Derivatives (2A)

- Partial Derivative
- Directional Derivative
- Tangent and Normal Planes

Young Won Lim
9/12/12

Copyright (c) 2012 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Partial Derivatives

Function of one variable $y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Function of two variable $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

treating y as a constant

$$\frac{\partial z}{\partial y} = \lim_{\Delta x \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

treating x as a constant

Partial Derivatives Notations

Function of one variable $y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Function of two variable $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = z_x = f_x$$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

treating y as a constant

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = z_y = f_y$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta x \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

treating x as a constant

Higher-Order & Mixed Partial Derivatives

Second-order Partial Derivatives

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Third-order Partial Derivatives

$$\frac{\partial^3 z}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) \qquad \frac{\partial^3 z}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y^2}\right)$$

Third-order Partial Derivatives

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

Chain Rule (1)

Function of two variable

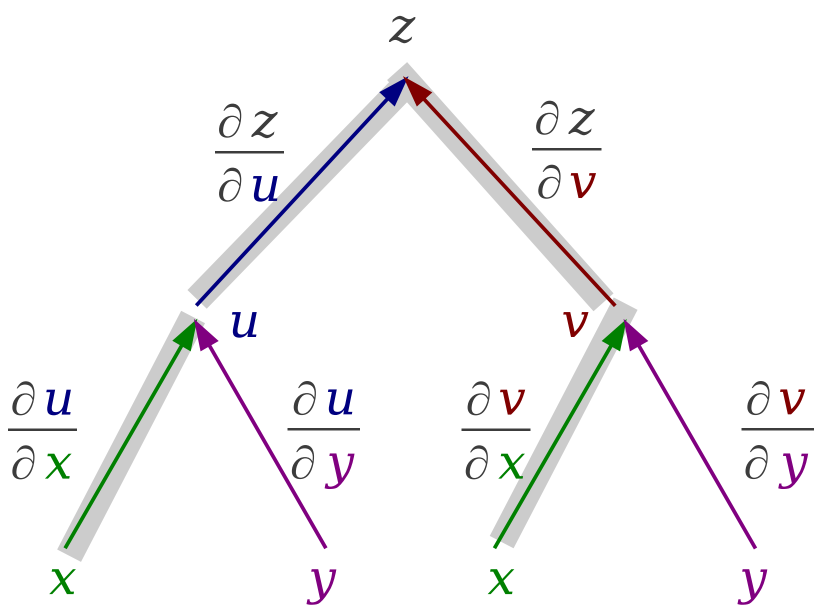
$$z = f(u, v)$$

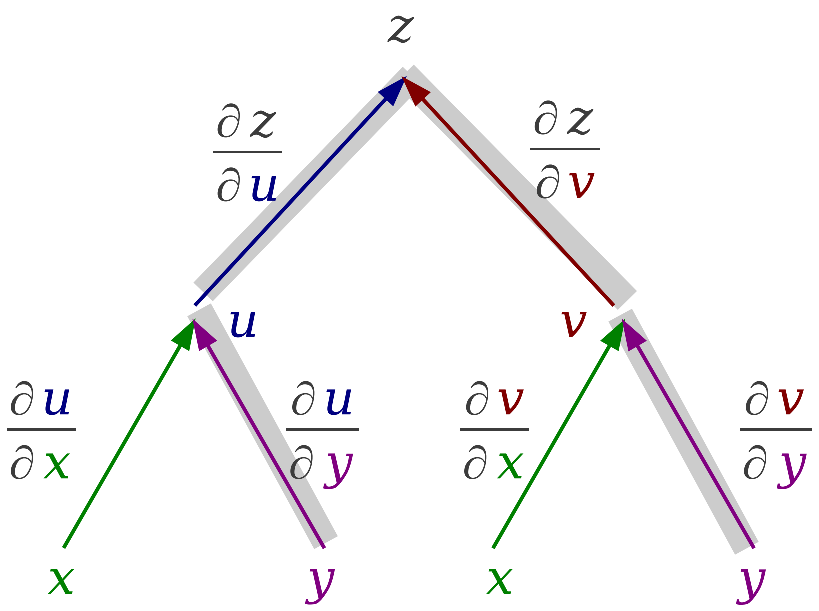
$$u = g(x, y) \qquad v = h(x, y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

