# General Vector Space (3A)

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11/21/12

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# Vector Space

V: non-empty set of objects

defined operations:

| | |
|---|---|
| addition | $\mathbf{u} + \mathbf{v}$ |
| scalar multiplication | $k\,\mathbf{u}$ |

if the following axioms are satisfied

for all object $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and all scalar $k$, $m$

V: vector space

objects in V: vectors

1. if $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ (zero vector)
5. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + (\mathbf{u}) = \mathbf{0}$
6. if $k$ is any scalar and $\mathbf{u}$ is objects in V, then $k\mathbf{u}$ is in V
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

# Test for a Vector Space

1. Identify the set $V$ of objects
2. Identify the addition and scalar multiplication on $V$
3. Verify $\mathbf{u} + \mathbf{v}$ is in $V$ and $k\mathbf{u}$ is in $V$

    **closure** under **addition** and **scalar multiplication**

4. Confirm other axioms.

1. if $\mathbf{u}$ and $\mathbf{v}$ are objects in $V$, then $\mathbf{u} + \mathbf{v}$ is in $V$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ (zero vector)
5. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + (\mathbf{u}) = \mathbf{0}$
6. if $k$ is any scalar and $\mathbf{u}$ is objects in $V$, then $k\mathbf{u}$ is in $V$
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

# Subspace

a subset W of a vector space V

If the subset W is itself a vector space → the subset W is a **subspace** of V

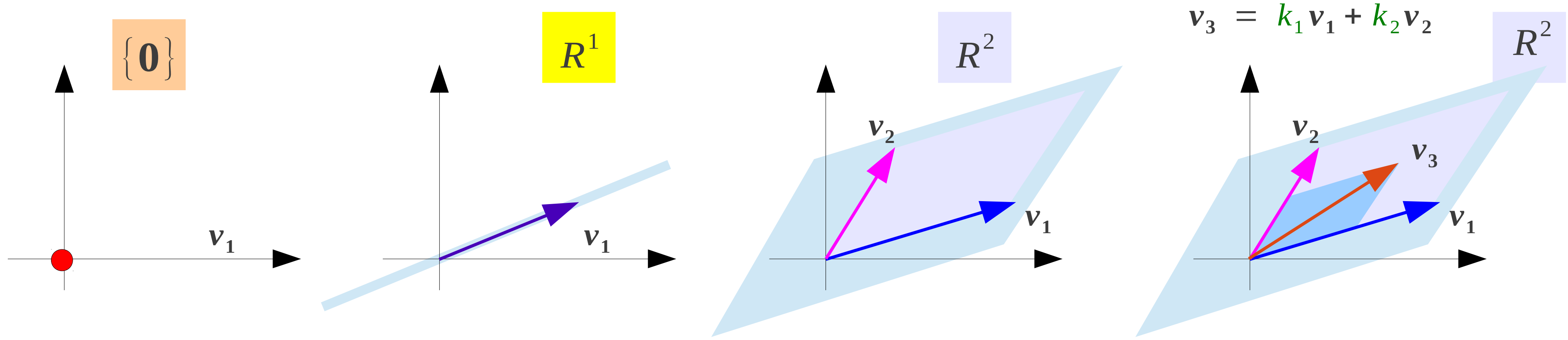
1. if $\mathbf{u}$ and $\mathbf{v}$ are objects in W, then $\mathbf{u} + \mathbf{v}$ is in W
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ (zero vector)
5. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + (\mathbf{u}) = \mathbf{0}$
6. if $k$ is any scalar and $\mathbf{u}$ is objects in W, then $k\mathbf{u}$ is in W
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

# Subspace Example (1)

**In vector space** $R^2$

| | | | | |
|---|---|---|---|---|
| any one vector | (linearly indep.) | **spans** | $R^1$ | line through **0** |
| any two non-collinear vectors | (linearly indep.) | **spans** | $R^2$ | plane |
| any three or more vectors | (linearly dep.) | **spans** | $R^2$ | plane |

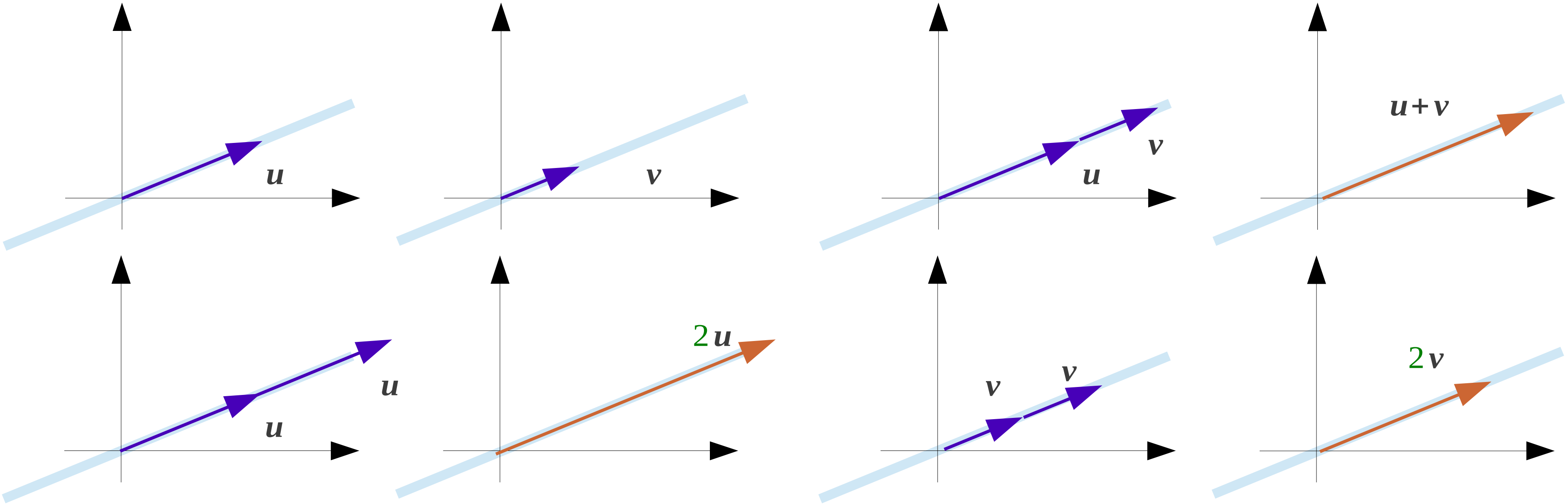
**Subspaces of** $R^2$

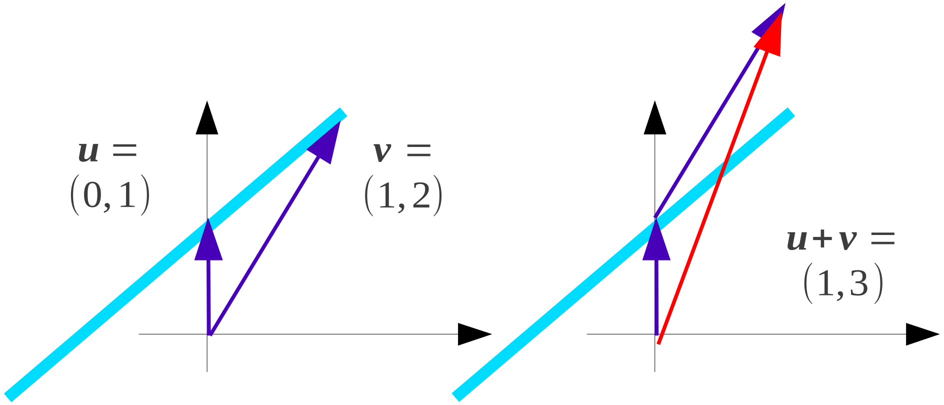


# Subspace Example (2)

**In vector space** $R^2$

any one vector (linearly indep.) **spans** $R^1$ line through **0**





~~**vector space**~~

# Subspace Example (3)

**In vector space** $R^3$

| | | | | |
|---|---|---|---|---|
| any one vector | (linearly indep.) | **spans** | $R^1$ | line through **0** |
| any two non-collinear vectors | (linearly indep.) | **spans** | $R^2$ | plane through **0** |
| any three vectors<br>non-collinear, non-coplanar | (linearly indep.) | **spans** | $R^3$ | 3-dim space |
| any four or more vectors | (linearly dep.) | **spans** | $R^3$ | 3-dim space |

