Group & Phase Velocities (2A)

- 1-D Group & Phase Velocities

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7/23/12

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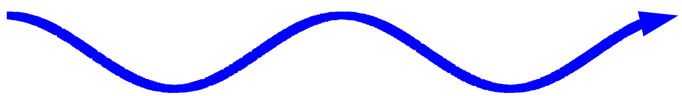
Wave Equation

$$A(x, t) = A_0 \cos(kx - \omega t)$$

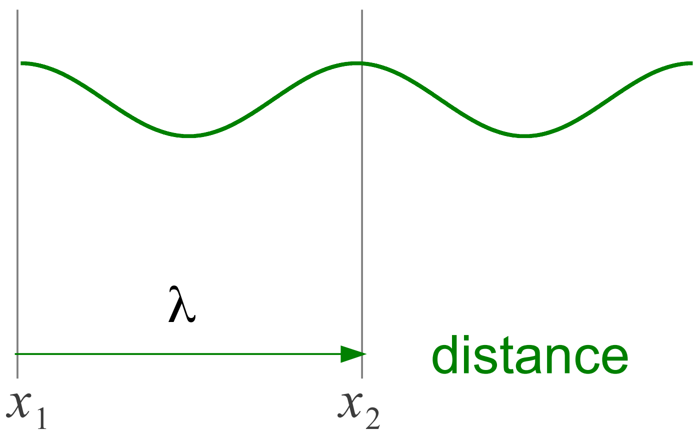


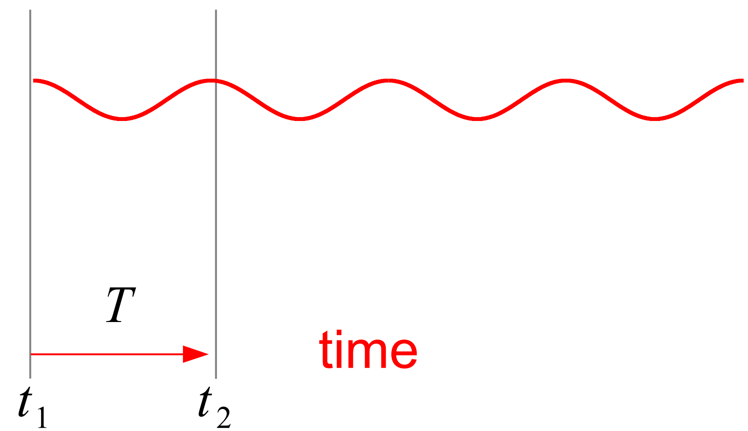
$A_0 \cos(kx - \omega t_0)$

At the snapshot of the time t_0

$A_0 \cos(kx_0 - \omega t)$

At the fixed site of the distance x_0

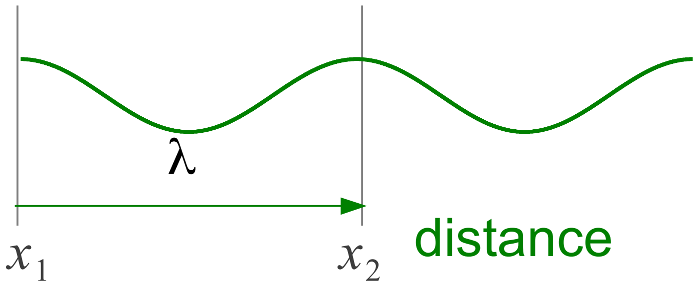




Wavelength, Frequency

$A_0 \cos(k\,x - \omega t_0)$

At the snapshot of the time t_0

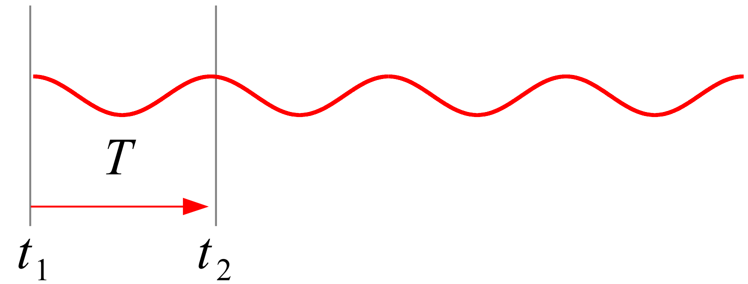


wavelength $\lambda = \dfrac{2\pi}{k}$

wave number $k = \dfrac{2\pi}{\lambda}$

$A_0 \cos(k\,x_0 - \omega t)$

At the fixed site of the distance x_0



time

frequency $f = \dfrac{\omega}{2\pi}$

period $T = \dfrac{2\pi}{\omega}$

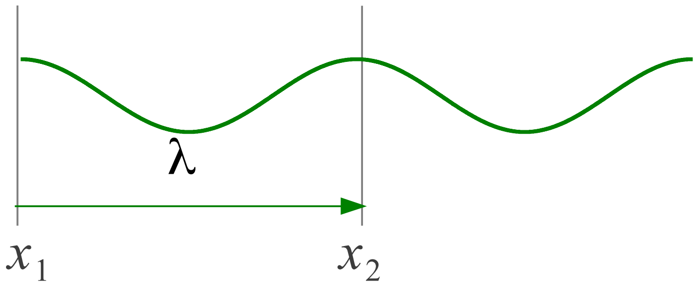
angular frequency $\omega = 2\pi f$

angular frequency $\omega = \dfrac{2\pi}{T}$

Wave Number, Angular Frequency

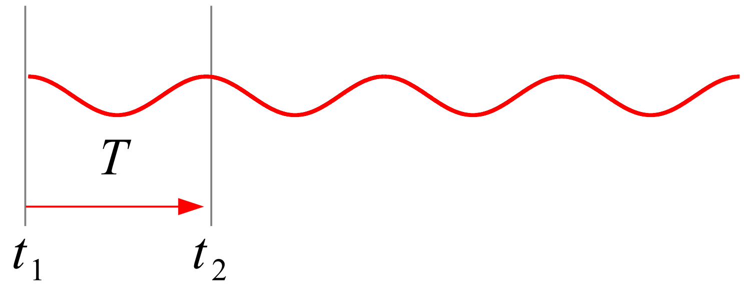
$A_0 \cos(k x - \omega t_0)$

At the snapshot of the time t_0



$A_0 \cos(k x_0 - \omega t)$

