Matrix Transformation (2A)

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11/7/12

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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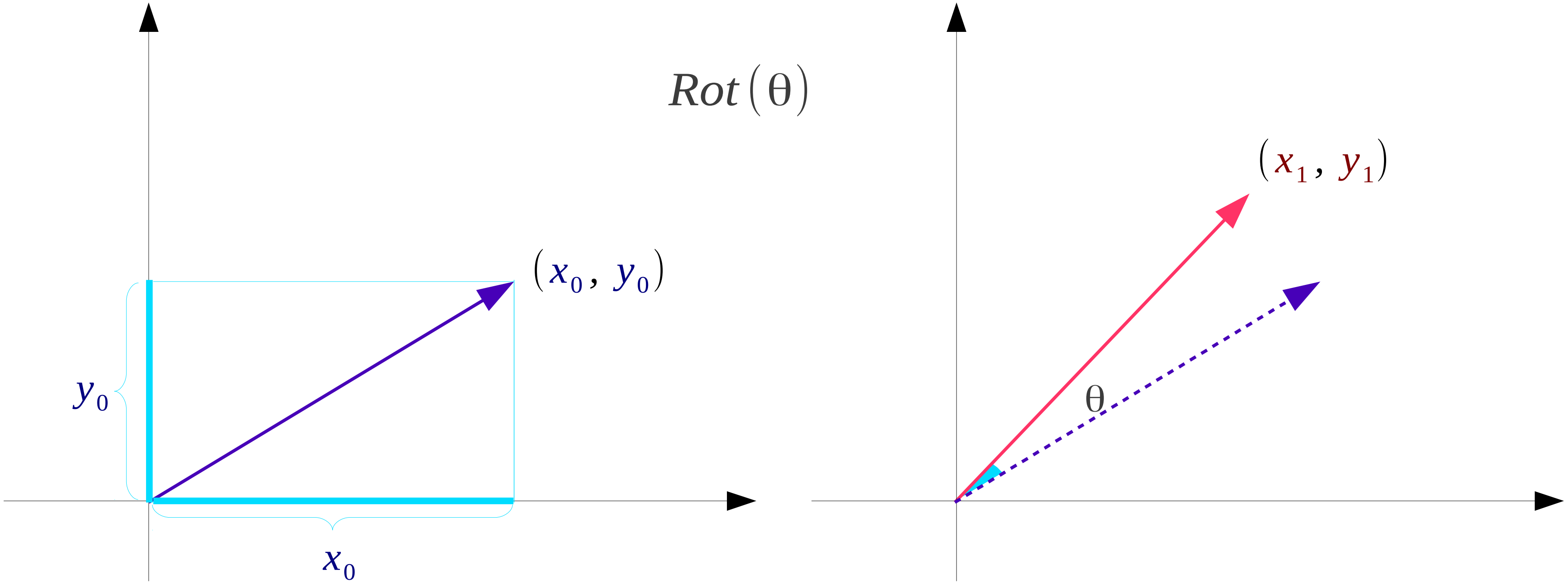
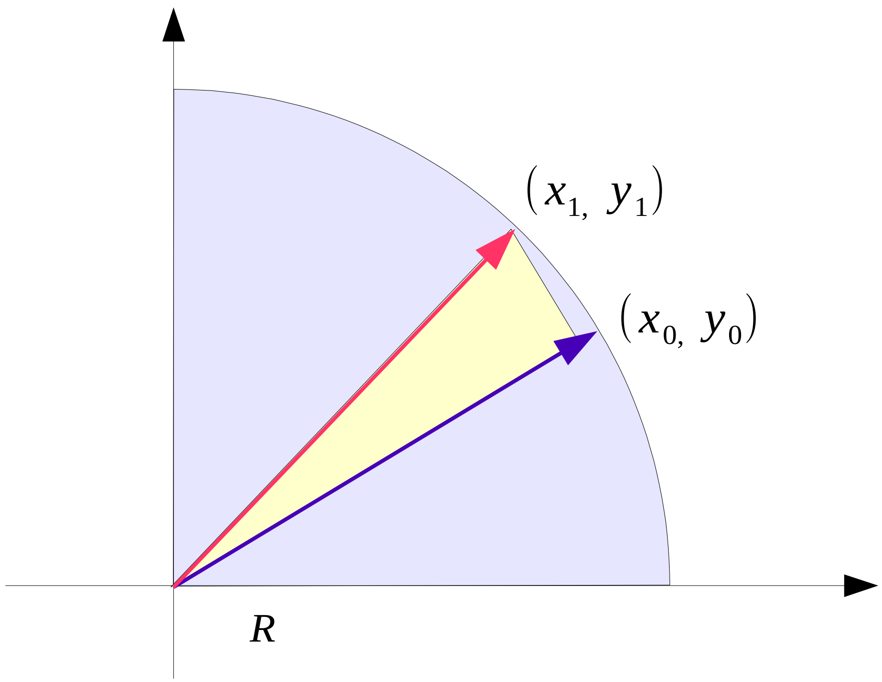
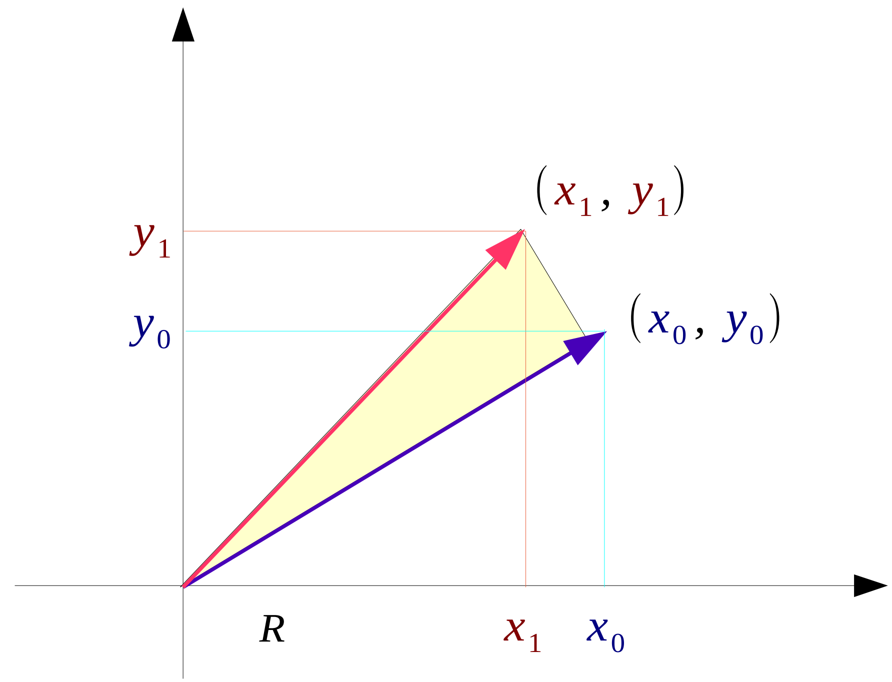
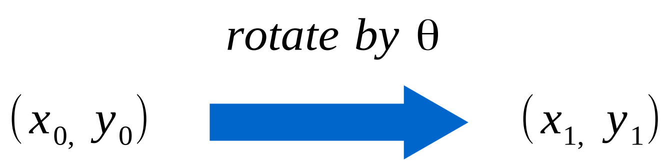
Vector Rotation (1)



Vector Rotation (2)



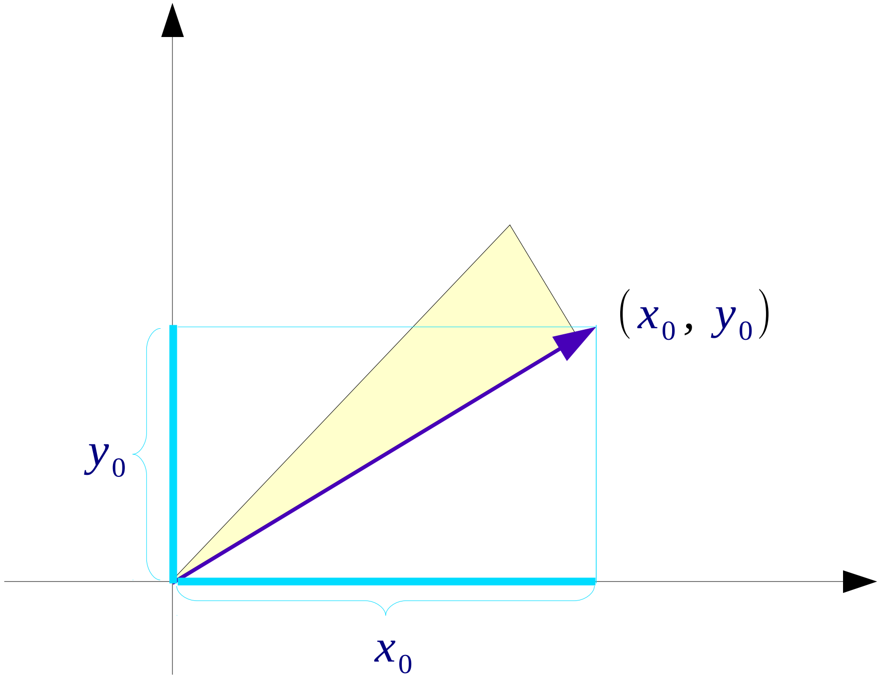


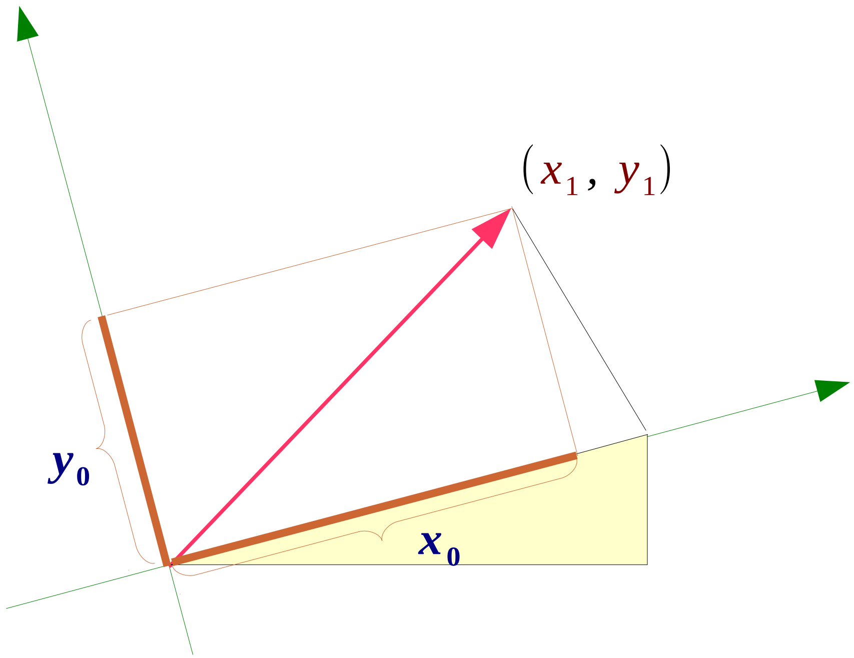


$$x_1 = x_0 \cos\theta - y_0 \sin\theta$$

$$y_1 = x_0 \sin\theta + y_0 \cos\theta$$

Vector Rotation (3)





