Matched Filter (3B)

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12/5/12

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Gaussian Random Process

Thermal Noise   zero-mean white Gaussian random process

$n(t)$ random function
the value at time t is characterized by
Gaussian probability density function

➡ $z(t) = a + n(t)$

$$p(n) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{n}{\sigma}\right)^2\right]$$

➡ $$p(z) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{z-a}{\sigma}\right)^2\right]$$

σ^2 variance of n

$\sigma = 1$ normalized (standardized)
Gaussian function

Central Limit Theorem   sum of statistically independent random variables
approaches Gaussian distribution
regardless of individual distribution functions
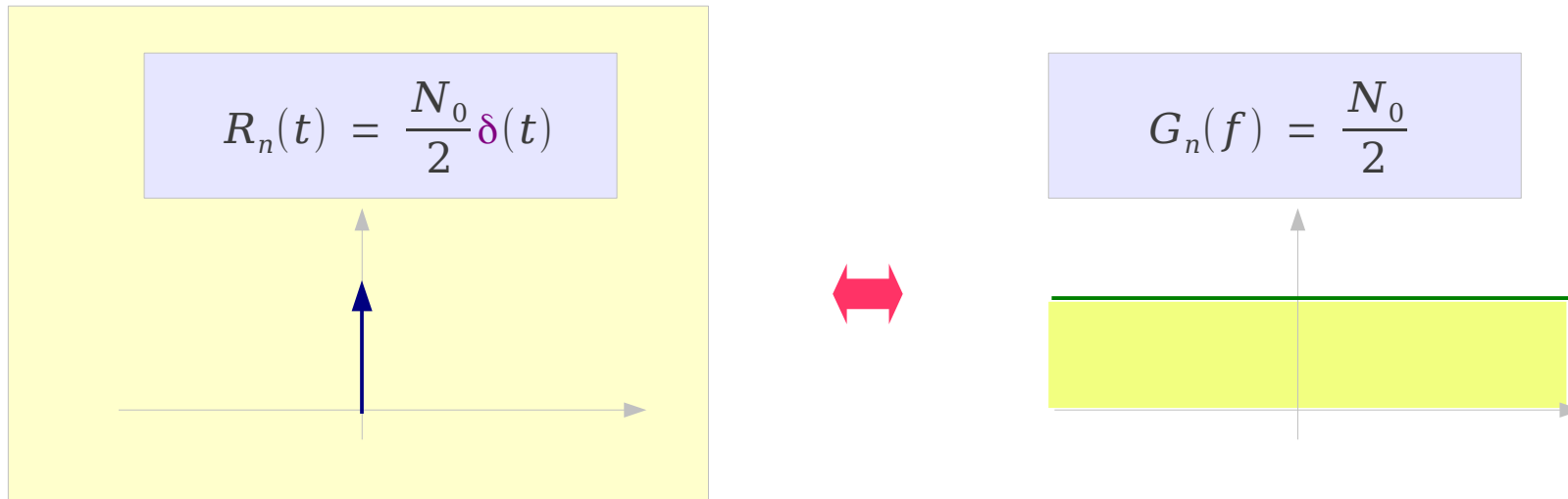
White Gaussian Noise (1)

$n($

Thermal Noise

power spectral density is the same for all frequencies

$$G_n(f) = \frac{N_0}{2} \quad \text{watts / hertz}$$

equal amount of noise power per unit bandwidth

uniform spectral density ➡ White Noise

$$R_n(t) = \frac{N_0}{2}\delta(t)$$

$$G_n(f) = \frac{N_0}{2}$$

⟷

$\delta(t)$ totally <u>uncorrelated</u>, noise samples are independent
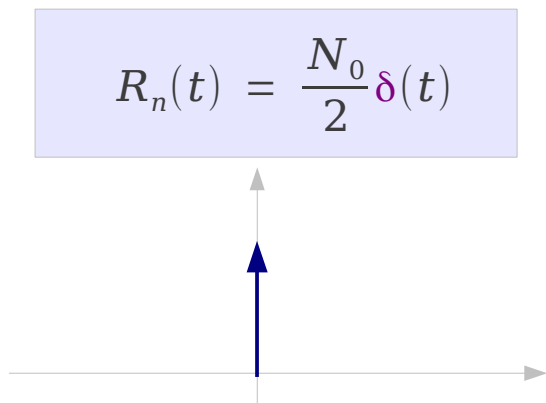
memoryless channel

White Gaussian Noise (2)

Thermal Noise

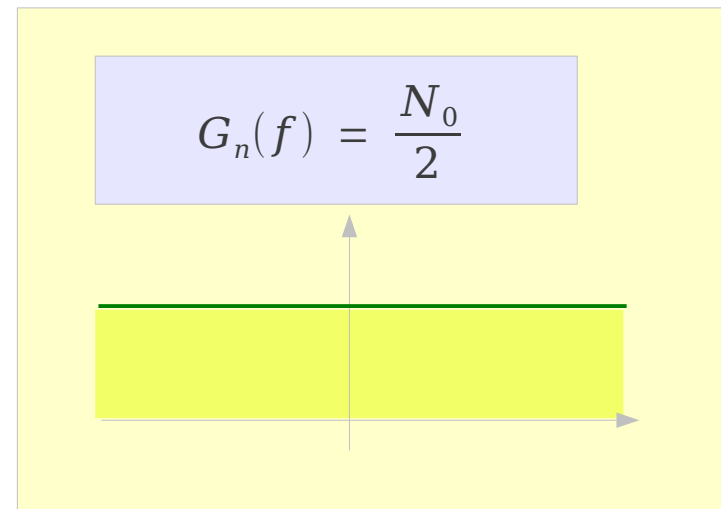power spectral density is the same for all frequencies

