

CTFS (1B)

- Continuous Time Fourier Series

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Fourier Series

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx dx$$

$$k = 1, 2, 3, \dots$$



one-sided spectrum
only positive frequencies

Trigonometric Identities

$$\cos \theta \cos \phi = \frac{1}{2} (\cos(\theta - \phi) + \cos(\theta + \phi))$$

$$\sin \theta \sin \phi = \frac{1}{2} (\cos(\theta - \phi) - \cos(\theta + \phi))$$

$$\sin \theta \cos \phi = \frac{1}{2} (\sin(\theta + \phi) + \sin(\theta - \phi))$$

$$\cos \theta \sin \phi = \frac{1}{2} (\sin(\theta + \phi) - \sin(\theta - \phi))$$

$$\frac{1}{2} (1 + \cos(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\frac{1}{2} (1 - \cos(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\frac{1}{2} (\sin(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\frac{1}{2} (\sin(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\int_{-\pi}^{+\pi} \cos nx \cos mx \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \sin mx \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \cos mx \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos nx \sin mx \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos nx \cos mx \, dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin nx \sin mx \, dx = \pi \quad (n = m)$$

n, m : integer

Trigonometric Orthogonality

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} \underline{f(x) \cos kx} dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} \underline{f(x) \sin kx} dx$$

$k = 1, 2, 3, \dots$

$$\int_{-\pi}^{+\pi} \cos nx \cos mx dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \sin mx dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \cos mx dx = 0$$

$$\int_{-\pi}^{+\pi} \cos nx \sin mx dx = 0$$

$$\int_{-\pi}^{+\pi} \underline{\cos nx \cos mx} dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \underline{\sin nx \sin mx} dx = \pi \quad (n = m)$$

n, m : integer

$$a_k \leftarrow \underline{f(x) \cdot \cos kx} = a_0 \cdot \cos kx + \sum_{m=1}^{\infty} (a_m \underline{\cos mx \cdot \cos kx} + b_m \sin mx \cdot \cos kx)$$

$$b_k \leftarrow \underline{f(x) \cdot \sin kx} = a_0 \cdot \sin kx + \sum_{m=1}^{\infty} (a_m \cos mx \cdot \sin kx + b_m \underline{\sin mx \cdot \sin kx})$$

Any Period $p = 2L$

$$g(v) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kv + b_k \sin kv)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} g(v) dv$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \cos kv dv$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \sin kv dv$$

$k = 1, 2, \dots$

$$v: [-\pi, +\pi]$$

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{L} x + b_k \sin \frac{k\pi}{L} x \right)$$

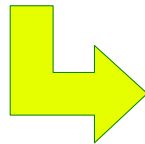
$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x) dx$$

$$a_k = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{k\pi x}{L} dx$$

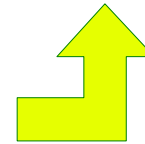
$$b_k = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{k\pi x}{L} dx$$

$k = 1, 2, 3, \dots$

$$x: [-L, +L]$$



$$v = \frac{\pi}{L} x$$
$$dv = \frac{\pi}{L} dx$$



Time and Frequency

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{L} x + b_k \sin \frac{k\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x) dx$$

$$a_k = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{k\pi x}{L} dx$$

$$b_k = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{k\pi x}{L} dx$$

$k = 1, 2, 3, \dots$

$$x: [-L, +L]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{2\pi k}{T} t + b_k \sin \frac{2\pi k}{T} t \right)$$

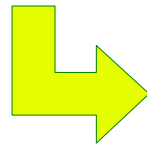
$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos \frac{2\pi k t}{T} dt$$

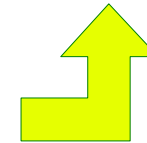
$$b_k = \frac{2}{T} \int_0^T x(t) \sin \frac{2\pi k t}{T} dt$$

$k = 1, 2, \dots$

$$t: [0, T]$$



$$2L = T$$



Continuous Time Periodic Signal $x(t)$

Harmonic Frequency

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{2\pi k}{T} t + b_k \sin \frac{2\pi k}{T} t \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos \frac{2\pi k t}{T} dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin \frac{2\pi k t}{T} dt$$

$k = 1, 2, \dots$

$$t: [0, T]$$

resolution frequency

n-th harmonic frequency

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(2\pi k f_0 t) + b_k \sin(2\pi k f_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(2\pi k f_0 t) dt \quad k = 1, 2, \dots$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(2\pi k f_0 t) dt \quad k = 1, 2, \dots$$

$$t: [0, T]$$

$$f_0 = \frac{1}{T}$$

$$f_n = n f_0 = n \frac{1}{T}$$

Radial Frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(k 2\pi f_0 t) + b_n \sin(k 2\pi f_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k 2\pi f_0 t) dt \quad k = 1, 2, \dots$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k 2\pi f_0 t) dt \quad k = 1, 2, \dots$$

$$t: [0, T]$$

linear frequency

angular (radial) frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(\mathbf{k} \omega_0 t) + b_n \sin(\mathbf{k} \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(\mathbf{k} \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(\mathbf{k} \omega_0 t) dt$$
$$k = 1, 2, \dots$$

$$t: [0, T]$$

f

$$\omega = 2\pi f$$

Complex Fourier Series Coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k \omega_0 t) dt$$

$k = 1, 2, \dots$

$$t: [0, T]$$

Real coefficients

$$a_0, a_k, b_k, k = 1, 2, \dots$$

Complex coefficients

$$A_0, A_k, B_k, k = 1, 2, \dots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{jk \omega_0 t} + B_k e^{-jk \omega_0 t})$$

$$A_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$A_k = \frac{1}{T} \int_0^T x(t) e^{-jk \omega_0 t} dt$$

$$B_k = \frac{1}{T} \int_0^T x(t) e^{+jk \omega_0 t} dt$$

$$t: [0, T]$$

one-sided spectrum

only positive frequencies

two-sided spectrum

Both pos and neg frequencies

Euler Equation (1)

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k \omega_0 t) dt$$

$k = 1, 2, \dots$

$$e^{+j\omega t} = \cos \omega t + j \sin \omega t$$

$$e^{-j\omega t} = \cos \omega t - j \sin \omega t$$

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$\begin{aligned} & a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t) \\ &= a_k \frac{1}{2} (e^{jk\omega_0 t} + e^{-jk\omega_0 t}) + b_k \frac{1}{2j} (e^{jk\omega_0 t} - e^{-jk\omega_0 t}) \\ &= \frac{(a_k - jb_k)}{2} e^{jk\omega_0 t} + \frac{(a_k + jb_k)}{2} e^{-jk\omega_0 t} \\ &= A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \end{aligned}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

zero freq	→	A_0	=	a_0	}	only positive frequencies
pos freq	→	A_k	=	$\frac{1}{2} (a_k - jb_k)$		
neg freq	→	B_k	=	$\frac{1}{2} (a_k + jb_k)$		

Euler Equation (2)