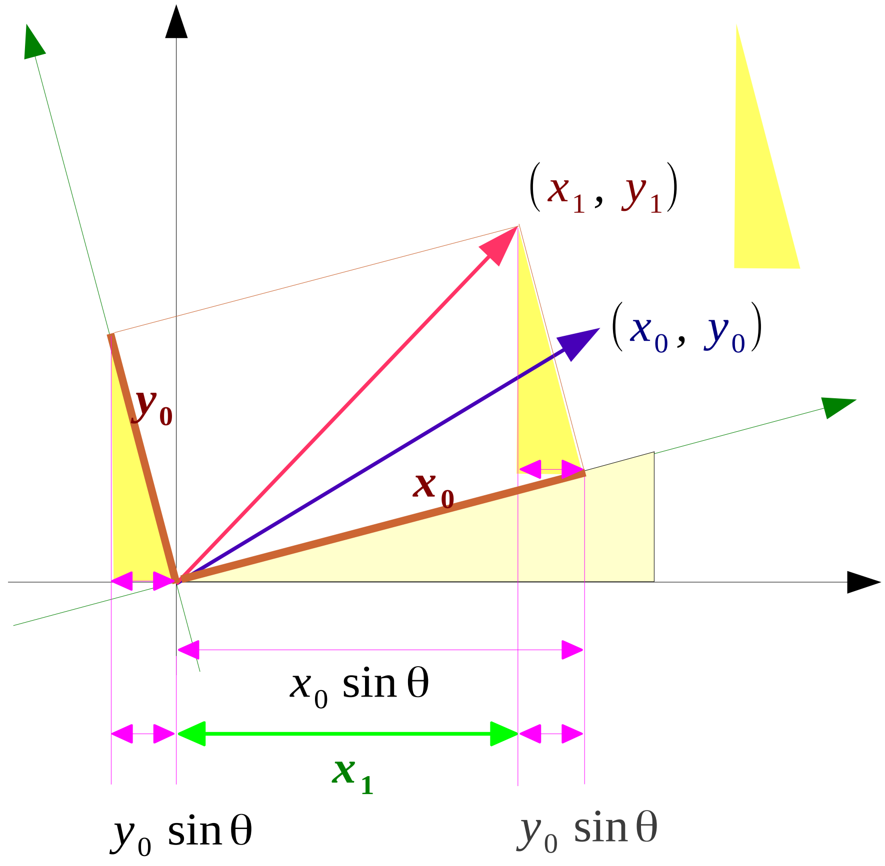
In the rotated coordinate

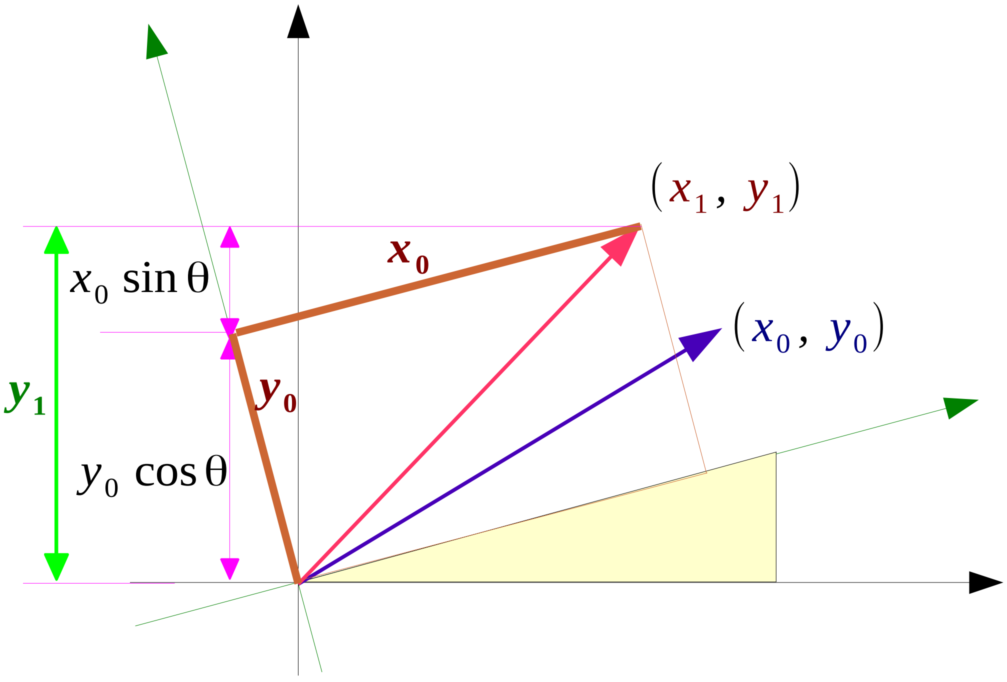
invariant length x_0, y_0

Vector Rotation (4)

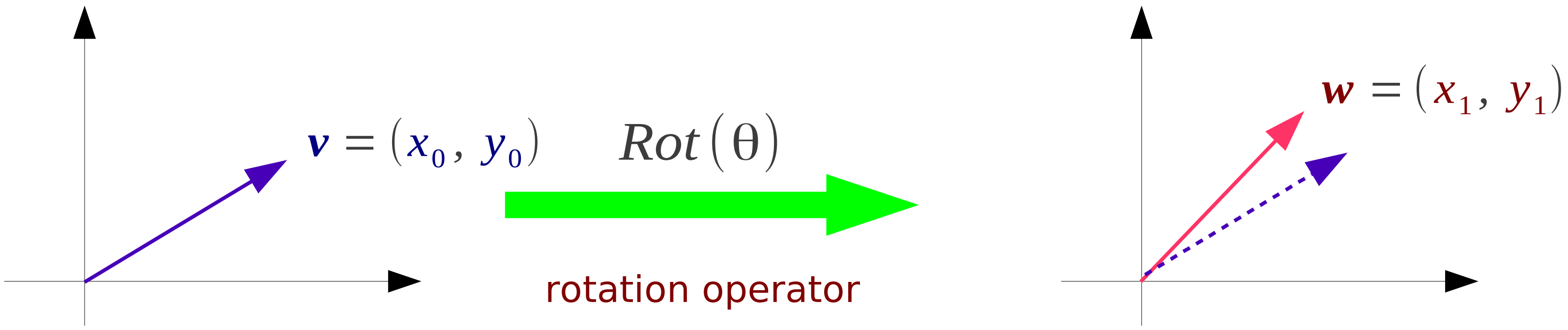
$$x_1 = x_0 \cos\theta - y_0 \sin\theta$$

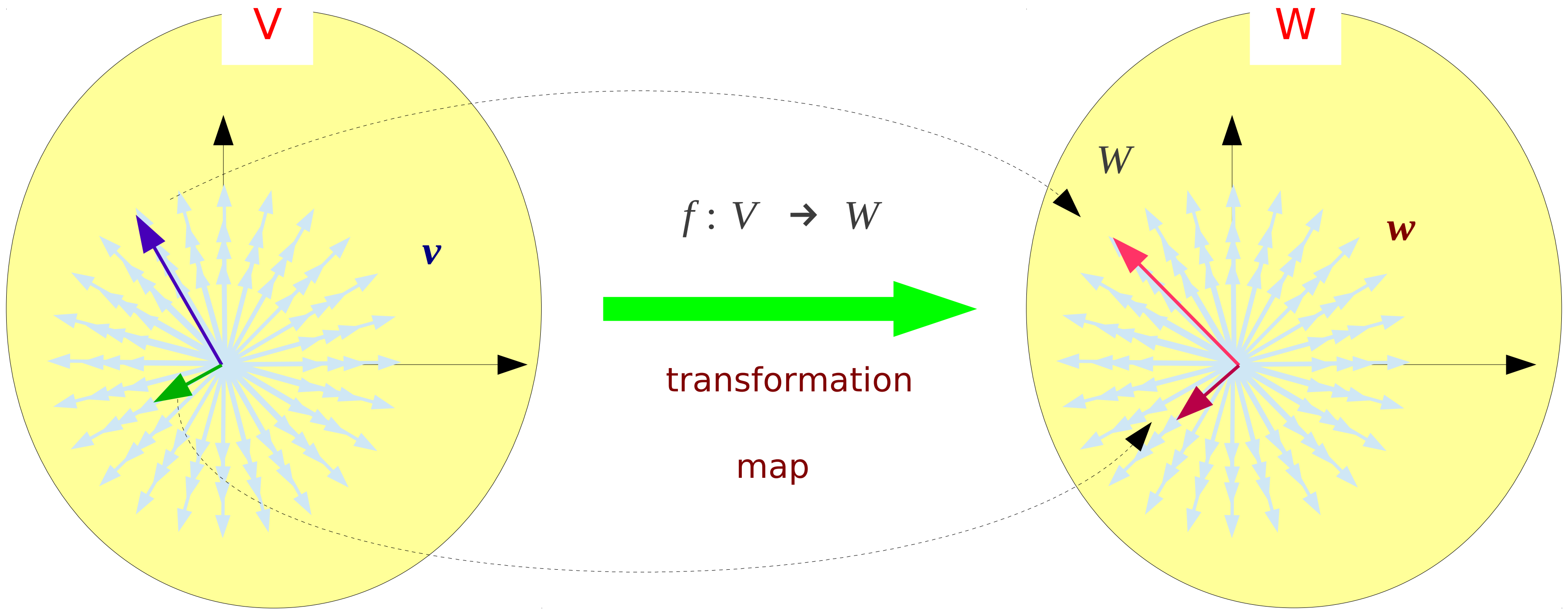
$$y_1 = x_0 \sin\theta + y_0 \cos\theta$$