$$G_n(f) = \frac{N_0}{2} \quad \text{watts / hertz}$$

equal amount of noise power per unit bandwidth

uniform spectral density ➡ White Noise

$$R_n(t) = \frac{N_0}{2}\delta(t)$$

$$G_n(f) = \frac{N_0}{2}$$

average power

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt = \int_{-\infty}^{+\infty} G_x(f)\, df$$

$$P_n = \int_{-\infty}^{+\infty} \frac{N_0}{2}\, df = \infty$$
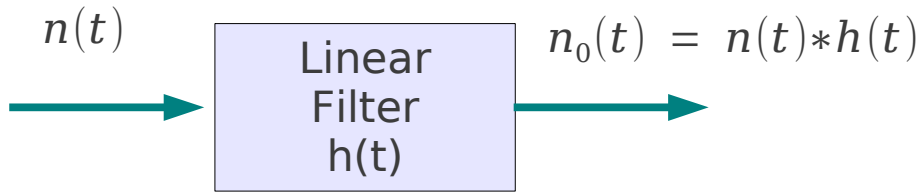
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White Gaussian Noise (3)

$n($

Additive White Gaussian Noise (AWGN)

additive and no multiplicative mechanism

average power

$$P_n = \infty$$

$$R_n(t) = \frac{N_0}{2}\delta(t)$$

$\updownarrow$

$$G_n(f) = \frac{N_0}{2}$$

R.V
$X(t)$ → $h(t)$ → R.V $Y(t)$

$R_{xx}(\tau)$ → $h(t)*h^*(-t)$ → $R_{yy}(\tau)$

$\updownarrow$

$S_{xx}(\omega)$ → $H(\omega)H^*(\omega)$ → $S_{yy}(\omega)$

average power

$$P_{n0} = \int_{-B}^{+B} \frac{N_0}{2}\,df$$

$$= \frac{N_0}{2}\cdot 2B$$

$$= N_0 B$$

$-B$ $+B$

$-B$ $+B$

White Gaussian Noise (4)

$n(t)$

$$n_0(t) = n(t) * h(t)$$

Linear Filter h(t)
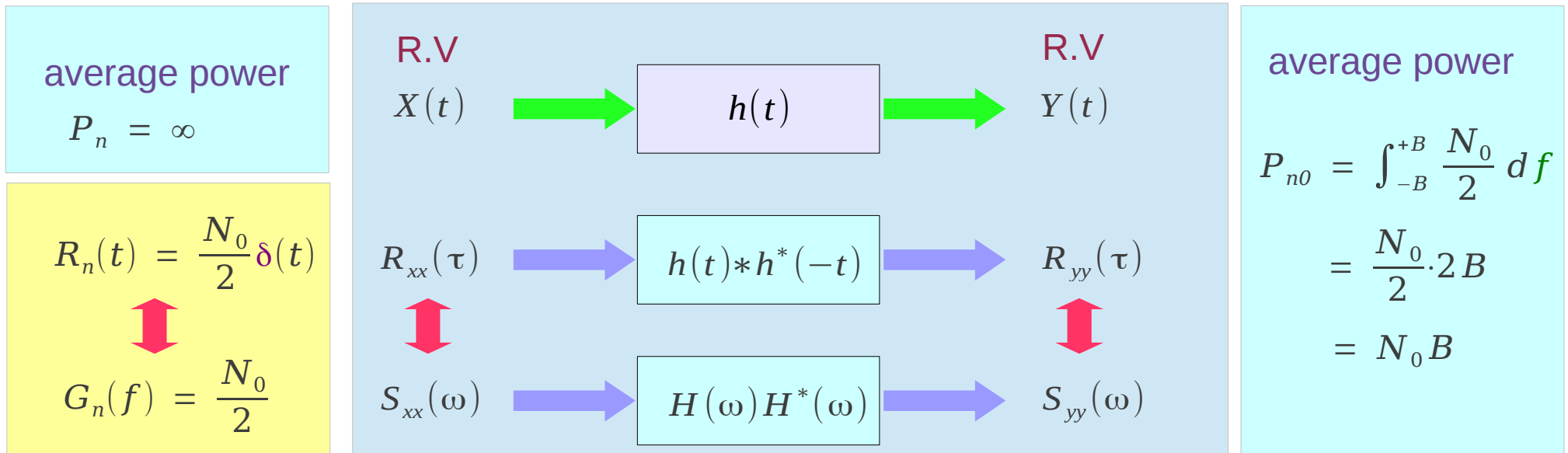
$$G_n(f) = \frac{N_0}{2}$$