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k \omega_0 t) dt$$

$$k = 1, 2, \dots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

zero freq	→	$A_0 = a_0$	} only positive frequencies
pos freq	→	$A_k = \frac{1}{2} (a_k - jb_k)$	
neg freq	→	$B_k = \frac{1}{2} (a_k + jb_k)$	

$$A_k = \frac{1}{T} \int_0^T x(t) (\cos(k \omega_0 t) - j \sin(k \omega_0 t)) dt$$

$$B_k = \frac{1}{T} \int_0^T x(t) (\cos(k \omega_0 t) + j \sin(k \omega_0 t)) dt$$



$$A_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$B_k = \frac{1}{T} \int_0^T x(t) e^{+jk\omega_0 t} dt$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$



$$x(t) = \sum_{k=0}^{\infty} (A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

Complex Fourier Series

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k \omega_0 t) dt$$
$$k = 1, 2, \dots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

$$A_0 = a_0$$

$$A_k = \frac{1}{2} (a_k - jb_k)$$

$$B_k = \frac{1}{2} (a_k + jb_k)$$

$$k = 1, 2, \dots$$

$$x(t) = \sum_{k=0}^{\infty} (A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

$$A_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$
$$k = 0, 1, 2, \dots$$

$$B_k = \frac{1}{T} \int_0^T x(t) e^{+jk\omega_0 t} dt$$
$$k = 1, 2, \dots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$
$$k = -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} A_0 & (k = 0) \\ A_k & (k > 0) \\ B_k & (k < 0) \end{cases}$$

Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k \omega_0 t) dt$$

$$k = +1, +2, \dots$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k \omega_0 t + \phi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1} \left(-\frac{b_k}{a_k} \right)$$

$$k = +1, +2, \dots$$

$$\cos(\alpha + \beta) = \underline{\cos(\alpha) \cos(\beta)} - \underline{\sin(\alpha) \sin(\beta)}$$

$$g_k \cos(k \omega_0 t + \phi_k) = \underline{g_k \cos(\phi_k) \cos(k \omega_0 t)} - \underline{g_k \sin(\phi_k) \sin(k \omega_0 t)}$$

$$\underline{a_k \cos(k \omega_0 t)} + \underline{b_k \sin(k \omega_0 t)}$$

$$a_k = g_k \cos(\phi_k)$$

$$-b_k = g_k \sin(\phi_k)$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

Phasor Representation (1)

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k \omega_0 t) dt$$
$$k = 1, 2, \dots$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k \omega_0 t + \phi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1} \left(-\frac{b_k}{a_k} \right)$$

$$k = 1, 2, \dots$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k \omega_0 t + \phi_k)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \Re \{ e^{+j(k \omega_0 t + \phi_k)} \}$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \Re \{ g_k \cdot e^{+j \phi_k} \cdot e^{+j k \omega_0 t} \}$$

$$x(t) = X_0 + \sum_{k=1}^{\infty} \Re \{ X_k e^{+j k \omega_0 t} \}$$

$$X_0 = g_0$$

$$X_k = g_k \cdot e^{+j \phi_k}$$

$$k = 1, 2, \dots$$

Phasor Representation (2)

$$x(t) = g_0 + \sum_{k=1}^{\infty} \frac{g_k}{2} \cdot \left(e^{+j(k\omega_0 t + \phi_k)} + e^{-j(k\omega_0 t + \phi_k)} \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \left(\frac{g_k}{2} e^{+j\phi_k} e^{+jk\omega_0 t} + \frac{g_k}{2} e^{-j\phi_k} e^{-jk\omega_0 t} \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \left(\frac{g_k e^{+j\phi_k}}{2} e^{+jk\omega_0 t} + \frac{g_k e^{-j\phi_k}}{2} e^{-jk\omega_0 t} \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1} \left(-\frac{b_k}{a_k} \right)$$

$$k = 1, 2, \dots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$x(t) = X_0 + \sum_{k=1}^{\infty} \Re \{ X_k e^{+jk\omega_0 t} \}$$

$$C_k = \frac{g_k e^{+j\phi_k}}{2} \quad (k > 0)$$

$$C_{-k} = \frac{g_k e^{-j\phi_k}}{2} \quad (k < 0)$$

$$k = -2, -1, 0, +1, +2, \dots$$

$$X_0 = g_0$$

$$X_k = g_k e^{+j\phi_k}$$

$$k = 1, 2, \dots$$

Two-Sided Spectrum

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \neq 0) \end{cases}$$

$$\text{Arg}(C_k) = \begin{cases} \tan^{-1}(-b_k/a_k) & (k > 0) \\ \tan^{-1}(+b_k/a_k) & (k < 0) \end{cases}$$

Power Spectrum *Two-Sided*

$$\underline{|C_k|^2 + |C_{-k}|^2} = \frac{1}{2}|g_k|^2 = \frac{1}{2}(a_k^2 + b_k^2)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}g_k e^{+j\phi_k} & (k > 0) \\ \frac{1}{2}g_k e^{-j\phi_k} & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \neq 0) \end{cases}$$

$$\text{Arg}(C_k) = \begin{cases} +\phi_k & (k > 0) \\ -\phi_k & (k < 0) \end{cases}$$

Periodogram *One-Sided*

$$2 \cdot |C_k| = \underline{|g_k|} = \underline{\sqrt{a_k^2 + b_k^2}}$$

CTFS of Impulse Train (1)

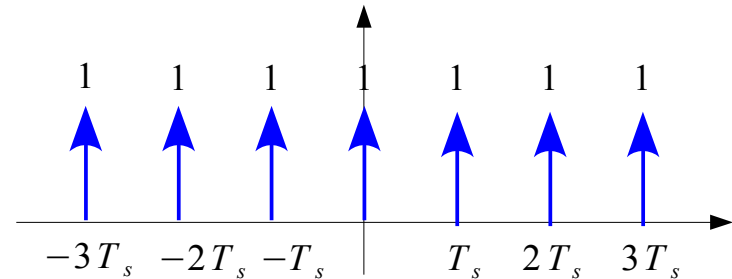
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Fourier Series Expansion of Impulse Train

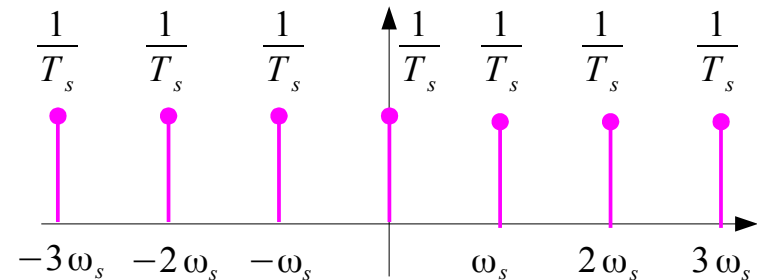
$$p(t) = \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t}$$

Fourier Series Coefficients

$$\begin{aligned} a_k &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jk\omega_s 0} dt \\ &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) dt = \frac{1}{T_s} \end{aligned}$$