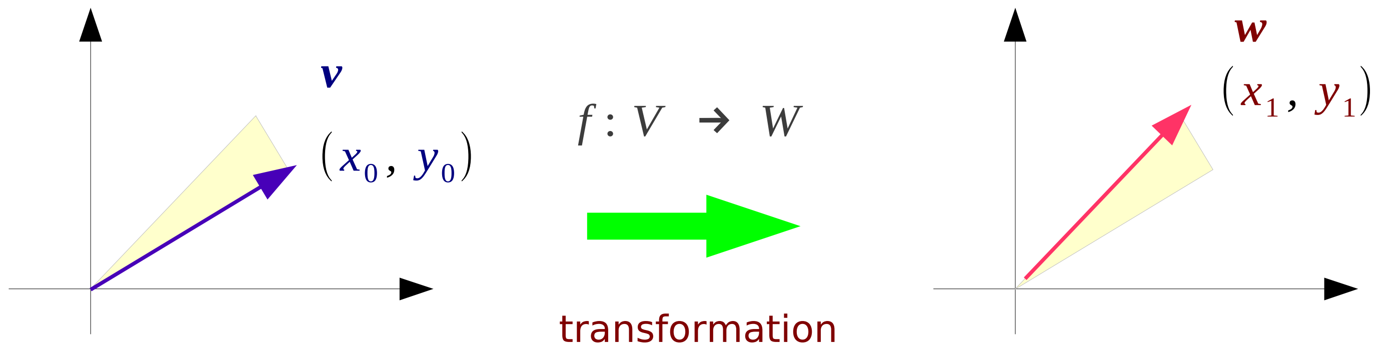


Transformation





Matrix Transformation



$$x_1 = x_0 \cos\theta - y_0 \sin\theta$$
$$y_1 = x_0 \sin\theta + y_0 \cos\theta$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_1 \end{bmatrix}$$

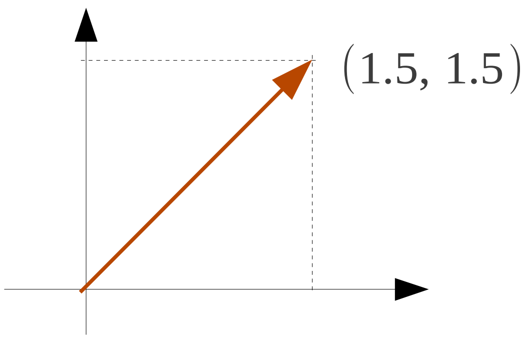
$$\boldsymbol{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

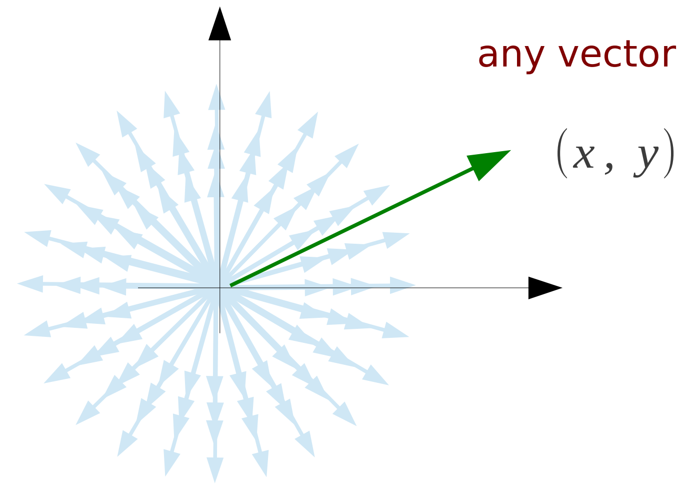
$$\boldsymbol{w} = \boldsymbol{A} \qquad \boldsymbol{x}$$
$$\boldsymbol{w} = T_{\boldsymbol{A}}(\boldsymbol{x})$$

$$\boldsymbol{x} \xrightarrow{T_{\boldsymbol{A}}} \boldsymbol{w}$$

Coordinates and Coordinates Systems

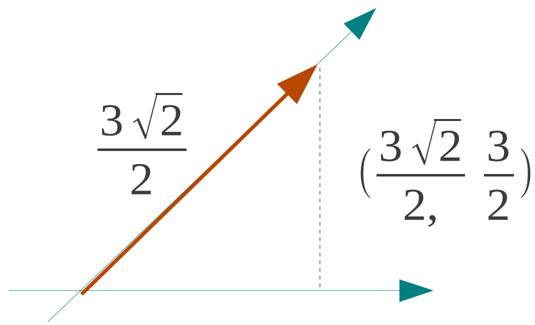
Rectangular Coordinate System

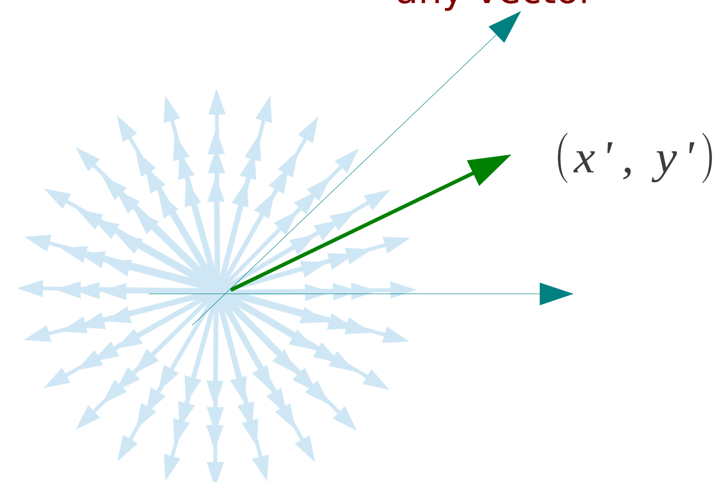




coordinates $(1.5 \,,\ 1.5)$

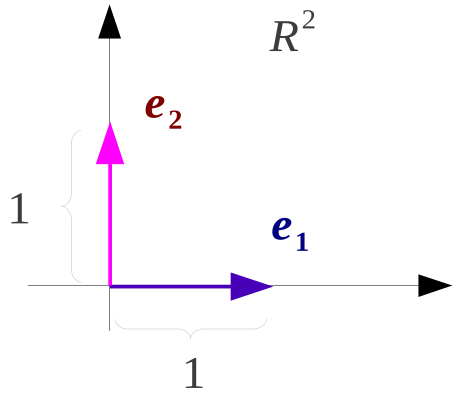
Non-Rectangular Coordinate System



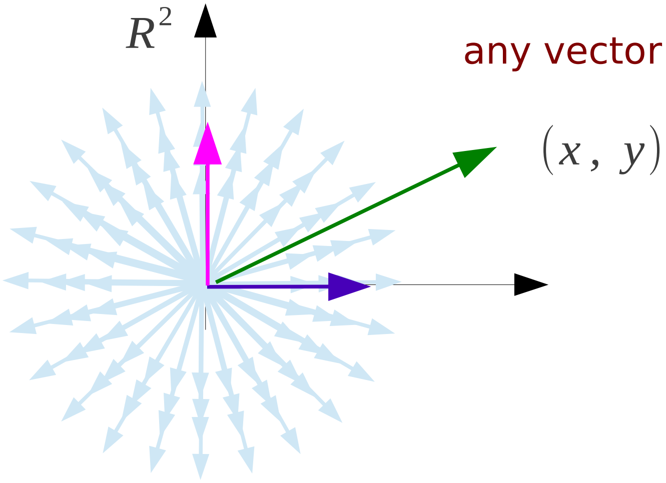


coordinates $\left(\frac{3\sqrt{2}}{2}, \frac{3}{2}\right)$

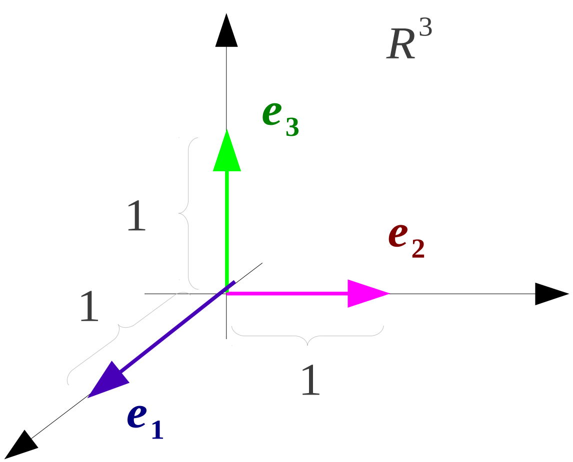
Standard Basis



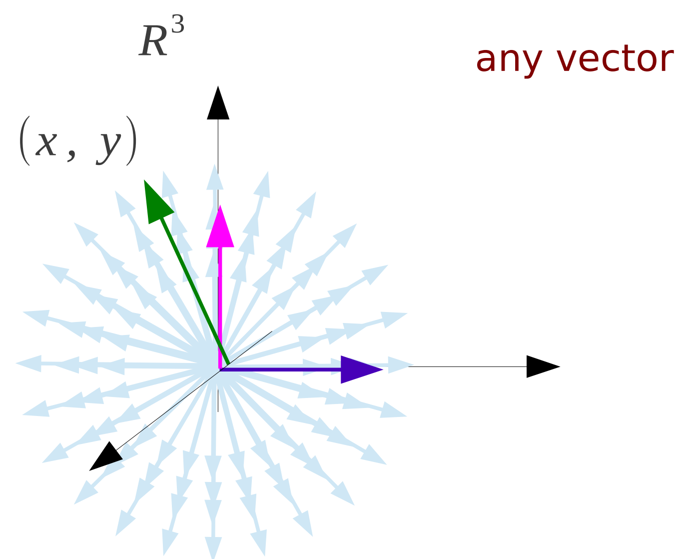
$$e_1 = (1,\ 0)$$
$$e_2 = (0,\ 1)$$



standard basis $\{e_1,\ e_2\}$ **spans** R^2

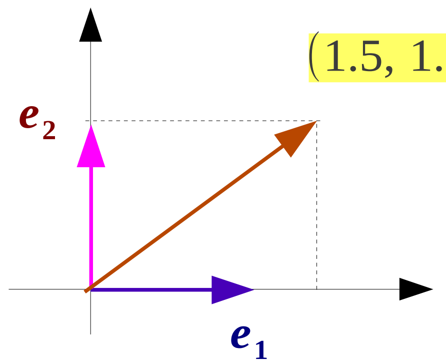


$$e_1 = (1,\ 0,\ 0)$$
$$e_2 = (0,\ 1,\ 0)$$
$$e_3 = (0,\ 0,\ 1)$$



standard basis $\{e_1,\ e_2,\ e_3\}$ **spans** R^3

Basis and Coordinates



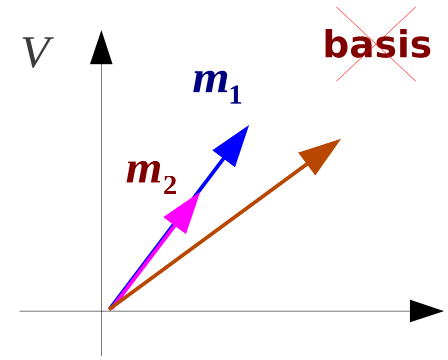
$$(1.5,\ 1.0) = 1.5\,\boldsymbol{e}_1 + 1.0\,\boldsymbol{e}_2$$