$$G_{n0}(f) = G_n(f)|H(f)|^2 = \begin{cases} \dfrac{N_0}{2}|H(f)|^2 & \text{for } |f| < f_u \\ 0 & \text{otherwise} \end{cases}$$
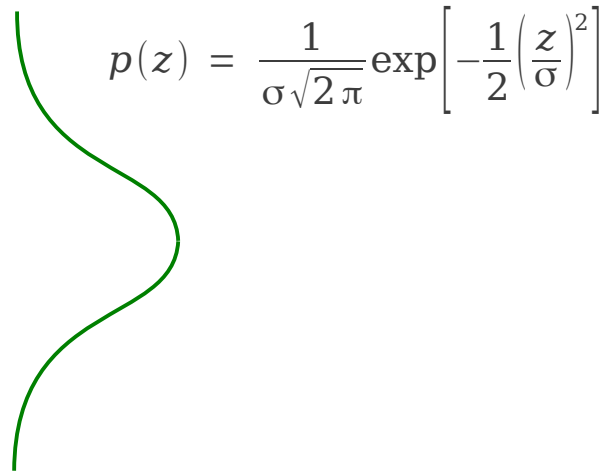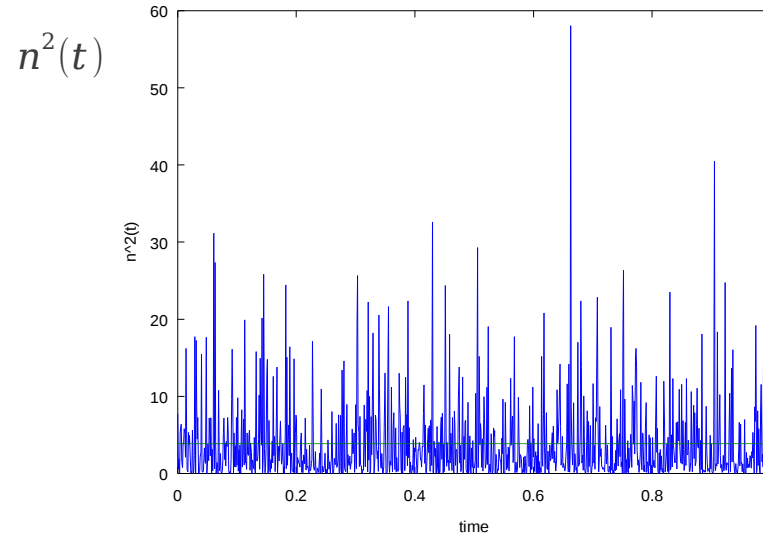
Average output noise power
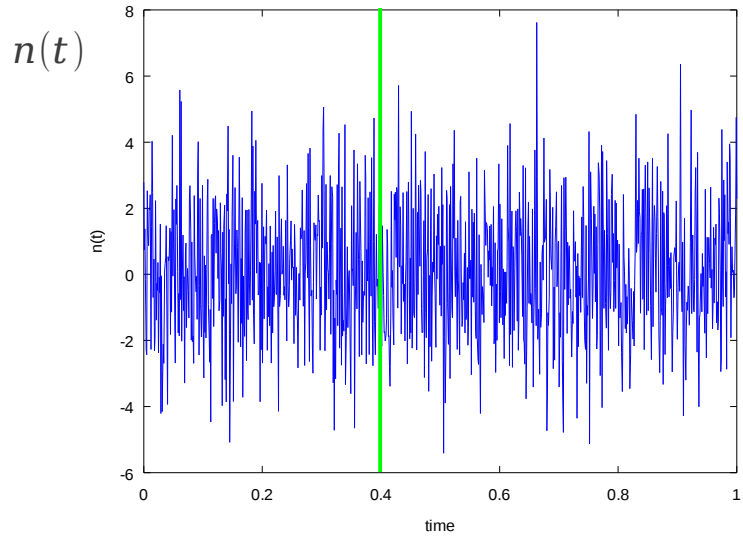
$$\sigma_0^2 = \overline{n_0^2(t)} = \frac{N_0}{2} \int\limits_{-\infty}^{+\infty} |H(f)|^2 \, df$$

RMS

$$\sigma_0 = \sqrt{\overline{n_0^2(t)}} = \sqrt{\frac{1}{T} \int\limits_{-T/2}^{+T/2} n_0^2(t) \, dt}$$
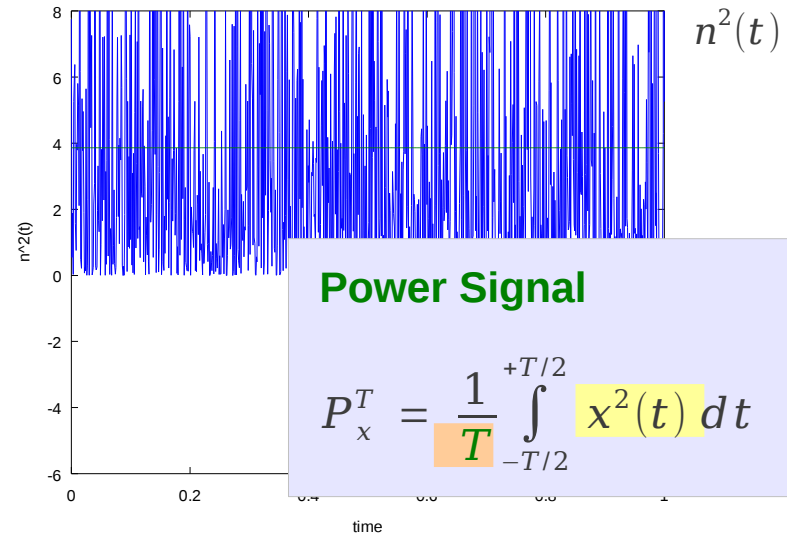
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Gaussian Random Process

$n(t)$

$n^2(t)$

$$p(z) \;=\; \frac{1}{\sigma\sqrt{2\,\pi}}\exp\left[-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2\right]$$
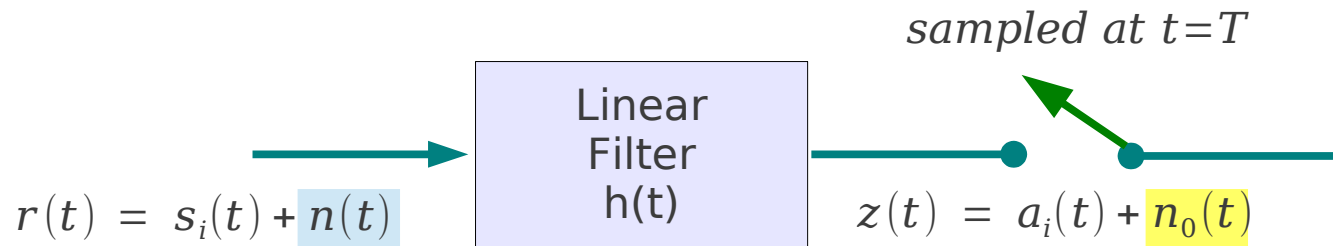
$m = 0$