At the fixed site of the distance x_0



wave number $k = \frac{2\pi}{\lambda}$

radians per unit distance

angular frequency $\omega = \frac{2\pi}{T}$

radians per unit time

Phase Velocity (1)

wave number $k = \frac{2\pi}{\lambda}$

radians per unit *distance*

angular frequency $\omega = \frac{2\pi}{T}$

radians per unit *time*

Phase Velocity $v_p = \frac{\lambda}{T} = \frac{2\pi/k}{2\pi/\omega} = \frac{\omega}{k}$

$$v_p = \frac{\omega}{k}$$

Phase Velocity (2)

Phase Velocity $v_p = \dfrac{\omega}{k}$

$$A\cos(kx - \omega t)$$

*Given time **t**,* ⇒ ωt *oscillations*

*Corresponding distance **x**,* ⇒ *the same oscillations*

$$kx = \omega t$$

$$v_p = \frac{x}{t} = \frac{\omega}{k}$$

Phase Velocity, Group Velocity

Phase Velocity $$v_p = \frac{\omega}{k}$$

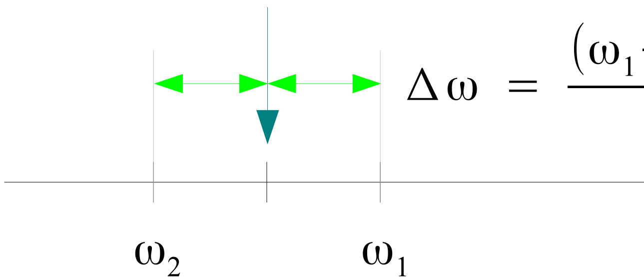
Group Velocity $$v_g = \frac{\partial \omega}{\partial k}$$

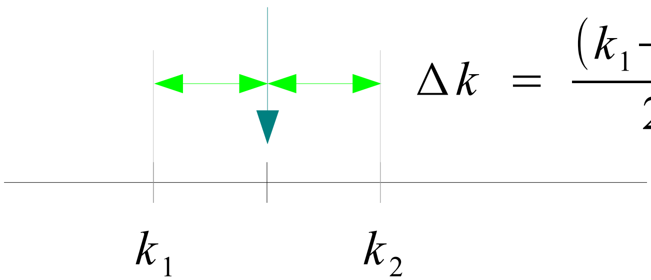
Group Velocity Explanation (1)

$$\omega_1 > \omega_2 \qquad\qquad k_1 > k_2$$

$$\omega = \frac{(\omega_1 + \omega_2)}{2} \qquad\qquad k = \frac{(k_1 + k_2)}{2}$$

$$\Delta\omega = \frac{(\omega_1 - \omega_2)}{2} \qquad\qquad \Delta k = \frac{(k_1 - k_2)}{2}$$