**Subspaces of** $R^3$

| | | | |
|---|---|---|---|
| $\{\mathbf{0}\}$ | $R^1$ | $R^2$ | $R^3$ |
| | line through **0** | plane through **0** | 3-dim space |

# Row & Column Spaces

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

**ROW Space** **subspace** of $R^n$

$$= span\{\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_m\}$$

**COLUMN Space** **subspace** of $R^m$

$$= span\{\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n\}$$

$$\begin{aligned} \boldsymbol{r}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \boldsymbol{r}_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ & \quad\ \ \vdots \qquad \vdots \qquad\qquad\quad \vdots \\ \boldsymbol{r}_m &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ & \qquad \underset{\boldsymbol{n}}{\xleftrightarrow{\qquad\qquad\qquad}} \end{aligned}$$

$$\boldsymbol{r}_i \in R^n$$

$$\begin{array}{ccccc} \boldsymbol{c}_1 & \boldsymbol{c}_2 & & \boldsymbol{c}_n & \\ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} & \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} & \updownarrow \boldsymbol{m} \end{array}$$

$$\boldsymbol{c}_i \in R^m$$

# Row Space

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

**ROW Space** **subspace** of $R^n$

$$= span\{\boldsymbol{r_1}, \boldsymbol{r_2}, \cdots, \boldsymbol{r_m}\}$$

$$= \{\boldsymbol{w}\}$$

$$\boldsymbol{r}_i \in R^n$$

$$\boldsymbol{w} = k_1\boldsymbol{r_1} + k_2\boldsymbol{r_2} + \cdots + k_m\boldsymbol{r_m}$$

$$\boldsymbol{r}_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

$$\boldsymbol{r}_2 = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$

$$\vdots \quad \vdots \qquad \vdots$$

$$\boldsymbol{r}_m = \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\xleftrightarrow{\qquad\qquad\qquad}$$

$$\boldsymbol{n}$$

$$= k_1 \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

$$+ k_2 \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$

$$\vdots \quad \vdots \qquad \vdots$$

$$+ k_m \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

# Column Spaces

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

**COLUMN Space** **subspace** of $R^m$

$$= span\{\boldsymbol{c_1}, \boldsymbol{c_2}, \cdots, \boldsymbol{c_n}\}$$

$$= \{\boldsymbol{w}\}$$

$$\boldsymbol{c}_i \in R^m \quad \begin{array}{ccccc} & \boldsymbol{c_1} & \boldsymbol{c_2} & & \boldsymbol{c_n} \\ \boldsymbol{m} \updownarrow & \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} & \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array}$$

$$\boldsymbol{w} = k_1\boldsymbol{c_1} + k_2\boldsymbol{c_2} + \cdots + k_n\boldsymbol{c_n}$$

$$= k_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + k_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots + k_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

# Null Space

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} m$$

**NULL Space**

**subspace** of $R^n$

**solution space**

$$= \begin{bmatrix} a_{11}x_1 + & a_{12}x_2 + & \cdots & a_{1n}x_n \\ a_{21}x_1 + & a_{22}x_2 + & \cdots & a_{2n}x_n \\ \vdots & \vdots & & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \cdots & a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$\boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots + x_n\boldsymbol{c}_n = \boldsymbol{0} \qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$$

$$\boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots + x_n\boldsymbol{c}_n = \boldsymbol{b} \qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

# Null Space

$$m \; \begin{array}{c} n \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \; m$$

**NULL Space** **subspace** of $R^n$

**solution space** $\boldsymbol{A x = 0}$

**Invertible A** $\boldsymbol{x = A^{-1} 0 = 0}$ only trivial solution $\{\mathbf{0}\}$

| | | | | |
|---|---|---|---|---|
| **Non-invertible A**<br>$\cancel{A^{-1}}$ | zero row(s) in a RREF | free variables | parameters *s, t, u, …* | |
| | one | one | a line through the origin | $R^1$ |
| | two | two | a plane through the origin | $R^2$ |
| | three | three | a 3-dim space through the origin | $R^3$ |

# Solution Space of **Ax=b** (1)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

$$\xcancel{0 = 1}$$

$$\begin{aligned} 1 \cdot x_1 \qquad + 3 \cdot x_3 &= -1 \\ 1 \cdot x_2 - 4 \cdot x_3 &= 2 \end{aligned}$$

$$1 \cdot x_1 - 5 \cdot x_2 + 1 \cdot x_3 = 4$$

Solve for a leading variable

$$x_1 = -1 - 3 \cdot x_3 \qquad\qquad x_1 = 4 + 5 \cdot x_2 - 1 \cdot x_3$$

$$x_2 = 2 + 4 \cdot x_3$$

Treat a free variable as a parameter

$$x_3 = t \qquad\qquad x_2 = s \quad x_3 = t$$

$$\begin{cases} x_1 = -1 - 3\text{t} \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

# Solution Space of **Ax=b** (2)

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ \boxed{x_3 = t} \end{cases}$$

free variable

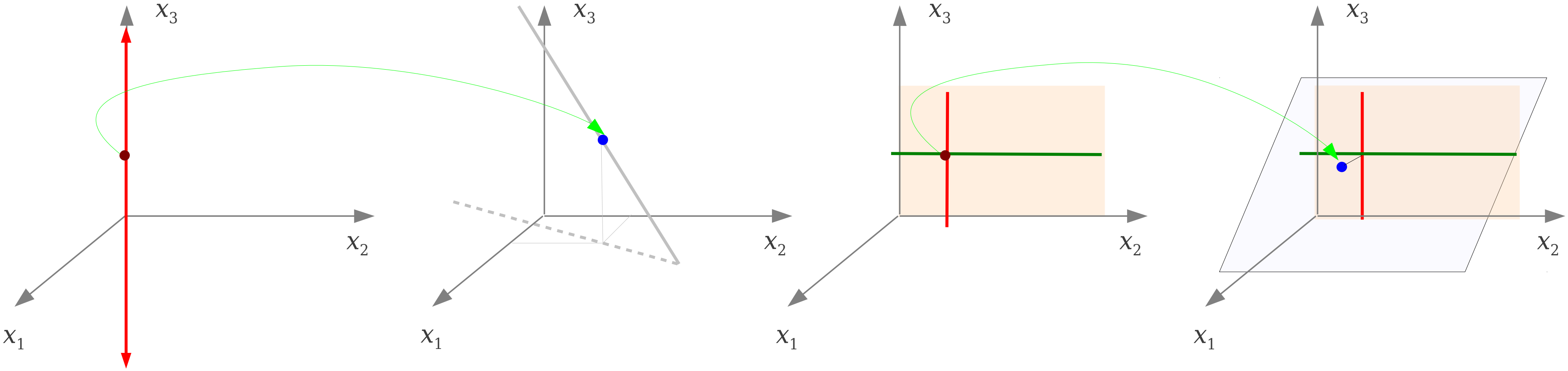
$$\begin{cases} x_1 = 4 + 5s - 1t \\ \boxed{\begin{aligned} x_2 &= s \\ x_3 &= t \end{aligned}} \end{cases}$$

free variable

free variable

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$



*infinitely many solutions*

*infinitely many solutions*

# Solution Space of **Ax=b** (3)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 = -1 - 3\mathrm{t} \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} + \boxed{t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}} + \boxed{s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

