General Vector Space (3A)

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Vector Space

V: non-empty set of objects

defined operations:

addition $\mathbf{u} + \mathbf{v}$

scalar multiplication $k\,\mathbf{u}$

if the following axioms are satisfied
for all object $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and all scalar k, m

⇒ V: vector space
objects in V: vectors

1. if $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ (zero vector)
5. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + (\mathbf{u}) = \mathbf{0}$
6. if k is any scalar and $\mathbf{u}$ is objects in V, then $k\mathbf{u}$ is in V
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

Test for a Vector Space

1. Identify the set V of objects
2. Identify the addition and scalar multiplication on V
3. Verify $\mathbf{u} + \mathbf{v}$ is in V and $k\mathbf{u}$ is in V

   closure under **addition** and **scalar multiplication**
4. Confirm other axioms.

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10. $1(\mathbf{u}) = \mathbf{u}$

Subspace

a subset W of a vector space V

If the subset W is itself a vector space ⇒ the subset W is a **subspace** of V

1. if **u** and **v** are objects in W, then **u** + **v** is in W
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. **0** + **u** = **u** + **0** = **u** (zero vector)
5. **u** + (–**u**) = (–**u**) + (**u**) = **0**
6. if *k* is any scalar and **u** is objects in W, then *k***u** is in W
7. *k*(**u** + **v**) = *k***u** + *k***v**
8. (*k* + *m*)**u** = *k***u** + *m***u**
9. *k*(*m***u**) = (*km*)**u**
10. *1*(**u**) = **u**

Subspace Test (1)

a subset W of a vector space V

If the subset W is itself a vector space ⇒ the subset W is a **subspace** of V

axioms not inherited by W

1. if **u** and **v** are objects in W, then **u** + **v** is in W
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. **0** + **u** = **u** + **0** = **u** (zero vector)
5. **u** + (–**u**) = (–**u**) + (**u**) = **0**
6. if *k* is any scalar and **u** is objects in W, then *k***u** is in W
7. *k*(**u** + **v**) = *k***u** + *k***v**
8. (*k* + *m*)**u** = *k***u** + *m***u**
9. *k*(*m***u**) = (*km*)**u**
10. *1*(**u**) = **u**

Subspace Test (2)

a subset W of a vector space V

if **u**, **v** ∈ W, then **u** + **v** ∈ W if k: a scalar, **u** ∈ W, then k**u** ∈ W	⟺ the subset W is a **subspace** of V

1. if **u** and **v** are objects in W, then **u** + **v** is in W
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. **0** + **u** = **u** + **0** = **u** (zero vector)
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6. if *k* is any scalar and **u** is objects in W, then *k***u** is in W
7. *k*(**u** + **v**) = *k***u** + *k***v**
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9. *k*(*m***u**) = (*km*)**u**
10. *1*(**u**) = **u**

Building Subspaces

if W_1, W_2, ... , W_n are subspaces of a vector space of V

the intersection of these subspaces are also a subspace of V

$S = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_r\}$ a nonempty set of a vector space V

the set W of all possible linear combination of the vectors in S

$\boldsymbol{w} = c_1 \boldsymbol{w}_1 + c_2 \boldsymbol{w}_2 + \cdots + c_r \boldsymbol{w}_r$

a subspace of V

the set W of is the smallest subspace of V that contains *all of the vectors* in S

any other subspace that contains those vectors contains W`

Spanning Set

$S_1 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r\}$ a nonempty set of a vector space V

$S_2 = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_k\}$ a nonempty set of a vector space V

$$span\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r\} = span\{\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_k\} \Longleftrightarrow$$