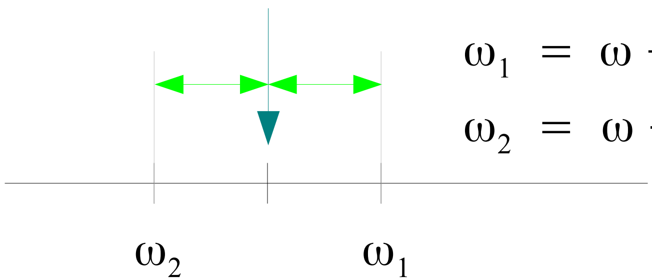


$$\omega_1 = \omega + \Delta\omega \qquad\qquad k_1 = k + \Delta k$$

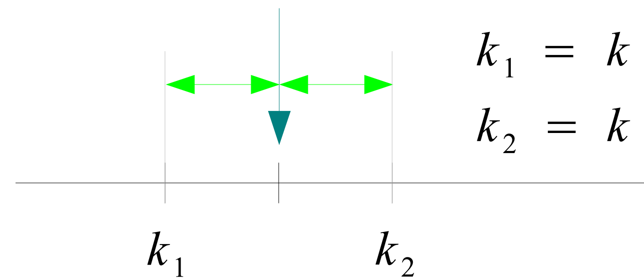
$$\omega_2 = \omega - \Delta\omega \qquad\qquad k_2 = k - \Delta k$$

$$e^{j(k_1 x - \omega_1 t)} \qquad\qquad e^{j(k_2 x - \omega_2 t)}$$

Group Velocity Explanation (2)



$$\omega_1 = \omega + \Delta\omega$$
$$\omega_2 = \omega - \Delta\omega$$



$$k_1 = k + \Delta k$$
$$k_2 = k - \Delta k$$

$$e^{j(k_1 x - \omega_1 t)} + e^{j(k_2 x - \omega_2 t)}$$

$$= e^{j\{(k+\Delta k)x - (\omega + \Delta\omega)t\}} + e^{j\{(k-\Delta k)x - (\omega - \Delta\omega)t\}}$$

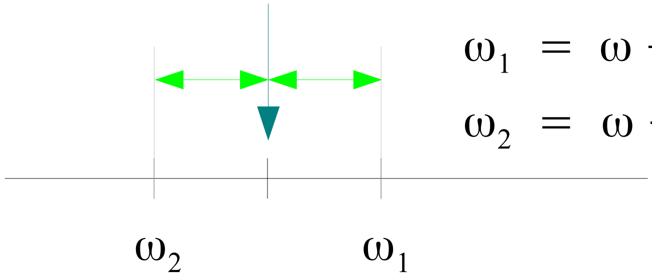
$$= e^{j\{(kx - \omega t) + (\Delta k x - \Delta\omega t)\}} + e^{j\{(kx - \omega t) - (\Delta k x - \Delta\omega t)\}}$$

$$= e^{j(kx - \omega t)}\left\{e^{j(\Delta k x - \Delta\omega t)} + e^{-j(\Delta k x - \Delta\omega t)}\right\}$$

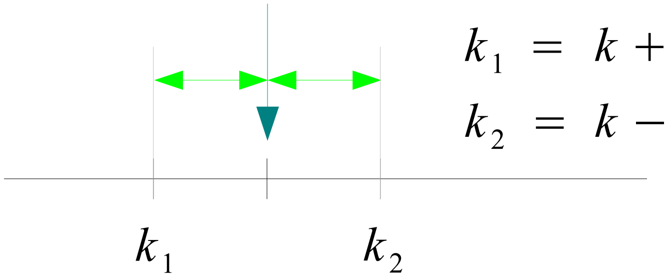
$$= 2\cos(\Delta k x - \Delta\omega t)\, e^{j(kx - \omega t)}$$