General Solution of $Ax = b$ →

Particular Solution of $Ax = b$

General Solution of $Ax = 0$

Particular Solution of $Ax = b$

General Solution of $Ax = 0$

# Linear System & Inner Product (1)

## Linear Equations

## Corresponding Homogeneous Equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

$$\boldsymbol{a} = (a_1 , a_2 , \cdots , a_n)$$

$$\boldsymbol{x} = (x_1 , x_2 , \cdots , x_n)$$

$$\boldsymbol{a} \cdot \boldsymbol{x} = b$$

normal vector

$$\boldsymbol{a} \cdot \boldsymbol{x} = 0$$

each **solution** vector $\boldsymbol{x}$ of a **homogeneous** equation

**orthogonal** to the coefficient vector $\boldsymbol{a}$

## Homogeneous Linear System

$$\begin{array}{l} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = 0 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = 0 \\ \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = 0 \end{array}$$

$$\begin{array}{l} \boldsymbol{r_1} \cdot \boldsymbol{x} = 0 \\ \boldsymbol{r_2} \cdot \boldsymbol{x} = 0 \\ \quad \cdots \\ \boldsymbol{r_m} \cdot \boldsymbol{x} = 0 \end{array}$$

# Linear System & Inner Product (2)

Homogeneous Linear System

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}$$

$$\begin{array}{l} \boldsymbol{r_1} \cdot \boldsymbol{x} = 0 \\ \boldsymbol{r_2} \cdot \boldsymbol{x} = 0 \\ \quad \cdots \\ \boldsymbol{r_m} \cdot \boldsymbol{x} = 0 \end{array}$$

each **solution** vector $\boldsymbol{x}$ of a **homogeneous** equation

**orthogonal** to the row vector $\boldsymbol{r_i}$ of the coefficient matrix

Homogeneous Linear System $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{0}$ $\boldsymbol{A} : m \times n$

**solution set** consists of all vectors in $R^n$

that are **orthogonal** to every row vector of $\boldsymbol{A}$

# Linear System & Inner Product (3)

Non-Homogeneous Linear System $A\cdot x = b$ $A : m\times n$

Homogeneous Linear System $A\cdot x = 0$

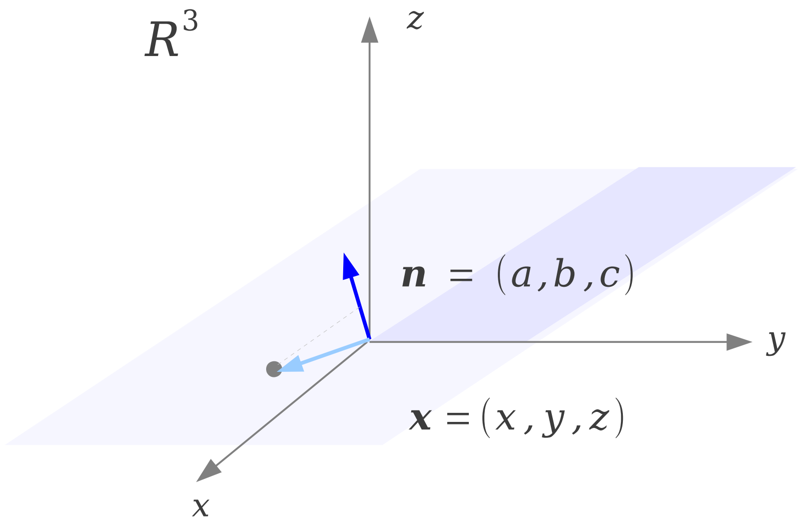
**a particular solution**

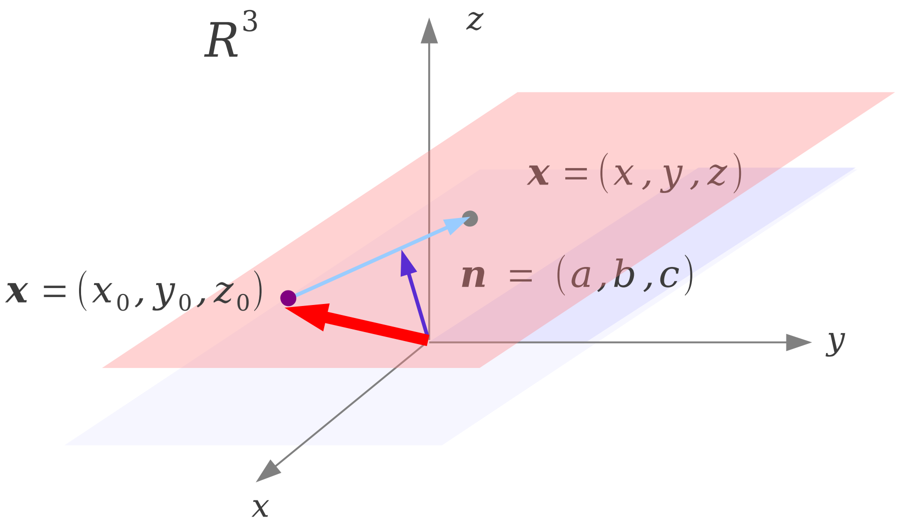
$$A\cdot x = b$$

**solution set** consists of all vectors in $R^n$
that are **orthogonal** to every row vector of $A$

+

**a particular solution** $x_0$ $A\cdot x_0 = b$





# Linear System & Inner Product (4)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$3 \begin{cases} 2 \begin{cases} \boldsymbol{r}_1 \cdot \boldsymbol{x} = 0 \\ \boldsymbol{r}_2 \cdot \boldsymbol{x} = 0 \end{cases} \\ 1 \begin{cases} \text{a line through the origin} \quad R^1 \end{cases} \end{cases} \qquad 3 \begin{cases} 1 \begin{cases} \boldsymbol{r}_1 \cdot \boldsymbol{x} = 0 \end{cases} \\ 2 \begin{cases} \text{a plane through the origin} \quad R^2 \end{cases} \end{cases}$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

# Consistent Linear System **Ax=b**

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + & a_{12}x_2 + & \cdots & a_{1n}x_n \\ a_{21}x_1 + & a_{22}x_2 + & \cdots & a_{2n}x_n \\ \vdots & \vdots & & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \cdots & a_{mn}x_n \end{pmatrix}$$

$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ consistent ⟺

$x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots + x_n\boldsymbol{c}_n = \boldsymbol{b}$

expressed in linear combination of column vectors

⟺ $\boldsymbol{b}$ is in the column space of $\boldsymbol{A}$

$$\boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots + x_n\boldsymbol{c}_n = \boldsymbol{b}$$

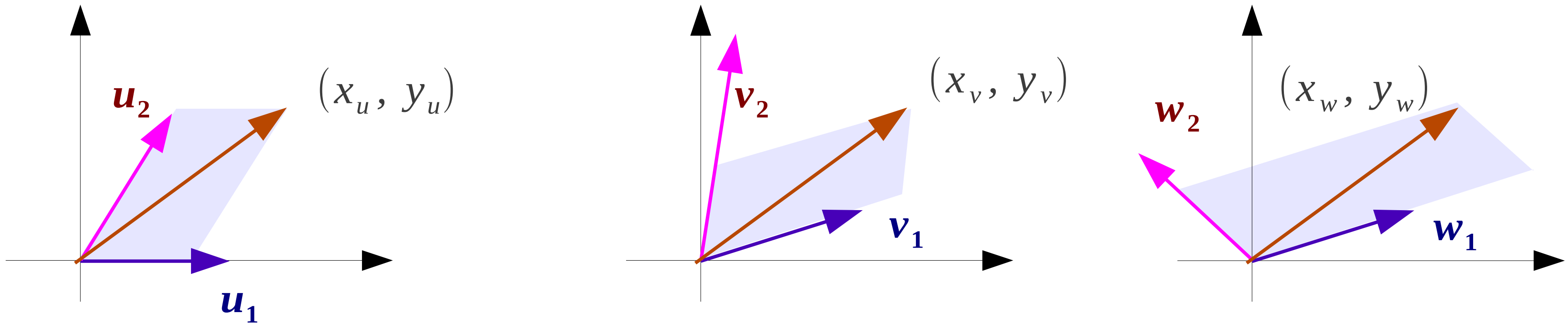
# Dimension

**In a finite-dimensional vector space** $R^n$ $\cancel{R^\infty}$

all bases ➡ the same number of vectors $n$

many bases but the same number of basis vectors

basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ $R^2$ basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ $R^2$ basis $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ $R^2$