Chain Rule

Function of two variable $z = f(u, v)$





$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

Directional Derivatives

Function of two variable $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

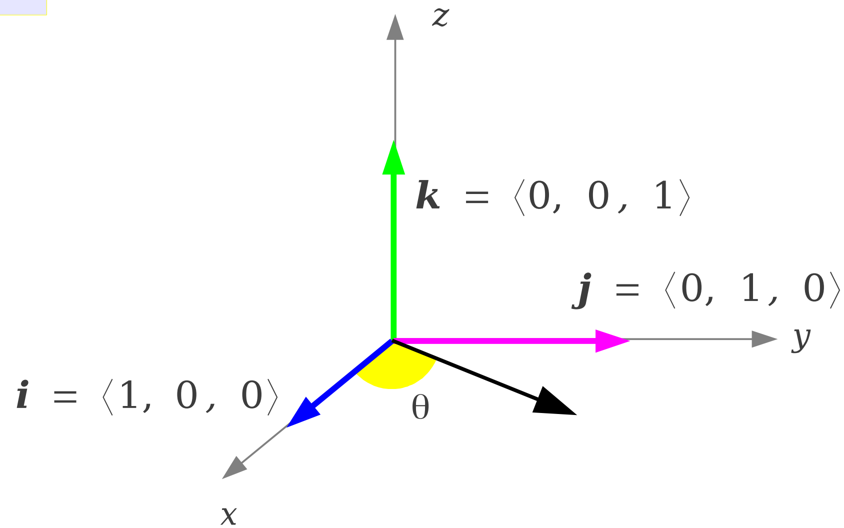
Rate of change of **f** in the **i** direction

$$\frac{\partial z}{\partial y} = \lim_{\Delta x \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

