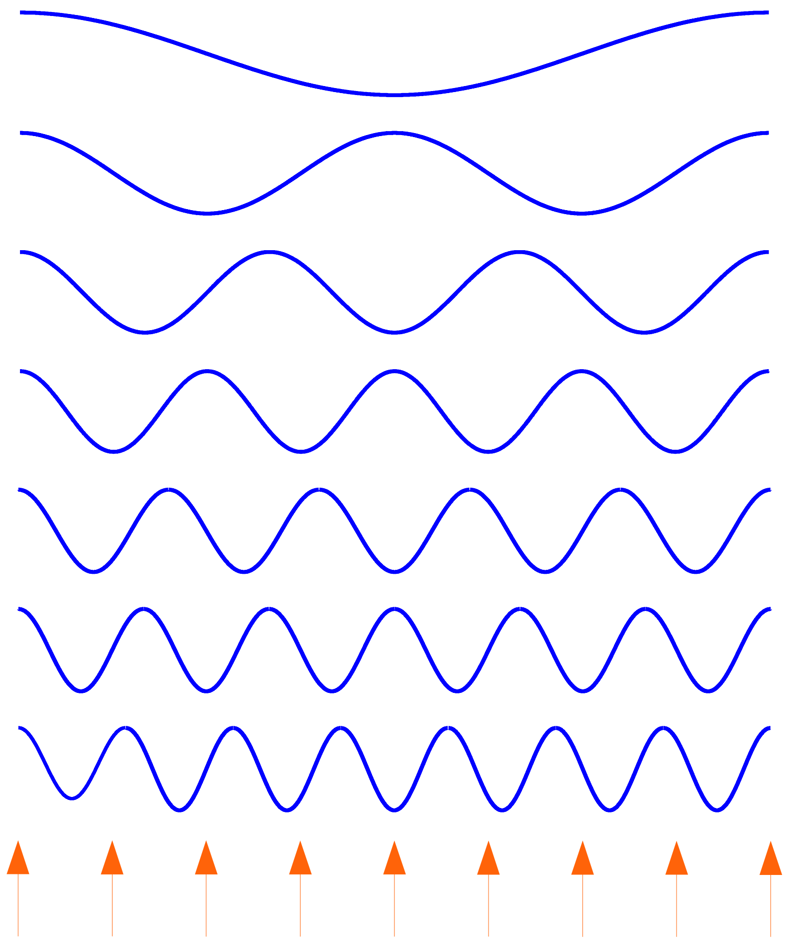
$$V(\omega) = V(e^{j\omega}) = \frac{1}{2}X(e^{j\omega}) + \frac{1}{2}X(e^{-j\pi}e^{j\omega}) = \frac{1}{2}X(\omega) + \frac{1}{2}X(\omega - \pi)$$

Measuring Rotation Rate





Signals with Harmonic Frequencies (1)