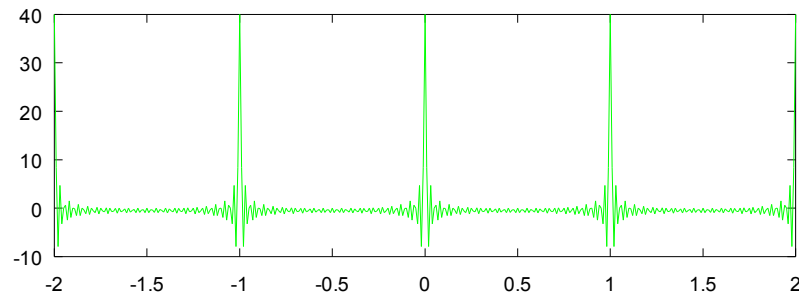
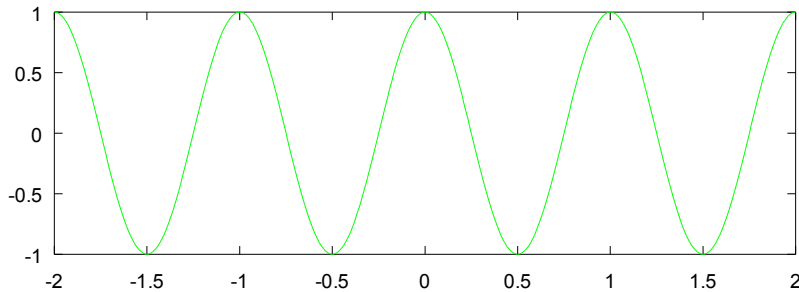
$$\omega_s = \frac{2\pi}{T_s}$$



CTFS of Impulse Train (2)

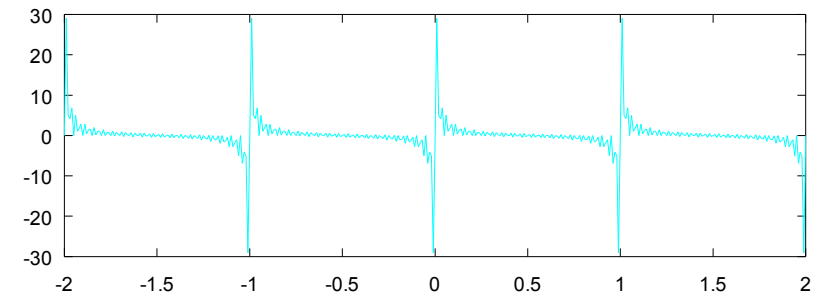
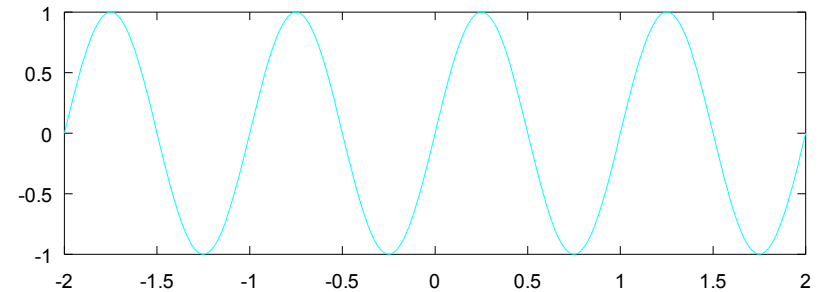
$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} (\cos k\omega_s t - j \sin k\omega_s t)$$

$\cos 2\pi \cdot 1 \cdot t$



$$\sum_{k=1}^{40} \cos 2\pi \cdot k \cdot t$$

$\sin 2\pi \cdot 1 \cdot t$

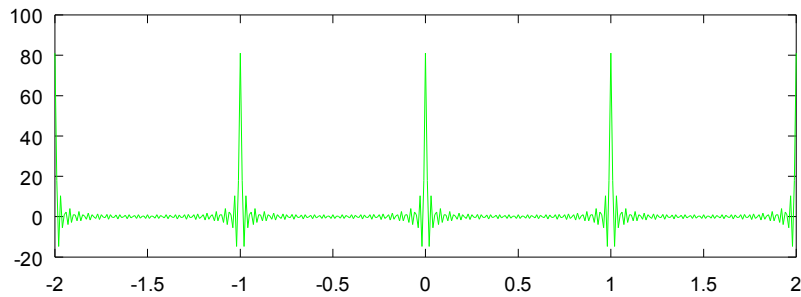
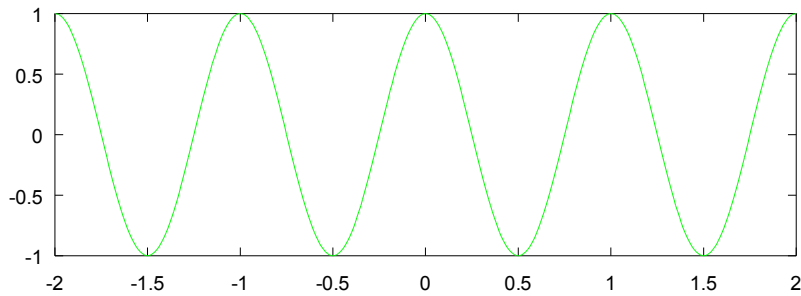


$$\sum_{k=1}^{40} \sin 2\pi \cdot k \cdot t$$

CTFS of Impulse Train (3)

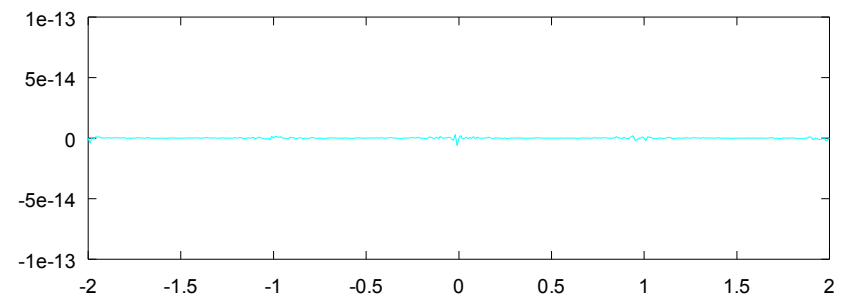
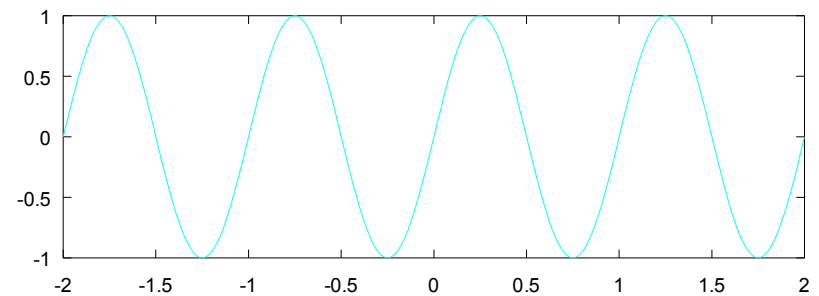
$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} (\cos k\omega_s t - j \sin k\omega_s t)$$

$\cos 2\pi \cdot 1 \cdot t$



$$\sum_{k=-40}^{40} \cos 2\pi \cdot k \cdot t$$

$\sin 2\pi \cdot 1 \cdot t$



$$\sum_{k=-40}^{40} \sin 2\pi \cdot k \cdot t$$

Square Wave CTFS (1)