Rate of change of **f** in the **j** direction

Rate of change of **f** in the **u** direction

→ value



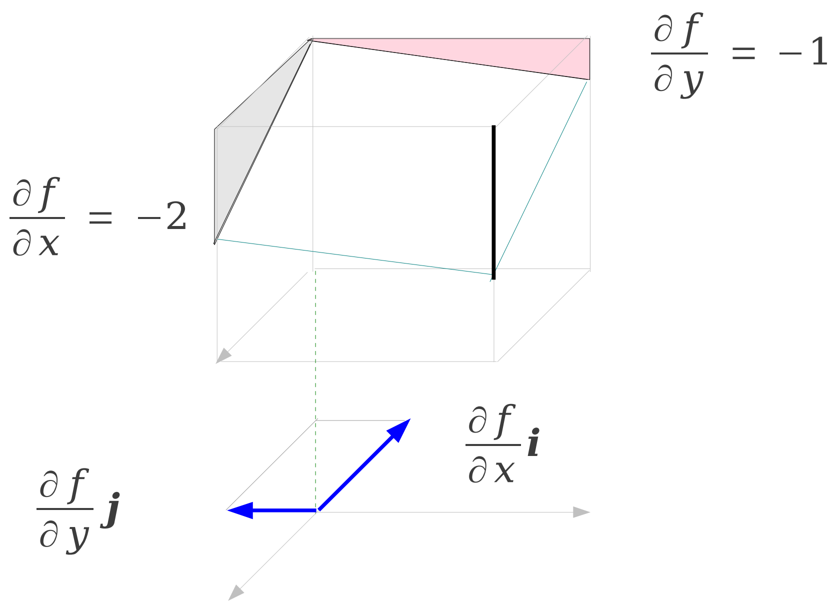
Gradient of a 2 Variable Function

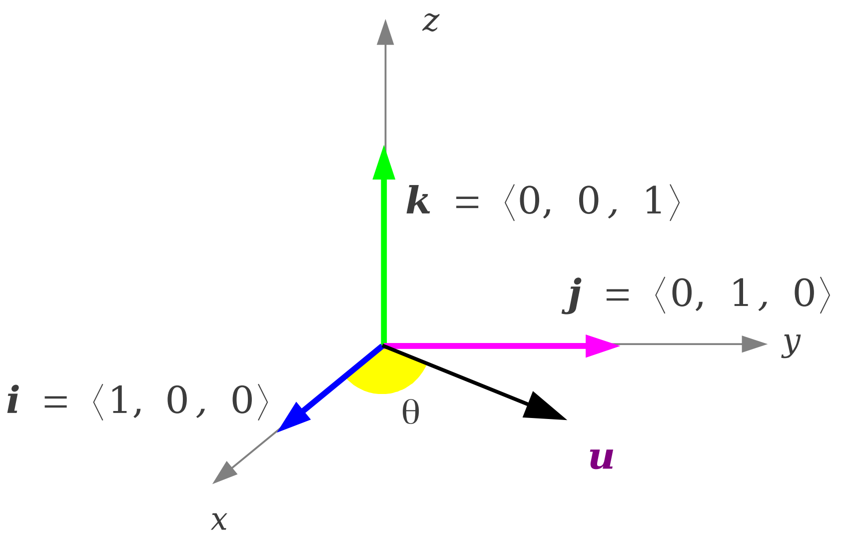
Function of two variable $f(x, y)$

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$$

⇨ vector

Rate of change of **f** in the **u** direction





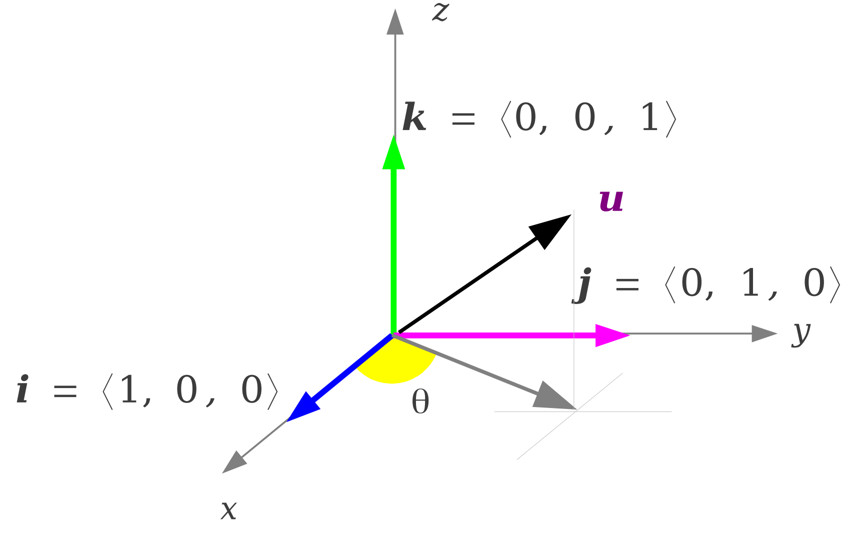
Gradient of a 3 Variable Function

Function of three variable $F(x, y, z)$

$$\nabla F(x,y,z) = \frac{\partial F}{\partial x}\boldsymbol{i} + \frac{\partial F}{\partial y}\boldsymbol{j} + \frac{\partial F}{\partial z}\boldsymbol{k}$$

⇨ vector

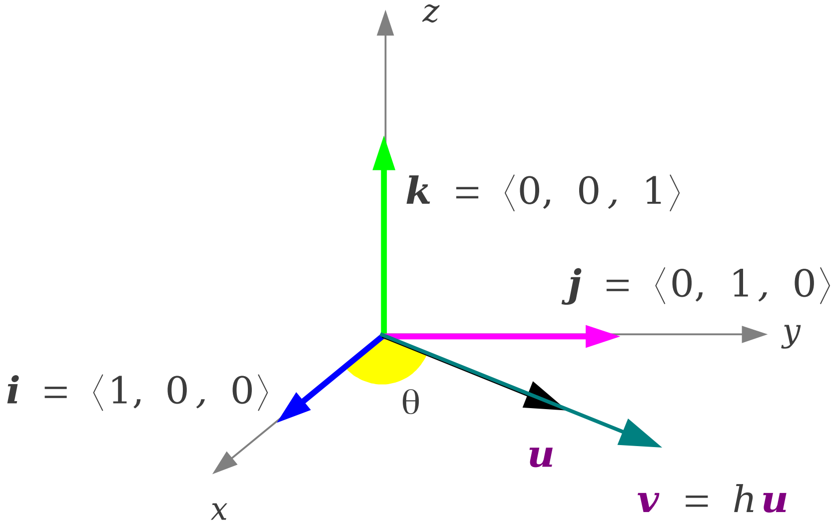
Rate of change of **f** in the **u** direction



General Partial Differentiation

Function of two variable $f(x, y)$

$$(x,\ y,\ 0) \xrightarrow[\mathbf{v} = h\mathbf{u}]{h = \sqrt{(\Delta x)^2 + (\Delta y)^2}} (x+\Delta x,\ y+\Delta y,\ 0)$$