each vector in S_1 is a linear combination of the vectors in S_2

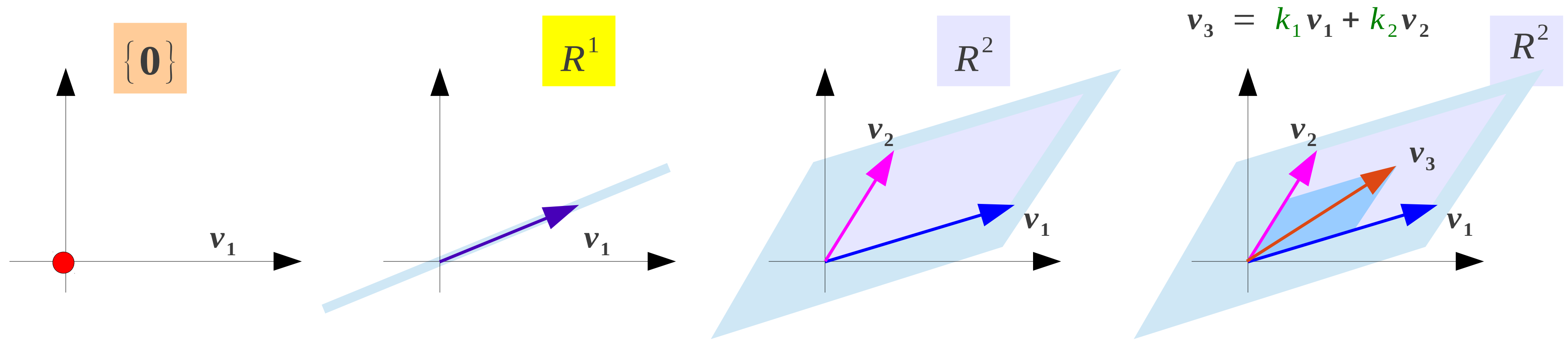
each vector in S_2 is a linear combination of the vectors in S_1

Subspace Example (1)

In vector space R^2

any one vector	(linearly indep.)	**spans**	R^1	line through **0**
any two non-collinear vectors	(linearly indep.)	**spans**	R^2	plane
any three or more vectors	(linearly dep.)	**spans**	R^2	plane