Envelope

Group Velocity Explanation (3)



$$\omega_1 = \omega + \Delta\omega$$
$$\omega_2 = \omega - \Delta\omega$$



$$k_1 = k + \Delta k$$
$$k_2 = k - \Delta k$$

$$e^{j(k_1 x - \omega_1 t)} + e^{j(k_2 x - \omega_2 t)}$$
$$= 2\cos(\Delta k\, x - \Delta\omega\, t)\, e^{j(k x - \omega t)}$$

Envelope

$\Delta k \ll k_{1,}\ k_2$ → *Small Wave number* → *Long Wavelength*

$\Delta\omega \ll \omega_{1,}\ \omega_2$ → *Small Frequency* → *Long Period*

Envelope Velocity $v_g = \dfrac{\Delta\omega}{\Delta k}$ Group Velocity $v_g = \dfrac{d\omega}{dk}$

Group Velocity & Fourier Transform (1)

A periodic function

$$f(\theta) = \sum_k a_k \sin(k\theta) + b_k \cos(k\theta)$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(\theta)\,\sin(k\theta)\,d\theta \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(\theta)\,\cos(k\theta)\,d\theta$$

A non-periodic function

$$f(x) = \int_{-\infty}^{+\infty} F(k)\,e^{jkx}\,dk$$

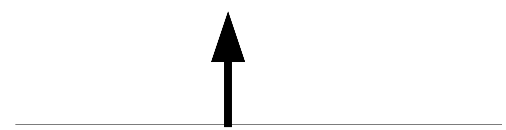
$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\,e^{-jkx}\,dx$$

Group Velocity & Fourier Transform (2)

A non-periodic function

$$f(x) = \int_{-\infty}^{+\infty} F(k)\, e^{jkx}\, dk \qquad F(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\, e^{-jkx}\, dx$$

Infinite number of sine waves
Well-defined wavelength
Well-defined k (wave number)



Lots of sine waves of diff wavelengths
Short wave packet

A long wave packet

A small spread in k
Sharp peak

A non-periodic function

$$f(x) = \int_{-\infty}^{+\infty} F(k)\, e^{jkx}\, dk \qquad F(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)\, e^{-jkx}\, dx$$

A long wave packet

A small spread in k
Sharp peak at k_0

At some initial time $t = t_0$

$$f(x,0) = \int_{-\infty}^{+\infty} F(x) e^{jkx} dk$$

After time t

$$f(x,t) = \int_{-\infty}^{+\infty} F(x) e^{j(kx - \omega(k)t)} dk$$

$\omega(k)$ ***different wavelength components have different frequencies***

Taylor expansion to 1st order

$$\omega(k) = \omega_0 + \frac{d\omega}{dk}(k - k_0)$$

$$f(x,t)$$

$$= \int_{-\infty}^{+\infty} F(x) e^{j(kx - (\omega_0 + \frac{d\omega}{dk}(k-k_0))t)} dk$$

Group Velocity & Fourier Transform (3)

A long wave packet

A small spread in k
Sharp peak at k_0

After time t

$$f(x,t) = \int_{-\infty}^{+\infty} F(x) e^{j(kx - \omega(k)t)} dk$$

Taylor expansion to 1st order

$$\omega(k) = \omega_0 + \frac{d\omega}{dk}(k - k_0)$$

$$f(x,t) = \int_{-\infty}^{+\infty} F(k) e^{j(kx - (\omega_0 + \frac{d\omega}{dk}(k-k_0))t)} dk$$

$$= \int_{-\infty}^{+\infty} F(k) e^{j(k_0 x + kx - k_0 x - (\omega_0 + \frac{d\omega}{dk}(k-k_0))t)} dk$$

$$= e^{j(k_0 x - \omega_0 t)} \int_{-\infty}^{+\infty} F(k) e^{j((k-k_0)x - \frac{d\omega}{dk}(k-k_0)t)} dk$$

$$= e^{j(k_0 x - \omega_0 t)} \int_{-\infty}^{+\infty} F(k) e^{j(k-k_0)\left(x - \frac{d\omega}{dk}t\right)} dk$$

Group Velocity $\quad v_g = \dfrac{d\omega}{dk} \quad \Leftarrow \quad \left(x - \dfrac{d\omega}{dk} t\right)$

Dispersion

Phase Velocity $v_p = \frac{\omega}{k}$

Dispersion : The angular frequency depends on the wave number (or wavelength) $\omega(k)$