$$= 1.5\begin{bmatrix}1\\0\end{bmatrix} + 1.0\begin{bmatrix}0\\1\end{bmatrix}$$

$$= \begin{bmatrix}1.5 & 1.0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

basis $\{\boldsymbol{e}_1,\ \boldsymbol{e}_2\}$

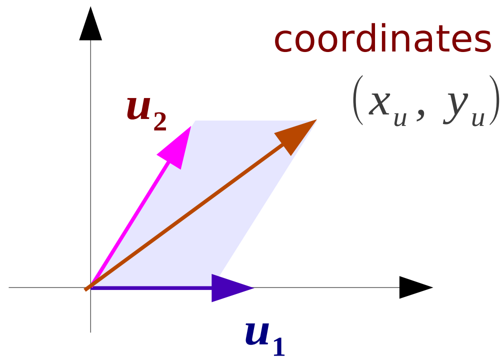
coordinates $(1.5,\ 1.0)$



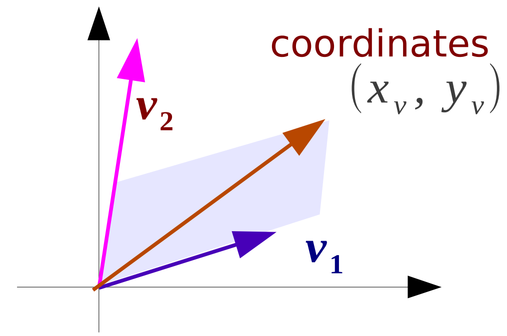
collinear vectors → linearly dependent vectors

many bases but the same number of basis vectors

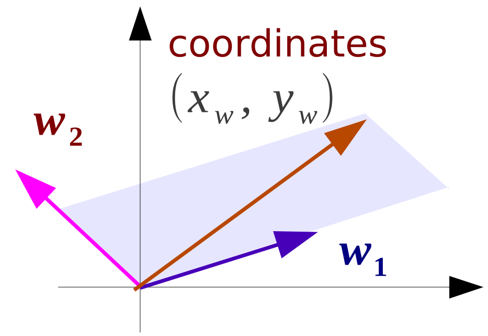
basis $\{\boldsymbol{u}_1,\ \boldsymbol{u}_2\}$



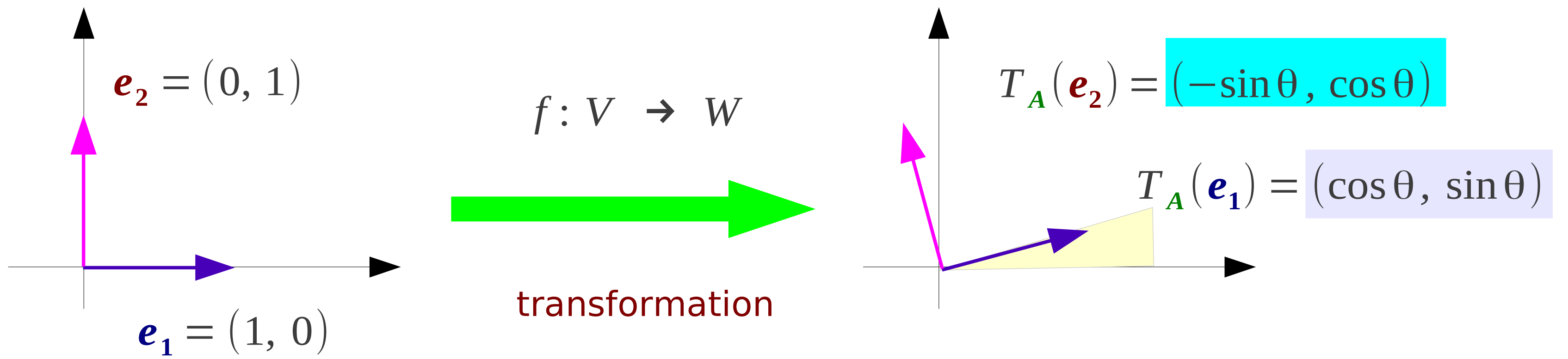
basis $\{\boldsymbol{v}_1,\ \boldsymbol{v}_2\}$



basis $\{\boldsymbol{w}_1,\ \boldsymbol{w}_2\}$



Standard Basis & Standard Matrix



standard basis $\{e_1, \ e_2\}$

standard matrix $A = [\, T_A(e_1) \mid T_A(e_2) \,]$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_1 \end{bmatrix}$$

$$w = A \quad x$$

$$w = T_A(x)$$

$$x \xrightarrow{T_A} w$$

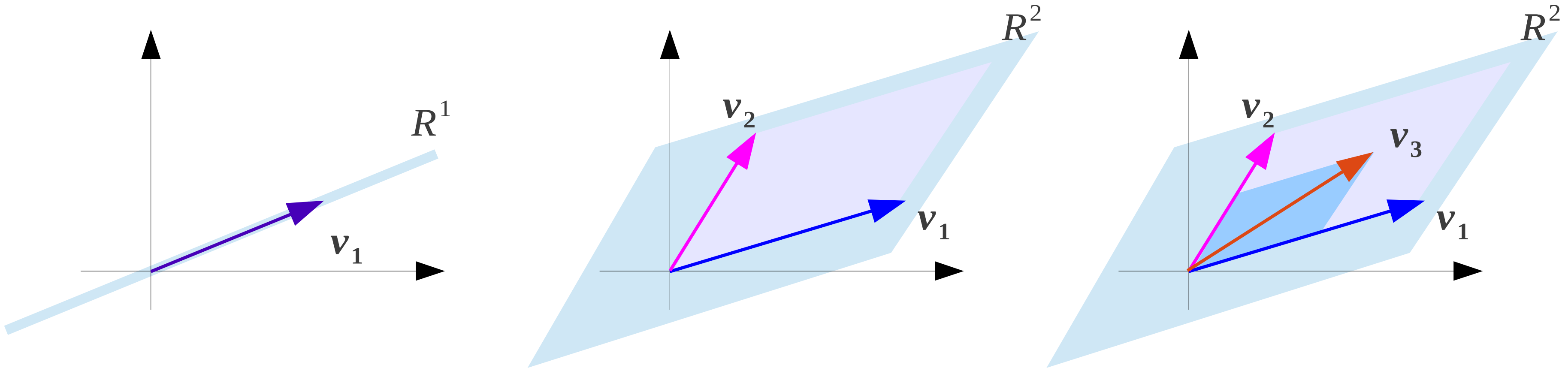
$$A = \begin{pmatrix} T_A(e_1) & T_A(e_2) & \cdots & T_A(e_n) \end{pmatrix}$$

Dimension

In vector space R^2

any one vector	line	R^1	linearly independent
any two non-collinear vectors	plane	R^2	linearly independent
any three or more vectors			linearly dependent

$$v_3 = k_1 v_1 + k_2 v_2$$

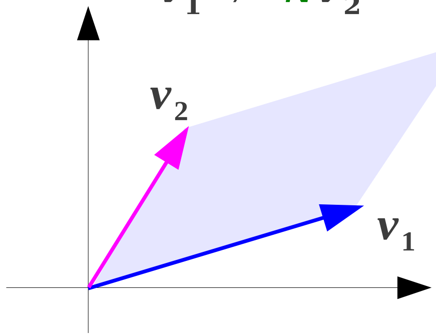


Basis

$S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ non-empty finite set of vectors in V

S is a basis ⟺

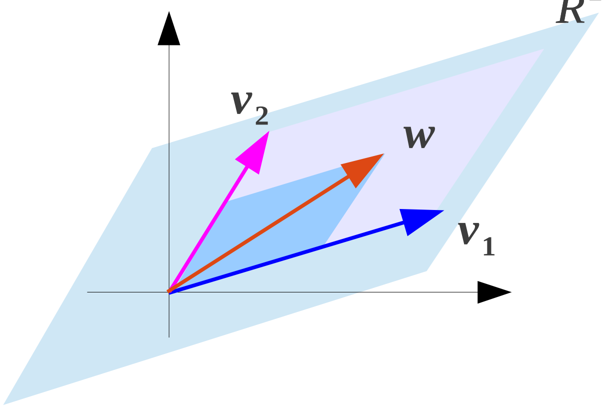
- S linearly independent
- S spans V



span(S) $= span\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ ⟺

all possible linear combination of the vectors in in S

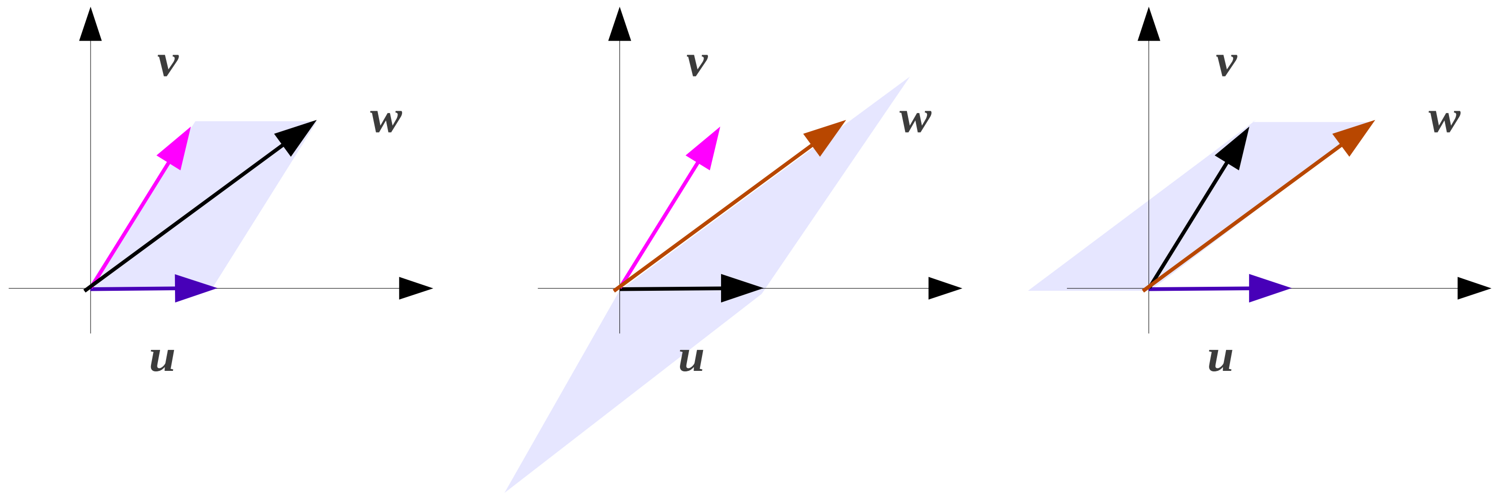
$\{\ \boldsymbol{w} = k_1\boldsymbol{v_1} + k_2\boldsymbol{v_2} + \cdots + k_n\boldsymbol{v_n}\ \}$