**The dimension of a finite-dimensional vector space V**

dim(V) ⟺ the number of vectors in a basis

# Dimension of a Basis (1)

**In vector space** $R^2$

| | | | | | | |
|---|---|---|---|---|---|---|
| | any one vector | (linearly indep.) | ~~**spans**~~ | $\xcancel{R^2}$ | line through **0** | |
| basis | any two non-collinear vectors | (linearly indep.) | **spans** | $R^2$ | plane | ⬅ |
| | any three or more vectors | ~~(linearly indep.)~~ | **spans** | $R^2$ | plane | |

**In vector space** $R^3$

| | | | | | | |
|---|---|---|---|---|---|---|
| | any one vector | (linearly indep.) | ~~**spans**~~ | $\xcancel{R^3}$ | line through **0** | |
| | any two non-collinear vectors | (linearly indep.) | ~~**spans**~~ | $\xcancel{R^3}$ | plane through **0** | |
| basis | any three vectors<br>non-collinear, non-coplanar | (linearly indep.) | **spans** | $R^3$ | 3-dim space | ⬅ |
| | any four or more vectors | ~~(linearly indep.)~~ | **spans** | $R^3$ | 3-dim space | |

# Dimension of a Basis (2)

**In vector space** $R^n$

| | | | | |
|---|---|---|---|---|
| | any n-1 vectors | (linearly indep.)? | ~~spans $R^n$~~ | line through **0** |
| basis | n vectors of a basis | (linearly indep.) | **spans** $R^n$ | plane |
| | any n+1 vectors | ~~(linearly indep.)~~ | **spans**? $R^n$ | plane |

a finite-dimensional vector space $V$

a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$

- a set of more than n vectors ⇒ ~~(linearly indep.)~~
- a set of less than n vectors ⇒ ~~**spans** $V$~~

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ non-empty finite set of vectors in $V$

$S$ is a basis ⇔
- $S$ linearly independent
- $S$ spans $V$

# Basis Test

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ non-empty finite set of vectors in $V$

$S$ is a basis ⟺ $\begin{cases} S \text{ linearly independent} \\ S \text{ spans } V \end{cases}$

$V$ an **n**-dimensional vector space

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ a set of **n** vectors in V

$S$ linearly independent ⇒ $S$ is a basis

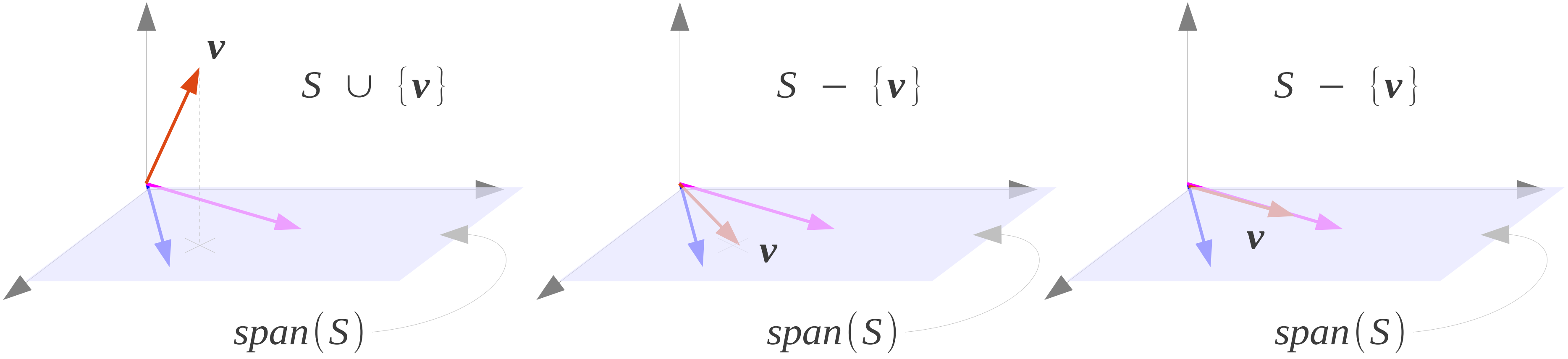
$S$ spans $V$ ⇒ $S$ is a basis

# Plus / Minus Theorem

$S$ a nonempty set of vectors in a vector space $V$

$\left\{ \begin{array}{l} S \text{ : linear independent} \\ \boldsymbol{v} \text{ a vector in V but outside of span(S)} \end{array} \right.$ ➡ $S \cup \{\boldsymbol{v}\}$ : linear independent

$\left\{ \begin{array}{l} \boldsymbol{v}, \boldsymbol{u_i} \in S \quad \text{linear combination} \\ \boldsymbol{v} = k_1 \boldsymbol{u_1} + k_2 \boldsymbol{u_2} + \cdots + k_n \boldsymbol{u_n} \end{array} \right.$ ➡ $span(S) = span(S - \{\boldsymbol{v}\})$



# Finding a Basis

$S$ a nonempty set of vectors in a vector space $V$

$\begin{cases} S & \text{: linear independent} \\ \boldsymbol{v} & \text{a vector in V but outside of span(S)} \end{cases}$ ➡ $S \cup \{\boldsymbol{v}\}$ : linear independent

if S is a *linearly independent* set that is *not already a basis* for V,
then S can be *enlarged* to a basis for V
by ***inserting*** appropriate vectors into S

$\begin{cases} \boldsymbol{v},\ \boldsymbol{u_i} \in S & \text{linear combination} \\ \boldsymbol{v} = k_1\boldsymbol{u_1} + k_2\boldsymbol{u_2} + \cdots + k_n\boldsymbol{u_n} \end{cases}$ ➡ $span(S) = span(S - \{\boldsymbol{v}\})$

if S *spans* V but is *not a basis* for V,
then S can be *reduced* to a basis for V
by ***removing*** appropriate vectors from S

# Vectors in a Vector Space

$S$ a nonempty set of vectors in a vector space $V$

if S is a *linearly independent* set that is *not already a basis* for V,
then S can be *enlarged* to a basis for V
by ***inserting*** appropriate vectors into S

Every *linearly independent* set in a subspace is
either a **basis** for that subspace
or can be **extended to a basis** for it

if S *spans* V but is *not a basis* for V,
then S can be *reduced* to a basis for V
by ***removing*** appropriate vectors from S

Every *spanning set* for a subspace is
either a **basis** for that subspace
or has a **basis as a subset**

# Dimension of a Subspace

$W$ a subspace of a finite-dimensional vector space $V$

W is *finite-dimensional*

dim(W) ≤ dim(V)

W = V ⟺ dim(W) = dim(V)

# Rank and Nullity

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

**ROW Space** **subspace** of $R^n$

$= span\{\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_m\}$

**COLUMN Space** **subspace** of $R^m$

$= span\{\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n\}$

**NULL Space** **subspace** of $R^n$ **solution space** $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$

**Invertible A** $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{0} = \boldsymbol{0}$ only trivial solution

**Non-invertible A** zero row(s) in a RREF free variables parameters *s, t, u, …*

dim(row space of A) = dim(column space of A) = **rank**(A)

dim(null space of A) = **nullity**(A)