1 Hz
1 cycle / sec

$$\cos(1\cdot 2\pi t) = \frac{e^{+j(1\cdot 2\pi)t} + e^{-j(1\cdot 2\pi)t}}{2}$$

2 Hz
2 cycles / sec

$$\cos(2\cdot 2\pi t) = \frac{e^{+j(2\cdot 2\pi)t} + e^{-j(2\cdot 2\pi)t}}{2}$$

3 Hz
3 cycles / sec

$$\cos(3\cdot 2\pi t) = \frac{e^{+j(3\cdot 2\pi)t} + e^{-j(3\cdot 2\pi)t}}{2}$$

4 Hz
4 cycles / sec

$$\cos(4\cdot 2\pi t) = \frac{e^{+j(4\cdot 2\pi)t} + e^{-j(4\cdot 2\pi)t}}{2}$$

5 Hz
5 cycles / sec

$$\cos(5\cdot 2\pi t) = \frac{e^{+j(5\cdot 2\pi)t} + e^{-j(5\cdot 2\pi)t}}{2}$$

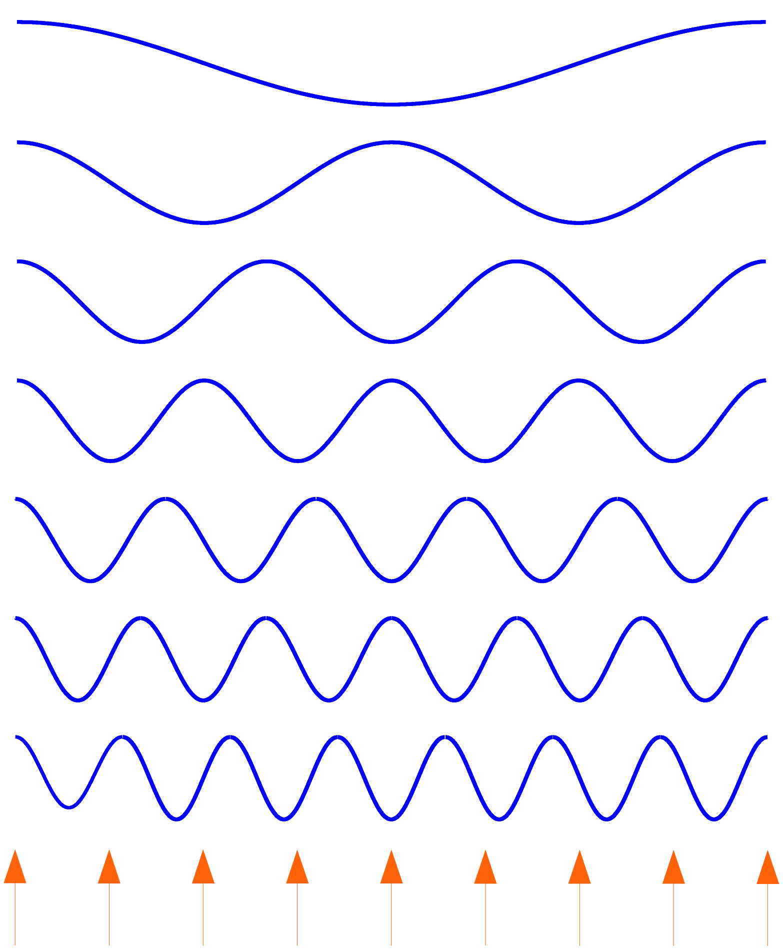
6 Hz
6 cycles / sec

$$\cos(6\cdot 2\pi t) = \frac{e^{+j(6\cdot 2\pi)t} + e^{-j(6\cdot 2\pi)t}}{2}$$

7 Hz
7 cycles / sec

$$\cos(7\cdot 2\pi t) = \frac{e^{+j(7\cdot 2\pi)t} + e^{-j(7\cdot 2\pi)t}}{2}$$

Signals with Harmonic Frequencies (2)