Linear Dependent (1)

$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ linearly dependent

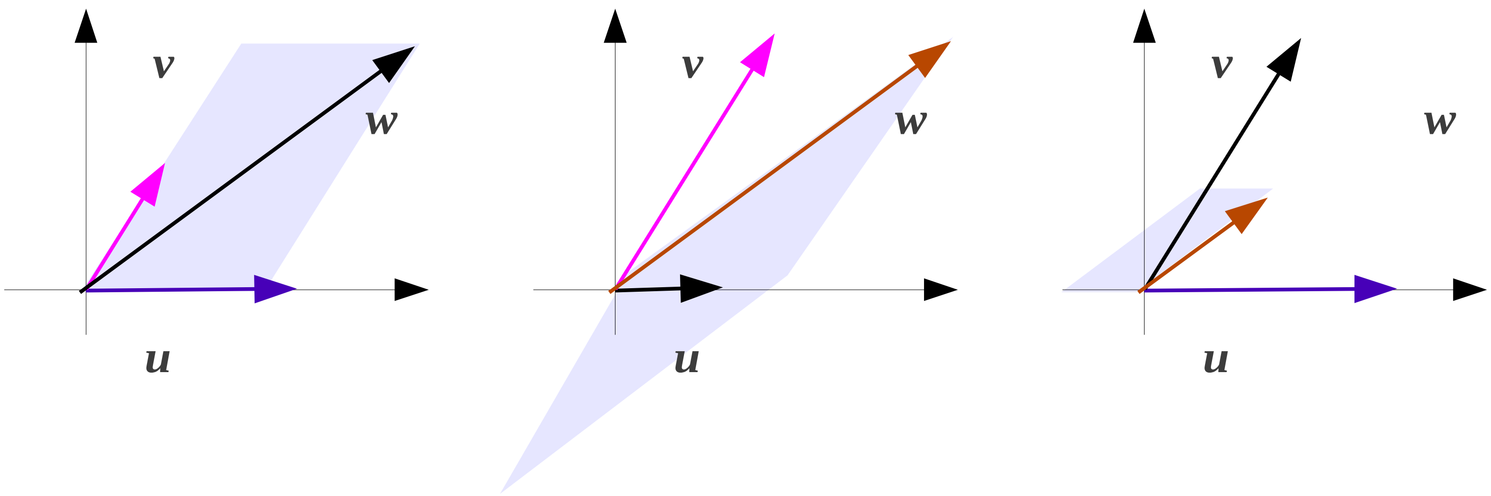
$\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ $\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{v}$ $\boldsymbol{v} = \boldsymbol{w} - \boldsymbol{u}$



$\boldsymbol{u} + \boldsymbol{v} - \boldsymbol{w} = \boldsymbol{0}$ $\boldsymbol{u} + \boldsymbol{v} - \boldsymbol{w} = \boldsymbol{0}$ $\boldsymbol{u} + \boldsymbol{v} - \boldsymbol{w} = \boldsymbol{0}$

Linear Dependent (2)

$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ linearly dependent



$k_1\boldsymbol{u} + k_2\boldsymbol{v} + k_3\boldsymbol{w} = \boldsymbol{0}$

$(k_1 = 0) \wedge (k_2 = 0) \wedge (k_3 = 0)$
$(k_1 \neq 0) \vee (k_2 \neq 0) \vee (k_3 \neq 0)$

$m_1\boldsymbol{u} + m_2\boldsymbol{v} + m_3\boldsymbol{w} = \boldsymbol{0}$

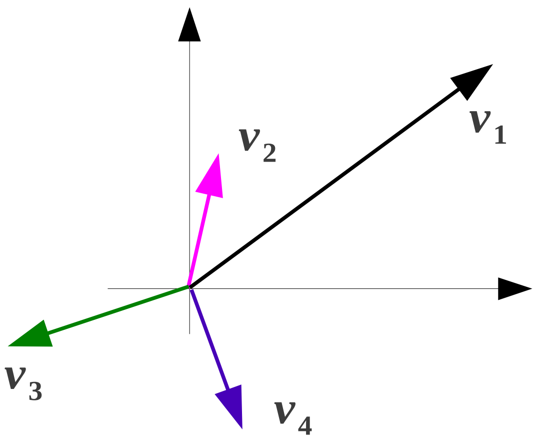
$(m_1 = 0) \wedge (m_2 = 0) \wedge (m_3 = 0)$
$(m_1 \neq 0) \vee (m_2 \neq 0) \vee (m_3 \neq 0)$

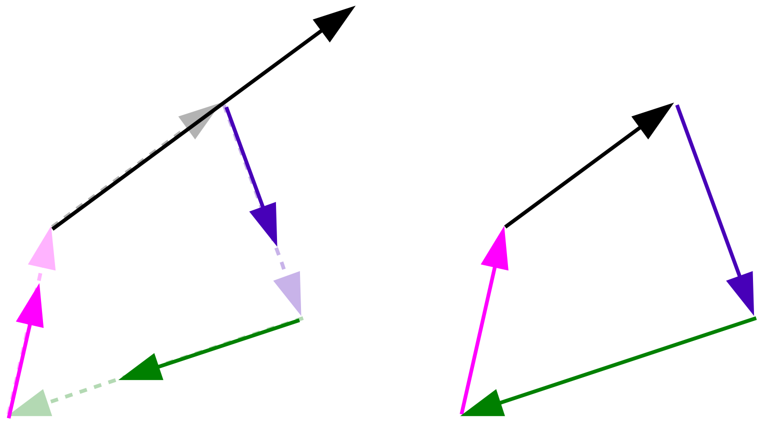
$n_1\boldsymbol{u} + n_2\boldsymbol{v} + n_3\boldsymbol{w} = \boldsymbol{0}$

$(n_1 = 0) \wedge (n_2 = 0) \wedge (n_3 = 0)$
$(n_1 \neq 0) \vee (n_2 \neq 0) \vee (n_3 \neq 0)$

Linear Dependent (3)

$\{\mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , \mathbf{v}_4\}$ linearly dependent



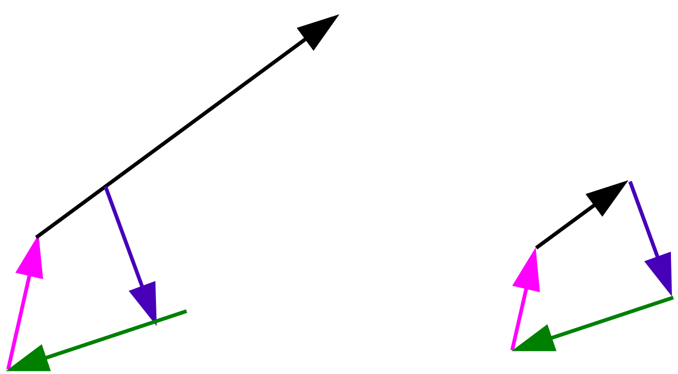


$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 = \mathbf{0}$$

$$(k_1 \neq 0)\vee(k_2 \neq 0)\vee(k_3 \neq 0)\vee(k_4 \neq 0)$$

$$0\,\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 + m_4\mathbf{v}_4 = \mathbf{0}$$

$$(m_1 = 0)\vee(m_2 \neq 0)\vee(m_3 \neq 0)\vee(m_4 \neq 0)$$

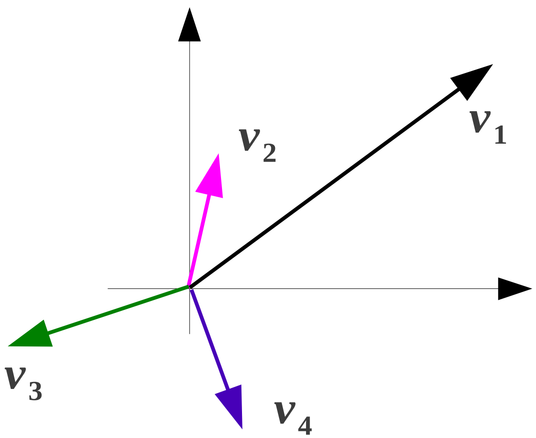


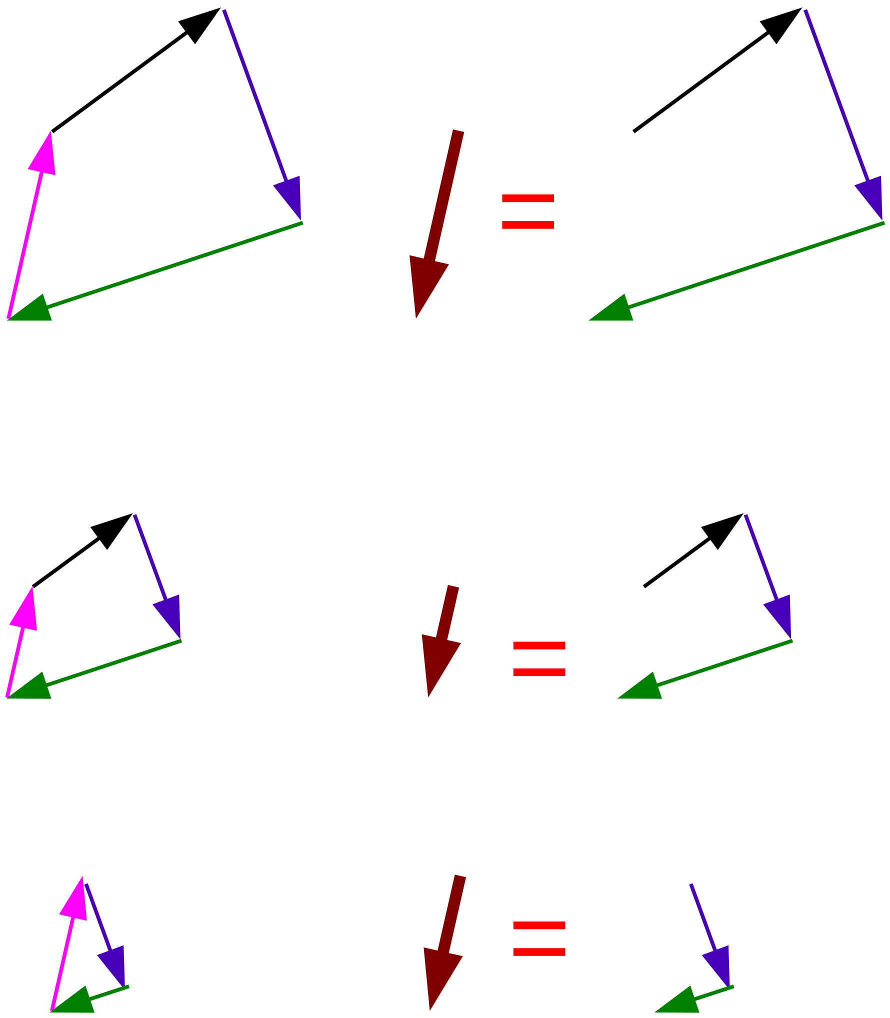
$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 + m_4\mathbf{v}_4 = \mathbf{0}$$

$$(m_1 \neq 0)\vee(m_2 \neq 0)\vee(m_3 \neq 0)\vee(m_4 \neq 0)$$

Linear Dependent (4)