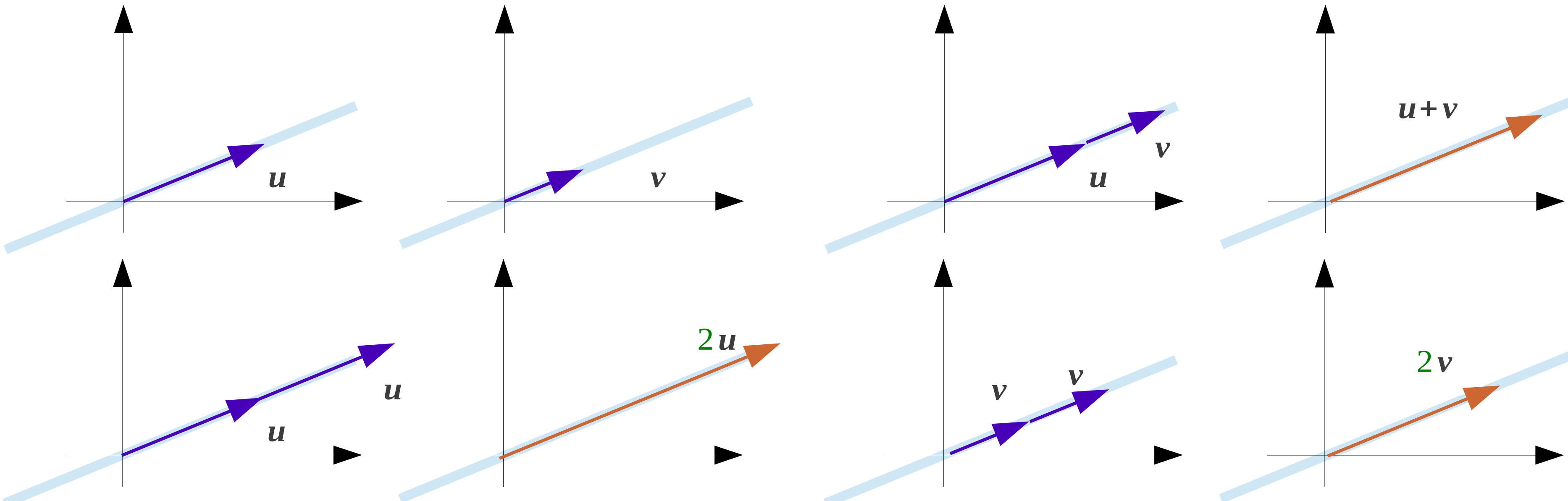
Subspaces of R^2

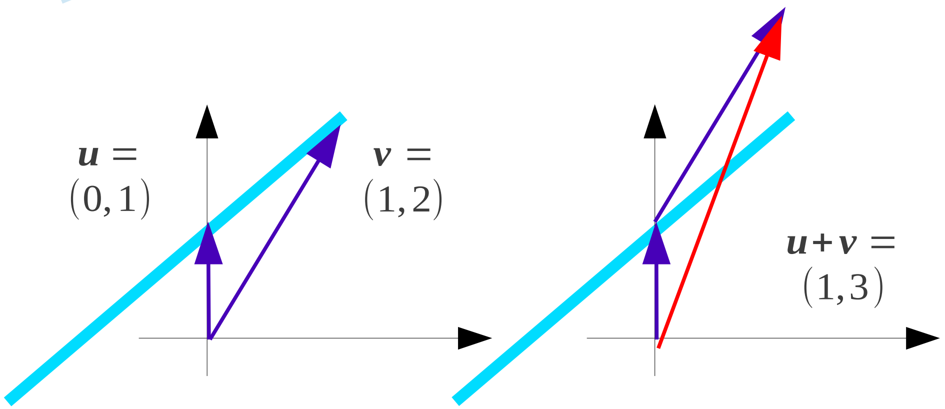


Subspace Example (2)

In vector space R^2

any one vector (linearly indep.) **spans** R^1 line through **0**





~~vector space~~

Subspace Example (3)

In vector space R^3

any one vector	(linearly indep.)	**spans**	R^1	line <u>through **0**</u>
any two non-collinear vectors	(linearly indep.)	**spans**	R^2	plane <u>through **0**</u>
any three vectors non-collinear, non-coplanar	(linearly indep.)	**spans**	R^3	3-dim space
any four or more vectors	(linearly dep.)	**spans**	R^3	3-dim space

Subspaces of R^3

$\{\mathbf{0}\}$	R^1	R^2	R^3
	line <u>through **0**</u>	plane <u>through **0**</u>	3-dim space

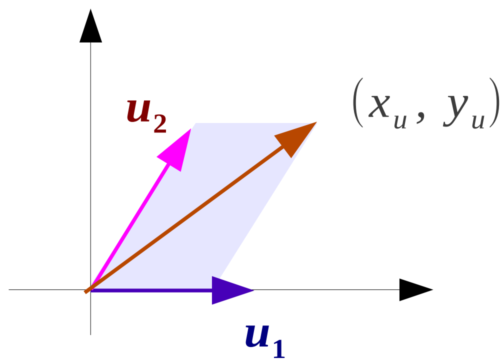
Dimension

In a finite-dimensional vector space R^n $\cancel{R^\infty}$

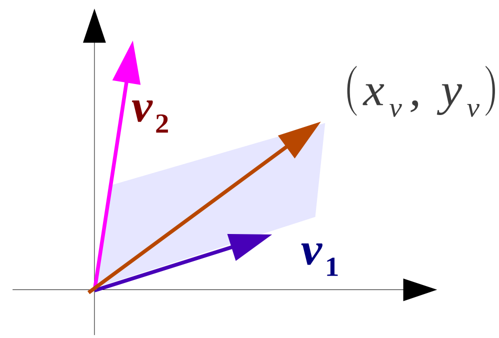
all bases ⇒ the same number of vectors n

many bases but the same number of basis vectors

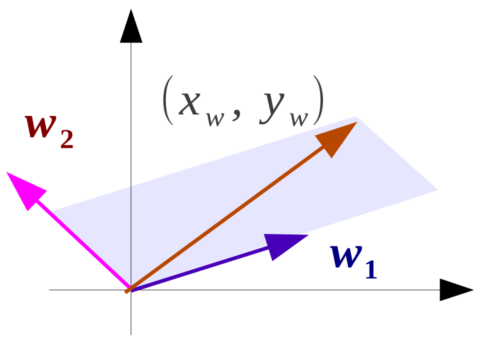
basis $\{\mathbf{u}_1, \ \mathbf{u}_2\}$ R^2



basis $\{\mathbf{v}_1, \ \mathbf{v}_2\}$ R^2



basis $\{\mathbf{w}_1, \ \mathbf{w}_2\}$ R^2



The dimension of a finite-dimensional vector space V

dim(V) ⇔ the number of vectors in a basis

Dimension of a Basis (1)