1 Hz
1 cycle / sec

2 Hz
2 cycles / sec

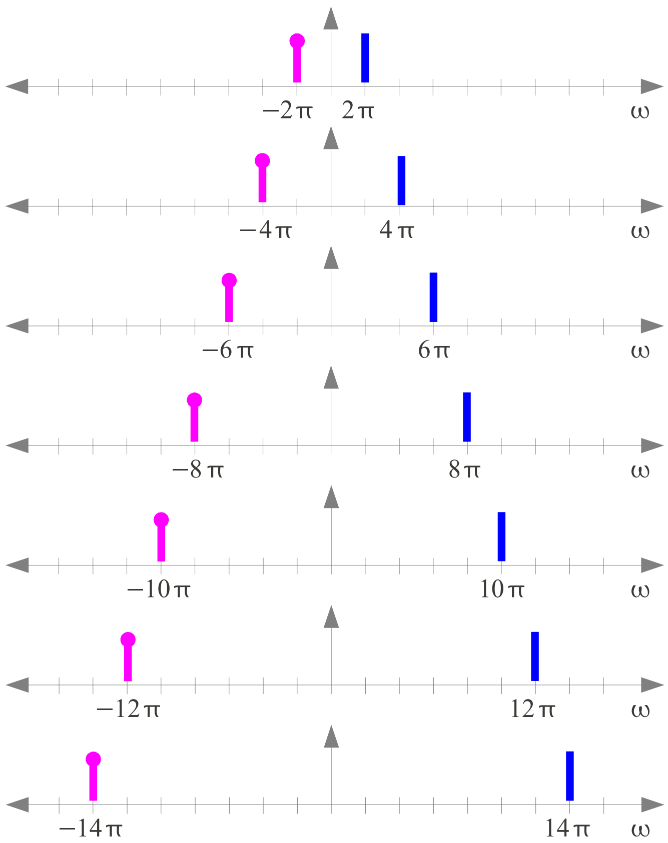
3 Hz
3 cycles / sec

4 Hz
4 cycles / sec

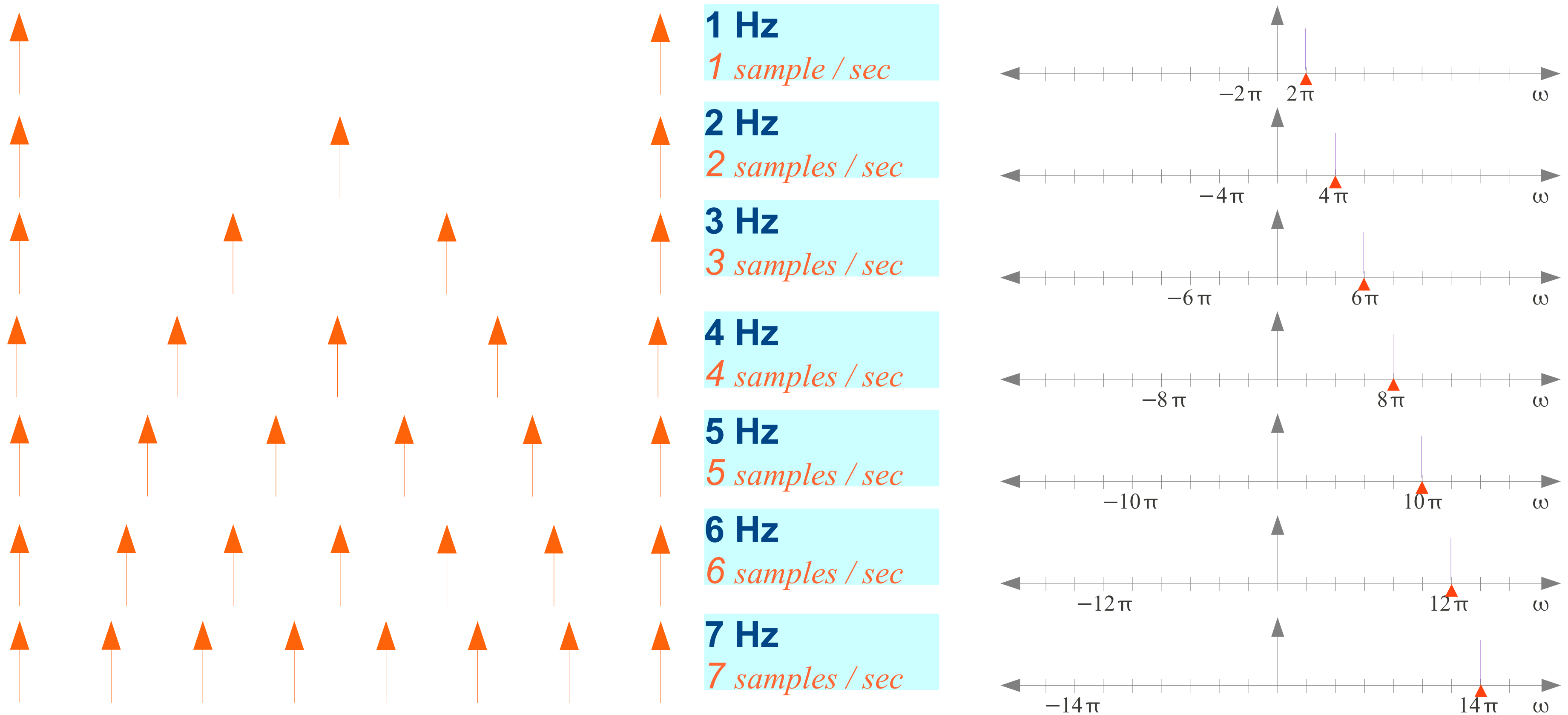
5 Hz
5 cycles / sec

6 Hz
6 cycles / sec

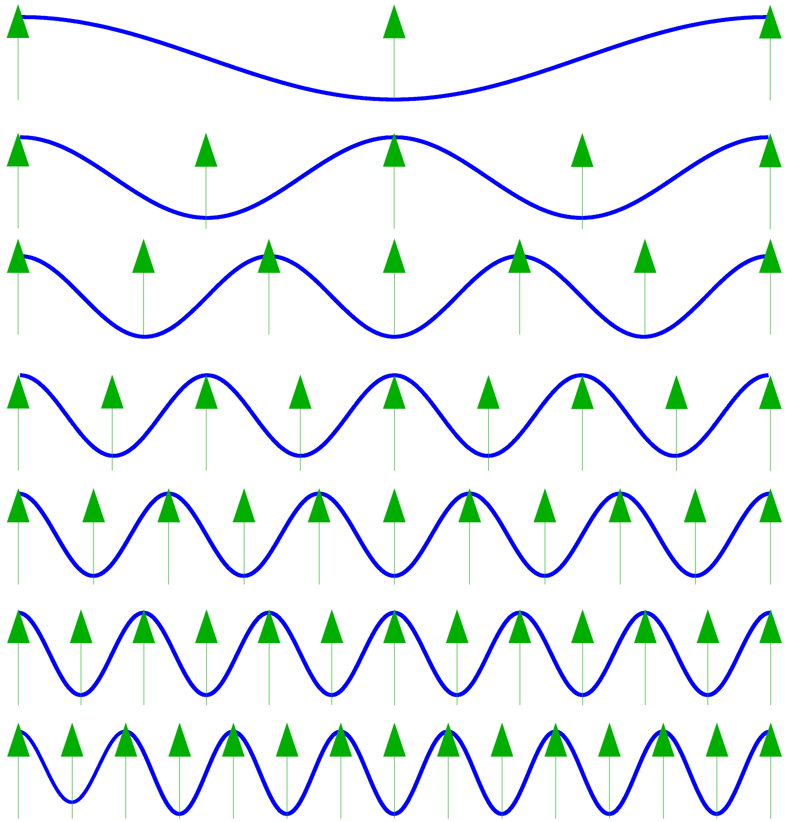
7 Hz
7 cycles / sec



Sampling Frequency