# Solution Space of **Ax=0**

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

the same case

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

General Solution of $\mathbf{A\,x = 0}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

dim(row space of A)
dim(col space of A)

**rank**(A) = 2

**rank**(A) = 1

dim(null space of A)

**nullity**(A) = 1

**nullity**(A) = 2

# Elementary Row Operation (1)

**ROW Space** **subspace** of $R^n$

$= span\{\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m\}$

**COLUMN Space** **subspace** of $R^m$

$= span\{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}$

**NULL Space** **subspace** of $R^n$

**solution space** $A\mathbf{x} = \mathbf{0}$

free variables parameters *s, t, u, …*

Elementary row operations do not change the **null space** of a matrix

Elementary row operations do not change the **row space** of a matrix

Elementary row operations ***do* change** the **col space** of a matrix

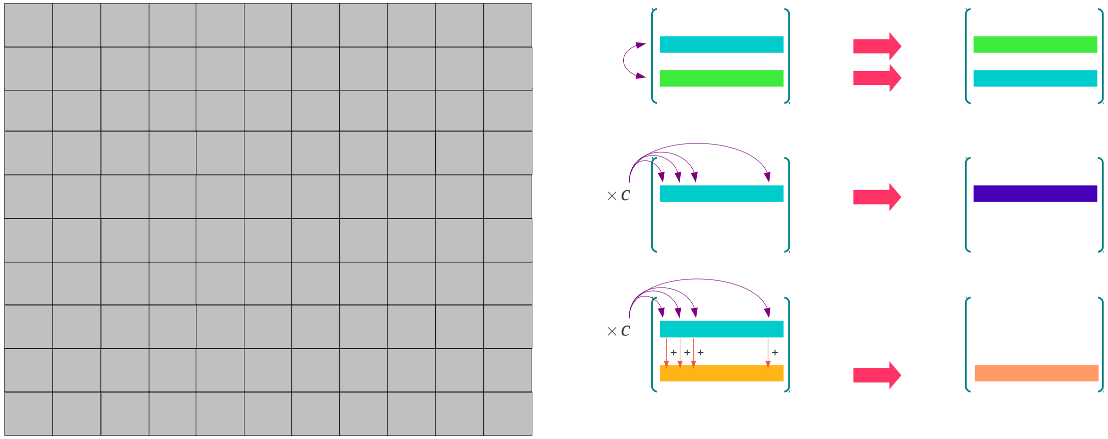
Elementary row operations do not change
the **linear dependence** and **linear independence** relationship
among column vectors

# Elementary Row Operation (2)

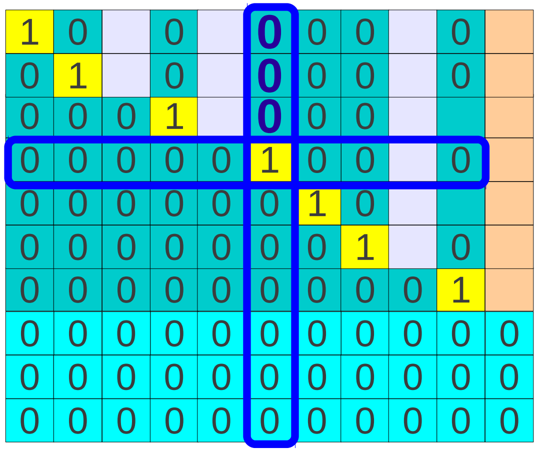
Elementary row operations do <u>not change</u> the **null space** of a matrix
Elementary row operations do <u>not change</u> the **row space** of a matrix
Elementary row operations do <u>not change</u> the **linear dependence** and **linear independence** relationship among column vectors

Elementary row operations <u>***do*** **change**</u> the **col space** of a matrix

**A**



**R**



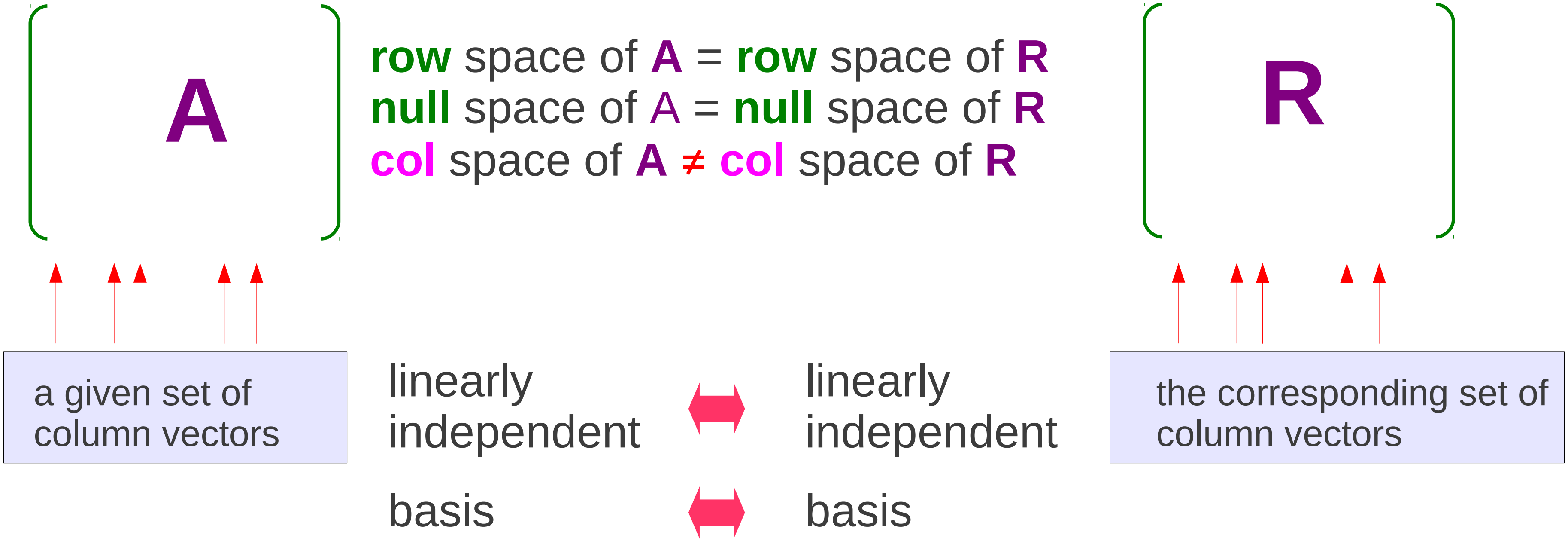
| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | | 0 | | 0 | 0 | 0 | | 0 | |
| 0 | 1 | | 0 | | 0 | 0 | 0 | | 0 | |
| 0 | 0 | 0 | 1 | | 0 | 0 | 0 | | | |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | | 0 | |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | 0 | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

# Elementary Row Operation (3)

Elementary row operations

- do not change the **null space** of a matrix
- do not change the **row space** of a matrix
- do not change the **linear dependence** and **linear independence** relationship among column vectors

- ***do* change** the **col space** of a matrix



**row** space of **A** = **row** space of **R**
**null** space of A = **null** space of **R**
**col** space of **A** ≠ **col** space of **R**