In vector space R^2

	any one vector	(linearly indep.)	~~spans~~	$\cancel{R^2}$	line through **0**
basis	any two non-collinear vectors	(linearly indep.)	spans	R^2	plane
	any three or more vectors	~~(linearly indep.)~~	spans	R^2	plane

In vector space R^3

	any one vector	(linearly indep.)	~~spans~~	$\cancel{R^3}$	line through **0**
	any two non-collinear vectors	(linearly indep.)	~~spans~~	$\cancel{R^3}$	plane through **0**
basis	any three vectors non-collinear, non-coplanar	(linearly indep.)	spans	R^3	3-dim space
	any four or more vectors	~~(linearly indep.)~~	spans	R^3	3-dim space

Dimension of a Basis (2)

In vector space R^n

	any n-1 vectors	(linearly indep.)?	~~spans~~	$\cancel{R^n}$	line <u>through **0**</u>
basis	n vectors of a basis	(linearly indep.)	**spans**	R^n	plane
	any n+1 vectors	~~(linearly indep.)~~	**spans**?	R^n	plane

a finite-dimensional vector space V

a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$

- a set of more than n vectors ➡ ~~(linearly indep.)~~
- a set of less than n vectors ➡ ~~**spans**~~ $\cancel{V}$

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ non-empty finite <u>set</u> of vectors in V

S is a basis ⇔
- S linearly independent
- S spans V

Basis Test

$S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ non-empty finite <u>set</u> of vectors in V

S is a basis $\Leftrightarrow$ $\begin{cases} S \text{ linearly independent} \\ S \text{ spans } V \end{cases}$

V an **n**-dimensional vector space

$S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ a set of **n** vectors in V

S linearly independent $\Rightarrow$ S is a basis

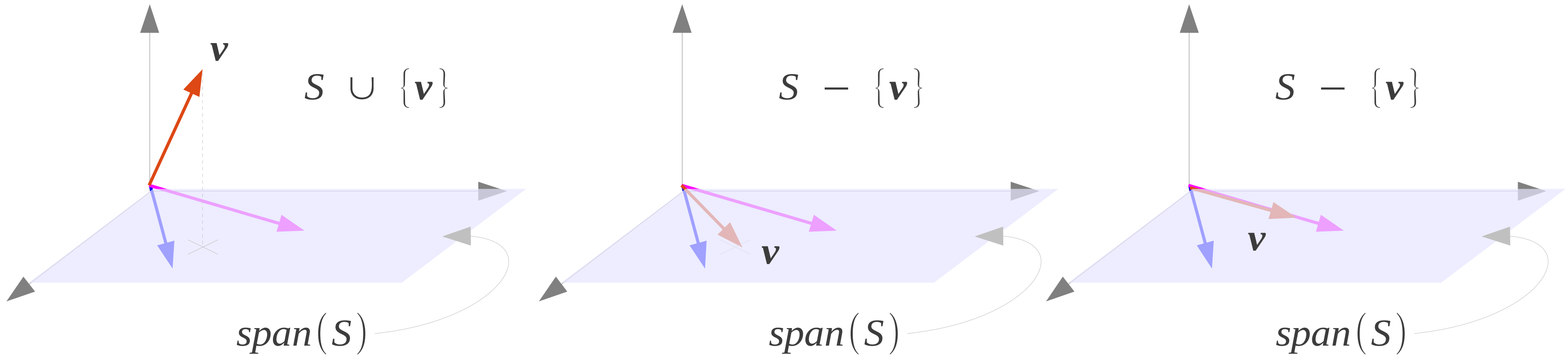
S spans V $\Rightarrow$ S is a basis

Plus / Minus Theorem

S a nonempty set of vectors in a vector space V

$\begin{cases} S & \text{: linear independent} \\ \mathbf{v} & \text{a vector in V but outside of span(S)} \end{cases} \Rightarrow S \cup \{\mathbf{v}\}$: linear independent