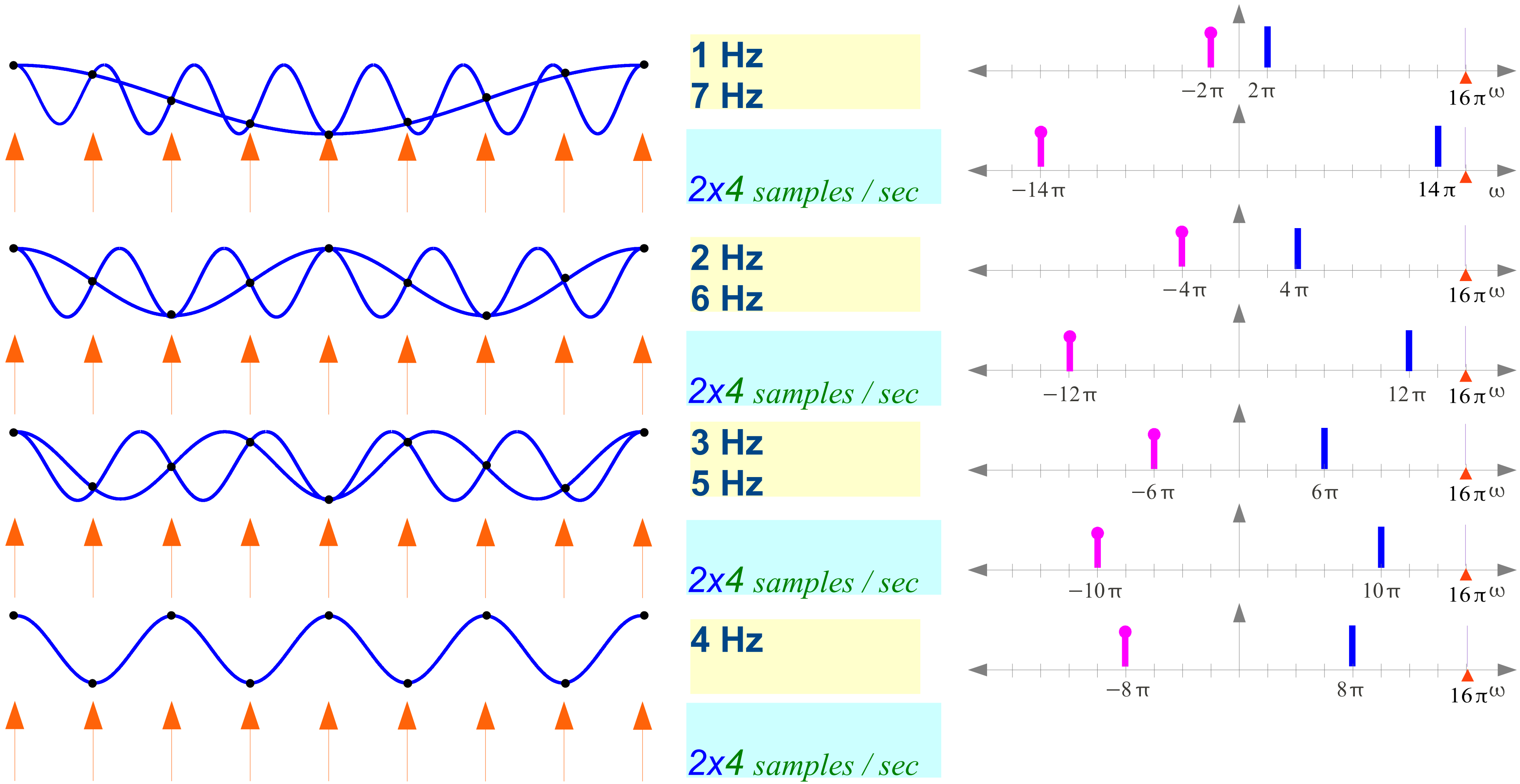


Nyquist Frequency



1 Hz *1 cycle / sec*	*2x1 sample / sec*
2 Hz *2 cycles / sec*	*2x2 samples / sec*
3 Hz *3 cycles / sec*	*2x3 samples / sec*
4 Hz *4 cycles / sec*	*2x4 samples / sec*
5 Hz *5 cycles / sec*	*2x5 samples / sec*
6 Hz *6 cycles / sec*	*2x6 samples / sec*
7 Hz *7 cycles / sec*	*2x7 samples / sec*

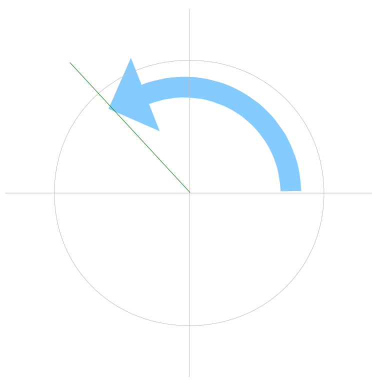
Aliasing



1 Hz
7 Hz
2x4 samples / sec
2 Hz
6 Hz
2x4 samples / sec
3 Hz
5 Hz
2x4 samples / sec
4 Hz
2x4 samples / sec
−2π 2π 16π ω
−14π 14π ω
−4π 4π 16π ω
−12π 12π 16π ω
−6π 6π 16π ω
−10π 10π 16π ω
−8π 8π 16π ω