$\sigma^2 \neq 0$

$n^2(t)$

Power Signal

$$P_x^T \;=\; \frac{1}{T}\int\limits_{-T/2}^{+T/2} x^2(t)\,dt$$

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Matched Filter (1)

to find a filter h(t) that gives max signal-to-noise ratio

sampled at t=T

$$r(t) = s_i(t) + n(t)$$

Linear Filter h(t)

$$z(t) = a_i(t) + n_0(t)$$

variance of $n_0(t)$ ➡ σ_0^2 avg noise power

$$\frac{a_i^2(T)}{n_0^2(t)}$$

$$\frac{\text{instantaneous signal power}}{\text{average noise power}} \quad ➡ \quad \left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2}$$

assume $H_0(f)$ a filter transfer function that maximizes $\left(\dfrac{S}{N}\right)_T$

Matched Filter (2)

$s(t)$ → **Linear Filter h(t)** → $a(t) = s(t)*h(t)$
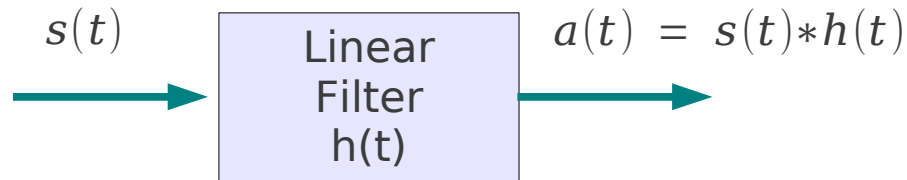
$S(f)$ → $A(f) = \boxed{S(f)H(f)}$ ⬌ $a(t) = \int\limits_{-\infty}^{+\infty} S(f)H(f)e^{j2\pi ft}\, df$

$n(t)$ → **Linear Filter h(t)** → $n_0(t) = n(t)*h(t)$

$$G_n(f) = \frac{N_0}{2}$$

$$G_{n0}(f) = G_n(f)|H(f)|^2 = \begin{cases} \dfrac{N_0}{2}|H(f)|^2 & \text{for } |f| < f_u \\[2mm] 0 & \text{otherwise} \end{cases}$$

Average output noise power $\sigma_0 = \dfrac{N_0}{2}\int\limits_{-\infty}^{+\infty} |H(f)|^2\, df$

Matched Filter (3)

instantaneous signal power $\quad a_i^2 \quad \longleftarrow \quad a(t) = \int_{-\infty}^{+\infty} S(f)H(f)e^{j2\pi ft}\, df$

average output noise power $\quad \sigma_0 = \dfrac{N_0}{2} \int_{-\infty}^{+\infty} |H(f)|^2\, df$

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{\left|\int_{-\infty}^{+\infty} H(f)S(f)e^{+j2\pi fT}\, df\right|^2}{N_0/2 \int_{-\infty}^{+\infty} |H(f)|^2\, df}$$

Does not depend on the particular shape of the waveform

Cauchy Schwarz's Inequality

$$\left|\int_{-\infty}^{+\infty} f_1(x)f_2(x)\, dx\right|^2 \leq \int_{-\infty}^{+\infty} |f_1(x)|^2\, dx \int_{-\infty}^{+\infty} |f_2(x)|^2\, dx \qquad '=' \; holds \; when \; f_1(x) = k f_2^*(x)$$

$$\left|\int_{-\infty}^{+\infty} H(f)S(f)e^{+j2\pi ft}\, dx\right|^2 df \leq \int_{-\infty}^{+\infty} |H(f)|^2\, df \int_{-\infty}^{+\infty} |S(f)e^{+j2\pi fT}|^2\, df \qquad \left|e^{+j2\pi fT}\right| = 1$$

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{\left|\int_{-\infty}^{+\infty} H(f)S(f)e^{+j2\pi fT}\, df\right|^2}{N_0/2 \int_{-\infty}^{+\infty} |H(f)|^2\, df} \leq \frac{\int_{-\infty}^{+\infty} |H(f)|^2\, df \int_{-\infty}^{+\infty} |S(f)e^{+j2\pi fT}|^2\, df}{N_0/2 \int_{-\infty}^{+\infty} |H(f)|^2\, df} = \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2\, df$$

Matched Filter (4)

Two-sided power spectral density of input noise $\longrightarrow$ $\dfrac{N_0}{2}$

Average noise power $\qquad \sigma_0 = \dfrac{N_0}{2} \displaystyle\int_{-\infty}^{+\infty} |H(f)|^2 \, df$

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{\left| \displaystyle\int_{-\infty}^{+\infty} H(f) S(f) e^{+j2\pi fT} \, df \right|^2}{N_0/2 \displaystyle\int_{-\infty}^{+\infty} |H(f)|^2 \, df}$$

Cauchy Schwarz's Inequality

$$\left(\frac{S}{N}\right)_T \leq \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 \, df$$

$$max \left(\frac{S}{N}\right)_T = \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 \, df = \frac{2E}{N_0}$$

input signal energy

power spectral density
of input noise

does not depend on the particular
shape of the waveform

Matched Filter (5)

$$\left| \int_{-\infty}^{+\infty} H(f) S(f) e^{+j2\pi ft} \, dx \right|^2 df \; \leqq \; \int_{-\infty}^{+\infty} |\boxed{H(f)}|^2 df \int_{-\infty}^{+\infty} |\boxed{S(f) e^{+j2\pi fT}}|^2 df \qquad \left(\frac{S}{N}\right)_T \leq \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 df$$

$$max \left(\frac{S}{N}\right)_T \; = \; \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 df \; = \; \frac{2E}{N_0}$$

when complex conjugate relationship exists

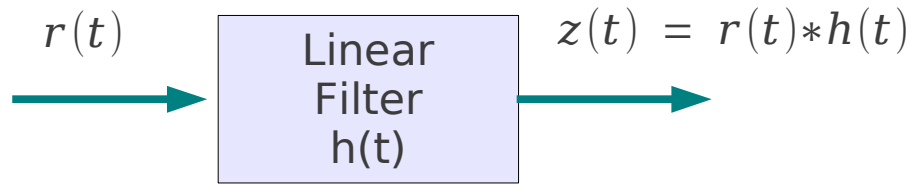$$H(f) \; = \; H_0(f) \; = \; k S^*(f) e^{-j2\pi fT}$$

$$h(t) \; = \; h_0(t) \; = \; \begin{cases} k s(T-t) & 0 \leq t \leq T \\ 0 & elsewhere \end{cases}$$