Rate of change of **f** in the **u** direction

$$\frac{f(x+\Delta x,\ y+\Delta y) - f(x,\ y)}{h}$$

$$\frac{f(x + h\cos\theta,\ y + h\sin\theta) - f(x,\ y)}{h}$$

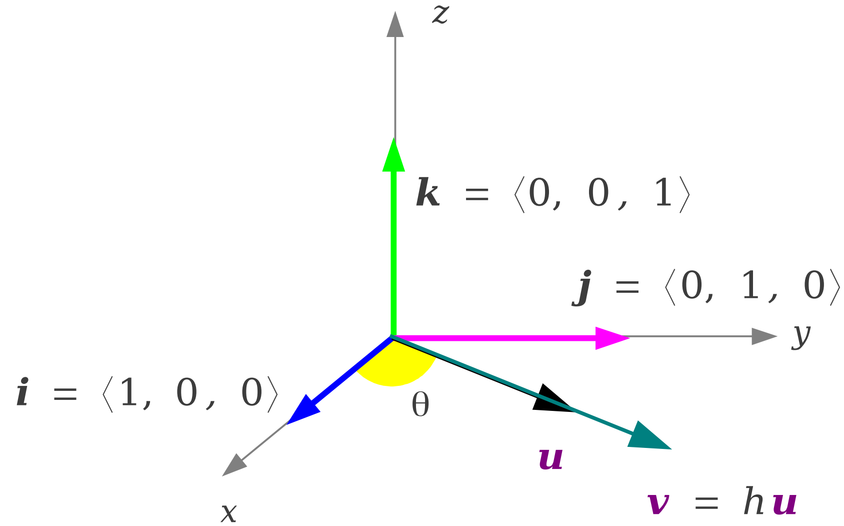
Directional Derivative

Function of two variable $f(x, y)$

$$h = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$(x,\ y,\ 0) \longrightarrow (x+\Delta x,\ y+\Delta y,\ 0)$$

$$\boldsymbol{v} = h\,\boldsymbol{u}$$



Rate of change of **f** in the **u** direction

$$D_{\boldsymbol{u}}\, f(x,y) = \lim_{h \to 0} \frac{f(x + h\cos\theta,\ y + h\sin\theta) - f(x,\ y)}{h}$$

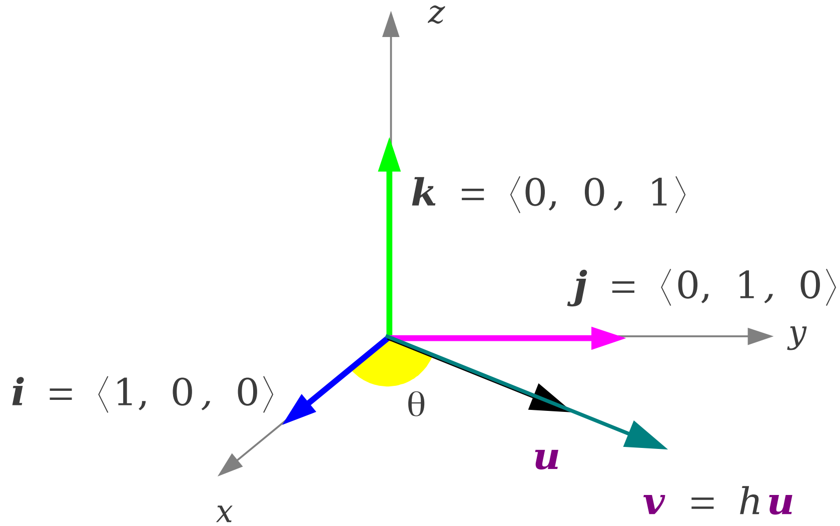
Computing Directional Derivative

Function of two variable $f(x, y)$

$$h = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$(x,\ y,\ 0) \longrightarrow (x+\Delta x,\ y+\Delta y,\ 0)$$

$$\boldsymbol{v} = h\boldsymbol{u}$$



$$\boldsymbol{u} = \cos\theta\ \boldsymbol{i} + \sin\theta\ \boldsymbol{j}$$

$$D_{\boldsymbol{u}} f(x,y) = \nabla f(x,y) \cdot \boldsymbol{u}$$

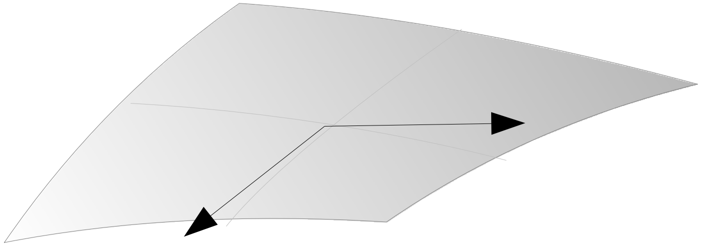
⇨ value

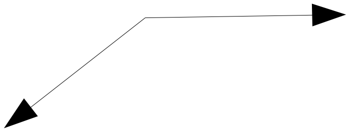
Rate of change of **f** in the **u** direction

Chain Rule

Function of two variable $y = f(u, v)$

$u = g(x, y)$ $v = h(x, y)$





Line Equations (2)

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] D.G. Zill, “Advanced Engineering Mathematics”