Continuous Time Fourier Series

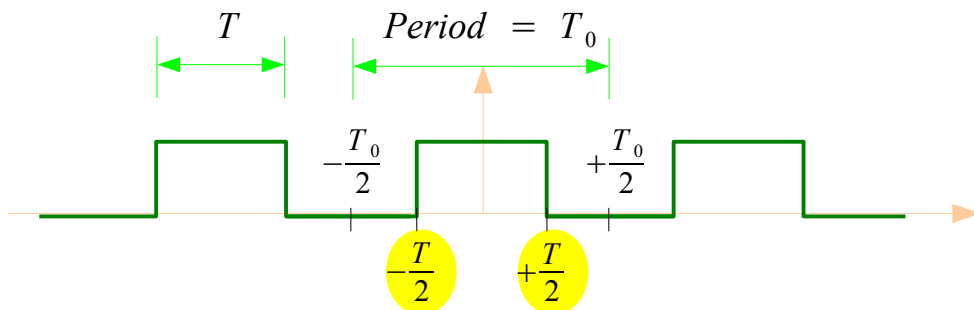
$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt \quad \longleftrightarrow \quad x(t) = \sum_{n=-\infty}^{\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(t) e^{-jk\omega_0 t} dt$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t) e^{-jk\omega_0 t} dt$$

$$= \int_{-T_0/2}^{+T_0/2} e^{-jk\omega_0 t} dt = \left[\frac{-1}{jk\omega_0} e^{-jk\omega_0 t} \right]_{-T_0/2}^{+T_0/2}$$

$$= -\frac{e^{-jk\omega_0 T_0/2} - e^{+jk\omega_0 T_0/2}}{jk\omega_0} = \frac{\sin(k\omega_0 T_0/2)}{k\omega_0/2}$$

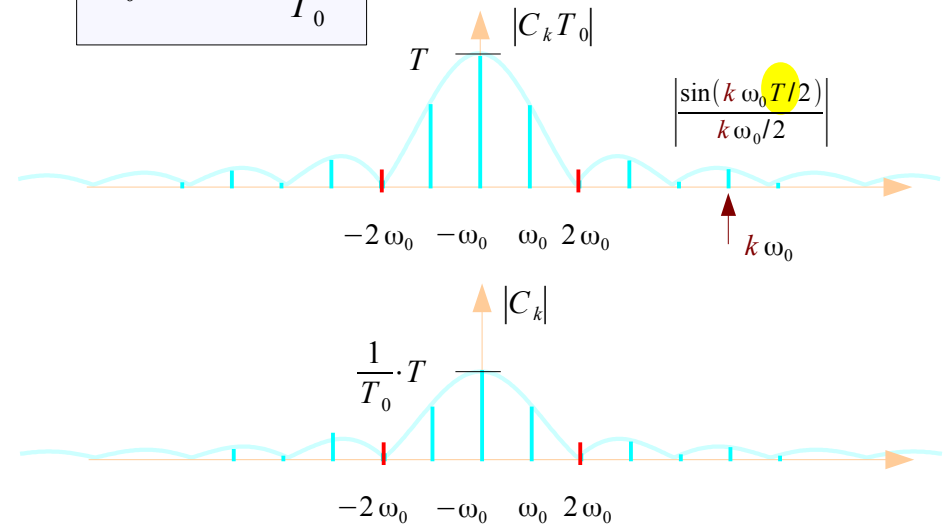


Fundamental Frequency

$$\omega_0 = \frac{2\pi}{T_0} \quad \omega_0 T_0 = 2\pi \quad \leftarrow \quad \frac{T_0}{T} = \frac{2}{1}$$

$$\omega_0 T = \pi$$

$$\omega_0 T = 2\pi \frac{T}{T_0}$$



Square Wave CTFS (2)

$$C_k = \frac{1}{T_0} \cdot \frac{\sin(k \omega_0 T/2)}{k \omega_0/2} = \frac{1}{T_0} \cdot \frac{\sin(T k \omega_0/2)}{k \omega_0/2}$$

$$C_k = 0 \quad \rightarrow \quad \sin(k \omega_0 T/2) = 0$$

$$\sin\left(k \frac{2\pi T}{T_0} \frac{T}{2}\right) = 0 \quad \rightarrow \quad \sin(\pm n \pi) = 0$$

$$k = \pm n \frac{T_0}{T} \quad \rightarrow \quad \omega = k \omega_0 = \pm n \frac{T_0}{T} \omega_0$$

$$C_0 = \lim_{k \rightarrow 0} \frac{1}{T_0} \cdot \frac{\sin(T k \omega_0/2)}{k \omega_0/2}$$

$$C_0 = \lim_{k \rightarrow 0} \frac{1}{T_0} \cdot \frac{(T \omega_0/2) \cos(T k \omega_0/2)}{\omega_0/2} = \frac{T}{T_0}$$

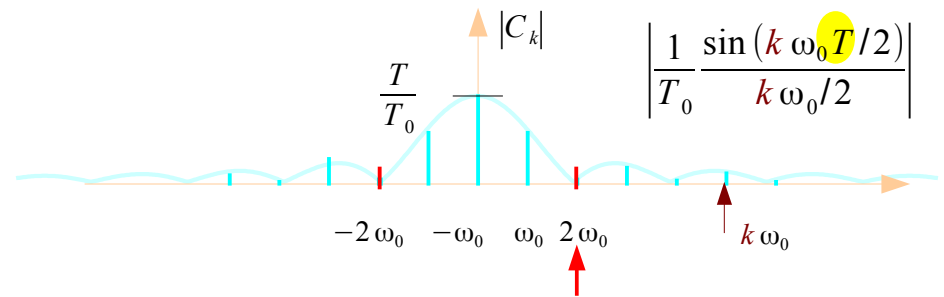
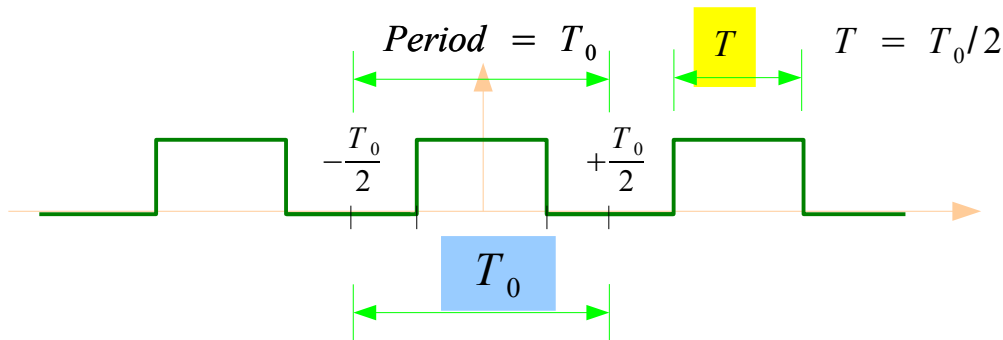
$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_0 \frac{T_0}{T} = \frac{2\pi}{T}$$

$$\omega_0 T = 2\pi \frac{T}{T_0}$$

$$C_k = \frac{1}{T_0} \cdot \frac{\sin(k \omega_0 T/2)}{k \omega_0/2}$$

$$C_0 = \frac{T}{T_0}$$



Square Wave CTFS (3)