$H_0(f)$ a filter transfer function that maximizes $\left(\dfrac{S}{N}\right)_T$

impulse response : <u>delayed</u> version of
the <u>mirror</u> image of the <u>signal</u> waveform

Convolution vs. Correlation Realization

$r(t)$ → Linear Filter $h(t)$ → $z(t) = r(t)*h(t)$

$$z(t) = \int_0^t r(\tau)h(t-\tau)\,d\tau$$

$$= \int_0^t r(\tau)s(T-(t-\tau))\,d\tau$$

$$= \int_0^t r(\tau)s(T-t+\tau)\,d\tau$$

$$z(T) = \int_0^T r(\tau)s(\tau)\,d\tau$$

shift position

convolution $\quad z(t) = \int_0^t r(\tau)s(T-t+\tau)\,d\tau \qquad z(T) = \int_0^T r(\tau)s(\tau)\,d\tau$
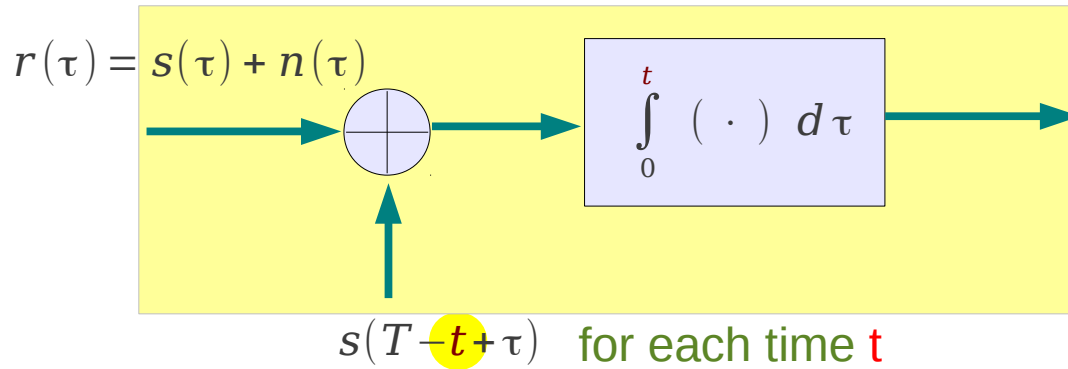
a sine-wave amplitude modulated
by a linear ramp

correlation $\quad z(t) = \int_0^t r(\tau)\,\boxed{s(\tau)}\,d\tau \qquad z(T) = \int_0^T r(\tau)\,\boxed{s(\tau)}\,d\tau$

a linear ramp output

fixed position

Convolution Realization

$$r(\tau) = s(\tau) + n(\tau)$$

$$\int_0^t (\ \cdot\ )\, d\tau$$

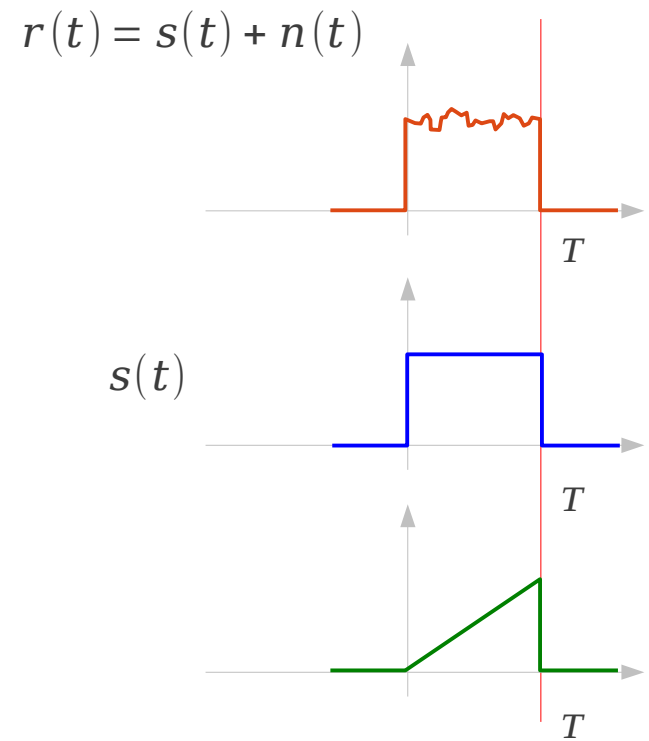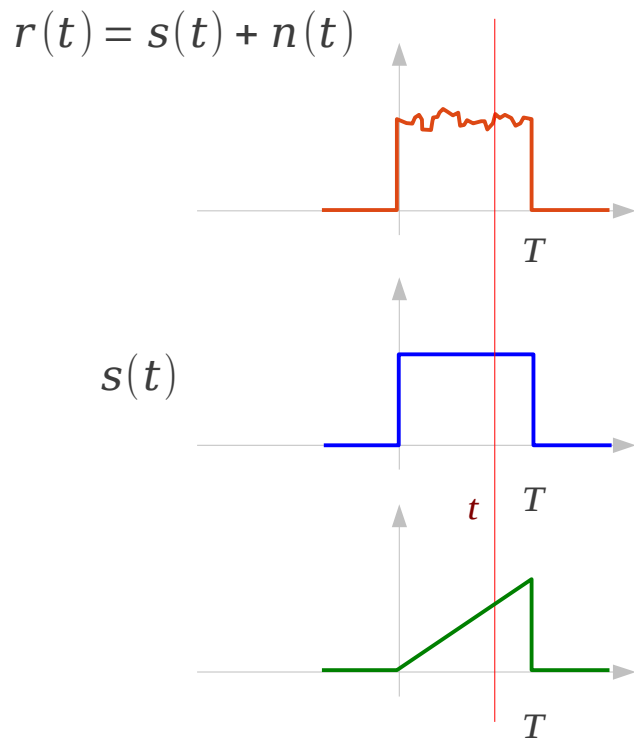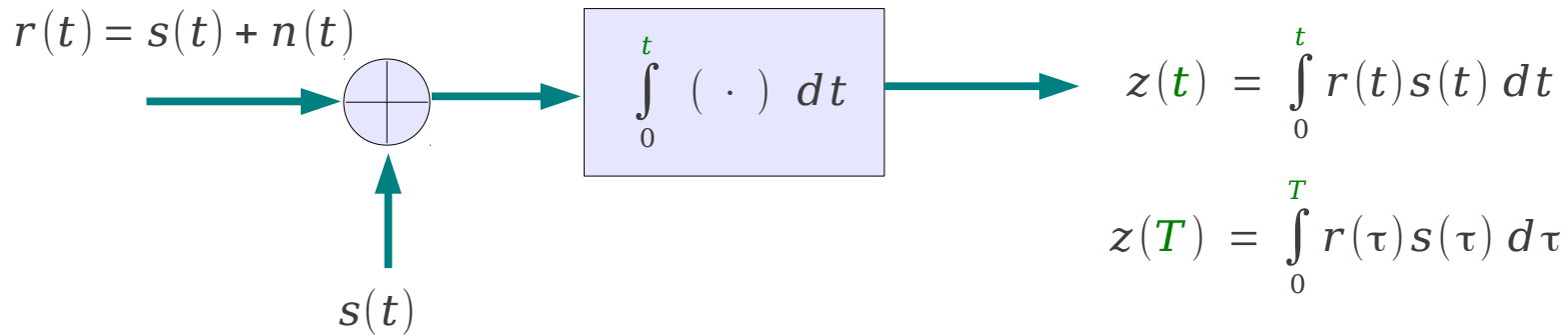$$s(T - t + \tau) \quad \text{for each time } t$$