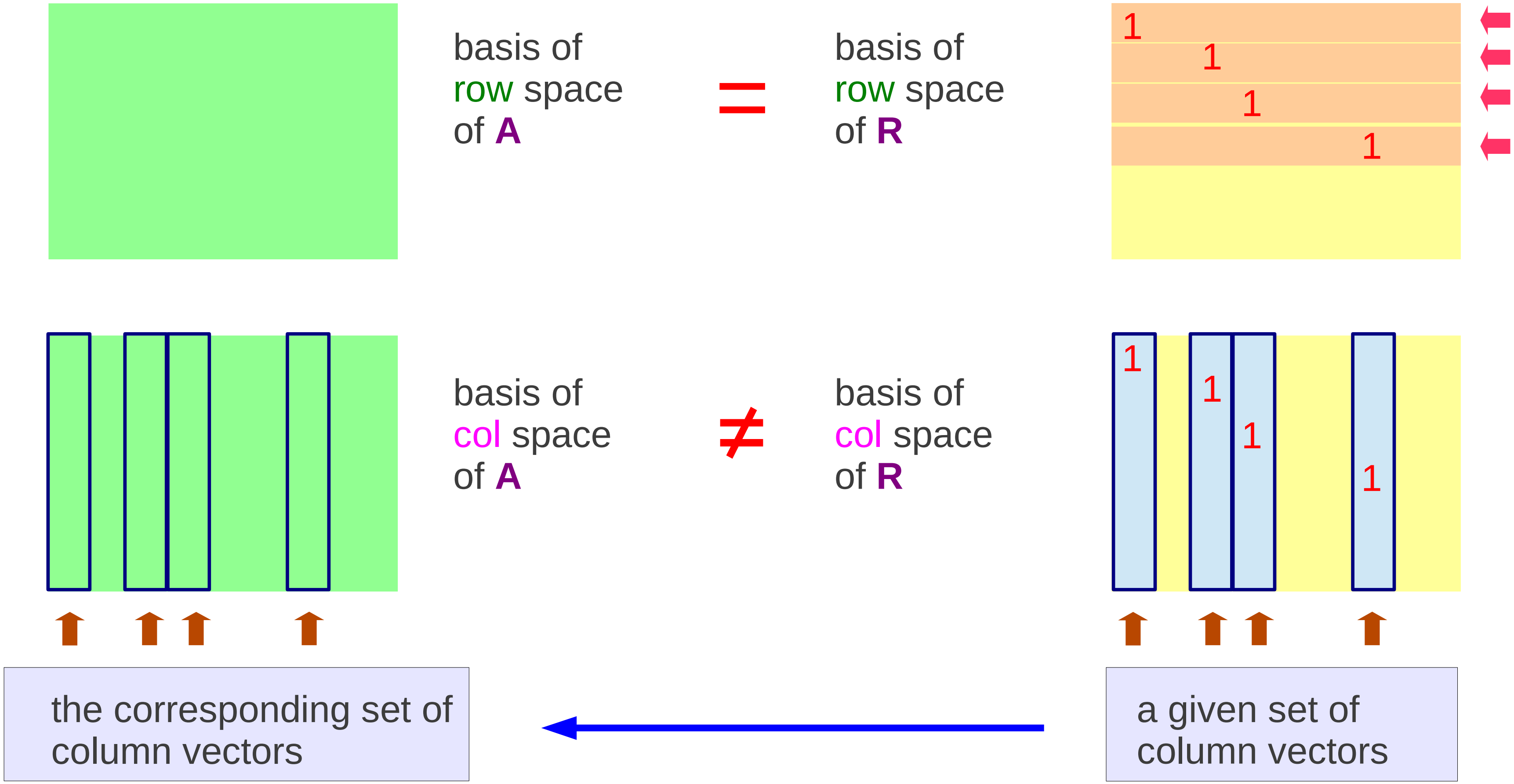
a given set of column vectors

linearly independent ↔ linearly independent

basis ↔ basis

the corresponding set of column vectors

# Bases of Row & Column Spaces (1)



basis of
row space
of **A**

=

basis of
row space
of **R**

basis of
col space
of **A**

≠

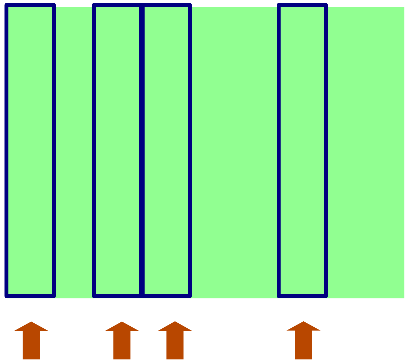
basis of
col space
of **R**

the corresponding set of
column vectors

a given set of
column vectors

dim(row space of A) = dim(column space of A) = **rank**(A)

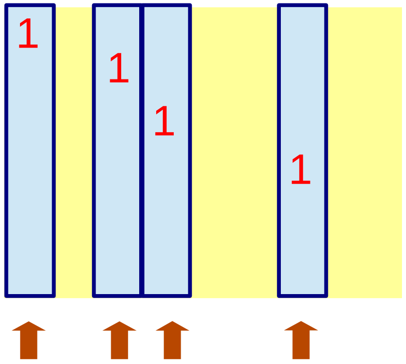
# Bases of Row & Column Spaces (2)



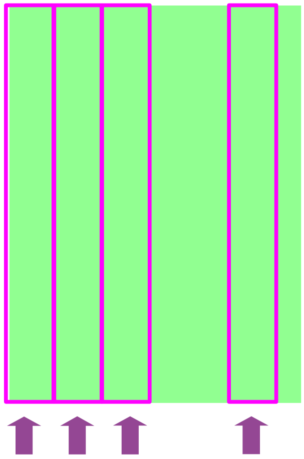
basis of
col space
of **A**

$\neq$

basis of
col space
of **R**



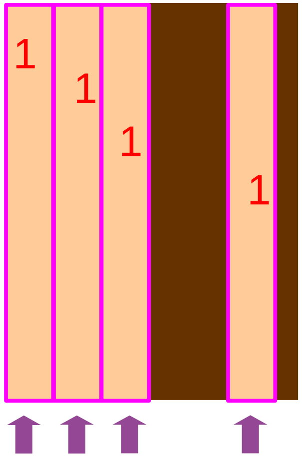
the basis consisting of columns of **A**