$$C_k = \frac{1}{T_0} \cdot \frac{\sin(k \omega_0 T/2)}{k \omega_0/2}$$

$$C_0 = \frac{T}{T_0}$$

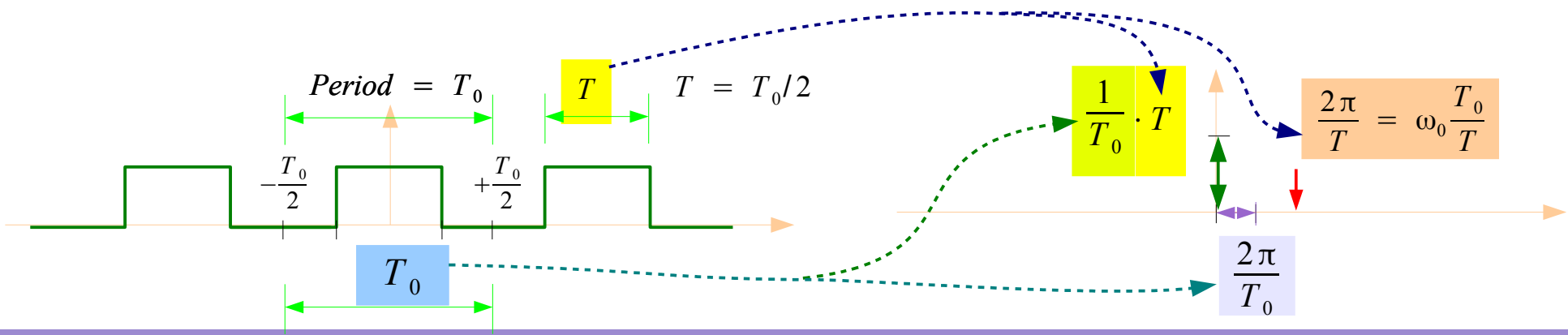
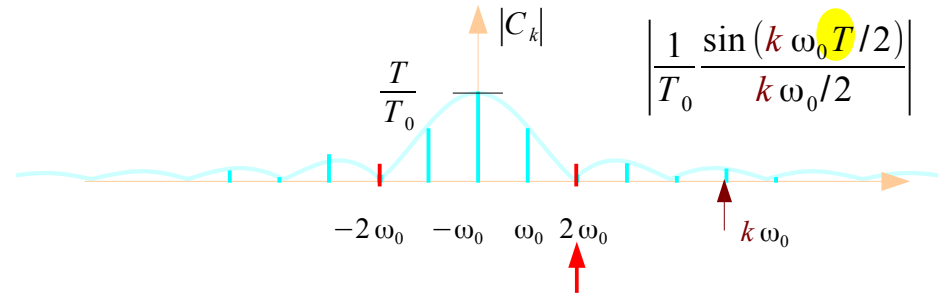
$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_0 \frac{T_0}{T} = \frac{2\pi}{T}$$

$$\omega_0 T = 2\pi \frac{T}{T_0}$$

$$\omega = k \omega_0 = \pm n \frac{T_0}{T} \omega_0 \quad \rightarrow$$

$$C_k = 0 \quad \left(k = \pm n \frac{T_0}{T} \right) \quad \uparrow$$



Square Wave CTFS (4)

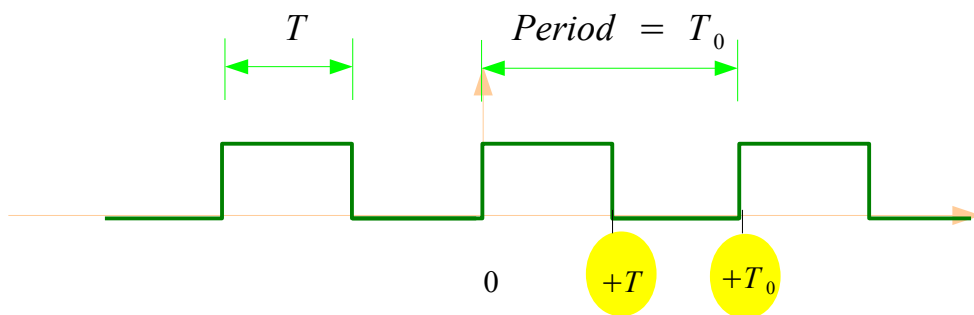
Continuous Time Fourier Series

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt \quad \longleftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0} \int_0^{+T_0} x_{T_0}(t) e^{-jk\omega_0 t} dt$$

$$C_k T_0 = \int_0^{+T_0} x_{T_0}(t) e^{-jk\omega_0 t} dt$$

$$\begin{aligned} &= \int_0^{+T} e^{-jk\omega_0 t} dt = \left[\frac{-1}{jk\omega_0} e^{-jk\omega_0 t} \right]_0^{+T} \\ &= -\frac{e^{-jk\omega_0 T} - e^0}{jk\omega_0} = \frac{1 - e^{-jk\omega_0 T}}{jk\omega_0} = \frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi/T_0} \end{aligned}$$



Fundamental Frequency

$$\omega_0 = \frac{2\pi}{T_0} \quad \omega_0 T_0 = 2\pi \quad \frac{T_0}{T} = \frac{2}{1}$$

$$\omega_0 T = \pi$$

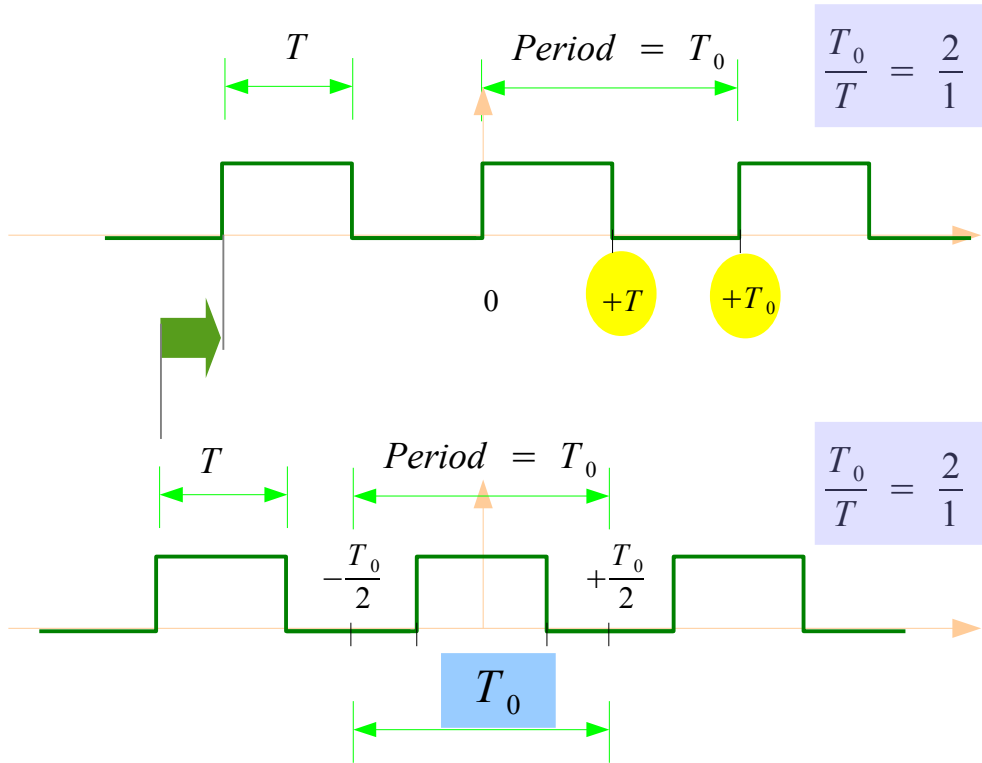
$$\omega_0 T = 2\pi \frac{T}{T_0}$$

$$C_k T_0 = \frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi/T_0} \quad \rightarrow \quad C_k = \frac{1 - (-1)^k}{j2\pi k}$$