Sampling

$$\omega_s = 2\pi f_s \ (rad/sec)$$



$$\omega_1 = 2\pi f_1$$

$$\omega_1 = \frac{\omega_s}{2} \ (rad/sec)$$

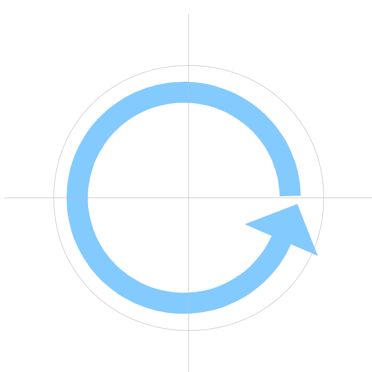
$$f_1 = \frac{f_s}{2} \ (rad/sec)$$

$$\omega_2 = 2\pi f_2$$

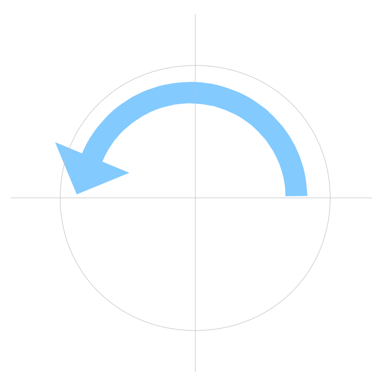
$$\omega_2 = -\frac{\omega_s}{2} \ (rad/sec)$$