$$z(t) = \int_0^t r(\tau) h(t - \tau)\, d\tau$$

$$= \int_0^t r(\tau) s(T - t + \tau)\, d\tau$$

$$z(T) = \int_0^T r(\tau) s(\tau)\, d\tau$$

$$r(\tau) = s(\tau) + n(\tau)$$

$$s(T - t + \tau)$$

$$r(\tau) = s(\tau) + n(\tau)$$

$$s(T + \tau)$$

$$t = 0$$

$$r(\tau) = s(\tau) + n(\tau)$$

$$s(\tau)$$

$$t = T$$

Correlation Realization (1)

$$r(t) = s(t) + n(t)$$

$$z(t) = \int_0^t (\cdot)\, dt$$

$$s(t)$$

$$z(t) = \int_0^t r(t)s(t)\, dt$$

$$z(T) = \int_0^T r(\tau)s(\tau)\, d\tau$$

$$r(t) = s(t) + n(t)$$

$$s(t)$$

$$r(t) = s(t) + n(t)$$

$$s(t)$$
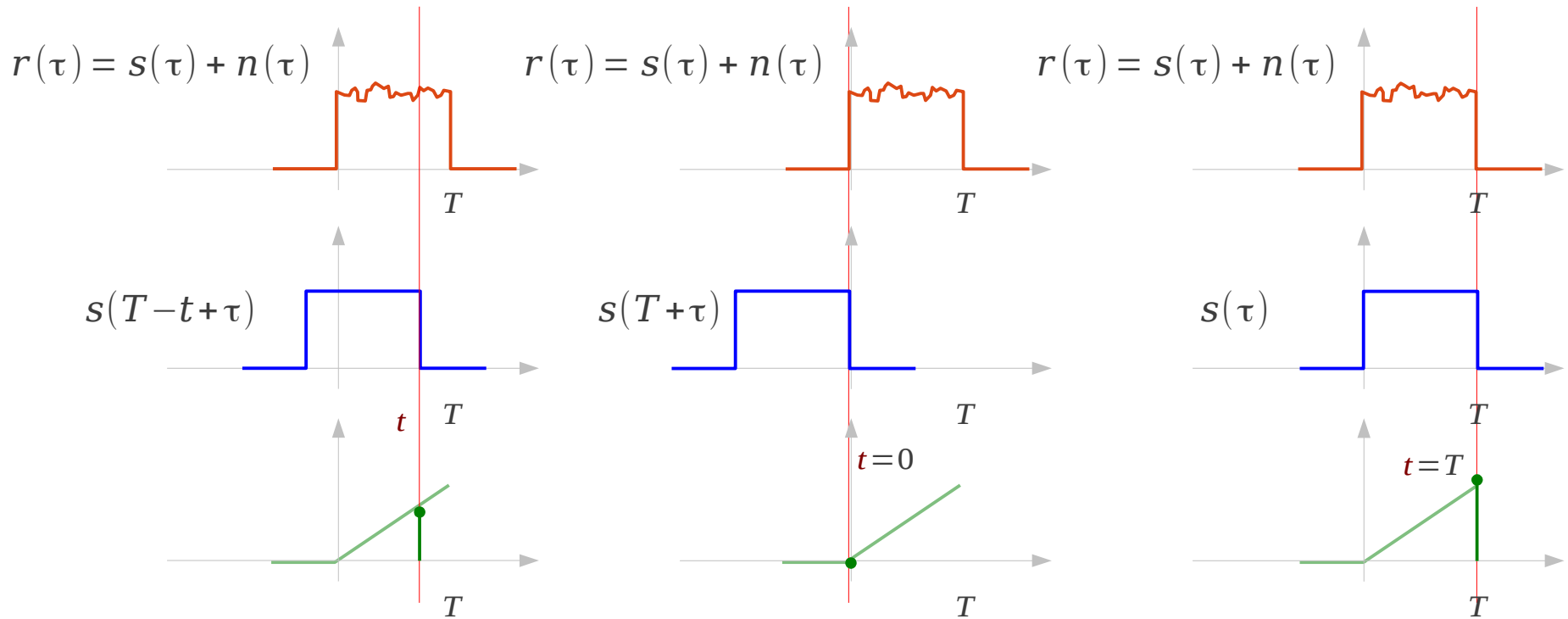
Correlation Realization (2)