$$C_0 T_0 = \int_0^{+T} e^{-j0\omega_0 t} dt = T \quad \rightarrow \quad C_0 = \frac{1}{2}$$

C_{-4}	C_{-3}	C_{-2}	C_{-1}	C_0	C_1	C_2	C_3	C_4
0	$\frac{-1}{j3\pi}$	0	$\frac{-1}{j\pi}$	$\frac{1}{2}$	$\frac{1}{j\pi}$	0	$\frac{1}{j3\pi}$	0

Square Wave CTFS (5)



$$C_k = \frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi}$$

$$C_0 = \frac{T}{T_0}$$

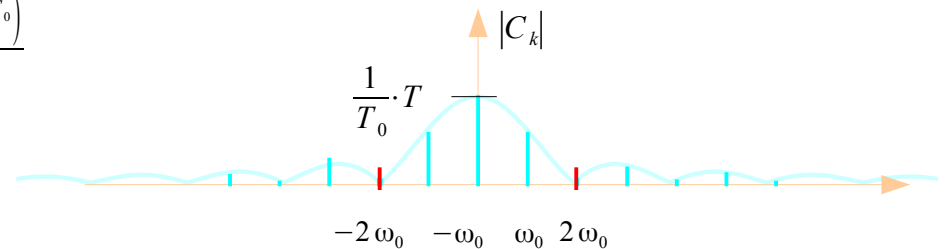
$$e^{+jk\omega_0 T/2}$$

$$C_k = \frac{1}{T_0} \cdot \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$C_0 = \frac{T}{T_0}$$

$$\frac{1}{T_0} \cdot \frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{\sin(k\pi T/T_0)}{k\pi} = \frac{(e^{+jk\pi T/T_0} - e^{-jk\pi T/T_0})}{jk2\pi}$$

$$= e^{+jk\pi T/T_0} \left(\frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi} \right) = e^{+jk\omega_0 T/2} \left(\frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi} \right)$$



Inner Product Space

Hilbert Space real / complex inner product space

$$\langle f, g \rangle = \int_a^b f(t) \overline{g(t)} dt$$

complex conjugate

$$\langle y, x \rangle = \overline{\langle x, y \rangle}$$

linear

$$\langle a x_1 + b x_2, y \rangle = a \langle x_1, y \rangle + b \langle x_2, y \rangle$$

positive semidefinite

$$\langle x, x \rangle \geq 0$$

Norm

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Cauchy-Schwartz Inequality

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Orthogonality

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

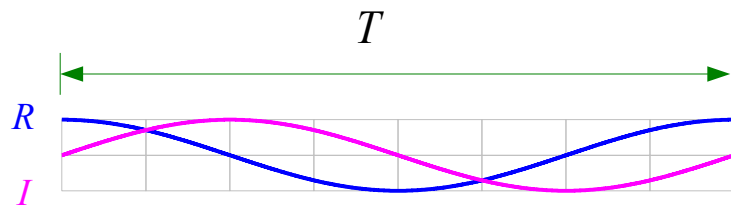
$$k = \dots, -2, -1, 0, +1, +2, \dots$$

fundamental frequency $f_0 = \frac{1}{T}$ $\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$

n-th harmonic frequency $f_n = n f_0$ $\omega_n = 2\pi f_n = \frac{2\pi n}{T}$

$$\langle e^{jm\omega_0 t}, e^{jn\omega_0 t} \rangle = \int_0^T e^{+j(m-n)\omega_0 t} dt = \begin{cases} 0 & (m \neq n) \\ T & (m = n) \end{cases} \quad m, n : \text{integer}$$

Inner Product Examples

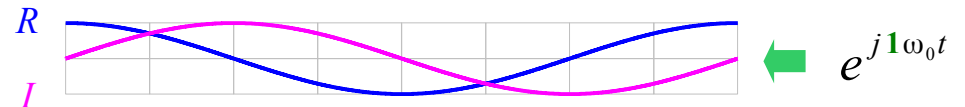


$$f_0 = 1/T$$

$$\omega_0 = 2\pi/T$$

$$\leftarrow e^{j1\omega_0 t}$$

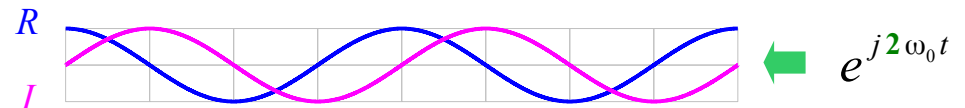
$$\langle e^{j1\omega_0 t}, e^{j1\omega_0 t} \rangle = \int_0^T e^{+j(1-1)\omega_0 t} dt = T \quad \leftarrow$$



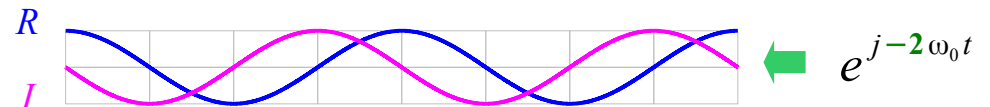
$$\langle e^{j1\omega_0 t}, e^{j-1\omega_0 t} \rangle = \int_0^T e^{+j(1+1)\omega_0 t} dt = 0 \quad \leftarrow$$



$$\langle e^{j1\omega_0 t}, e^{j2\omega_0 t} \rangle = \int_0^T e^{+j(1-2)\omega_0 t} dt = 0 \quad \leftarrow$$



$$\langle e^{j1\omega_0 t}, e^{j-2\omega_0 t} \rangle = \int_0^T e^{+j(1+2)\omega_0 t} dt = 0 \quad \leftarrow$$



Cauchy-Schwartz Inequality

For all vectors \mathbf{x} and \mathbf{y} of an inner product space

$$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \cdot \langle \mathbf{y}, \mathbf{y} \rangle$$