$\{v_1 , v_2 , v_3 , v_4\}$ linearly dependent





Linear Independent (1)

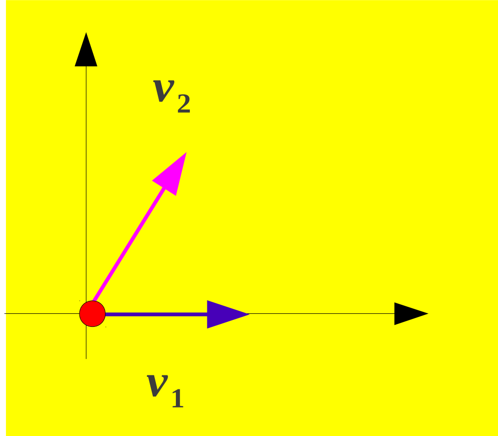
$S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ non-empty set of vectors in V

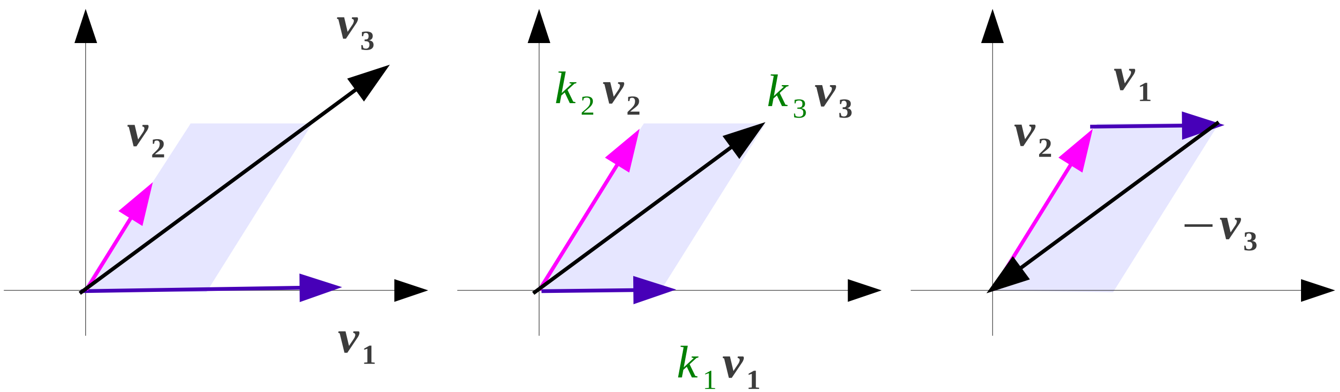
$k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_n\mathbf{v_n} = \mathbf{0}$

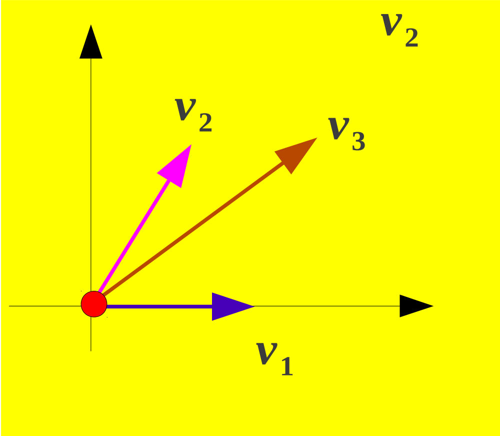
the solution of the above equation

trivial solution: $k_1 = k_2 = \cdots = k_n = 0$

- if other solution exists — S linearly dependent
- if no other solution exists — S linearly independent







Linear Independent (2)

$S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ non-empty set of vectors in V

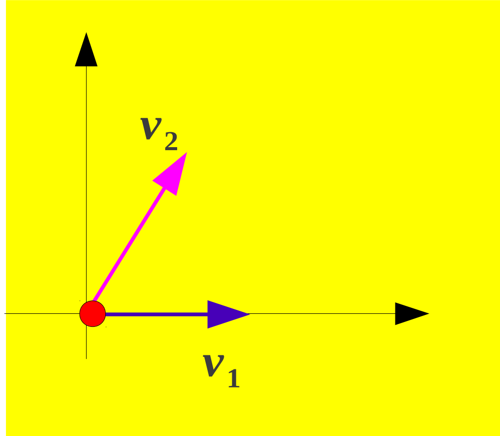
$$k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_n\mathbf{v_n} = \mathbf{0}$$

the solution of the above equation

$$k_1 = k_2 = \cdots = k_n = 0$$

if other solution exists ⟺ S linearly dependent

if no other solution exists ⟺ S linearly independent



at least one vector in S is a linear combination of the other vectors in S

$$\mathbf{v_i} = k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_{i-1}\mathbf{v_{i-1}} + k_{i+1}\mathbf{v_{i+1}} + \cdots + k_n\mathbf{v_n}$$

⟺ S linearly dependent

no vector in S is a linear combination of the other vectors in S

$$\mathbf{v_i} \neq k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_{i-1}\mathbf{v_{i-1}} + k_{i+1}\mathbf{v_{i+1}} + \cdots + k_n\mathbf{v_n}$$

⟺ S linearly independent

Linear Independent (3)

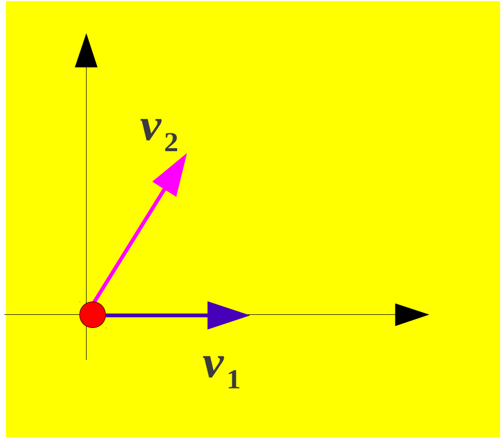
$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ non-empty set of vectors in V

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{0}$$

the solution of the above equation

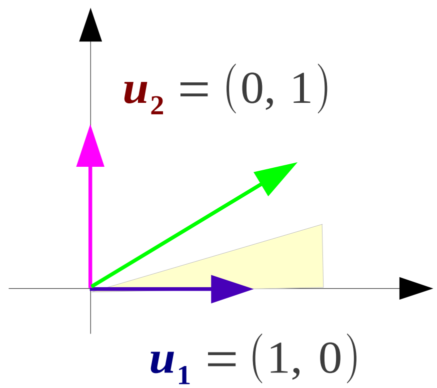
$$k_1 = k_2 = \cdots = k_n = 0$$

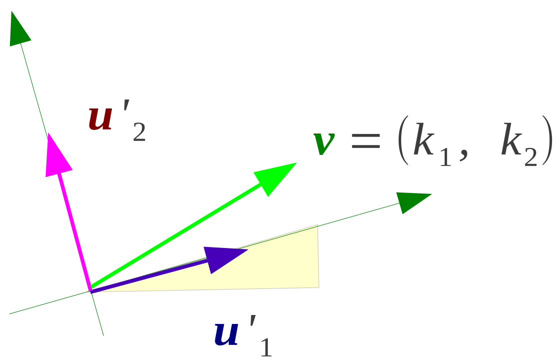
- if other solution exists ⇔ S linearly dependent
- if no other solution exists ⇔ S linearly independent



$S = \{ \mathbf{0} \}$		linearly dependent
$S = \{ \mathbf{v}_1 \}$		linearly independent
$S = \{ \mathbf{v}_1, \mathbf{v}_2 \}$	$\mathbf{v}_1 \neq k\,\mathbf{v}_2$	linearly independent

Change of Basis





Old Basis $B = \{u_1, u_2\}$

New Basis $B' = \{u'_1, u'_2\}$

$$[u'_1]_B = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$ coordinate of u'_1 with respect to B

$$[u'_2]_B = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$ coordinate of u'_2 with respect to B

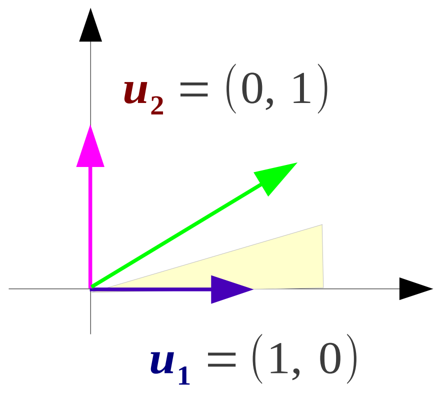
$$u'_1 = \cos\theta\, u_1 + \sin\theta\, u_2$$
$$u'_2 = -\sin\theta\, u_1 + \cos\theta\, u_2$$

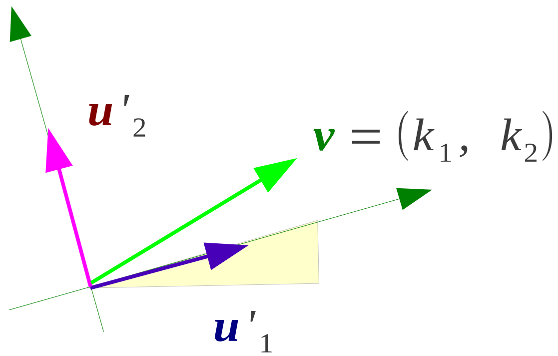
$$v = k_1 u'_1 + k_2 u'_2$$
$$= k_1(\cos\theta\, u_1 + \sin\theta\, u_2) + k_2(-\sin\theta\, u_1 + \cos\theta\, u_2)$$
$$= (k_1\cos\theta - k_2\sin\theta)u_1 + (k_1\sin\theta + k_2\cos\theta)u_2$$

$$[v]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$ coordinate of v with respect to B'

$$[v]_B = \begin{bmatrix} k_1\cos\theta - k_2\sin\theta \\ k_1\sin\theta + k_2\cos\theta \end{bmatrix}$$ coordinate of v with respect to B