basis of
col space
of **A**

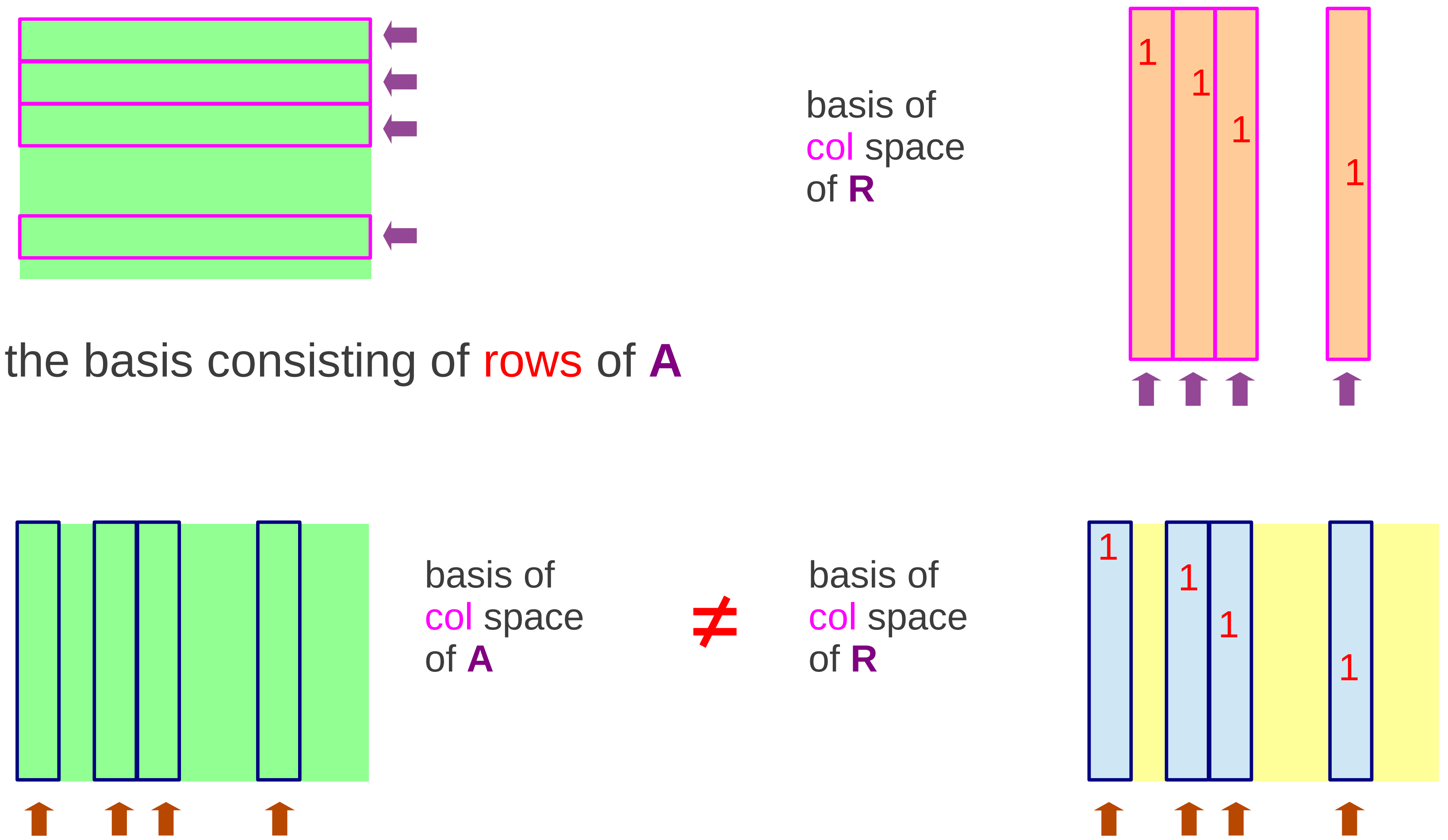
$\neq$

basis of
col space
of **R**



the basis consisting of rows of **A**

# Bases of Row & Column Spaces (3)



basis of
col space
of **R**

the basis consisting of rows of **A**

basis of
col space
of **A**

≠

basis of
col space
of **R**

the basis consisting of columns of **A**

# General Solution of **Ax=b** (1)

Non-Homogeneous Linear System $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$

Homogeneous Linear System $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{0}$

$\boldsymbol{A} : m \times n$

**a particular solution**

$\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$

**solution set** consists of all vectors in $R^n$ that are **orthogonal** to every row vector of $\boldsymbol{A}$

**+**

**a <u>particular</u> solution** $\boldsymbol{x}_0$ $\boldsymbol{A} \cdot \boldsymbol{x}_0 = \boldsymbol{b}$

The general solution of a consistent linear system can be written as

the sum of <u>a particular solution</u> of **Ax=b** and <u>the general solution</u> of **Ax=0**

# General Solution of **Ax=b** (2)

Any solution of a **consistent** linear system $A \cdot x = b$ $\quad x_0$

A **basis** for the **null space** (solution space $A \cdot x = 0$ ) $\quad S = \{ v_1, v_2, \cdots, v_k \}$

Every solution of $A \cdot x = b$

⇒ in the form $x = x_0 + c_1 v_1 + c_2 v_2 + \cdots , c_1 v_k$

$x$ is a solution of $A \cdot x = b$

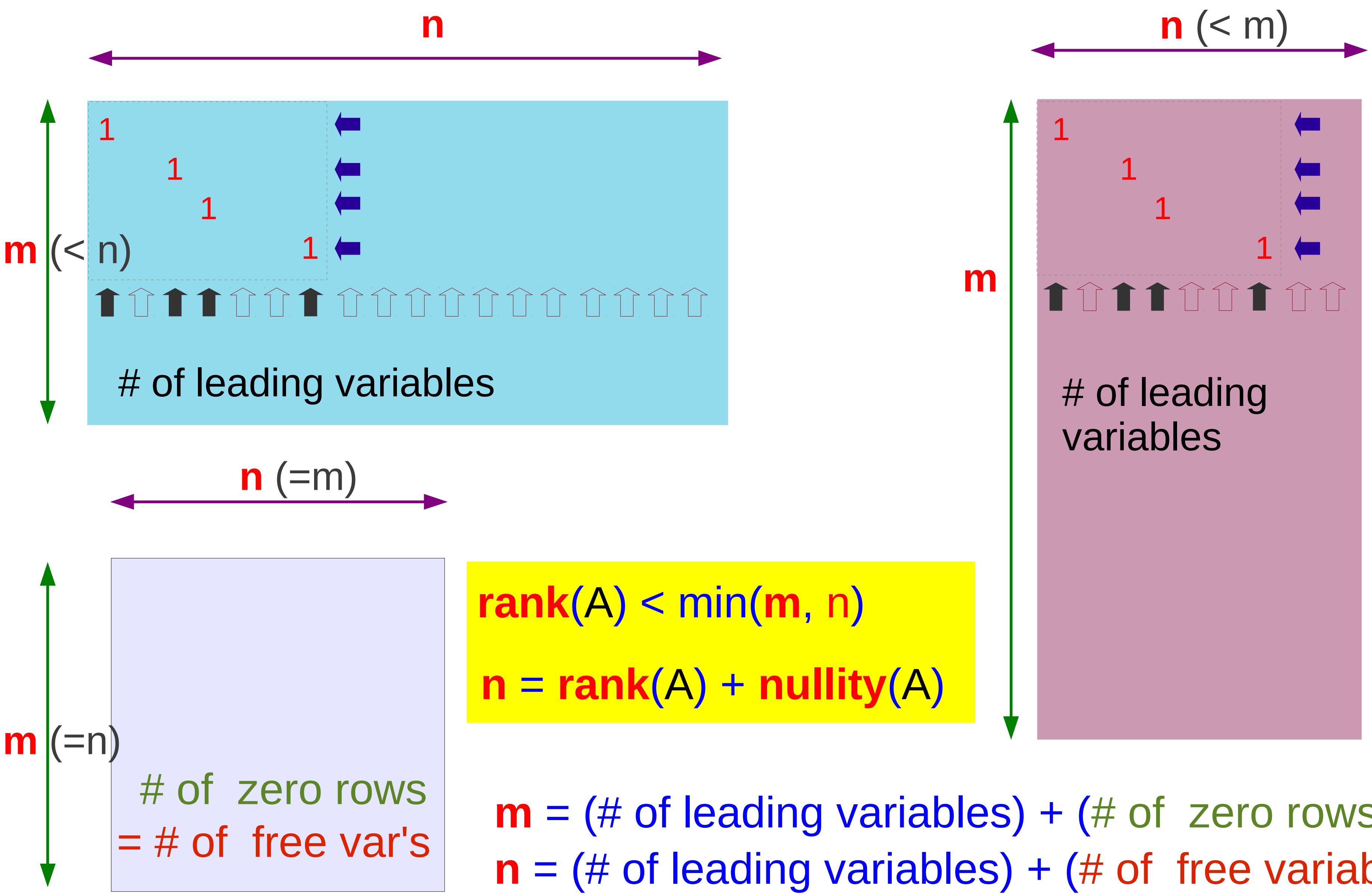
⇐ $x = x_0 + c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$

for all choices of scalars $c_1, c_2, \cdots c_k$

The general solution of a consistent linear system can be written as

the sum of a particular solution of **Ax=b** and the general solution of **Ax=0**