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$$

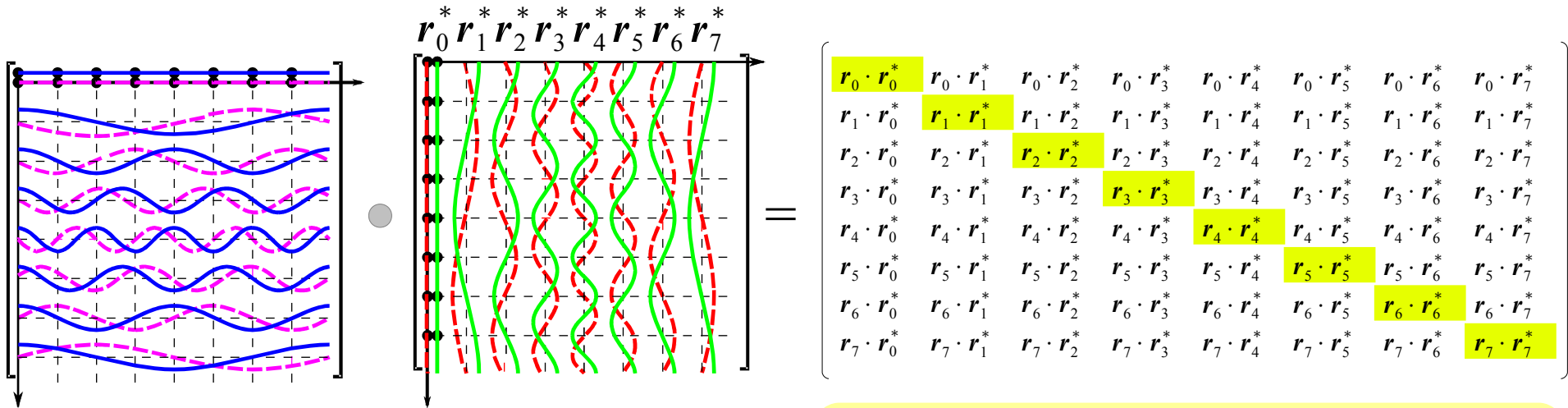
The equality holds if and only if \mathbf{x} and \mathbf{y} are linearly dependent \Rightarrow maximum

$$\left| \int_a^b x(t) \overline{y(t)} dt \right| \leq \sqrt{\int_a^b x(t) \overline{x(t)} dt} \sqrt{\int_a^b y(t) \overline{y(t)} dt}$$

Inner product is maximum

when $y = kx$

Orthogonality



$$\langle \mathbf{r}_i^H, \mathbf{r}_i^H \rangle = \mathbf{r}_i \cdot \mathbf{r}_i^* = N$$

$$\langle \mathbf{r}_i^H, \mathbf{r}_j^H \rangle = \mathbf{r}_i \cdot \mathbf{r}_j^* = 0 \quad (i \neq j)$$

Complex Vector Inner Product

Hermitian inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^H \cdot \mathbf{y} = \sum x_i^* y_i \quad \mathbf{x}^H : \text{conjugate transpose}$$

Norm of Hermitian inner products

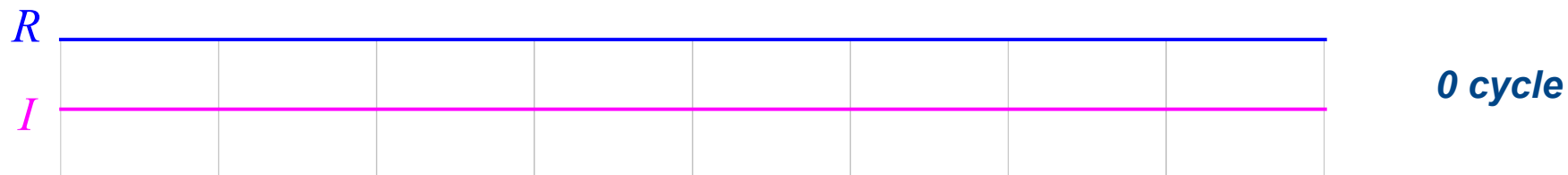
$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\mathbf{x}^H \cdot \mathbf{x}} = \sqrt{\sum x_i^* x_i} \quad \text{the length of a vector}$$

$$\mathbf{x} = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ \vdots \\ a_n + jb_n \end{pmatrix}$$

$$\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^H \cdot \mathbf{x} = \sum x_i^* x_i$$

$$\begin{pmatrix} a_1 - jb_1 & a_2 - jb_2 & \cdots & a_n - jb_n \end{pmatrix} \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ \vdots \\ a_n + jb_n \end{pmatrix} = \sum_{i=1}^n a_i^2 + b_i^2$$

The 1st Row of the DFT Matrix



$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \quad k = 0, \quad n = 0, 1, \dots, 7$$

$$\begin{array}{l}
 R \rightarrow \text{samples of } \cos(-\omega t) = \cos(\omega t) \\
 I \rightarrow \text{samples of } \sin(-\omega t) = -\sin(\omega t)
 \end{array}
 \left. \vphantom{\begin{array}{l} R \\ I \end{array}} \right\} \xrightarrow{\text{measure}} \begin{array}{l} \omega t = 2\pi f t \\ 2\pi \cdot \left(\frac{0}{8}\right) \cdot f_s \cdot t \end{array}$$

$X[0]$ measures how much of the $+0 \cdot \omega$ component is present in x .

The 3rd Row of the DFT Matrix

R

I

2 cycles

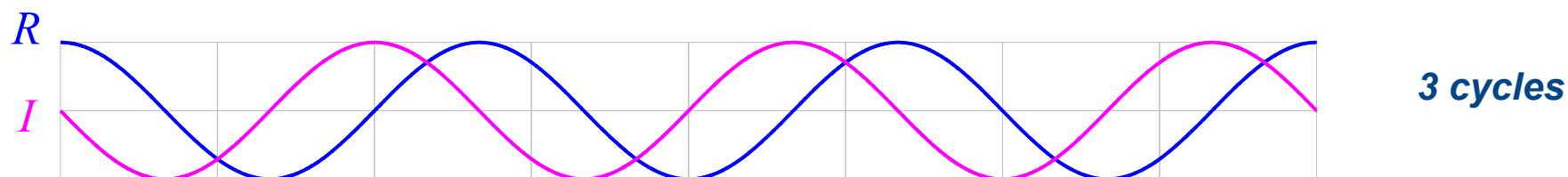
$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \quad k = 2, \quad n = 0, 1, \dots, 7$$

$R \rightarrow$ samples of $\cos(-2\omega t) = \cos(2\omega t)$
 $I \rightarrow$ samples of $\sin(-2\omega t) = -\sin(2\omega t)$ } *measure* \rightarrow

$$\begin{aligned} \omega t &= 2\pi f t \\ &= 2\pi \cdot \left(\frac{2}{8}\right) \cdot f_s \cdot t \end{aligned}$$

$X[2]$ measures how much of the $+2 \cdot \omega$ component is present in x .