Change of Basis





Old Basis $B = \{u_1,\ u_2\}$

New Basis $B' = \{u'_1,\ u'_2\}$

$$[v]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

coordinate of **v** with respect to B'

$$[u'_1]_B = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

coordinate of **u'**$_1$ with respect to B

$$[u'_2]_B = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

coordinate of **u'**$_2$ with respect to B

$$[v]_B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

coordinate of **v** with respect to B

$$[v]_B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} [v]_{B'}$$

$$[v]_B = P_{B' \to B} [v]_{B'}$$

$$P_{B' \to B} = \left[\ [u'_1]_B \quad [u'_2]_B \right]$$

Transition Matrix

$$P_{B' \to B} = \left[\; [\boldsymbol{u}'_1]_B \;\; [\boldsymbol{u}'_2]_B \;\; \cdots \;\; [\boldsymbol{u}'_n]_B \; \right]$$

$$[\boldsymbol{v}]_B = P_{B' \to B} [\boldsymbol{v}]_{B'}$$

$[\boldsymbol{u}'_1]_B$ coordinate of $\mathbf{u'_1}$ with respect to B

$[\boldsymbol{u}'_2]_B$ coordinate of $\mathbf{u'_2}$ with respect to B

$[\boldsymbol{v}]_{B'}$ coordinate of **v** with respect to B'

↓

$[\boldsymbol{v}]_B$ coordinate of **v** with respect to B

$$P_{B \to B'} = \left[\; [\boldsymbol{u}_1]_{B'} \;\; [\boldsymbol{u}_2]_{B'} \;\; \cdots \;\; [\boldsymbol{u}_n]_{B'} \; \right]$$

$$[\boldsymbol{v}]_{B'} = P_{B \to B'} [\boldsymbol{v}]_B$$

$[\boldsymbol{u}_1]_{B'}$ coordinate of $\mathbf{u_1}$ with respect to B'

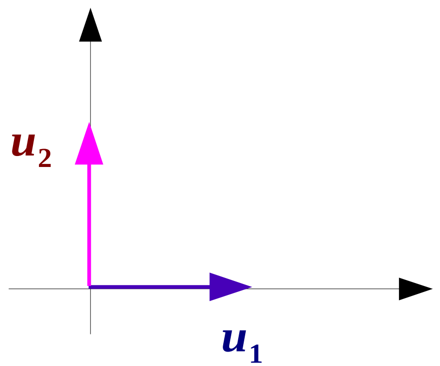
$[\boldsymbol{u}_2]_{B'}$ coordinate of $\mathbf{u_2}$ with respect to B'

$[\boldsymbol{v}]_B$ coordinate of **v** with respect to B

↓

$[\boldsymbol{v}]_{B'}$ coordinate of **v** with respect to B'

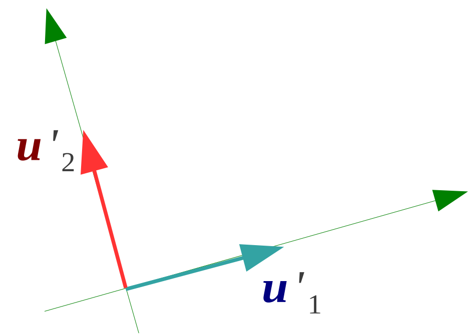
Change of Basis Example (1)



$$Rot(30°)$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Rot(-30°)$$



$$P_{B\to B'} = \begin{bmatrix} [u_1]_{B'} & [u_2]_{B'} & \cdots & [u_n]_{B'} \end{bmatrix}$$

$[u_1]_{B'}$ coordinate of $\mathbf{u}_1$ with respect to B' $\begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}$

$[u_2]_{B'}$ coordinate of $\mathbf{u}_2$ with respect to B' $\begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$

$$P_{B\to B'} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

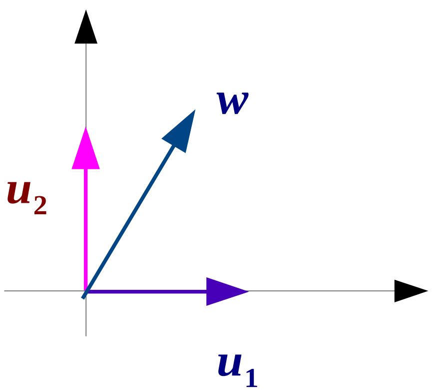
$$P_{B'\to B} = \begin{bmatrix} [u'_1]_{B} & [u'_2]_{B} & \cdots & [u'_n]_{B} \end{bmatrix}$$

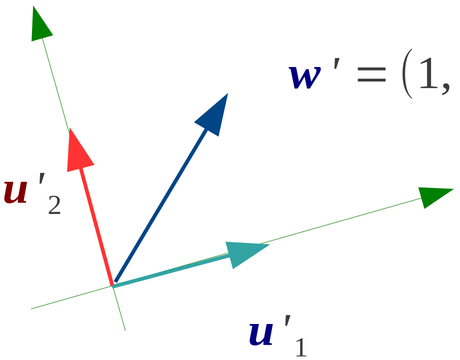
$[u'_1]_{B}$ coordinate of $\mathbf{u'}_1$ with respect to B $\begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$

$[u'_2]_{B}$ coordinate of $\mathbf{u'}_2$ with respect to B $\begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$

$$P_{B'\to B} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Change of Basis Example (2)





$[\mathbf{v}]_B = P_{B' \to B} [\mathbf{v}]_{B'}$

$[\mathbf{v}]_{B'}$ coordinate of **v** with respect to B'

$[\mathbf{v}]_B$ coordinate of **v** with respect to B

$$\mathbf{w} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1+\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} \end{bmatrix}$$

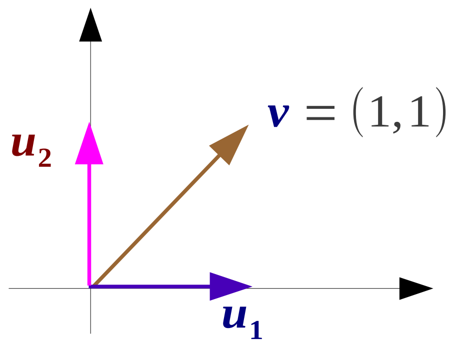
$P_{B' \to B} = \left[[\mathbf{u}'_1]_B \quad [\mathbf{u}'_2]_B \quad \cdots \quad [\mathbf{u}'_n]_B \right]$

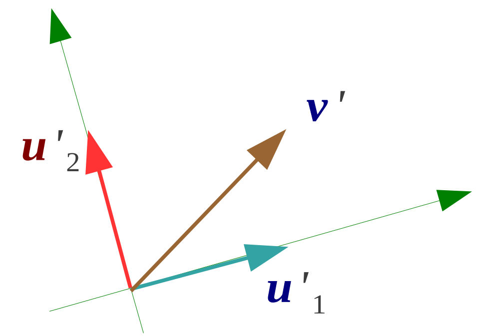
$[\mathbf{u}'_1]_B$ coordinate of **u'**$_1$ with respect to B $\begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$

$[\mathbf{u}'_2]_B$ coordinate of **u'**$_2$ with respect to B $\begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$

$$P_{B' \to B} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Change of Basis Example (3)





$$P_{B \to B'} = \left[\; [u_1]_{B'} \quad [u_2]_{B'} \quad \cdots \quad [u_n]_{B'} \;\right]$$

$$[v]_{B'} = P_{B \to B'} [v]_B$$

$[u_1]_{B'}$ coordinate of u_1 with respect to B' $\begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}$

$[u_2]_{B'}$ coordinate of u_2 with respect to B' $\begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$

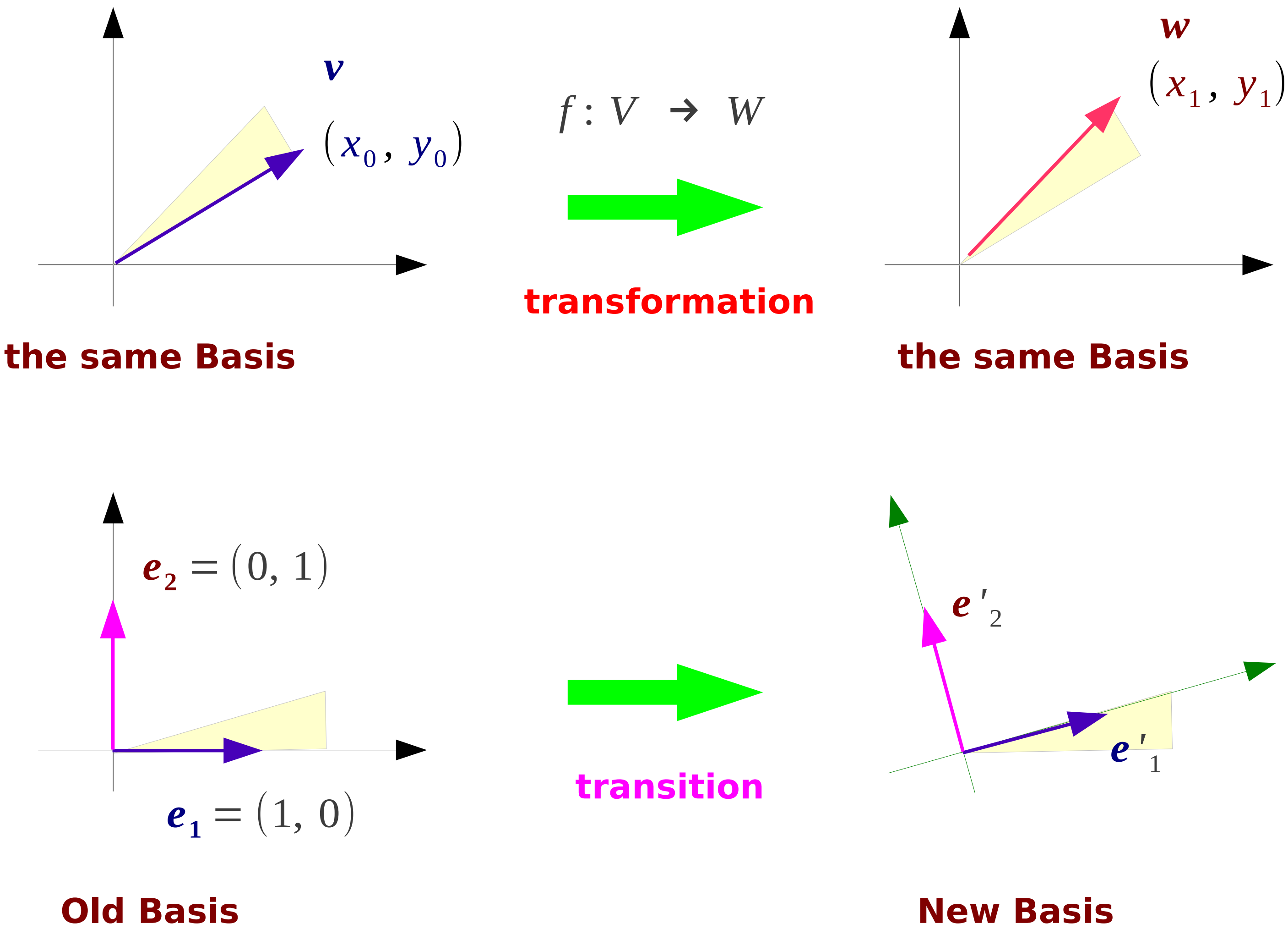
$[v]_B$ coordinate of v with respect to B