$$r(t) = s(t) + n(t)$$

$$z(t) = \int_0^t r(\tau)s(\tau)\,d\tau$$

$$s(t)$$
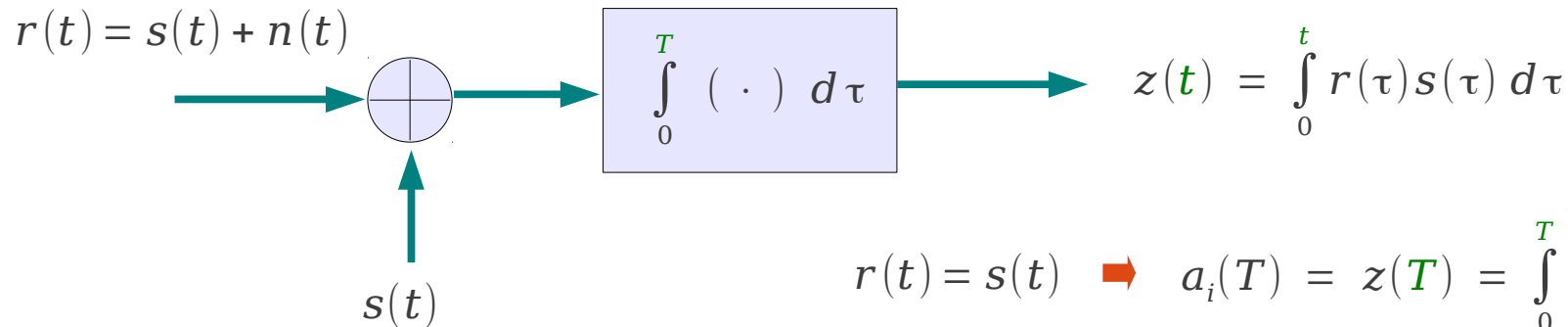
$$r(t) = s(t) \quad \Rightarrow \quad a_i(T) = z(T) = \int_0^T s^2(\tau)\,d\tau = E$$

$$\sigma_0^2 = E[n_o^2(t)] = E\left[\int_0^T n(t)s(t)\,dt \int_0^T n(\tau)s(\tau)\,d\tau\right]$$

$$= E\left[\iint_0^T n(t)n(\tau)\,s(t)s(\tau)\,dt\,d\tau\right]$$

$$= \iint_0^T E[n(t)n(\tau)]\,s(t)s(\tau)\,dt\,d\tau$$

$$= \iint_0^T \frac{N_0}{2}\delta(t-\tau)\,s(t)s(\tau)\,dt\,d\tau$$

$$= \frac{N_0}{2} \int_0^T s^2(t)\,dt = \frac{N_0}{2}E$$
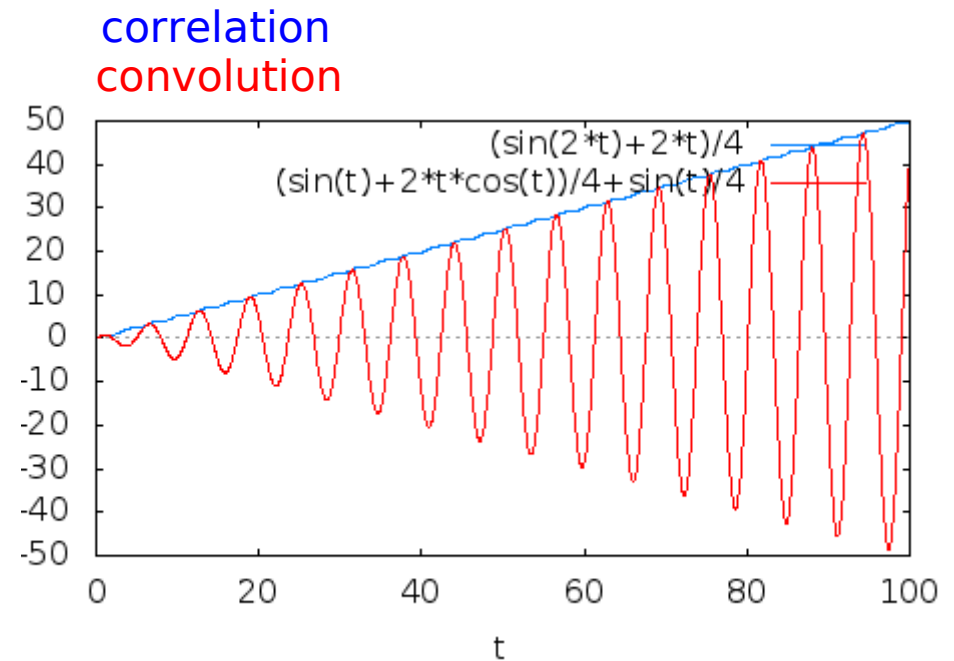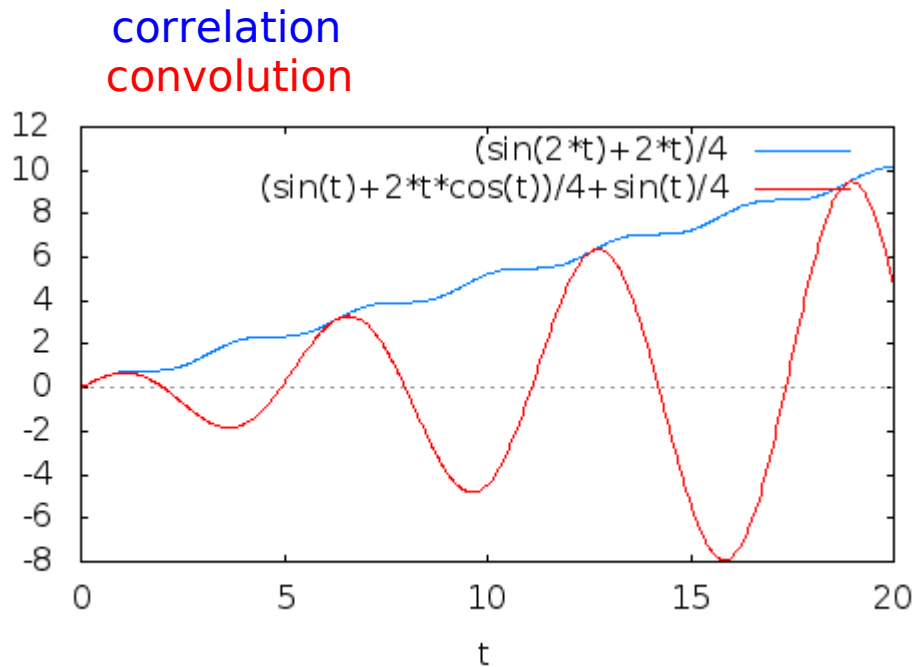
convolution