The 4th Row of the DFT Matrix

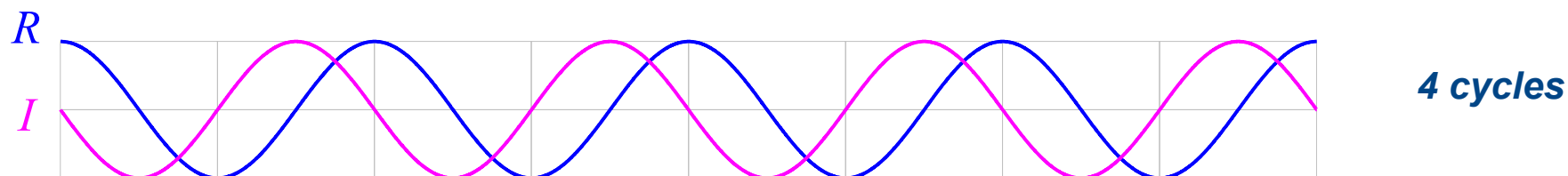


$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \quad k = 3, \quad n = 0, 1, \dots, 7$$

$$\begin{array}{l}
 R \rightarrow \text{samples of } \cos(-3\omega t) = \cos(3\omega t) \\
 I \rightarrow \text{samples of } \sin(-3\omega t) = -\sin(3\omega t)
 \end{array}
 \left. \vphantom{\begin{array}{l} R \\ I \end{array}} \right\} \xrightarrow{\text{measure}} \begin{array}{l} \omega t = 2\pi f t \\ 2\pi \cdot \left(\frac{3}{8}\right) \cdot f_s \cdot t \end{array}$$

$X[3]$ measures how much of the $+3 \cdot \omega$ component is present in x .

The 5th Row of the DFT Matrix

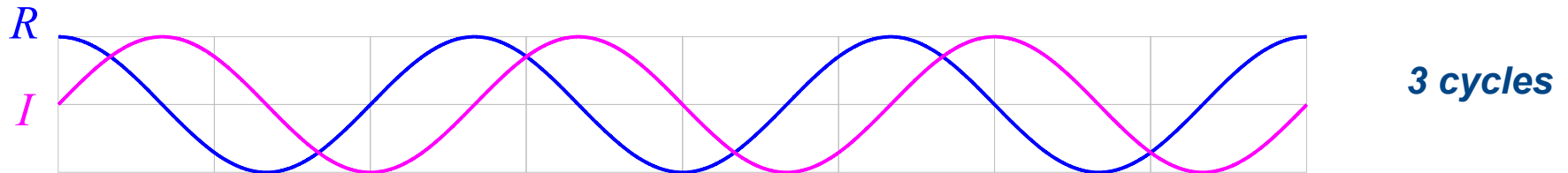


$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \quad k = 4, \quad n = 0, 1, \dots, 7$$

$$\begin{array}{l}
 R \rightarrow \text{samples of } \cos(-4\omega t) = \cos(4\omega t) \\
 I \rightarrow \text{samples of } \sin(-4\omega t) = -\sin(4\omega t)
 \end{array}
 \left. \vphantom{\begin{array}{l} R \\ I \end{array}} \right\} \text{measure} \rightarrow \begin{array}{l} \omega t = 2\pi f t \\ 2\pi \cdot \left(\frac{4}{8}\right) \cdot f_s \cdot t \end{array}$$

$X[4]$ measures how much of the $+4 \cdot \omega$ component is present in x .

The 6th Row of the DFT Matrix

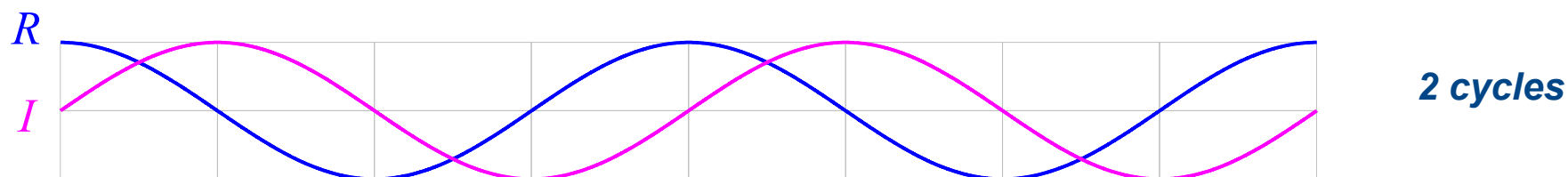


$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \quad k = 5, \quad n = 0, 1, \dots, 7$$

$$\begin{array}{l}
 R \rightarrow \text{samples of } \cos(-(-3\omega)t) = \cos(3\omega t) \\
 I \rightarrow \text{samples of } \sin(-(-3\omega)t) = \sin(3\omega t)
 \end{array}
 \left. \vphantom{\begin{array}{l} R \\ I \end{array}} \right\} \text{measure} \rightarrow \begin{array}{l} -\omega t = -2\pi f t \\ 2\pi \cdot \left(\frac{-3}{8}\right) \cdot f_s \cdot t \end{array}$$

$X[5]$ measures how much of the $-3 \cdot \omega$ component is present in x .

The 7th Row of the DFT Matrix

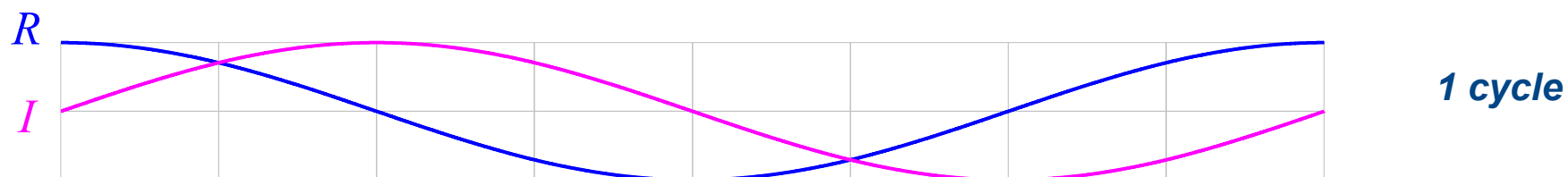


$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \quad k = 2, \quad n = 0, 1, \dots, 7$$

$$\begin{array}{l}
 R \rightarrow \text{samples of } \cos(-(-2\omega)t) = \cos(2\omega t) \\
 I \rightarrow \text{samples of } \sin(-(-2\omega)t) = \sin(2\omega t)
 \end{array}
 \left. \vphantom{\begin{array}{l} R \\ I \end{array}} \right\} \text{measure} \rightarrow \begin{array}{l} -\omega t = -2\pi f t \\ 2\pi \cdot \left(\frac{-2}{8}\right) \cdot f_s \cdot t \end{array}$$

$X[6]$ measures how much of the $-2 \cdot \omega$ component is present in x .

The 8th Row of the DFT Matrix



$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \quad k = 7, \quad n = 0, 1, \dots, 7$$

$R \rightarrow$ samples of $\cos(-(-\omega)t) = \cos(\omega t)$

$I \rightarrow$ samples of $\sin(-(-\omega)t) = \sin(\omega t)$

} *measure* \rightarrow

$$-\omega t = -2\pi f t$$

$$2\pi \cdot \left(\frac{-1}{8}\right) \cdot f_s \cdot t$$

$X[7]$ measures how much of the $-1 \cdot \omega$ component is present in x .

References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] G. Beale, http://teal.gmu.edu/~gbeale/ece_220/fourier_series_02.html
- [4] C. Langton, <http://www.complextoreal.com/chapters/fft1.pdf>