$$f_2 = -\frac{f_s}{2} \ (rad/sec)$$

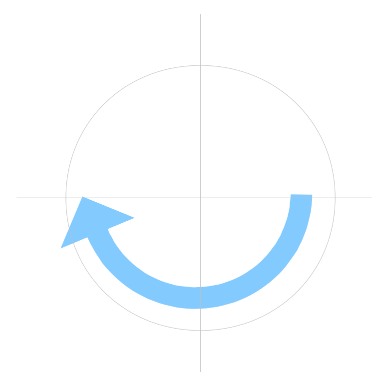
$$2\pi \ (rad) \ / \ T_s \ (sec)$$



$$\pi \ (rad) \ / \ T_s \ (sec)$$

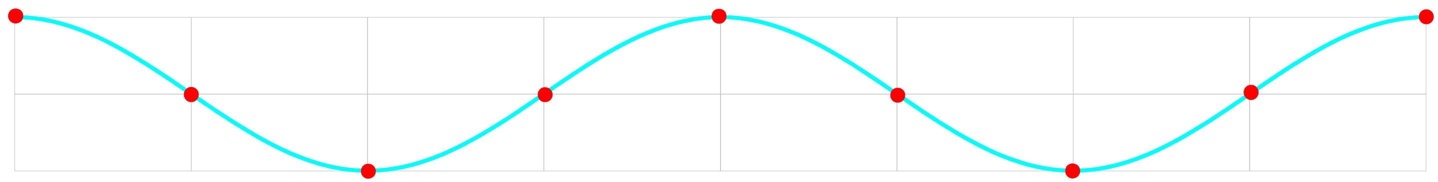


$$-\pi \ (rad) \ / \ T_s \ (sec)$$

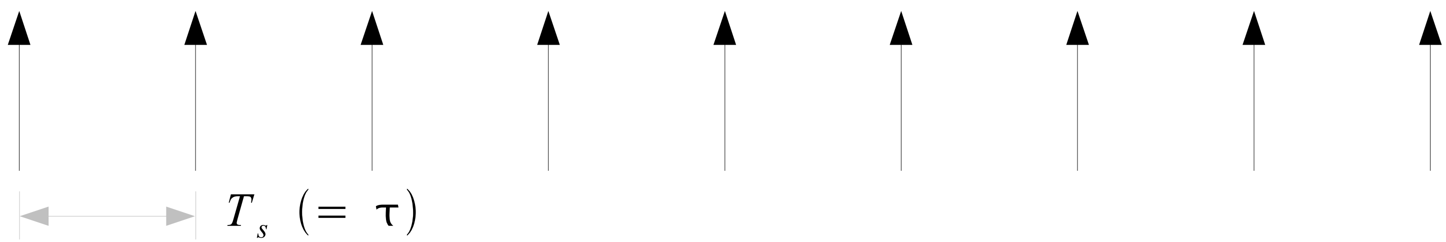


Sampling

$\omega_1 = 2\pi f_1 \ (rad/sec)$

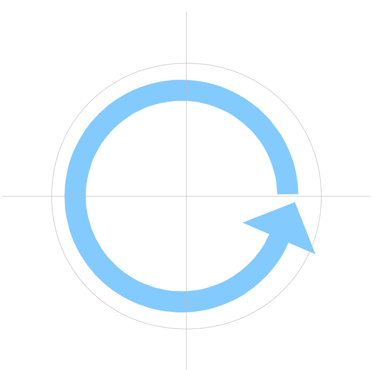


$\omega_s = 2\pi f_s \ (rad/sec)$

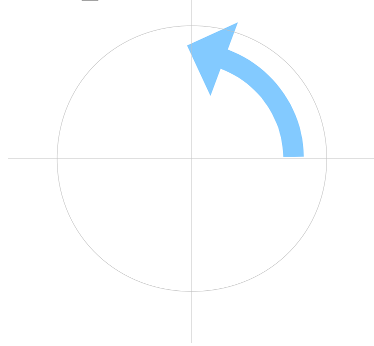


$T_s \ (= \tau)$

$2\pi \ (rad) \ / \ T_s \ (sec)$



$\frac{\pi}{2} \ (rad) \ / \ T_s \ (sec)$



For the period of T_s

Angular displacement $\frac{\pi}{2} \ (rad)$

$$\hat{\omega} = \omega \cdot T_s \ (rad)$$

$$= 2\pi f_1 \cdot T_s \ (rad)$$

$$= 2\pi \frac{f_s}{4} \cdot T_s \ (rad)$$

$$= \frac{\pi}{2} \ (rad)$$

Angular Frequencies in Sampling

continuous-time signals

Signal Frequency

$$f_0 = \frac{1}{T_0}$$

Signal Angular Frequency

$$\omega_0 = 2\pi f_0 \quad (rad/sec)$$

sampling sequence

Sampling Frequency

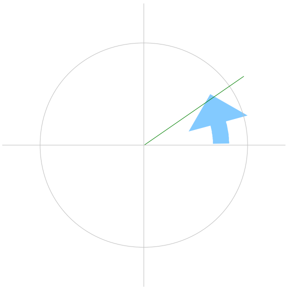
$$f_s = \frac{1}{T_s}$$

Sampling Angular Frequency

$$\omega_s = 2\pi f_s \quad (rad/sec)$$

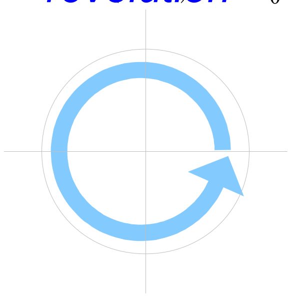
For 1 second

$$\omega_0 = 2\pi f_0 \quad (rad/sec)$$



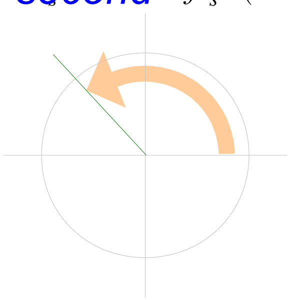
For 1 revolution

$$2\pi \,(rad) \,/\, T_0 \,(sec)$$



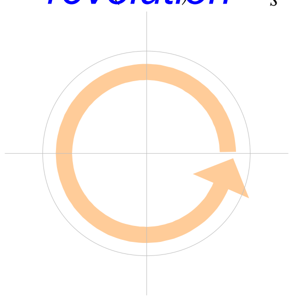
For 1 second

$$\omega_s = 2\pi f_s \quad (rad/sec)$$



For 1 revolution

$$2\pi \,(rad) \,/\, T_s \,(sec)$$



References

[1] http://en.wikipedia.org/
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