Group Velocity $v_g = \frac{\partial \omega}{\partial k} = \frac{\partial \omega(k)}{\partial k}$

$\omega(k) = k\,c$ — Light in vacuum

$\omega(k) = \frac{\hbar k^2}{2m}$ — Free, non-relativistic quantum mechanical particle of mass m

$\omega(k) = 2\sqrt{\frac{\gamma}{M}}\left|\sin \frac{k a}{2}\right|$ — Acoustic branch of vibrations in a crystal

Linear Dispersion

$$\omega(k) = k\,c$$

Light in vacuum

Quadratic Dispersion

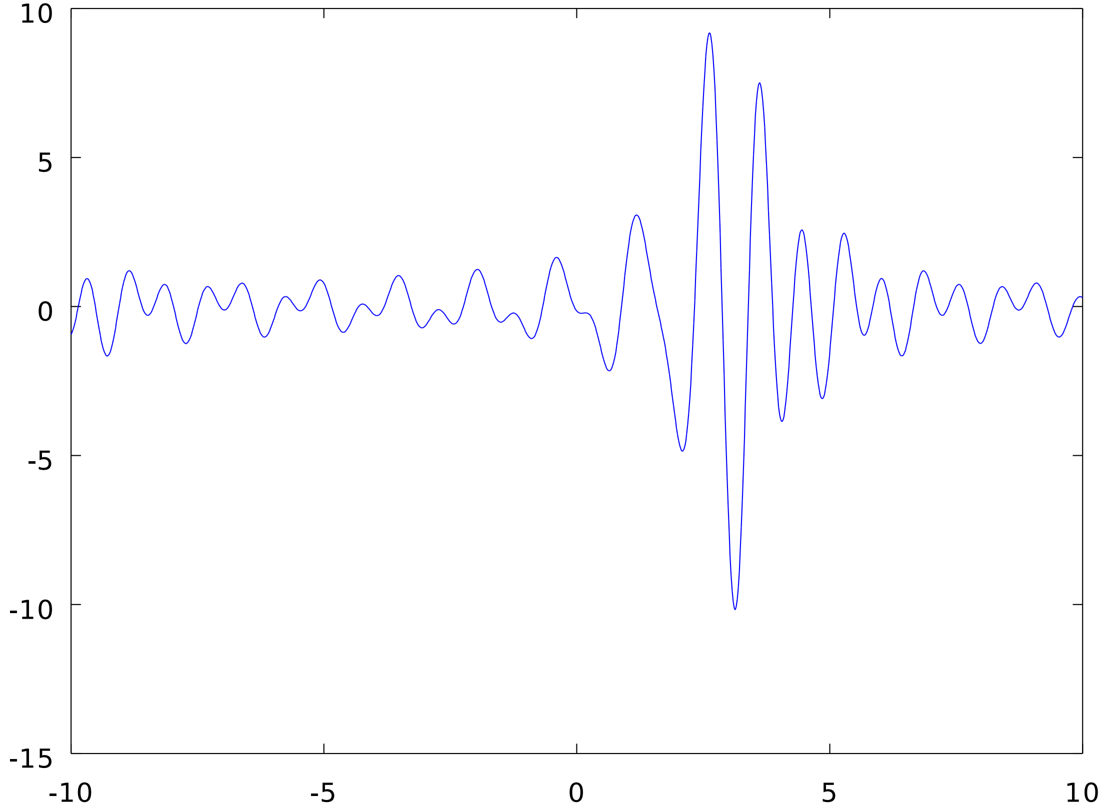
$$\omega(k) = \frac{\hbar k^2}{2m}$$

Free, non-relativistic quantum mechanical particle of mass m

Acoustic Phonon Dispersion

$$\omega(k) = 2\sqrt{\frac{\gamma}{M}}\left|\sin\frac{k\,a}{2}\right|$$

Acoustic branch of vibrations in a crystal




```
x = linspace(-10, +10, 1000);

y = zeros(1, 1000);
for k= 1.0:0.1:2.0
   y = y + cos(4*k*(x-k));
end
plot(x, y);
```

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] http://www.mathpages.com/, Phase, Group, and Signal Velocity
[4] R. Barlow, www.hep.man.ac.uk/u/roger/PHYS10302/lecture15.pdf
[5] P. Hofmann, www.philiphofmann.net/book_material/notes/groupphasevelocity.pdf