$\begin{cases} \mathbf{v}, \mathbf{u_i} \in S & \text{linear combination} \\ \mathbf{v} = k_1\mathbf{u_1} + k_2\mathbf{u_2} + \cdots + k_n\mathbf{u_n} \end{cases} \Rightarrow span(S) = span(S - \{\mathbf{v}\})$



Finding a Basis

S a nonempty set of vectors in a vector space V

$\begin{cases} S & \text{: linear independent} \\ \boldsymbol{v} & \text{a vector in V but outside of span(S)} \end{cases}$ $\Rightarrow$ $S \cup \{\boldsymbol{v}\}$: linear independent

if S is a *linearly independent* set that is *not already a basis* for V,
then S can be *enlarged* to a basis for V
by ***inserting*** appropriate vectors into S

$\begin{cases} \boldsymbol{v}\,,\ \boldsymbol{u_i} \in S & \text{linear combination} \\ \boldsymbol{v} = k_1 \boldsymbol{u_1} + k_2 \boldsymbol{u_2} + \cdots + k_n \boldsymbol{u_n} \end{cases}$ $\Rightarrow$ $span(S) = span(S - \{\boldsymbol{v}\})$

if S *spans* V but is *not a basis* for V,
then S can be *reduced* to a basis for V
by ***removing*** appropriate vectors from S

Vectors in a Vector Space

S a nonempty set of vectors in a vector space V

if S is a *linearly independent* set that is *not already a basis* for V,
then S can be *enlarged* to a basis for V
by ***inserting*** appropriate vectors into S

Every *linearly independent* set in a subspace is
either a **basis** for that subspace
or can be **extended to a basis** for it

if S *spans* V but is *not a basis* for V,
then S can be *reduced* to a basis for V
by ***removing*** appropriate vectors from S

Every *spanning set* for a subspace is
either a **basis** for that subspace
or has a **basis as a subset**

Dimension of a Subspace

W a subspace of a finite-dimensional vector space V

W is *finite-dimensional*

dim(W) ≤ dim(V)

W = V ⟺ dim(W) = dim(V)

Vector Space Examples

$\{\mathbf{0}\}$

R^n

M_{mn} mxn matrix

$F(-\infty, +\infty)$ real-valued functions in the interval $(-\infty, +\infty)$

$C(-\infty, +\infty)$ real-valued continuous functions in the interval $(-\infty, +\infty)$

$C^1(-\infty, +\infty)$ real-valued continuously differentiable functions in $(-\infty, +\infty)$

P_∞ $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$

the solution space $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ in n unknowns R^n

Real-Valued Functions (1)

V the set of real-valued functions

defined at every x in $(-\infty,\ +\infty)$

$$\boldsymbol{u} = u(x) \qquad \boldsymbol{u}+\boldsymbol{v} = u(x)+v(x)$$

$$\boldsymbol{v} = v(x) \qquad k\boldsymbol{u} = k\,u(x)$$

1. if **u** and **v** are objects in V, then **u** + **v** is in V
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. **0** + **u** = **u** + **0** = **u** (zero vector)
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6. if *k* is any scalar and **u** is objects in V, then *k***u** is in V
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10. *1*(**u**) = **u**

Real-Valued Functions (2)

V the set of real-valued functions $\{\sin(x),\ \sin(2x),\ \sin(3x),\ \cdots\}$

defined at every x in $[0,\ 2\pi]$

$$\boldsymbol{u_1} = \sin(x)$$
$$\boldsymbol{u_2} = \sin(2x)$$
$$\boldsymbol{u_3} = \sin(3x)$$
$$\cdots$$

$$\boldsymbol{u_m} + \boldsymbol{v_n} = \sin(mx) + \sin(nx)$$
$$k\,\boldsymbol{u_m} = k\sin(mx)$$

1. if **u** and **v** are objects in V, then **u** + **v** is in V
2. **u** + **v** = **v** + **u**
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V basis R^{∞}

linear independent

References

[1] http://en.wikipedia.org/
[2] Anton, et al., Elementary Linear Algebra, 10th ed, Wiley, 2011
[3] Anton, et al., Contemporary Linear Algebra,