$$\frac{a_i^2(T)}{n_0^2(t)} \qquad \left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2}$$

$$max\left(\frac{S}{N}\right)_T = \frac{2}{N_0}\int_{-\infty}^{+\infty}|S(f)|^2\,df = \frac{2E}{N_0}$$

correlation

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{E^2}{\dfrac{N_0}{2}E} = \frac{2E}{N_0}$$

Correlation and Convolution Examples (1)

correlation
convolution

correlation
convolution

z : integrate(cos(x)*cos(2*%pi - t +x), x, 0, t);

(sin(t)+2*t*cos(t))/4+sin(t)/4

z : integrate(cos(x)*cos(x), x, 0, t);

(sin(2*t)+2*t)/4

convolution

correlation

Correlation and Convolution Examples (2)

$$s(t) \qquad A\cos(\omega_0 t) \qquad 0 \le t < T$$
$$0 \qquad elsewhere$$

$$z(t) = \int_0^t r(\tau)h(t-\tau)\,d\tau$$

$$= \int_0^t r(\tau)s(T-(t-\tau))\,d\tau$$

$$= \int_0^t r(\tau)s(T-t+\tau)\,d\tau$$

$$z(t) = \int_0^t r(\tau)s(T-t+\tau)\,d\tau$$

when $r(t) = s(t)$

$$z(t) = \int_0^t s(\tau)s(T-t+\tau)\,d\tau$$

$$= A^2 \int_0^t \cos(\omega_0\tau)\cos(\omega_0(T-t+\tau))\,d\tau$$

$$= \frac{A^2}{2}\int_0^t \cos(\omega_0(T-t)) + \cos(\omega_0(T-t+2\tau))\,d\tau$$

$$= \frac{A^2}{2}\left[\cos(\omega_0(T-t))\tau - \frac{1}{2\omega_0}\sin(\omega_0(T-t+2\tau))\right]_0^t$$

$$= \frac{A^2}{2}\left[\cos(\omega_0(T-t))t - \frac{1}{2\omega_0}\{\sin(\omega_0(T+t)) - \sin(\omega_0(T-t))\}\right]$$

$$s(t) \qquad A\cos(\omega_0 t)$$

$$0 \qquad\qquad elsewhere$$

$$z(t) = \int_0^t r(\tau) h(t-\tau)\, d\tau$$

$$= \int_0^t r(\tau) s(T-(t-\tau))\, d\tau$$

$$= \int_0^t r(\tau) s(T-t+\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau) x(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} x(t_0-\tau) x(t-\tau)\, d\tau$$

$$when \quad r(t) = s(t) \qquad\qquad A=T=1, n=4, \omega_0=8\pi$$

$$y(t) = A^2 \int_0^t \cos[\omega_0(t-\tau)]\cos[\omega_0(-\tau+T)]\, d\tau$$

$$= \frac{A^2}{2} \int_0^t \cos[\omega_0(t-T)]+\cos[\omega_0(-2\tau+t+T)]\, d\tau$$

$$= \frac{A^2}{2} \{ t\cos[\omega_0(t-T)] - \frac{1}{2\omega_0}\cos[\omega_0(-2\tau+t+T)]_0^T \}$$

$$= \frac{A^2}{2} \{ t\cos(\omega_0 t) + \frac{1}{\omega_0}\sin(\omega_0 t) \} \qquad (0 \le t \le T)$$

$$= \frac{A^2}{2} \{ t(2T-t)\cos(\omega_0 t) - \frac{1}{\omega_0}\sin(\omega_0 t) \} \qquad (T \le t \le 2T)$$

$$y(t) = A^2 \int_0^t \cos(\omega_0 \tau)\, d\tau$$

$$= \frac{A^2}{2} \int_0^t [1+\cos(2\omega_0 \tau)]\, d\tau$$

$$= A^2 \frac{t}{2} + \frac{A^2}{4\omega_0}[\sin(2\omega_0 \tau)]_0^t$$

$$= A^2 \frac{t}{2} + \frac{A^2}{4\omega_0}[\sin(2\omega_0 t)] \qquad (0 \le t \le T)$$

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] B. Sklar, "Digital Communications: Fundamentals and Applications"
[4] W. Etten, "Introduction to Random Signals and Noise"