# Rank and Nullity



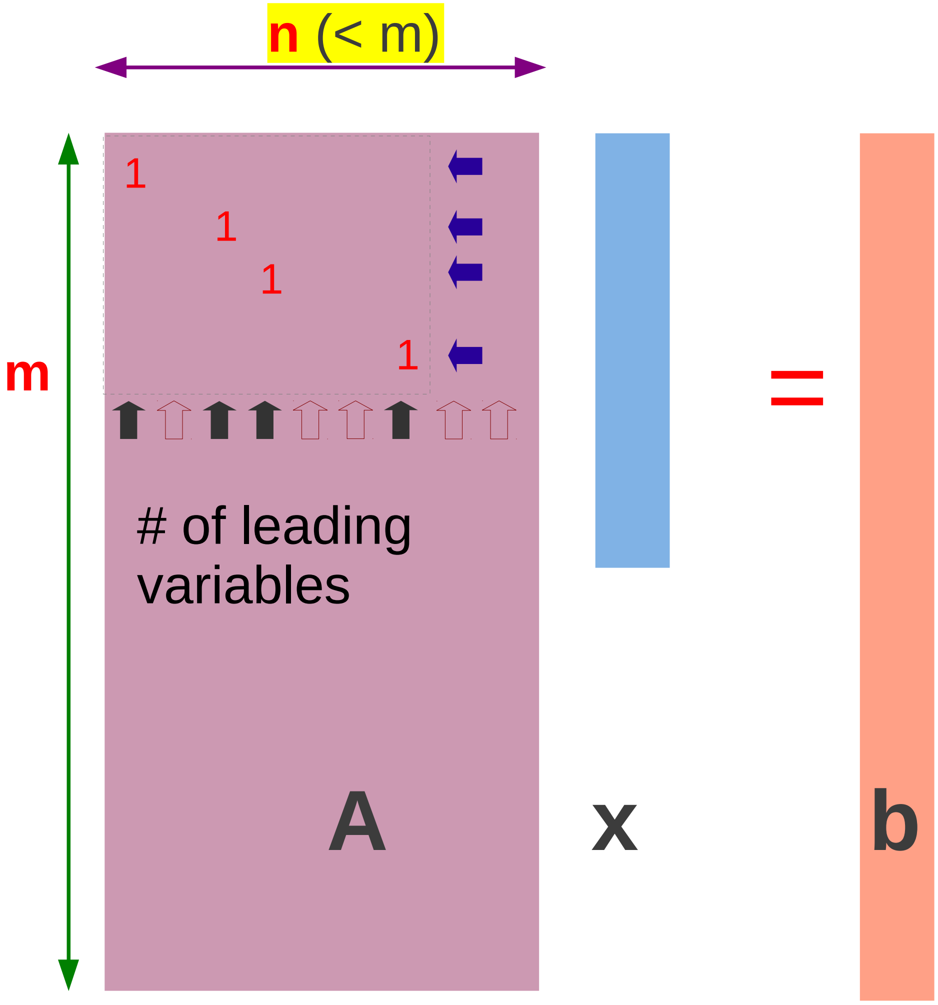
**rank**(A) < min(**m**, n)

**n** = **rank**(A) + **nullity**(A)

**m** = (# of leading variables) + (# of zero rows)

**n** = (# of leading variables) + (# of free variables)

# Overdetermined System



**n** column vectors
can span at most $R^n$

**b** is in $R^m$ $R^m \supset R^n$

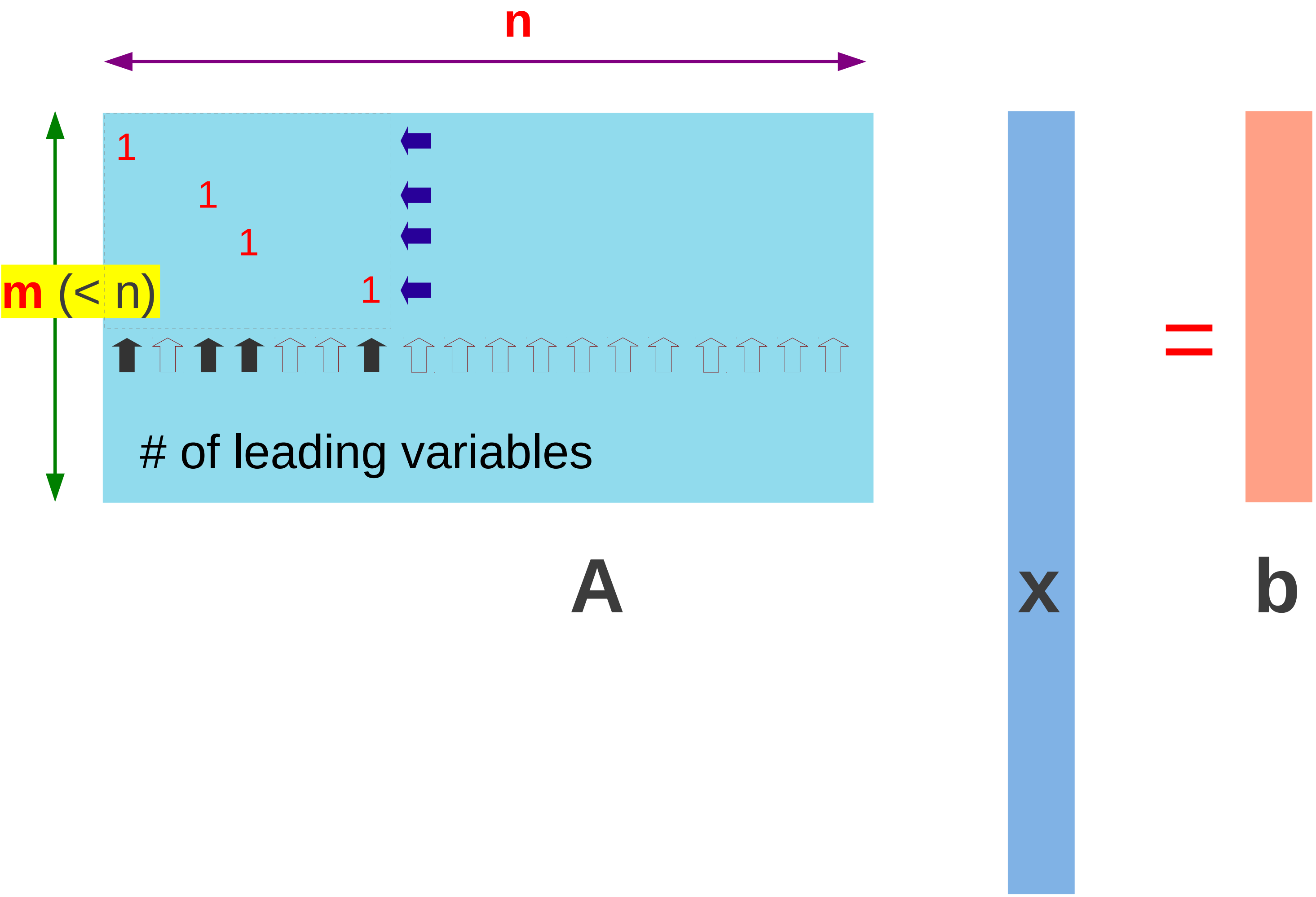
At least one vector **b** in $R^m$
does not lie in column space

For such **b** in $R^m$
**Ab = b** inconsistent

$$\boldsymbol{A}\boldsymbol{x} = x_1 \boldsymbol{c}_1 + x_2 \boldsymbol{c}_2 + \cdots + x_n \boldsymbol{c}_n \cancel{=} \boldsymbol{b}$$

**n** column vectors can span at most $R^n$ **b** is in $R^m$

# Underdetermined System



can span at most R$^n$

## References

[1] http://en.wikipedia.org/
[2] Anton, et al., Elementary Linear Algebra, 10th ed, Wiley, 2011
[3] Anton, et al., Contemporary Linear Algebra,