$[v]_{B'}$ coordinate of v with respect to B'

$$P_{B \to B'} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$v' = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{-1+\sqrt{3}}{2} \end{bmatrix}$$

Transformation & Transition Matrix



Vector Space

V: non-empty set of objects

defined operations: addition $\mathbf{u} + \mathbf{v}$

scalar multiplication $k\,\mathbf{u}$

if the following axioms are satisfied

for all object $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and all scalar k, m

⇒ V: vector space

objects in V: vectors

1. if $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ (zero vector)
5. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + (\mathbf{u}) = \mathbf{0}$
6. if k is any scalar and $\mathbf{u}$ is objects in V, then $k\mathbf{u}$ is in V
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

Test for a Vector Space

1. Identify the set V of objects
2. Identify the addition and scalar multiplication on V
3. Verify **u** + **v** is in V and *k***u** is in V

   closure under **addition** and **scalar multiplication**
4. Confirm other axioms.

1. if **u** and **v** are objects in V, then **u** + **v** is in V
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. **0** + **u** = **u** + **0** = **u** (zero vector)
5. **u** + (–**u**) = (–**u**) + (**u**) = **0**
6. if *k* is any scalar and **u** is objects in V, then *k***u** is in V
7. *k*(**u** + **v**) = *k***u** + *k***v**
8. (*k* + *m*)**u** = *k***u** + *m***u**
9. *k*(*m***u**) = (*km*)**u**
10. *1*(**u**) = **u**

Subspace

a subset W of a vector space V

If the subset W is itself a vector space ➡ the subset W is a **subspace** of V

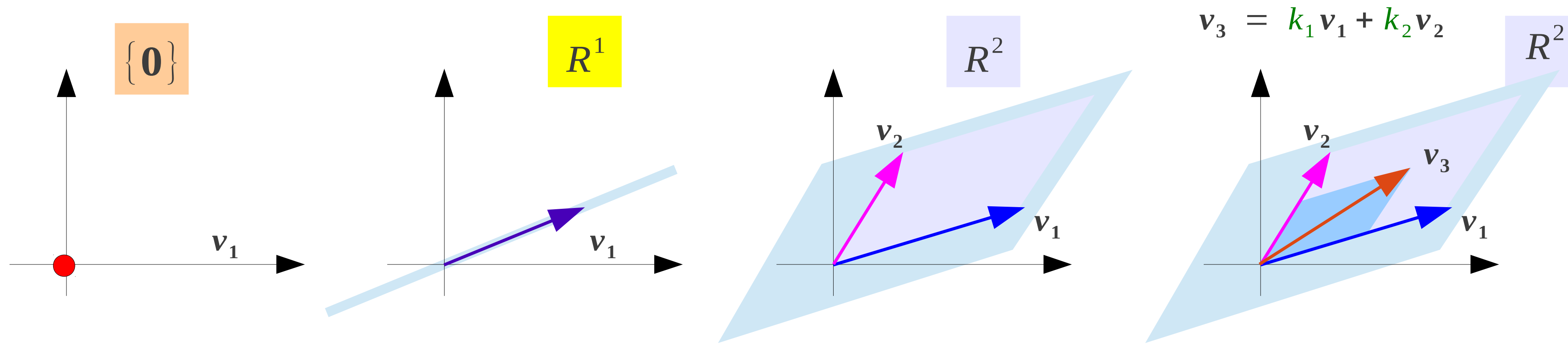
1. if **u** and **v** are objects in W, then **u** + **v** is in W
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. **0** + **u** = **u** + **0** = **u** (zero vector)
5. **u** + (–**u**) = (–**u**) + (**u**) = **0**
6. if *k* is any scalar and **u** is objects in W, then *k***u** is in W
7. *k*(**u** + **v**) = *k***u** + *k***v**
8. (*k* + *m*)**u** = *k***u** + *m***u**
9. *k*(*m***u**) = (*km*)**u**
10. *1*(**u**) = **u**

Subspace Example (1)

In vector space R^2

any one vector	(linearly indep.)	**spans**	R^1	line through **0**
any two non-collinear vectors	(linearly indep.)	**spans**	R^2	plane
any three or more vectors	(linearly dep.)	**spans**	R^2	plane

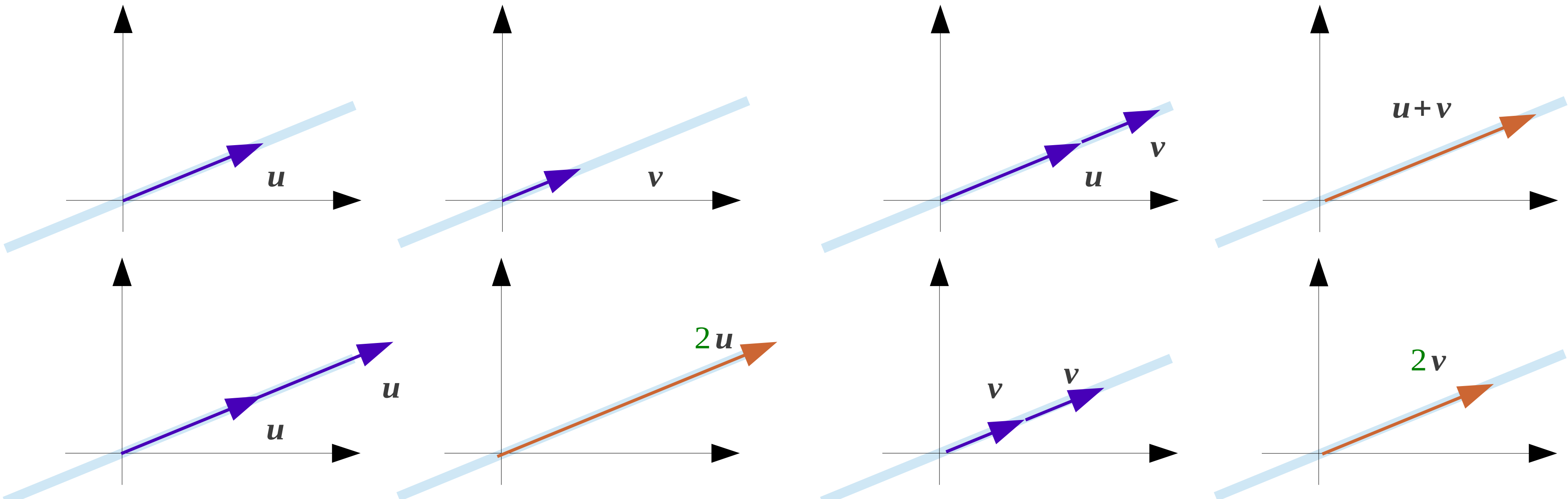
Subspaces of R^2

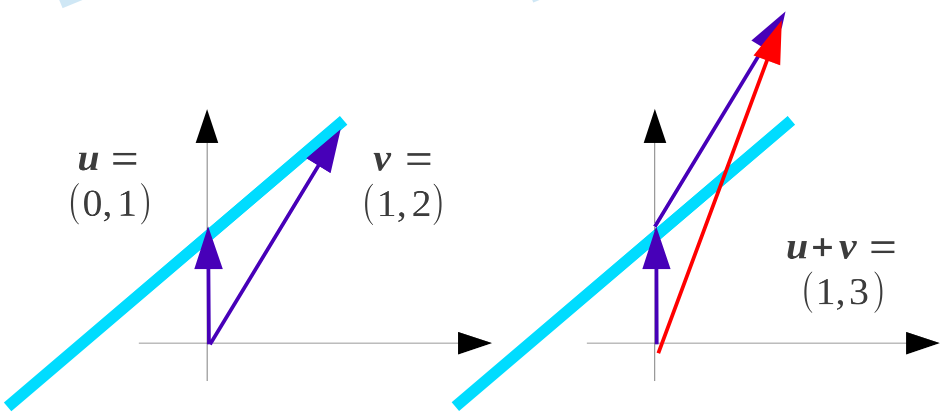


Subspace Example (2)

In vector space R^2

any one vector (linearly indep.) **spans** R^1 line through **0**





~~**vector space**~~

Subspace Example (3)

In vector space R^3

any one vector	(linearly indep.)	**spans**	R^1	line through **0**
any two non-collinear vectors	(linearly indep.)	**spans**	R^2	plane through **0**
any three vectors non-collinear, non-coplanar	(linearly indep.)	**spans**	R^3	3-dim space
any four or more vectors	(linearly dep.)	**spans**	R^3	3-dim space

Subspaces of R^2

$\{\mathbf{0}\}$	R^1	R^2	R^3
	line through **0**	plane through **0**	3-dim space

Row & Column Spaces

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

ROW Space R^m **subspace**

$$= span\{\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_m\}$$

COLUMN Space R^n **subspace**

$$= span\{\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n\}$$

$$\boldsymbol{r}_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

$$\boldsymbol{r}_2 = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$

$$\vdots$$

$$\boldsymbol{r}_m = \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\boldsymbol{c}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \quad \boldsymbol{c}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \quad \cdots \quad \boldsymbol{c}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Null Space

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$$

NULL Space R^n **subspace**

$$= span\{\boldsymbol{x_1}, \boldsymbol{x_2}, \cdots, \boldsymbol{x_n}\}$$

solution space

ROW Space R^m **subspace**

$$= span\{\boldsymbol{r_1}, \boldsymbol{r_2}, \cdots, \boldsymbol{r_m}\}$$

COLUMN Space R^n **subspace**

$$= span\{\boldsymbol{c_1}, \boldsymbol{c_2}, \cdots, \boldsymbol{c_n}\}$$

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] D.G. Zill, “Advanced Engineering Mathematics”